JTB-EPISTEMOLOGY AND THE GETTIER PROBLEM IN THE FRAMEWORK OF TOPOLOGICAL EPISTEMIC LOGIC

Abstract. Traditional epistemology of knowledge and belief can be succinctly characterized as JTB-epistemology, i.e., by the thesis that knowledge is justified true belief. Since Gettier's trail-blazing paper of 1963 this account has become under heavy attack. The aim of is paper is to study the Gettier problem and related issues in the framework of topological epistemic logic. In the framework of topological epistemic logic Gettier situation necessarily occur for formal reasons. On the other hand, there is a special class of topological models (based on so called nodec spaces) for which traditional JTB-epistemology is valid, i.e., for which no Gettier situations occur. Further, for each topological model of Stalnaker's combined logic KB of knowledge and belief a canonical JTB-model (its JTB-doppelganger) can be constructed that shares many structural properties with the original model but is free of Gettier situations. A topological model and its JTB-doppelganger both share the same justified belief operator and have very similar knowledge operators. Succinctly, the JTB-account of epistemology amounts to a simplification of a more complex epistemological account of knowledge and (justified true) belief that assumes that these two concepts may differ. The JTB-account of knowledge and belief assumes that the epistemic agent's cognitive powers are rather large. This is the reason why in the JTB-epistemology Gettier cases do not occur. Eventually, it is shown that for all topological models of Stalnaker's KB-logic Gettier situations are topologically characterized as nowhere dense situations. This entails that Gettier situations are epistemologically invisible in the sense that they can neither be known nor believed with respect to the knowledge operator and the belief operator of the models involved.

Key words: Topological epistemic logic, Idealized models, JTB-epistemology, Gettier problem, Gettier worlds, Justified belief, Epistemic and doxastic invisibility.

1. Introduction. The use of formal, often mathematical, models is ubiquitous in the natural and social sciences. There is no reason why in philosophy, understood as a science in a broad sense, this should be otherwise. According to Williamson:

> The aim of using models is to gain insight into phenomena by studying how they work under simplified, rigorously described conditions that enable us to apply mathematical or quasi-mathematical reasoning that we cannot apply directly to the phenomena in the wild. (Williamson 2013, 131)

The proof of the pudding is in the eating, of course. Using mathematical or other idealized models in philosophy is not a foolproof method for obtaining philosophically interesting results. Rather, mathematical philosophy is always in danger to indulge in mere mathematical window-dressing in order to appear "more scientific" without substantial philosophical content. This is a classical problem of any mathematical (and more generally formal) philosophy already the founding father of this philosophical current of was aware of:

> The acceptance or rejection of abstract linguistic forms, just as the acceptance of rejection of any other linguistic forms in any branch of science, will finally be decided by their efficiency as instruments, the ratio of the results achieved to the amount and complexity of the efforts required. (Carnap 1950, 40)

The aim of this paper is to investigate the fundamental epistemological problem of "What is Knowledge?" put on the agenda of modern epistemology by Gettier's classical paper Gettier (1963) by using the conceptual tools of topological epistemology.[1]

In the last fifty years Gettier's short paper has generated an extremally rich literature mainly engaged in the invention of ever more sophisticated thought experiments that aimed to refute the classical analysis of knowledge as true justified belief (cf. Turri 2012, Borges et al. 2017). The present paper proposes to tackle the issue from a different angle. Following Williamson (2013, 2015) I propose to address the Gettier problem and related issues from the perspective

[1] The topological concepts used in the following will be explained in detail in section 2.

of topological epistemology. This move provides a variety of formal models of knowledge and belief that can be used to deal with this kind of problems:

> Natural formal models of knowledge and justified belief (in a non-factive sense) provide robust evidence against JTB, independently of thought experiments in any distinctive sense, but in a way closely related to Gettier's original arguments. (Williamson 2015, 139)

Succinctly, JTB is the classical epistemological account according to which knowledge can be identified with justified true belief. How JTB is related to topological epistemology? The general answer is that topological epistemology investigates topological models of knowledge, belief, and possibly further epistemic concepts by modeling them as appropriately chosen topological operators. Then the basic question of how JTB can be dealt with in the framework of topological epistemology is to ask the question whether there are topological models for which knowledge is justified true belief or even all (reasonable) topological models satisfy this classical thesis.

Nowadays, the most prominent topological account of knowledge is Stalnaker's "combined logic of knowledge and belief" KB where knowledge is represented as the interior kernel operator Int of a topological space (X, OX), with OX = {Int(A); A∈PX}, PX being the power set of X and A ∈ PX to be conceived as a proposition of classical propositional logic. While rather unanimously knowledge is modeled by the interior operator Int, it is not so clear, how to define other epistemic operators such as belief. In Stalnaker's well-known account of combined logic KB of knowledge and belief (restricted to extremally disconnected spaces (ED-spaces)) belief is represented as ClInt[2]. As Baltag et al. (2019) have shown the operator ClInt works quite well as a belief operator for extremally disconnected spaces (ED-spaces). For general

[2] Here, Cl is the topological closure operator, defined by Cl(A) := Int(A^C)^C. IntClInt is the concatenation of the operators Int and Cl.

topological models, however, ClInt is not a good belief operator. Already in Stalnaker (2006) it is observed that for general topological spaces ClInt is not even a normal operator. Hence, as will be argued in this paper, for general topological models ClInt should be replaced by IntClInt. On ED-spaces, ClInt and IntClInt coincide and on general topological spaces IntClInt preserves almost all qualities of a plausible belief operator that ClInt exhibits on ED-spaces. In particular, as will be explained in more detail in section 3, IntClInt(A) can interpreted as justified belief B. This makes it possible to use to use topological epistemology to dealing with Gettier problem and related issues. Interpreting the operators Int and IntClInt as knowledge and justified belief, respectively, leads us to express the JTB-account of epistemology succinctly by the following identity:

(1.1) $$\mathrm{Int}(A) = A \cap \mathrm{IntClInt}(A) \qquad \text{(JTB)}$$

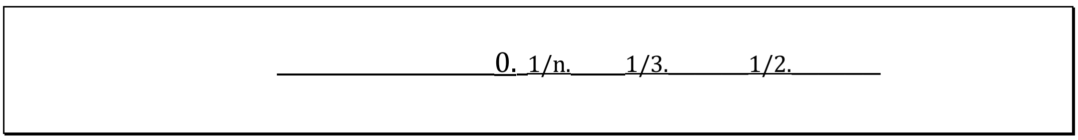
for all $A \in PX$. Informally expressed (1.1) asserts that knowing that A is equivalent to the conjunction that A obtains (is true) and that it is believed with justification that A. For the extreme situations of total ignorance and omniscience (1.1) turns out to be true.

(1.2) Fact. The trivial topological spaces $(X, \{\varnothing, X\})$ and (X, PX) satisfy JTB.

On the other hand, it is also easily shown that there are topological models of knowledge and belief for which JTB is not valid. Perhaps the simplest model is based on the familiar metrical (Euclidean) topology of the real line $(\mathbb{R}, O\mathbb{R})$. One easily calculates that for the topological model $(\mathbb{R}, O\mathbb{R})$ JTB does not hold for all A:

(1.3) Proposition. For the topological model of knowledge and belief based on the universe of possible worlds of the Euclidean line $(\mathbb{R}, O\mathbb{R})$. For $A := \mathbb{R} - \{1/n, n \geq 1\}$ one obtains

$$\mathrm{Int}(A) = A - \{0\} \neq A \cap \mathrm{IntClint}(A) = A.$$



Proof: One has $0 \notin \text{Int}(A)$ since every open neighborhood U(0) of 0 contains an element 1/n for some n that is not contained in A. On the other hand, $\text{IntClInt}(A) = \mathbb{R}$, since all open neighborhoods of 0 and 1/n contain elements of Int(A).

Hence, the sets Int(A) and $A \cap \text{IntClint}(A)$ are different in general. In other words, $(\mathbb{R}, O\mathbb{R})$ does not satisfy JTB (1.1).♦

Topological epistemology is, so to speak, "undecided" with respect to JTB: some topological models satisfy JTB, others do not. Admittedly, the trivial topological models (1.2) of JTB hardly provide a convincing explanation of the fact that JTB played such a prominent role in traditional epistemology. More is needed to argue that JTB be at least topologically plausible. A means to improve the situation for JTB has been proposed by Parikh et al. (2007). There it is observed that for the special class of DSO-spaces[3] knowledge is defined by knowledge as Int and belief as the co-derived set operator t of the topological structure of these spaces. For this tandem of operators, one obtains $\text{Int}(A) = A \cap t(A)$ (cf. Baltag et al. (2019), Parikh et al. (2007)). Thus, DSO-spaces may be considered as honest non-trivial topological models for which JTB is true. DSO spaces are, however, a far cry from any familiar class of topological spaces. In sum, a more balanced and more complete presentation of JTB-models in topological epistemology is called for. The following considerations aim to provide just this. A particular aim of this paper is to explain the fact why the JTB-account has played and still plays such a prominent role and at the same time adduce some reasons why it nevertheless seems irremediably flawed.

The organization of this paper is as follows. The next two sections are preparatory for these purposes. In section 2 we introduce the necessary formal apparatus to deal with concepts of

[3] DSO-spaces are defined as dense-in-themselves spaces for which all derived sets d(A) are open (DSO = D(erived)S(ets)O(pen). The technical details of this definition need not interest us for the moment. It suffices to know that DSO-spaces exist (cf. Parikh et al. (2007)).

knowledge and belief in a topological framework. In section 3 we recall the basics of a topological version of Stalnaker's combined logic KB of knowledge and belief. This has been elaborated in detail by Baltag et al. in various publications (cf. Baltag et al. (2014, 2015, 2021) and elsewhere). Moreover, it is shown that Stalnaker's belief operator is to be understood as an operator that conceptualizes justified belief in quite a strong sense. Thus, Stalnaker's KB logic is directly relevant to the issue of JTB.

The central section 4 of this paper deals with three issues: first we show that there exist plenty of topological JTB-models, i.e., models that satisfy the axiom (1.1) characteristic for JTB. In standard language of topology, JTB holds for topological models based on topological spaces (X, OX) that are nodec[4] spaces. These spaces are defined by the special feature that all their nowhere dense subsets are closed. In other words, the class of nodec spaces may be considered as a niche for the JTB-account. Second, we will show that for every topological space (X, OX) whatsoever there exists a canonical nodec space (X, $O_{nod}X$). This nodec space (X, $O_{nod}X$) is a kind of doppelganger of (X, OX). That is, in a precise topological sense, (X, $O_{nod}X$) is very similar to the original topological space (X, OX) from which it is derived. Most topological models, however, are not JTB-models. The logic of nodec spaces is characterized as an extension of the standard modal logic S4, namely, by the Zeman extension S4.Zem of S4 (cf. Zeman (1969), Bezhanishvili et al. (2004)). The last topic treated in section 4 is the discussion of a topological analogue of the double-luck construction that has been used in many thought-experiments proposed by the "Gettier industry" as a device for providing ever more sophisticated examples of Gettier situations refuting the classical JTB-account. This topological construction shows that the topological account is able to emulate important aspects of the standard informal thought-experiments leading to Gettier situations.

[4] A space (X, OX) is a nodec iff all nowhere dense subsets of X are closed (NODEC = NO(where)DE(nse)C(losed)) (cf. van Douwen (1993, Definition and Fact 1.14, p. 129)).

In section 5 we show that for general topological models (X, OX) for which Gettier propositions exist these propositions can neither be known nor believed with respect to the epistemological operators Int and IntClInt that characterize the topological model (X, OX): Gettier cases are epistemically and doxastically invisible, i.e., if w is a world, for which a Gettier situation holds with respect to A, an agent who relies on the operators Int and IntClInt, neither knows nor consistently believes that w is an A-world. This doxastic invisibility of Gettier cases for general topological models may have contributed to the impression that the classical JTB account (for which no Gettier cases exist) appears to be correct without being so.

We conclude with some general remarks on further possible directions of research for topological epistemology in section 6.

2. The Topology of Knowledge and Belief. To set the stage, in this section we recall the basics of elementary set-theoretical topology that is needed for the formulation of the interior semantics for epistemic logic of knowledge and belief as presented by Baltag, Bezhanishvili, Özgün, and Smets (cf. Baltag et al. (2013, 2015, 2016, 2019)). This semantics will be used throughout the rest of this paper. First of all, recall the definition of a topological space:

(2.1) Definition. Let X be a set with power set PX. A topological space is an ordered pair (X, OX) with OX $\subseteq$ PX that satisfies the following conditions:

(i) $\emptyset$, X $\in$ OX.

(ii) OX is closed under finite set-theoretical intersections $\cap$ and arbitrary set-theoretical unions $\cup$.♦

The elements of OX are called the open sets of the topological space (X, OX). The set-theoretical complement A^C of an open set A $\subseteq$ X is called a closed set. The set of closed subsets of (X, OX) is denoted by CX. The interior kernel operator Int and the closure operator Cl of (X, OX) are defined as usual: The interior kernel Int(A) of a set A $\in$ PX is the largest open set that is contained in A; the closure Cl(A) of A is the smallest closed set containing A. For details, see Willard (2004), Steen and Seebach Jr. (1982), or any other textbook on set-theoretical topology).

Topologies (X, OX) on a set X can be partially ordered set-theoretically:

(2.2) Definition. Let (X, OX) and (X, O'X) be two topologies on X. OX is said to be coarser than O'X iff OX is a subset of O'X, i.e., OX $\subseteq$ O'X. If OX is coarser than O'X this is also expressed by saying that O'X is finer than OX.♦

Clearly, the coarsest topology on X is O_0X = {Ø, X} and the finest topology is O_1X = PX. For all topologies OX one has $O_0X \subseteq OX \subseteq O_1X$. The set TOP(X) := {OX; OX topology on X} of topologies of X endowed with the partial order $\subseteq$ is well known to be a complete lattice (TOP(X), $\subseteq$). The infimum of (TOP(X), $\subseteq$) is just set-theoretical intersection of topologies. The bottom element of (TOP(X), $\subseteq$) is {X, Ø} and the top element is (X, PX).

The epistemological interpretation of TOP(X) works as follows: OX $\in$ TOP(X) is to be interpreted as a cognitive agent who uses the interior kernel operator Int of as knowledge operator for its epistemic activity. More precisely, A$\in$ PX is interpreted as a propositions A. A is true at a world w $\in$ X iff w $\in$ A, otherwise, A is false in w. A proposition A entails a proposition D iff A is a subset of D, A $\subseteq$ D. The other Boolean operators on PX are to be interpreted as usual. A proposition A is known at a world x iff x $\in$ Int(A). The fact w $\in$ Cl(A) is to be interpreted as the fact that w is considered conceptually possible to be an A-world. Or, in a more agent-centered language, an epistemic agent knows that x is in A iff w belongs to int(A). The assertion that w is an A-world is to be considered as equivalent to the assertion that the proposition A is true in the world w.

The partial order $\subseteq$ on the lattice (TOP(X), $\subseteq$) has an obvious epistemological interpretation: If OX $\subseteq$ O'X $\in$TOP(X) an agent who uses OX has less knowledge than an agent who uses O'X. Moving from (X, OX) to (X, O'X) may be conceived as a learning process in which the epistemic agent enlarges its cognitive powers by extending its knowledge from OX to O'X. The maximal (discrete) topology O_1X may be interpreted as (trivial) omniscience with respect to the universe of possible worlds X.

The partial order $\subseteq$ on TOP(X) will be important in later sections to assess the relation between traditional JTB-epistemology (for which no Gettier situations exists) and modern epistemology that recognizes the existence of Gettier cases.

The operators Int and Cl are well-known to satisfy the Kuratowski axioms (cf. Kuratowski and Mostowski (1976)):

<u>(2.3) Proposition (Kuratowski Axioms)</u>. Let (X, OX) be a topological space, A, D $\in$ PX. Define the interior kernel operator Int of (X, OX) by Int(A) := $\cup${G ; G $\in$ OX and G $\subseteq$ A}. Dually, the closure operator Cl is defined by Cl(A) := $\cap${K; K $\in$ CX and A $\subseteq$ K}. Then Int and Cl satisfy the following axioms:

(i)	Int(A $\cap$ D) = Int(A) $\cap$ Int(D).	Cl(A $\cup$ D) = Cl(A) $\cup$ Cl(D).
(ii)	Int(Int(A)) = Int(A).	Cl(Cl(A)) = Cl(A).
(iii)	Int(A) $\subseteq$ A.	A $\subseteq$ Cl(A).
(iv)	Int(X) = X.	Ø = Cl(Ø).♦

In the following these axioms are used without explicit mention. Moreover, we will use freely the fact that the operators Int and Cl are inter-definable: Int(A) = $Cl(A^C)^C$ and Cl(A) = $Int(A^C)^C$.

Further, it is often expedient to conceive the operators Int and Cl as operators Int:PX→PX and Cl:PX→PX defined on PX in the obvious way. Hence, the concatenation of these operators makes perfect sense. In the following, concatenations such as IntCl, IntClInt etc. will play an important role. For later use, we note:

<u>(2.4) Proposition</u>. Let (X, OX) be a topological space with interior kernel operator Int, closure operator Cl, and A, D $\subseteq$ X.

(i) IntClIntCl(A) = IntCl(A) and ClIntClInt(A) = ClInt(A).

(ii) IntCl(Int(A) $\cap$ D) = IntClInt(A) $\cap$ IntCl(D).

<u>Proof</u>. The identities (i) are well known, (ii) is also well known for A, D $\in$ OX. For the following, however, we also need the stronger, but lesser known (ii). The point is that the identity (ii) even holds if D is <u>not</u> open. The proof of (2.3)(ii) can be found in Kuratowski and Mostowski (1976, Ch.I, §8).♦

<u>(2.5) Definition</u>. A subset Z of a topological space (X, OX) is <u>nowhere dense</u> iff IntCl(Z)) = Ø.♦

Informally expressed, nowhere dense subsets of (X, OX) are topologically “small” or “negligible”. Epistemologically, nowhere dense sets may be interpreted as propositions the conceptual possibility of which cannot be known.

Before we come to this issue, however, it is expedient to dwell at bit more upon the general problem of how the epistemological concept of belief is to be explicated topologically. This issue is less clear than the corresponding problem for knowledge.[5] Since Kuratowski (1922) it

[5] In the following we need not consider Steinvold’s proposal to conceive belief as the coderived set operator t of the topological structure, since this proposal only yields JTB-systems. In contrast, we are interested in general systems.

is known that there are exactly seven different combinations[6] of the topological operators Int and Cl:

(2.6) Id[7], Int, Cl, IntCl, CIint, IntClInt, ClIntCl

It is not directly obvious, however, whether any of the combinations of operators listed in (2.6) can be meaningfully interpreted as a formal topological model of belief. For instance, the closure operator Cl is certainly not a plausible candidate for a belief operator since the inclusion $A \subseteq Cl(A)$ (required by (2.3) (iii)) had to be interpreted as the assertion that if w is an A-world, i.e., $w \in A$, then it would be believed that w is an A-world. This is certainly not true for a realistic concept of belief: There are many facts that are not believed to be facts.

Further, the following four intuitively plausible conditions may be required to be satisfied by a "good" belief operator:

(2.7) Definition (Adequacy conditions for belief operators). Let (X, OX) be topological space of possible worlds, A, D $\in$ PX propositions. An operator B:PX$\rightarrow$PX can be interpreted as a good belief operator only if it satisfies for all A, D the following (in)equalities:

(i) NOT($A \subseteq B(A)$): There is a world w that is an A-world but that is not believed to be an A-world.
(ii) NOT ($B(A) \subseteq A$): There is a world w that is believed to be an A-world, but w is not an A-world.
(iii) $B(B(A)) = B(A)$: The proposition A is believed iff it is believed that A is believed.
(iv) $B(A \cap D) = B(A) \cap B(D)$: The conjunction of the propositions A and B is believed iff A is believed and D is believed.♦

[6] This means that there are topological spaces (X, OX) and A $\in$ PX such that the operators of (2.6) yield seven different results: A, Int(A), Cl(A), IntCl(A), ClIint(A), IntClInt(A), and ClIntCl(A).
[7] Id is the identity map, Id(A) = A considered as the empty concatenation of Int and Cl.

A closer look on (2.6) reveals that there is indeed an operator in this list that scores quite well as a plausible candidate for the office of a good belief operator, namely, the operator IntClInt. As is easily checked by elementary examples and calculations, IntClInt satisfies (2.7) (i) – (iv). Even better, for all topological spaces (X, OX) the pair of operators (Int, IntClInt) satisfies all axioms of Stalnaker's combined logic KB of knowledge and belief except the axiom (NI) of negative introspection (cf. Stalnaker (2006), Baltag, Bezhanishvili, Özgün, Smets (2014, 2019)).

After having discussed in some detail the topological versions of the epistemological concepts of knowledge and belief it is possible to take into consideration a particularly interesting combination of them, namely, the Gettier propositions. In a topological setting these propositions may be precisely defined as follows:

(2.8) Definition. Let (X, OX) be a topological space to be conceived as a universe of possible worlds in which knowledge and belief are modeled by the topological operators Int and IntClInt, respectively. Let $A \in PX$ be interpreted as a proposition. Then define

$$G(A) := A \cap IntClInt(A) \cap Int(A)^C.$$

If $G(A) \neq Ø$ the proposition G(A) is called the Gettier-proposition of A for w and $w \in G(A)$ is called a Gettier-world for A. ♦

As it stands, definition (2.8) is not fully convincing since no argument has been given that IntClInt can be interpreted as justified belief. This gap will be filled in the next section by explaining in more detail that IntClInt can be interpreted as justified belief in a strong sense. This requires to dwell more closely on the justificatory qualities of the topological operators Int and Cl and that are inherited by the composition IntClInt of these components.

3. Stalnaker's Combined Logic KB of Knowledge and Justified Belief. First, for the sake of definiteness, let us recall the axioms and the inference rules of Stalnaker's system (cf. Stalnaker (2006), Baltag et al. (2017, 2019):

(3.1) Definition (Stalnaker's axioms and inference rules for knowledge and belief).

(CL)	All tautologies of classical propositional logic.	
(K)	$K(\varphi \rightarrow \psi) \rightarrow (K\varphi \rightarrow K\psi)$	(Knowledge is additive).
(T)	$K\varphi \rightarrow \varphi$	(Knowledge implies truth).
(KK)	$K\varphi \rightarrow KK\varphi$	(Positive introspection of K.
(CB)	$B\varphi \rightarrow \neg B \neg\varphi$	(Consistency of belief).
(PI)	$B\varphi \rightarrow KB\varphi$	(Positive introspection of B).
(NI)	$\neg B\varphi \rightarrow K\neg B\varphi$	(Negative introspection of B).
(KB)	$K\varphi \rightarrow B\varphi$	(Knowledge implies belief).
(FB)	$B\varphi \rightarrow BK\varphi$	(Full belief).

Inference Rules:

(MP)	From φ and $\varphi \rightarrow \psi$, infer ψ.	(Modus Ponens).
(NEC)	From φ, infer $K\varphi$.	(Necessitation).♦

For the topological approach to knowledge and belief, the axiom (NI) plays a special role. It has been shown that (NI) holds only for topological models of a very special kind, namely, models based on extremally disconnected spaces topological spaces (cf. Baltag et al (2019), Stalnaker (2006)). All other axioms and rules of KB are satisfied by all topological spaces. Thus, by giving up (NI) a lot of generality is gained. There is to pay a price, however. The validity of (NI) guarantees unique definability of the belief operator, i.e., for extremally disconnected[8] spaces, the belief operator is uniquely determined by the knowledge operator as IntClInt (cf. Baltag et al. (2019), Stalnaker (2006)). This no longer holds for topological models that are not extremally disconnected. Indeed, if (NI) is not assumed to hold, the belief operator B is no longer uniquely determined B = IntClInt. For the systems of knowledge and belief considered in this paper we will only require that they are weak Stalnaker systems in the following sense:

(3.2) Definition (Weak KB logic). A bimodal logic (with modal operators K and B) based on the bimodal language L_{KB} is a weak KB-logic iff it satisfies the conditions:

(i) The modal operator B satisfies the Kripke axiom: (K) $B(\varphi \rightarrow \psi) \rightarrow (B\varphi \rightarrow B\psi)$.

(ii) The modal operator B is idempotent: (4)* $BB\varphi \leftrightarrow B\varphi$.

(iii) For the tandem (K, B) all of Stalnaker's axioms and rules given in (3.1) are satisfied except the axiom (NI) of negative introspection.♦

It should be noted that the B-fragment of weak KB logic is a normal modal logic since the necessitation rule NEC for B is satisfied: from φ one may infer Bφ.[9] Even more, by (3.2) one has:

(3.3) Corollary. The B-fragment of weak KB-logic is a KD4-logic. More precisely, the following axioms hold in weak KB:

(K) $B(\varphi \rightarrow \psi) \rightarrow (B(\varphi) \rightarrow B(\psi))$.

(D) $B\varphi \rightarrow \neg B \neg \varphi$.

(4) $B\varphi \rightarrow BB\varphi$.♦

[8] Recall that a space (X, OX) is extremally disconnected iff the closure of every open set is open: Clint(A) = IntClint(A) (cf. Willard (2004, 15.G, p. 106)).

[9] For various equivalent definitions of a normal modal logic, see Chellas (1980, Theorem 4.3, p. 115).

This result may be compared with the corresponding result for full KB logic that the B-fragment of full KB logic is a KD45 system (cf. Baltag et al. (2019, Proposition 4), Stalnaker (2006)).[10]

It should be noted that a weak KB-system gives rise to a system that slightly stronger than just a KD4-system, since the idempotence (4)* of B ((3.2)(ii)) requires not only $B\varphi \rightarrow BB\varphi$, i.e., the axiom (4), but also its converse $BB\varphi \rightarrow B\varphi$. One immediately observes that also for (full) KB systems the operator B is idempotent. Hence, the B-fragment of full KB logic can be characterized more precisely by KD4*5.

The following proposition shows that definition (3.1) and (3.2) fit well together:

<u>(3.4) Proposition</u>. Weak KB logic is strictly weaker than KB logic.

<u>Proof</u>. First, we show that the modal operator B of KB logic satisfies the Kripke axiom (K) of (3.2)(i). According to Stalnaker (2006) and Baltag et al. (2019) in KB-logic one has

$$B \leftrightarrow \neg K \neg K \leftrightarrow K \neg K \neg K.$$

As is easily checked, $K\neg K\neg K$ is a normal operator, i.e., satisfies (K). Moreover, (3.2)(ii) is satisfied, i.e., B is idempotent. Hence, as it should be, KB logic is a weak KB logic. In order to show that weak KB is <u>strictly</u> weaker than KB, one has to find a formula that is valid for KB but not for weak KB. The formula $\neg K\neg K(\varphi \wedge \psi) \leftrightarrow \neg K\neg K\varphi \wedge \neg K\neg K\psi$ will do. ♦

Now let us recall the basics of the interior semantics for modal epistemic logic as presented by Baltag, Bezhanishvili, Özgün, and Smets in various recent publications (cf. Baltag et al. (2013, 2015, 2016, 2019)). In the rest of this paper this semantics will be used throughout.

We start with a standard unimodal epistemic language L_K with a countable set PROP of propositional letters, Boolean operators $\neg$, $\wedge$, and a modal operator K to be interpreted as a knowledge operator. The formulas of L_K are defined as usual by the grammar

(3.5) $$\varphi ::= p \mid \neg p \mid \phi \wedge \psi \mid K\varphi \quad , \qquad p \in \text{PROP}.$$

[10] Elementary examples based on the Euclidean line ($\mathbb{R}$, $O\mathbb{R}$) show that there are models of the weak KB logic the B-fragment of which are <u>not</u> KD45 models (cf. Proposition (4.11)).

The abbreviations for the Boolean connectives $\vee$, $\rightarrow$, and $\leftrightarrow$ are standard. Analogously to L_K, a bimodal epistemological language L_{KB} for operators K and B is defined. For a more detailed presentation of topological semantics, the reader may consult the recent papers of Baltag et alii (2019). The grammar and topological models for the modal language L_{KB} of knowledge and belief can be defined as follows:

(3.6) $$\varphi ::= p \mid \neg p \mid \phi\wedge\psi \mid K\varphi \mid B\psi$$

<u>(3.7) Definition</u>. Given a topological space (X, OX), we define a topo(logical) model for L_K as M = (X, OX, v), where v:PROP→PX is a valuation function from the set PROP of propositional letters to PX.♦

The function v:PROP→PX can be inductively extended to a function v:FORM(L_K)→PX (also denoted by v) of the set of well-formed formulas FORM(L_K) of L_K to PX in the usual way by defining:

<u>(3.8) Definition</u>. Let M = (X, OX, v) be a topological model of L_K. The interior semantics of L_{KB} with values in M is given by

(i) $v(p) \in PX$.

(ii) $v(\neg p) := v(p)^C$.

(iii) $v(\varphi\wedge\psi) := v(\varphi) \cap v(\psi)$.

(iv) $v(K\varphi) := Intv(\varphi)$.

(v) $v(B\varphi) := IntClIntv(\varphi)$.♦

Actually, this semantics is the semantics of the unimodal language L_K, since $v(B\varphi)$ is defined in terms of $v(K\varphi)$, namely, $v(B\varphi) = v(K\neg K\neg K\varphi)$.

Given a topological model M = (X, OX, v), in this paper the knowledge operator of M is always interpreted as the topological interior kernel operator Int of (X, OX) and the belief operator always as IntClInt.[11]

Checking the pertinent definitions of a topology (2.1), (2.3) and the axioms of weak KB-systems (3.2) one easily obtains the following:

(3.9) Proposition**.** Every topological model (X, OX, v) defines a model of a weak Stalnaker system KB in the sense of (3.2). ♦

As already mentioned, with more effort the following more specific theorem can be proved:

(3.10) Theorem. A topological model (X, OX, v) satisfies all rules and axioms (3.1) of a Stalnaker model (the axiom (NI) of negative introspection included) iff (X, OX) is an extremally disconnected space (ED-space) (cf. Baltag et al. (2021), Stalnaker (2006).♦

In order to deal with epistemological issues concerning Gettier problems and related questions the belief operator IntClInt has to be not just any kind of belief, but rather justified belief. For this claim we may argue as follows. First, it should be noticed that the interior semantics of knowledge offers, so to speak, a built-in evidential justification of knowledge. Eventually this gives rise to a strong justificatory component of the belief operator IntClInt as well. This may be explicated by interpreting the steps that go from knowledge (Int) over ClInt to belief (IntClInt) as follows.

For A ∈ PX the proposition Int(A) is to be interpreted as true at a world w ∈ X iff w ∈ Int(A). By definition (2.3) of Int, this means that there is an open neighborhood U(x) of x such that x

[11] Other definitions for belief operators are possible but they will not be considered in this paper.

$\in U(x) \subseteq Int(A)$. By definition of the topological structure OX of (X, OX) one has $U(w) \in OX$. Epistemically, U(w) may be interpreted as a piece of observable evidence that the cognitive agent possesses, who uses the knowledge operator Int to carry its epistemic activities.

In order to render plausibel the interpretation of the belief operator IntClInt as justified belief it is expedient to dwell more on the definition of Int and its justificatory aspects. For this purpose, the following definition is useful:

(3.11) Definition. A subbase of a topological space (X, OX) is subset $SX \subseteq OX$ such that every element of OX is the set-theoretical union of a finite intersection of elements of SX. A base of BX of (X, OX) is a subbase SX of (X, OX) such that every element of OX is a set-theoretical union of elements of SX.♦

Informally expressed, a subbase of (X, OX) "generates" the topology. Clearly, every subbase SX defines a base by taking all finite intersections of elements of SX as elements of that base. Every topology (X, OX) has OX as its largest base. Often, it is convenient to look for smaller (sub)bases of OX. For instance, a useful subbase for the topological space of the Euclidean line ($\mathbb{R}$, $O\mathbb{R}$) is the set of all open rational intervals $\{(a, b); a < b, a, b \in \mathbb{Q}\}$. Epistemologically interpreted, a subbase SX of OX may be considered as the class of (propositions of) directly observable evidences that are available to a cognitive agent whose epistemic activity is characterized by OX (cf. Baltag et al. 2019)). Correspondingly, a base generated by subbase SX of OX may be considered as the class of finitely constructed (indirect) evidences that the cognitive agent carries out in the ongoing process of his research.

Defining knowledge topologically by the interior kernel operator Int comes with the conceptual advantage that knowledge is naturally correlated with appropriate evidence. That it is known that w is an A-world, $w \in Int(A)$ is true iff there is an open neighborhood U(w) of w such that $w \in U(w) \subseteq A$. Bringing into play the concepts of bases and subbases we may say that a cognitive agent, engaged in the task to claim with justification that he knows that w is an A-world, has to find a finitely constructed piece of evidence U(w) from a subbasis SX of OX such that $w \in U(w) \in OX$ such that $U(w) \subseteq A$. This account of knowledge based on evidential justification can be expanded to a justificatory account of belief. According to (3.6)(v) belief of a proposition A is defined as IntClInt(A). By (3.7) the operator IntClInt is a good belief operator in the sense that the pair (Int, IntClInt) satisfies the rules and axioms of a weak Stalnaker system KB. Moreover, Int(A) and IntClInt(A) are extensionally very close to each other. One easily calculates:

(3.12) Lemma. The extensional difference between Int(A) (knowledge) and IntClInt(A) (justified belief) is topologically negligible, i.e., nowhere dense:

$$\mathrm{IntCl}(\mathrm{IntClInt}(A) \cap \mathrm{Int}(A)^C) = \emptyset \text{ for all } A \in PX.$$

Proof: By (2.4) we calculate $\mathrm{IntCl}(\mathrm{IntClInt}(A) \cap \mathrm{Int}(A)^C) = \mathrm{IntClInt}(A) \cap \mathrm{IntCl}(\mathrm{Int}(A)^C)$

$$= \mathrm{IntClInt}(A) \cap \mathrm{IntCl}(\mathrm{Int}(A)^C) \subseteq \mathrm{ClInt}(A) \cap (\mathrm{ClIntInt}(A))^C = \emptyset. \blacklozenge$$

Now, Int(A) as knowledge of A is certainly justified, since knowledge as such "by definition" is evidentially justified. Hence, the proposition IntClInt(A) that differs from Int(A) by a topologically negligible difference may be considered also as justified as well, although not as knowledge but at least as justified belief. Elementary examples that justified belief may differ from (true) knowledge are easily given:

(3.13) Example. For the Euclidean line $(\mathbb{R}, O\mathbb{R})$ consider the set $A := [-1, 1] - \{0\}$. One calculates $\mathrm{Int}(A) = (-1, 1) - \{0\}$ and $\mathrm{IntClInt}(A) = (-1, 1)$, i.e., $\mathrm{Int}(A) \neq \mathrm{IntClInt}(A)$. ♦

Moreover, (3.13) suggests that extensionally Int(A) and IntClInt(A) differ only in rare exceptional cases. This is indeed the case: One calculates that the set-theoretical difference is between these sets is topologically small, i.e., nowhere dense:

$$\mathrm{IntCl}(\mathrm{IntClInt}(A) - \mathrm{Int}(A)^{C}) = \varnothing. \blacklozenge$$

For the special case of topological model based on extremally disconnected spaces treated in detail by Baltag et al. (2019) the operators IntClInt and ClInt coincide. Recalling that in modal logic the closure operator Cl is naturally interpreted as possibility, on the class of extremally disconnected spaces "belief" may be elegantly characterized as "possibility of knowledge" (cf. Stalnaker (2006).

On general topological models, however, ClInt is no longer a good belief operator. Among other deficiencies, ClInt is not a normal operator (as was already observed by Stalnaker (2006)). If ClInt is replaced by IntClInt most of the qualities of ClInt as a good belief operator are preserved. Only negative introspection (NI) is no longer valid. Instead of interpreting belief as "possibility of knowledge", for the general case IntClInt belief is to be interpreted as "knowledge of possibility of knowledge".

In sum, we may consider the operator IntClInt as an example of a justified belief operator. Reading the operator IntClInt as justified belief allows us to use the apparatus of topological epistemology for elucidating the Gettier problem that may be considered as the problem of how knowledge and justified belief are related.

The problem of structurally elucidating the concept of justified belief has been tackled by many scholars, Stalnaker's evidential approach has not been the only game in town. Hence it may be expedient to briefly compare the topological approach with some other existing structural accounts of justification. The first one I'd like to mention is Ichikawa's account JPK of "justification is possibility of knowledge" (Ichikawa (2014, 2017)). Ichikawa is a vigorous partisan of the "knowledge first" approach. That means, justification and other epistemic notions are defined in terms of knowledge and not the other way round.

For general topological spaces (X, OX) justification J in Ichikawa's JPK (taken as "possibility of knowledge" J = ClInt) is neither normal nor factive, but at least idempotent, i.e., $J^2 = J$. For extremally disconnected spaces (X, OX) Ichikawa's justification J = ClInt is even normal. Further, it is, of course, idempotent, but still not factive, i.e., $J\varphi \rightarrow \varphi$ does not hold for JPK.

In Rosenkranz (2018) the author offers a non-topological structural approach that, despite fundamental differences with the topological one, possesses some surprising analogies. Rosenkranz's structural approach of justification J is based on the basic modal concept of "being in a position to know" to be denoted by *K*. The concept J of justification is defined as $J := \neg K \neg K$, with *K* being the non-normal operator "being in a position to know". The modal operators K and J are assumed to satisfy the following axioms:

(3.14) Definition (Structural Justification operator according to Rosenkranz (2018)).

(E)	$J\varphi \leftrightarrow \neg K \neg K$
$(T)_K$	$K\varphi \rightarrow \varphi$
$(RN_K$	If $\vdash \varphi$, then $\vdash K\varphi$.
RM_K	If If $\vdash \varphi \rightarrow \psi$, then $\vdash K\varphi \rightarrow K\psi$.
LUM	$\neg K \neg K\varphi \rightarrow K \neg K \neg K\varphi$.♦

The following comments on these axioms may be in order. Rosenkranz's modal logic of justification J is a D45 logic, but actually a D4*5 logic, since $J \leftrightarrow J^2$. Rosenkranz subscribes to a knowledge first account, but in a somewhat different way than usual, since not knowledge as such, i.e., knowledge simpliciter, is "first" (among all other epistemic concepts, but "being in a position to know" is in this position. Rosenkranz assumes that "being in a position to know" is weaker than knowledge proper in that to know entails being in a position to know but not vice versa. Moreover, "being in a position to know" is stronger than Ichikawa's "possibility of knowledge" as justification since the former is explicitly factive but the latter is clearly not: For the real line ($\mathbb{R}$, $O\mathbb{R}$) and $A = (0, 1)$ one clearly has that $ClInt(A) \rightarrow A$ is not valid, since $Cl(A) = [0, 1]$ is not a subset of $A = (0, 1)$. On the other hand, Rosenkranz's operator *K* "being in a position to know" is not normal, while the operator Int of the topological approach is normal.

4. Topological Models of JTB and JTB-doppelgangers. In the previous sections we have established the elementary fact that all topological spaces (X, OX) can serve as the carriers of topological models of knowledge and belief, knowledge being represented by the topological interior operator Int and belief being represented by IntClInt. As has been shown, the tandem (Int, IntClInt) of operators validates many plausible features of the concepts of knowledge and belief and their relations. In particular, all axioms of Stalnaker's KB logic hold except the axiom of (NI) of negative introspection. On the other hand, we already know that not all topological spaces (X, OX) support the JTB-account of knowledge. Already for the topological model line ($\mathbb{R}$, $O\mathbb{R}$) of the real line JTB is not valid (see (1.1)). Clearly, the counterexample of JTB given in proposition (1.3) is typical and could be multiplied ad libitum. This fact may be considered as sufficient to lay to rest the issue of JTB (cf. Williamson (2013)): JTB turns out to be falsified not only by countless informal counterexamples (cf. Turri (2012), Borges et al. (2017), Machery (2017)) but also for general formal reasons. For Williamson the existence of formal models of Gettier situations is a sufficient reason to lay to rest JTB.

But a closer look reveals that things are more complex. It is worth the effort to pursue this issue a bit further. First of all, it should be taken into account that JTB is trivially valid at least for some topological models. As already mentioned, the coarse topology (X, {Ø, X}) and the discrete topology (X, PX) are JTB-systems, not to forget the DSO-models introduced by Parikh et al. (2007). Actually, we can do better than to rely on these either trivial or rather contrived models of JTB. In this section we will show that for any topological space (X, OX) whatsoever we can canonically construct a topological space (X, O_{nod}X) "in the neighborhood of (X, OX)" that is a JTB system.

(4.1) Proposition. Let (X, OX) be a topological space with interior operator Int. Define a topological space (X, O_{nod}X) with interior operator Int_{nod} as follows:

$$Int_{nod}(A) := A \cap IntClInt(A).$$

Then Int_{nod} is the interior operator of a topology on X that is at least as fine as OX, i.e., OX $\subseteq$ O_{nod}X. Moreover, $O_{nod}X = O_{nodnod}X$. The topological closure operator Cl_{nod} corresponding to Int_{nod} is given by

$$Cl_{nod}(A) = A \cup ClIntCl(A).$$

Proof. See Njåstad (1965, Proposition 2, p. 962), or just check that the Kuratowski axioms are satisfied for Int_{nod} and Cl_{nod}.♦

(4.2) Theorem. Let (X, O_{nod}X) be the topological space defined in (4.1) by Int_{nod}, A $\in$ PX. Denote the belief operator defined on (X, O_{nod}X) by $Int_{nod}Cl_{nod}Int_{nod}$. Then (X, O_{nod}X) is a JTB-system with respect to Int_{nod} and $Int_{nod}Cl_{nod}Int_{nod}$, i.e.:

$$Int_{nod}(A) = A \cap Int_{nod}Cl_{nod}Int_{nod}(A).$$

Proof: The proof consists of an elementary, but somewhat tedious calculation using well known results of the concatenations of the topological operators Int and Cl and the not-so-well-known technical result of Kuratowski and Mostowski (2.3). For the sake of simplifying the denotation the following abbreviations are used: I := Int, C := Cl, I' := Int_{nod}, and C' := Cl_{nod}. Then by definition we have $C_{nod}(A) = A \cup CIC(A)$ and we can prove for all A $\in$ PX:

$I'(A) \;=\; A \cap I'C'I'(A)$ iff

$= A \cap I'C'(A \cap ICI(A)) = A \cap I'((A \cap ICI(A)) \cup CIC(A \cap ICI(A)))$

$= A \cap I'((A \cap ICI(A)) \cup C(IC(A) \cap ICICI(A)))$

$= A \cap I'((A \cap ICI(A)) \cup CICI(A))) = A \cap I'((A \cap ICI(A)) \cup CI(A)))$

$= A \cap I'(A \cup CI(A) \cap CI(A)) = A \cap I'(CI(A))$

$= A \cap CI(A) \cap ICICI(A) = A \cap ICI(A).$ ♦

(4.2) shows that there are plenty of JTB systems: Each ordinary topological system (X, OX) gives rise to a topological JTB-system (X, $O_{nod}X$) defined on the same underlying set X by changing slightly the original topological operator Int by the finer topological operator Int_{nod}.

Theorem (4.2) can be further improved:

(4.3) Theorem. Let (X, $O_{nod}X$) be the nodec doppelganger of (X, OX) defined in (4.1). Then

$$Int_{nod}Cl_{nod}Int_{nod} = IntClInt.$$

Hence $Int_{nod}(A) = A \cap IntClInt(A)$ and therefore $Int_{nodnod}(A) = Int_{nod}(A)$.

Proof. The proof consists of a direct calculation analogous to that of the proof of (4.2):

$I'C'I'(A) = I'C'(A \cap ICI(A)) = I'((A \cap ICI(A)) \cup CIC(A \cap ICI(A)))$

$= I'((A \cap ICI(A)) \cup C(IC(A) \cap ICIICI(A)))$

$= I'((A \cap ICI(A)) \cup CICI(A))) = I'((A \cap ICI(A) \cup CI(A))$

$= I'CI(A) = CI(A) \cap ICICI(A) = ICI(A).$ ♦

(4.2) and (4.3) show that the topological spaces (X, OX) and (X, $O_{nod}X$) are a very similar structure. On the other hand, we obtain:

(4.4) Corollary. (i) Let (X, $O_{nod}X$) be the nodec companion (4.1) of (X, OX). The topological model (X, $O_{nod}X$) is free of Gettier situations, i.e., for all $A \in PX$ one has $G(A) = Ø$.

(ii) The spaces (X, OX) and (X, O_{nod}X) have the same Boolean algebra of regular open regular sets O*X = O_{nod}*X.

(iii) The extensional difference between Int(A) and Int_{nod}(A) is nowhere dense for all A ∈ PX.

Proof. (i) Using the same abbreviations as in (4.2) by definition one has

$$G(A) = A \cap Int_{nod}Cl_{nod}Int_{nod}(A) \cap I_{nod}(A)^C$$

$$= A \cap ICI(A) \cap (A \cap ICI(A))^C$$

$$= (A \cap ICI(A)) \cap (A^C \cup ICI(A)^C)$$

$$= (A \cap ICI(A)) \cap A^C) \cup (A \cap ICI(A)) \cap ICI(A)^C) = \emptyset \cup \emptyset = \emptyset.$$

(ii) By (4.3) and the definition of the Boolean algebras of regular open subsets of OX and O_{nod}X* one has

$$O^*X = \{IntClInt(A);\ A \in PX\} = \{Int_{nod}Cl_{nod}Int_{nod}(A);\ A \in PX\} = O_{nod}X^*.$$

(iii) By (2.3) and (2.4) one calculates

$$IntCl(Int_{nod}(A) \cap Int(A)^C) = IntCl(IntClInt(A) \cap A \cap Int(A)^C) = IC(ICI(A) \cap A \cap I(A)^C)$$

$$= ICI(A) \cap IC(A \cap I(A)^C) = IC(I(A) \cap I(A)^C) = IC(\emptyset) = \emptyset. \blacklozenge$$

Corollary (4.4) offers – at least theoretically – a method to get rid of the Gettier cases that beset general topological models (X, OX): If the cognitive agent, who employs Int and IntClInt for her doxastic and epistemic actions, is able to change her knowledge operator from Int to Int_{nod} while keeping her method of justified belief as IntClInt then she can avoid the appearance of Gettier cases. In other words, Gettier anomalies could be avoided by improving the agent's epistemic powers. The spaces (X, OX) and (X, O_{nod}X) may be conceived as the two stages of a learning process: This process starts from an initial stage topologically characterized by (X, OX) that is beset with epistemic anomalies aka Gettier cases G(A) ≠ Ø. It finds its ideal end in the Gettier-free universe (X, O_{nod}X) that is luminous for the cognitive agent who uses Int_{nod} for his cognitive endeavours.

Nodec spaces are epistemically characterized as the appropriate class of topological models for traditional JTB-epistemology which does not recognize Gettier cases. This may be considered as a partial rehabilitation of the JTB-account by topological epistemology.

The relation between the class of all topological spaces TOP(X) and the class of nodec spaces $TOP_{nod}(X)$ defined on X may be described as follows: $TOP_{nod}(X)$ is a subclass to TOP(X), i.e., there is a natural embedding $i:TOP_{nod}(X) \to TOP(X)$. Moreover, there is a left inverse $j:TOP(X) \to TOP_{nod}(X)$ that maps every topological space (X, OX) onto its nodec doppelganger $j(X, OX) := (X, O_{nod}X)$ such that $ji:T_{nod}(X) \to TOP_{nod}(X)$ is the identity map id_{TOPnod} on $TOP_{nod}(X)$.

It is worth the effort to elaborate the logical relation between these two classes of spaces further in terms of modal logics as follows. Let $K \subseteq$ TOP be any class of topological spaces (for instance, the class of all topological spaces, the class of extremally disconnected spaces, or the class of nodec spaces, etc.). Recall that S4 is the least set of formulas of the basic unimodal language L with basic modal operator $\Box$ satisfying the axioms

(4.5) (i) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$, (ii) $\Box\varphi \to \varphi$, (iii) $\Box\varphi \to \Box\Box\varphi$.

and closed under modus ponens, substitution, and necessitation ($\varphi/\Box\varphi$). For a class of *K* of topological spaces, let L(*K*) denote the set of formulas of L that are valid in *K* interpreting the formulas of L in the familiar way (see 3.4). L(K) is called the modal logic of K. Since the classical result of McKinsey and Tarski (1944) it is well known that the modal logic of the class of all topological spaces TOP is S4.

Meanwhile, the modal logics of many subclasses of TOP have been determined, see Bezhanishvili et al. (2004). For topological epistemology particularly interesting is the extension S4.2 of S4. As has been shown by Stalnaker (2006) and Baltag et al. (2019), S4.2 is the logic of extremally disconnected topological spaces (X, OX). Topologically, these spaces may be characterized as spaces the closure operator of which satisfies $Cl(A \cap D) = Cl(A) \cap$

Cl(D) for open sets A and B (cf. Footnote 7). Expressed in epistemological terms, (X, OX) is extremally disconnected essentially iff the axiom (NI) of negative introspection is satisfied (Baltag et al. (2019)).

The interesting point is that the class of nodec spaces – as the class of topological spaces that satisfy JTB – also corresponds to a well-known extension of S4, namely, S4.Zem (cf. Bezhanishvili et al. (2004) and Zeman (1969)). S4.Zem is the logic of JTB, i.e., the logic of the classical account of knowledge and belief for which knowledge is equivalent to justified true belief, or, in other words, for which Gettier propositions do not exist.

Thus, the status of JTB in topological epistemology may be understood as analogous to that of the extension S4.2 of S4, namely, as a special modal logic in which a special axiom holds, namely, the axiom S4.Zem (Bezhanishvili et al. (2004), Theorem (3.4)), L(N) = S4.Zem). Hence, from the point of view of formal topological epistemology, one should not ask simpliciter whether JTB is true or not. Rather, a more appropriate question is to ask for which class of topological models JTB is true. The neat answer to this question is that JTB - as corresponding to the class of nodec spaces – holds for nodec models. This is analogical to the statement that the topological knowledge operator Int satisfies the axioms of Stalnaker's combined logic KB (the axiom of negative introspection (NI) included) iff the topological universe of possible worlds (X, OX) has the topological structure of an extremally disconnected space. Analogously, a topological model (X, OX) satisfies the classical JTB-account of knowledge as justified true belief iff it also satisfies the Zeman axiom $(\Box\neg\Box\neg\Box p)\rightarrow(p\rightarrow\Box(p))$:

<u>(4.6) Proposition</u>. Interpreting $\Box$ as the topological interior operator Int and the belief operator as IntClInt, for topological models (X, OX, v) the Zeman axiom $(\Box\neg\Box\neg\Box p)\rightarrow(p\rightarrow\Box(p))$ holds iff the JTB-axiom (1.1) is valid:

$$\mathrm{Int}(A) = A \cap \mathrm{IntClInt}(A)$$

Proof: By definition of the topological interpretation of S4 we have that the characteristic axiom $\Box\neg\Box\neg\Box p)\rightarrow(p\rightarrow\Box(p))$ for S4.Zem holds for a topological model (X, OX, v) iff

$$\mathrm{IntClInt}(A) \cap A \subseteq \mathrm{Int}(A) \text{ for all } A \in PX.$$

On the other hand, the inequality $\mathrm{Int}(A) \subseteq A \cap \mathrm{IntClInt}(A)$ holds by definition of Int and well-known properties of IntClInt for all topological models, see (2.3). Hence $\mathrm{Int}(A) = A \cap \mathrm{IntClInt}(A)$. In other words, (X, OX) is a nodec space.♦

The logical characterization of nodec spaces as the class of spaces that satisfy S4.Zem establishes an analogy between JTB and other epistemologically interpretable modal logics that are defined as extensions of S4:

(4.7) Some epistemically interpretable modal logics and their classes of topological spaces.

(i) The "logic of clearness" (Bobzien (2012)) is the logic $S4.1 = S4 + (\Box\neg\Box\neg p) \rightarrow (\neg\Box\neg\Box p)$. The class of topological models of S4.1 is the class of McKinsey spaces.

(ii) Stalnaker's combined logic of knowledge and belief KB (satisfying the rules and axioms (3.1) with (NI) included)) is the logic S4.2. The class of topological models of S4.2 is the class of extremally disconnected spaces.

(iii) The traditional epistemic logic JTB of knowledge as justified true belief is the logic $S4.Zem = S4 + \Box\neg\Box\neg\Box p\rightarrow(p\rightarrow\Box p)$. The class of topological models of S4.Zem is the class of nodec spaces.♦

In sum, topological epistemology suggests a relativization of the question whether the traditional thesis that knowledge is justified true belief is correct or not. The answer of this question depends on the class of topological models chosen. Thus, traditionalists like Sellars (cf. Sellars (1975)) who insisted that traditional JTB is essentially correct should subscribe to S4.Zem as the appropriate logic for the epistemology of knowledge and belief, those who acknowledge Gettier counterexamples will have to choose another extension of S4 as the appropriate logic of knowledge and belief. For instance, if they insist on (NI) as a necessary condition for a good belief operator they will choose S4.2.

5. Topology of Gettier Cases: A Topological "Double Luck" Construction and the Dox–astic Invisibility of Gettier Situations. The previous section dealt with the topological and logical problems of a very special class of topological epistemological systems, namely, systems for which the traditional account of knowledge as justified true belief is valid. As said, since Gettier's paper this account has come under heavy attack. Nevertheless, till today some philosophers doubt that Gettier cases definitively have refuted the traditional JTB-account. Nodec spaces may be considered as a niche where the traditional JTB-account can survive. Topological epistemology provides a relative and partial justification of JTB.

This section intends to show that topological epistemology is also useful to bring to the fore some interesting formal aspects of Gettier situations that have seldom or never been noticed in the decades of the flourishing production of ever more sophisticated Gettier examples. First of all, let us consider the obvious, but nevertheless somewhat enigmatic aspect of Gettier cases that they are exceptional and rare cases. Topology is an expedient device to render precise this impression.

Let $A \in PX$ be a proposition that describes a situation as a set of possible worlds. Assume that $G(A) := A \cap IntClInt(A) \cap Int(A)^C \neq \emptyset$ is the set of possible worlds $w \in X$ for which A describes a Gettier situation, namely, (i) $w \in G(A)$ is an A-world, (ii) w is believed with justification to be an A-world, and (iii) w is not known to be an A-world. This conjunction of qualities (i) – (iii) turns out to be a rare event in a precise topological sense:

(5.1) Theorem. (i) For all topological spaces (X, OX) and all $A \in PX$ the set G(A) of Gettier worlds for A is nowhere dense, i.e., IntCl(G(A) = Ø:

$$\mathrm{IntCl}(G(A)) = \mathrm{IntCl}(A \cap \mathrm{IntClInt}(A) \cap \mathrm{Int}(A)^C) = \varnothing.$$

(ii) For nodec spaces (X, OX) one obtains the stronger result G(A) = Ø for all $A \in PX$.

Proof. (i) Using once again the abbreviations of (4.2) and the technical Lemma (2.3) one calculates:

$$\begin{aligned} IC(G(A)) &= IC(A \cap ICI(A) \cap I(A)^C) = ICICI(A) \cap IC(A \cap I(A)^C) \\ &= ICICI(A) \cap IC(A \cap I(A)^C) = ICI(A) \cap IC(A \cap I(A)^C) \\ &= IC(A \cap IC(A) \cap I(A)^C) = IC(\varnothing) = \varnothing. \end{aligned}$$

(ii) This has already been proved in (4.5).♦

(5.1) (i) and (ii) confirm the intuitive impression that Gettier situations are rare events. In the special case of nodec spaces, Gettier situations are extremally rare events – they never occur. The fact that Gettier situation are nowhere dense events affects their epistemic status:

(5.2) Corollary. Let (X, OX) be a topological model of knowledge and belief where these epistemic operators are represented by Int and IntClInt, respectively, and G(A) ≠ Ø is the Gettier proposition of $A \in PX$.

(i) The Gettier proposition G(A) cannot be known with respect to the knowledge operator Int, i.e., Int(G(A)) = Ø.

(ii) The Gettier proposition G(A) cannot be not believed with respect to the belief operator IntClInt: IntClInt(G(A)) = Ø.[12]

Proof. Clearly, $\mathrm{Int}(G(A)) \subseteq \mathrm{IntClInt}(G(A)) \subseteq \mathrm{IntCl}(G(A)) = \varnothing$ by Theorem (4.4).♦

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[12] More generally it can be proved that Gettier propositions cannot be believed with respect to any (reasonable) consistent belief operator.

For all topological models all Gettier situations are topologically rare situations. They may be overcome by improving the cognitive agent's epistemic capacities, namely, by replacing the knowledge operator Int by the finer operator Int_{nod}. This replacement dissolves the cognitive anomalies exemplified by G(A). Moreover, the move from Int to Int_{nod} is topologically small, since the extensional difference of $Int_{nod}(A)$ and Int(A) is nowhere dense. Of course, an omniscient cognitive agent is not plagued with Gettier propositions. But omniscience is not a realistic option. In contrast, the move from (X, OX) to (X, $O_{nod}X$) is a rather modest cognitive improvement. To avoid Gettier cases it is already sufficient to move from (X, OX) to (X, $O_{nod}X$) considered as the JTB-doppelganger of (X, OX). Remarkedly, both topological models share the same operator of justified true belief IntClInt, only their knowledge operators Int and Int_{nod} differ slightly. This could be interpreted as the fact that in combined systems of knowledge and belief the most important ingredient is not the knowledge operator Int and Int_{nod} but the belief operator IntClInt.

As has been observed by the "Gettier industry" of the past decades Gettier examples may be constructed according to certain general recipes. As Turri put it:

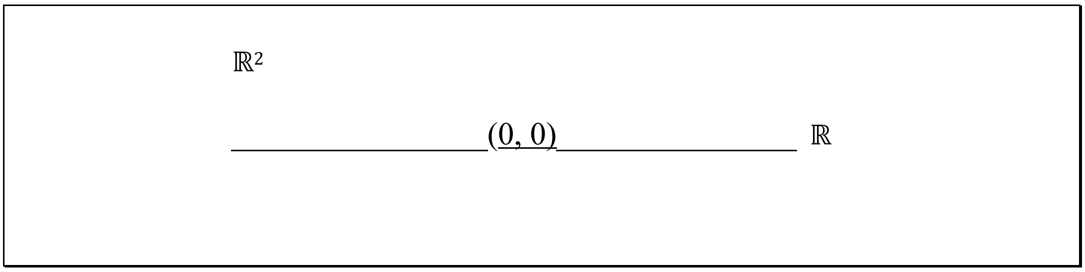
> Gettier cases are constructed by a recipe. Start with a belief sufficiently justified to meet the the justification requirement for knowledge. Then add an element of bad luck that would normally prevent the justified belief form being true. Lastly add a dose of good luck that "cancels out the bad", so the belief ends up true anyhow. Turri (2012, 248))

This recipe has a topological analogue. This may be taken as evidence that the topological model offers a quite faithful representation of the epistemological phenomena.

To keep matters as simple and intuitive as possible, consider the following example based on the 2-dimensional Euclidean plane endowed with its familiar Euclidean topology. The plane may be identified in a natural way with a 2-dimensional vector space. Let the 1-dimensional line $\mathbb{R}$ embedded in $\mathbb{R}^2$ as $\mathbb{R}:=\{(x, 0); x \in \mathbb{R}\}$. Let $A := \mathbb{R}^2 - \mathbb{R} \cup \{(0, 0)\}$. Intuitively, the construction of A may be described as follows: One begins with a class of "ordinary" situations $\mathbb{R}^2$, removes a subclass of "exceptional" situations $\mathbb{R}$ ("bad luck") and finally adds a class of "exceptional exceptions" $\{(0, 0)\}$ ("good luck"). The resulting set may be considered as a topological version of the "double luck construction" that Turri (and others) describe as a general recipe for constructing Gettier situations. Indeed, A turns out to be a Gettier situation since one calculates for G(A):

$$G(A) = A \cap \mathrm{IntClInt}(A) \cap \mathrm{Int}(A)^C = \{(0, 0)\} \neq \emptyset.\blacklozenge$$ [13]

Hence, (0, 0) is a "Gettier world" with respect to A, i.e., A is a justified true belief at (0, 0) but not is not known there, since $\mathrm{Int}(G(A)) = \emptyset$. In an elementary geometric model ($\mathbb{R}^2$, $O\mathbb{R}^2$) this "double-luck" situation looks as follows:



After knowing that JTB is false in general, it is natural to look for an explanation why JTB has found and still finds so much attraction in epistemology and beyond. Somehow, JTB seems close of being true. Topology offers a formal explanation for this fact. As I want to explicate in the following, a reason why JTB has the appeal of being essentially true is that counterexamples to JTB, namely, Gettier cases are - as "anomalies" for JTB - epistemically invisible in a precise

[13] A moment's reflection reveals that this construction has not much to do with the specific structure of ($\mathbb{R}^2$, $O\mathbb{R}^2$) but can be considerably generalized to arbitrary Hausdorff spaces (X, OX), i.e., if (X, OX) is a Hausdorff space, $w_G \in X$, the subset $A := X \times X - D(X) \cup \{w_G\}$ is a Gettier proposition, i.e., $G(A) = \{w_G\}$ is a Gettier world where A is true, believed with justification, but not known. Here, of course, D is the diagonal function defined as $D: X \to X \times X$ by $D(x) := (x, x)$ for $x \in X$, and $x \in X$.

sense. Thus, it is not very surprising that many people consider JTB as “essentially” correct though the existence of Gettier situations cannot be denied. Moving from (X, OX) to (X, $O_{nod}X$), thereby eliminating the Gettier situations existing in (X, OX), amounts to an extensionally small cognitive improvement by which JTB is rendered true. The knowledge operators Int and Int_{nod} of these models are very similar, their justified belief operators IntClInt and $Int_{nod}Cl_{nod}Int_{nod}$ are even identical.

On the one hand, the topological account of knowledge and belief confirms Williamson‘s thesis according to which JTB can be refuted not only by an abundance of counterexamples, its failure can also be predicted on general theoretical grounds. On the other hand, topological epistemology also offers some arguments for the assessment that JTB is almost true. Topological epistemology offers a niche for JTB on general theoretical grounds, and, at the same time, how to overcome this approach. As Williamson put it, Gettier cases can indeed be predicted within the general framework of (topological) epistemic logic. Depending on the structure of the universes of possible worlds, Gettier cases are bound to occur. And, complementarily, they can be avoided if one subscribes to certain requirements for conceptual spaces and the cognitive capacities of cognitive agents.

The proof that Gettier cases cannot be known by Int is elementary and amounts to a simple calculation using some axioms of KB:

<u>(5.3) Proposition</u>. Let (X, OX) be a topological model of KB-logic, $A \in PX$, and let

$$G(A) := IntClInt(A) \cap A \cap Int(A)^C$$

a Gettier case of A, i.e., $G(A) \neq Ø$. Then G(A) cannot be known by a cognitive agent who uses the knowledge operator Int, i.e., $Int(G(A)) = Ø$.

Proof. By definition the Gettier proposition G(A) is known at a world w ∈ X iff w ∈Int(G(A)). An elementary calculation using the Kuratowski axiom (2.3) shows that this is impossible since Int(G(A) is empty:

$$\begin{aligned} \mathrm{Int}(G(A)) &= I(ICI(A) \cap A \cap I(A)^C) \\ &= IICIA) \cap IA) \cap I(I(A)^C) = I(A) \cap I(I(A)^C) \\ &= I(I(A)) \cap I(I(A)^C) = I(I(A) \cap I(A)^C) = I(\emptyset) = \emptyset. \blacklozenge \end{aligned}$$

The theorem (5.3) can be strengthened by replacing Int (knowledge) by IntClInt (justified belief):

(5.4) Theorem. Let (X, OX) be a topological model of the weak KB-logic of knowledge and belief for the operators Int and IntClInt. Let G(A) := IntClInt(A) ∩ A ∩ $\mathrm{Int}(A)^C$. Then Gettier propositions G(A) cannot be believed consistently, i.e., there is no world where they can be believed by a cognitive agent who relies on IntClInt due to the fact tha IntClInt(G(A)) = Ø.

Proof. Suppose that w ∈X is a world in which the Gettier case G(A) ≠ Ø is believed with respect to the belief operator IntClInt, i.e., w∈ IntClInt(IntClInt(A) ∩ A ∩ $\mathrm{Int}(A)^C$) ≠ Ø. Then, using the axioms of KB-logic and (2.3) one calculates:

$$\begin{aligned} \mathrm{IntClInt}(G(A)) &= ICI(ICI(A) \cap A \cap I(A)^C) \\ &= ICIICI)(A) \cap ICI(A) \cap ICI(I(A)^C) \\ &= ICI\ A \cap ICI(I(A)^C) = IC(I(A) \cap I(A)^C) = I(\emptyset) = \emptyset. \end{aligned}$$

Thus, IntClInt(G(A)) = Ø for any A ∈ PX, i.e., there is no world w in which any G(A) can be believed.♦

As it stands, theorem (5.4) may be considered as slightly unsatisfying: It is formulated for the tandem of operators (Int, IntClInt) of general topological models of weak KB-logic. As can be shown, for weak KB logic a belief operator is no longer uniquely determined by Int (cf.

Mormann 2023), i.e., there are more operators B of justified belief than IntClInt such that the pairs (Int, B) satisfy the axioms of weak KB logic. This is in stark contrast to topological models of full KB logic. Indeed, for weak KB logic there may be several different belief operators related to Int. Consequently, for any given belief operator *B* one may define a Gettier proposition

(5.5) $$G_B(A) := A \cap B(A) \cap Int(A)^C$$

for any $A \in PX$ and ask whether there are worlds $w \in X$ for which $G_B(A)$ is true and can either be known (wrt to Int) or believed (wrt to *B*). (5.4) only asserts something for *B* = IntClInt. What about the analogues of (5.4) for those other belief operators? It goes without saying that these questions can only be answered if we know something about the operators *B*. Fortunately, and this may be considered as one of the real virtues of the nucleus-based approach, one of the fundamental theorems of this approach – Isbell's theorem – provides sufficient information about the variety of exiting consistent belief operators to prove an elegant theorem about the doxastic invisibility of all Gettier propositions that can be defined for them. For our purposes Isbell's theorem can be formulated as follows:

(5.6) Theorem (Isbell's theorem). Let (X, OX) be a topological space such that (Int, *B*) defines a model for weak KB logic. Then one has for all belief operators *B:*

$$Int(A) \subseteq B(A) \subseteq IntClInt(A) \qquad \text{for all } A \in PX$$

i.e., all consistent belief operators *B* are equal or smaller than Stalnaker's belief operator IntClInt.

Proof. For a proof of Isbell's theorem the reader may consult Johnstone (1982, 2.4 Lemma, p. 50/51), or Picado and Pultr (2012, 8.3., Proposition, p. 40)).♦

(5.6) entails the following generalization of (5.4):

(5.7) Corollary. For all consistent belief operators *B*, Gettier propositions $G_B(A)$ are epistemically and doxastically invisible:

$$\text{Int}((G_B(A)) = B(G_B(A)) = \emptyset.$$

Proof: By (5.6) and (5.4) we have $G_B(A) \subseteq G(A)$ and *a fortiori*

$$B(G_B(A)) \subseteq \text{IntClInt}(G(A)) = \emptyset. \blacklozenge$$

Succinctly: Even if we don't know almost nothing about the class of belief operators related to Int, due to Isbell's theorem we can conclude that Gettier propositions are doxastically invisible to all of them.

6. Concluding Remarks. This paper has dealt with two main complementary issues:

(1) How Gettier cases can be constructed systematically in topological epistemology?

(2) How Gettier cases can be avoided systematically for appropriate topological universes of possible worlds?

On general theoretical grounds one can predict that Gettier cases show up for general topological models. (cf. Williamson 2015, 139). Thus, topological models of knowledge and justified belief provide robust evidence that traditional JTB-epistemology is in general false, independently of more or less contrived thought experiments.

On the one hand, the existence of JTB-doppelgangers for all topological models whatsoever suggests that the classical JTB-account should not simply dismissed as an obsolete erroneous theory. Rather, JTB is to be considered as a simplified account of knowledge and belief that works quite well in most cases, but fails in exceptional cases. The conceptual surgery that is necessary to remove Gettier situations from a topological universe (X, OX) of possible worlds is extensionally small in the sense that for all propositions A the difference between Int(A) and its JTB-counterpart doppelganger $\text{Int}_{nod}(A)$ is topologically negligible, i.e., nowhere dense. This fact may be interpreted as a partial rehabilitation of the classical JTB-account.

Unknowability and unbelievability of Gettier propositions confirm the impression that Gettier cases are exceptional. Topological epistemology has, so to speak, a janus face with respect to JTB: on the one hand it offers a justification to it, on the other hand, it provides a strict general refutation by formal mathematically valid arguments. Thereby topological epistemology may be considered as a useful addition to the many informal arguments that often rely on rather contrived thought experiments. In any case, topological epistemology strengthens a systematic conception of philosophy.

Sellars in his classical paper *Epistemic Principles* (Sellars 1975) asserted:

> The explication of knowledge as "justified true belief" though it involves many pitfalls [,] … is, I believe, essentially sound (Sellars (1975, p. 99).

Sellars did not give arguments for his traditionalist assessment of this issue. He simply assumed it:

> In the present lecture I shall assume that it can be formulated in such a way as to be immune from the type of counterexamples with which it has been bombarded since Gettier's pioneering paper in *Analysis*. Sellars (1975, ibid.)

Almost 50 years have passed since Sellars put forward his optimistic assessment that eventually a formulation of JTB would be found that is immune to Gettier's criticism. Today, Sellars's hope seems to be less realistic than ever. Since then, the production of ever more sophisticated counterexamples went on without interruption (cf. for instance Turri (2012), Borges et al. (2017)). Worse, many philosophers have given up the hope and even lost interest in this issue. Topological epistemology offers a way out of this deadlock. The topological account proposes to conceive JTB as one among many possible topological versions of epistemological logic each of which is characterized by some specific axiom(s). According to it, one should no longer ask the simple question: "Is knowledge justified true belief?" but rather "What type of topological models validates JTB?” This modified, simultaneously more modest and more sophisticated question has a neat and satisfying answer:

(6.1) Theorem. (Validity of the traditional JTB-account of knowledge as justified true belief): The JTB-account is valid for topological models based on nodec spaces (X, $O_{nod}X$). It is not valid for models that are not nodec.♦

Epistemically, the move from (X, OX) to (X, $O_{nod}X$) that amounts to the elimination of all Gettier cases {G(A); G(A) ≠ Ø, A ∈ PX} can be characterized as a learning process that improves the cognitive powers of the cognitive agent from an initial state of knowledge characterized by the operator Int to a more comprehensive knowledge characterized by the finer operator Int_{nod}.

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