Topological Games, Supertasks, and (Un)determined Experiments

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Abstract. The general aim of this paper is to introduce some ideas of the theory of infinite topological games into the philosophical debate on supertasks. First, we discuss the elementary aspects of some infinite topological games, among them the Banach-Mazur game. Then it is shown that the Banach-Mazur game may be conceived as a Newtonian supertask. In section 4 we propose to conceive physical experiments as infinite games. This leads to the distinction between determined and undetermined experiments and the problem of how it is related to that between determinism and indeterminism. Finally the role of the Axiom of Choice as a source of indeterminacy of supertasks is discussed.

Keywords: Infinite Topological Games, Banach Mazur Game, Supertasks, Newtonian Mechanics, Axiom of Choice, Axiom of Determinacy, (Un)determined Experiments.

1. Supertasks in Physics and Mathematics
2. Topological Games
3. Newtonian Supertasks
4. Experiments as Games
5. The Axiom of Choice and the Axiom of Determinacy

References

1. Supertasks in Physics and in Mathematics. A supertask (of order type ω) may be defined as an infinite sequence s_n of actions s_n, n = 1, 2, 3, ... carried out one after another in a finite time. For instance, Achilles's running a distance of 100 meters with constant speed may be considered as such a supertask: the first task s_1 consists in running the first 50 meters in five seconds, the second task s_2 in running the next 25 meters in 2.5 seconds, and so on, reaching the finish in exactly 10 seconds.

In the literature, usually only individual supertasks, carried out by one agent alone have been considered. There is no need to such a restriction, however. An interesting and natural generalization of individual supertasks are collective supertasks in which two (or more) agents are involved. Often such collective supertasks do not amount to a peaceful collaboration of the participating agents as, for instance, in a relay race, but rather the collective supertask is a competitive undertaking, in which each agent seeks to realize his own interests that often run contrary to that of the other. In other words, often a collective supertask is a competitive game in which one player aims to win against the other. Indeed, the most famous classical supertask was such as competitive game, to wit, the footrace between Achilles and the tortoise: Achilles wins, if he catches up with the tortoise before a certain time has elapsed, otherwise, the tortoise wins.

In mathematical terms, the game of Achilles and the Tortoise can be described in the following way. Let **R** denote the real line, and $[0,1] \subseteq$ **R** the interval of real number x, $0 \leq x \leq 1$. Let a_0, a_1, a_2, ... be the positions that Achilles occupies at the times t_0, t_1, t_2, ..., with $t_n = 1 - ½^{n-1}$; analogously the positions b_0, b_1, b_2, ... of the tortoise are defined. Let us further assume that the tortoise is given a head start, so that it starts the race at $b_0 = 1/2$, while Achilles starts at $a_0 = 0$. Denote the limit points that Achilles and the tortoise reach at time $t_\infty := 1$ by $a_{oo} := \lim_{n\to oo} a_n$ and $b_{oo} := \lim_{n\to oo} b_n$, respectively. Assuming that Achilles and the tortoise run with constant velocity such that Achilles runs twice as fast as the (rather fast) tortoise the collective supertask carried out by them can be described by the following diagram:

a_0 $\qquad\qquad\qquad\qquad$ a_1 $\qquad\qquad\qquad$ a_2 $\qquad\qquad$ a_3 $\quad$ a_4 ... a_∞

0----------------------------------1/2-----------------------3/4--------------7/8-------------1

$\qquad\qquad\qquad\qquad\quad$ b_0 $\qquad\qquad\qquad$ b_1 $\qquad\qquad$ b_2 $\quad$ b_3 b_4...b_∞

Achilles catches up with the tortoise at $t_\infty = 1$, since then $b_{oo} = \lim_{n\to oo} b_n = a_{oo} = \lim_{n\to oo} a_n$. If we do not fix the velocities of the competitors in advance, we may stipulate that Achilles is the winner if and only if he catches up with the tortoise or even overtakes it before the time 1 has elapsed, i.e. $\lim_{n\to oo} b_n \leq$ and $\lim_{n\to oo} a_n$. Otherwise the tortoise wins.

Zeno argued that Achilles did not have a winning strategy, since he would always stay behind the tortoise, since $a_n < b_n$ for all n implied that $a_{oo} = b_{oo}$. This is clearly fallacious. More generally, many arguments against the feasibility of supertasks rely on a fallacious argument that legitimates the inference from what is happening at the times t_n to what is happening at time t_{oo} (cf. Laraudogoitia 2004). This fallacy not only affects infinite games with two or more players, it is present already in those supertasks in which at least *prima facie* only one agent is involved, e.g. in Achilles's race without intervention of the tortoise:

a_0 a_1 a_2 a_3 a_4 ... a_∞

0---------------------------------1/2-----------------------3/4-------------7/8------------1

For this version of the race Zeno claimed that one could infer from $a_n < 1$ for all n that $a_\infty = \lim a_n < 1$.

Supertasks are usually assumed to belong to the realm of a more or less idealized physics, in particular Newtonian mechanics. Nevertheless, the ideal worlds, where supertasks take place closely resemble with the realm of pure mathematics. Of course, as long as we consider the universe of mathematics as a timeless realm it does not make much sense to speak of supertasks in mathematics. But at least for some tasks that require only the realization of countable many mathematical operations, it is not too far-fetched to conceive them as supertasks which an ideal mathematician carries out in a finite time much in the same way as Achilles did. A general pattern for such mathematical supertasks is provided by the countable axiom of choice (CAC):

(1.1) Countable Axiom of Choice (CAC). For every countable family $F := \{S; S \in F\}$ of non-empty sets S there is a function $C: F \longrightarrow \cup\{S; S \in F\}$ such that $f(S) \in S$ for each $S \in F$.♦

This axiom may be reformulated as the assertion that the "ideal mathematical agent" Mat has the capacity of carrying out supertasks of the following kind: Given a countable family $F = \{S_1, S_2, ...\}$ of nonempty sets, let s_i be the task of choosing an element from S_i. Then the countable axiom of choice asserts that Mat (cf. Kitcher 1984, Hoffman 2004) is able to perform the totality of the countable many tasks s_i in a finite time. To be specific, one may assume that Mat works in a similar fashion as Achilles in that he needs for the first choice s_1 the time s (s = second), for the second choice s_2 the time ½s, for the third choice s_3 the time ¼s, and so on. Thereby, the construction of a choice function $F \xrightarrow{C} \cup F$ can be conceived as supertask that is carried out in 2 seconds. Something like this is assumed, at least implicitely, when "real mathematicians" invoke (CAC) in their proofs which are thought to be carried out in a finite amount of time. Using (CAC) they feel entitled to the contention of having available a choice function C for any countable family F of non-empty sets. Analogously, the general axiom of choice (AC) may be conceived as the assertion that the “hypertasks” of constructing choice functions for arbitary families F of nonempty sets in a finite amount of time are possible. Worlds, in which such hypertasks are possible, are, of course, very remote from the real world and are rather remote from physically possible worlds. Thus, in this paper, we’ll leave aside hypertasks considering only countable supertasks that have at least some flavour of physical realisability. More precisely, we intend to enlarge the arsenal of Newtonian supertasks by some infinite topological games that can be realized in continuous Newtonian worlds (cf. Davies 2001).

An “ideal mathematical agent” with the capacity of carrying out countable mathematical supertasks may be considered as a rather fancy idea, but the impressive progress in the calculatory capacity of computers in the last decades may be regarded as a reason for taking the idea of an “ideal mathematician” or “ideal calculator” more seriously, at least as a regulative concept. If today a real world computer can carry out certain mathematical operations in finite time t, it seems plausible that at least in principle a more powerful computer in the future will be able to carry out these operations in half of the time. Pushing this idea to the limit, one is lead to the idea that an “ideal computer” would be able to carry out countable many operations a_n , n = 1,2 , ..., in a finite time. Of course, for various reasons this is impossible in our world. But at least, in a “continuous Newtonian universe” this may be possible (cf. Davies 2001). As Davies has explained in detail, on a more detailed view the intuitively appealing idea of an “ideal computer” needs elaboration but in the end the “ideal computer” and the “ideal mathematical agent” may be taken as sound and useful regulative ideas.

If we accept the idea that in our mathematical practice we assume the feasibility of countable (and even uncountable) infinite supertasks there is no reason to restrict one’s attention to the deeds of solitary ideal mathematical agents who perform their supertasks in isolation. Rather, it seems natural to broaden the perspective to include countable supertasks in which two (or more) ideal mathematical agents are involved somehow similar to the competitive game which Achilles and the Tortoise play since the times of Zeno. Then an individual supertask may be conceived as a degenerated game in which one player remains idle and does nothing. For instance, we may consider Achilles’s supertask of running a given distance in a given finite time as a game in which the tortoise is simply waiting at the goal:

a_0 a_1 a_2 a_3 a_4 … a_∞

0.--------------------------------1/2-------------------------3/4-------------7/8------------1

$b_0 = \ldots = b_\infty$

In this game Achilles wins if he catches up with the tortoise before a certain time, fixed in advance; otherwise, the tortoise wins.

Actually, infinite mathematical games appeared on the scene long before philosophers introduced "ideal mathematical agents" or pondered about the calculatory power of "ideal computers" that were able to carry out mathematical supertasks of one kind or another. The beginning of the theory of infinite mathematical games may be identified with a problem that the Polish mathematician Stanislaw Mazur proposed in the famous "Scottish Book" in 1935 in which the members of the mathematical community of the then Polish city of Lvow (formerly Lemberg, now Lviv) collected interesting mathematical problems and conjectures. Mazur's problem dealt with the winning strategies of an infinite game that later became known as the Banach-Mazur game (cf. Telgársky (1987), Oxtoby (1980)).

The general aim of this paper is to introduce some elementary ideas of the theory of infinite topological games into the ongoing philosophical debate on supertasks, which up to now is dealing mainly with supertasks from the realm of physics ignoring almost completely the many results that exist on supertasks in mathematics.

The outline of this paper is as follows. To set the stage, in the next section 2 we discuss the elementary aspects of some infinite topological games, among them the Banach-Mazur game (BM). In section 3 it is shown that the Banach-Mazur game may be conceived as a Newtonian supertask. This means that this game is a physically possible supertask in a suitably idealized continuous Newtonian world (cf. Davis 2001). In section 4 we propose to conceive physical experiments as infinite games between the two players Science and Nature. This leads to the distinction between determinated and undeterminated experiments and the problem of how this distinction is related to that between determinism and indeterminism. In section 5 the opposition of the Axiom of Choice and the Axiom of Determinacy for the determinacy of supertasks is discussed.

2. Topological Games. In this section we consider two typical infinite topological games needed in the following sections for the construction of some interesting new supertasks.

All games that we consider take place on the real line **R** or a part of it, most often on the unit interval I = [0, 1]. Throughout, **R** and I are assumed to be endowed with their standard (metrical) topology. Although most of the topological games to be considered in this section can be played on more general topological spaces, we concentrate on the special case of the real line and its standard topology. Many interesting problems arise already here. All topological and set-theoretical concepts used in the following are standard and may be found in any textbook on topology or (descriptive) set theory, see Willard (1970, Oxtoby 1980, Jech 1978, 2008) or Kechris (1995).

Let us start with a simple topological game recently proposed by Baker (cf. Baker 2007).[1] There are two players, Alice and Bob. Let T be a subset of the interval I = [0, 1] called the target set. The game consists in alternately choosing real numbers in the unit interval I. More precisely, it goes like this. Alice moves first, choosing any real number a_1 strictly between 0 and 1. Bob then chooses any real number b_1 strictly between a_1 and 1. On each subsequent turn, the players must choose a point strictly between the previous choices, i.e. Alice chooses her number a_n with $a_{n-1} < a_n < b_{n-1}$ and Bob can choose his number b_n under the condition that $a_n < b_n < b_{n-1}$. Since the unit interval I is a compact set and $(a_n)_{n \in \mathbf{N}}$ is a bounded monotonically increasing sequence it has a limit $a := \lim_{n \to \infty} a_n$. Alice wins if she suceeds in constructing a sequence (a_n) with limit point $a \in T$, and Bob wins if $a \notin T$.

(2.1) ... $\Rightarrow$ a b $\Leftarrow$ ...

0--------------------- a_1 ----------- a_2 --------------------- b_2 ---------- b_1 -----------1

Intuitively, Alice seeks to eventually reach a point inside T, while Bob seeks to hinder her in some way or other. For the time being, it may not be clear, how Bob can achieve this, since he cannot directly push out Alice of the target set. Note that according to the rules of the game one always has $a_n < b_n$. Nevertheless, as we shall see in a moment, he might be successful in his efforts to keep Alice out of the target if she plays in an unclever way or the target T is too small.

Baker's game can be interpreted as a collective supertask as follows: Alice's choice of a real number a_1 may be considered as the action that she moves from 0 to a_1 where she stops. Analogously, Bob's choice of b_1 can be interpreted as that he the moves from 1 to b_1 where he stops. Alice's next action is to move from a_1 to a_2 waiting there until Bob has moved from b_1 to b_2. After Bob has finished this action she moves from a_2 to a_3, and so on. All these actions are finite moves that can be carried out in a finite time. To be

[1] In the following this game is called "Baker's game", although I don't know whether Baker invented it or not.

specific, let us choose some finite time t and assume that Alice and Bob move from a_n to a_{n+1} in the time $t/2^n$ and from b_n to b_{n+1} in the time $t/2^n$, respectively. Analogously to the familiar calculations for the footrace of Achilles and the Tortoise one obtains that after the time $2 \sum_n t/2^n$ has elapsed Alice has carried out the supertask of passing through a_1, a_2, ... finally reaching a, while Bob has carried out the analogous supertask of passing through b_1, b_2, ... finally reaching a limit point b. In sum, the game between Alice and Bob may be described as a race somewhat similar to that between Achilles and the Tortoise. Only the winning conditions are different: Alice wins if her final position $a := \lim a_n$ which she eventually reaches, is a point of the target set T, whileBob wins if he can hinder Alice to end up in the target set T.

Since Alice and Bob are assumed to have full information about the moves of each other, whether one of them has a winning strategy depends on the target set T. In order to get a feeling for the questions that arise here, let us consider first some trivial cases for which Alice or Bob can be sure to win the game. Clearly, if T = [0, 1], Bob has no chance to win. He may carry out any move he likes, according to the rules of the game Alice is always able to construct a sequence (a_n) that converges to some limit point a located in T. On the other hand, if T = Ø Bob trivially wins, since any monotonically increasing sequence (a_n) of real numbers in a compact set such as [0, 1] always has a limit point a, but this point cannot be an element of the target Ø.

As the reader may check for himself, Bob always has a winning strategy if the target set is finite, i.e. $T = \{t_1 , ... , t_n\}$. Generalizing these rather trivial observations one may conjecture that Alice wins if the target set T is sufficiently large, while Bob wins, if T is sufficiently small. This is indeed true. Thus, the mathematical point of the game consists in determining what exactly is meant by “sufficiently large” and “sufficiently small”, respectively. In general, this is a highly non-trivial problem for which no general solution is known (cf. Telgársky 1987).

<u>(2.2) Proposition (Baker 2007, Proposition 1).</u> If T is countable, then Bob has a winning strategy.

<u>Proof.</u> Since T is countable, the elements of T may be enumerated as t_1 , t_2 , t_3 , ... Then, similar as for finite target sets, Bob has the following winning strategy. In his n-th move he

chooses $b_n = t_n$ if this is a legal move, otherwise he randomly chooses any allowable number. Since Alice's final destination $a := \lim a_n$ satisfies $a < b_n$ one has $a \neq b_n$, for all n, and thus $a \notin T$. In other words, Bob always wins.♦

Informally, a countable target is not sufficiently large to ensure that Alice has a winning strategy. As an example for (2.2) one may may take the set of rational numbers $\mathbf{Q} \cap [0,1]$. Although this target set is dense in [0, 1], Bob has a winning strategy. In order to characterize the target sets for which Alice has a winning strategy, one has to invest some more topology. Let us start with the following definition:

<u>(2.3). Definition.</u> Let X be a subset of [0,1]. A limit point of X is a point $x \in [0, 1]$ such that for every $\varepsilon > 0$, the open interval $(x - \varepsilon, x + \varepsilon)$ contains an element $a \in X$ other than x.

(i) X is closed if and only if each limit point of X is an element of X.

(ii) X is perfect if and only if it is closed and equal to the set of its limit points.♦

Points of X that are not limit points of X may be called isolated points. Thus, a perfect set may be characterized as a closed set that has no isolated points. Obviously, closed intervals [a, b], $0 \leq a < b \leq 1$ are closed. As is easily shown, they are even perfect. As is evidenced by finite sets, not all closed sets are perfect. On the other hand, the famous Cantor set C shows that there are perfect sets that are quite different from intervals in that C is nowhere dense in [0, 1].

After these preparatory definitions we can now formulate the following sufficient condition for Alice having a winning strategy:

<u>(2.4) Proposition (Baker 2007, Proposition 2).</u> If the target set T contains a perfect subset $T^* \subseteq T$, then Alice has a winning strategy.

<u>Proof.</u> Let $T^* \subseteq T$ be a perfect subset of the target T. Alice's strategy is to choose all her points a_n in a clever way from T*. If she succeeds in doing this she will win since $\lim a_n \in T^*$ due to the fact that T* is closed. Hence, in order to show that this strategy can be carried out, one has to show that Bob cannot hinder her to place her points appropriately in the perfect set T*. This is shown by induction on n.

For a perfect target set T* define the subset $T^{**} \subseteq T^*$ as the set of those points that are approachable from the right, i.e. $T^{**} := \{x; x \in T^*$ and for all $\varepsilon > 0$ there is a point y such that $y \in T^*$ and $y \in (x, x + \varepsilon)\}$. Since the infimum of T^* belongs to T^{**}, the set T^{**} is not empty, and Alice can choose $a_1 \in T^{**}$, since she is the first player. As can easily be shown that for $x \in T^{**}$ and any $\varepsilon > 0$ the open interval $(x, x + \varepsilon)$ contains an element of T^{**} (indeed, infinitely many). Assume that Alice has succeeded in placing the numbers a_j in T^{**} for $1 \leq i \leq n-1$, $n \geq 2$. Alice's only constraint on her nth move is that $a_{n-1} < a_n < b_{n-1}$. Due to the just mentioned feature of T^{**} she can find an element a_n of T^{**} that meets this constraint. This clinches the induction and shows that her strategy of choosing points from T^{**} is a feasible winning strategy.♦

An interesting instance of (2.5) is provided by the target set of irrational numbers IR = I - Q. Although IR itself is not a perfect set (it is not even closed, since its closure is the entire interval I), it can be shown by a non-trivial argument based on some surprising features of Cantor's discontinuum (cf. Oxtoby 1980, Lemma 5.1) that it contains a perfect subset. Hence, according to (2.5) Alice has a winning strategy by staying inside this set:

<u>(2.5) Corollary.</u> If the target set T is chosen to be irrational numbers, then Alice has a winning strategy.♦

This result can be generalized. Recall that for a topological space X the Borel algebra BX of X is the smallest σ-algebra that is generated by the open (or the closed) subsets of X by countable combinations of the set-theoretical operations of union, intersection and complement. Thus, BX contains the open sets OX, the closed sets CX, the countable intersections of open sets $G_\delta X$, the countable union of closed sets $F_\sigma X$, to mention the first classes in the Borel hierarchy of more and more complex subsets of X (cf. Jech 1978, Chapter 7, Kechris 1995, Chapter 2). A well-known result from set theory asserts:

<u>(2.6) Theorem.</u> Every uncountable Borel set contains a perfect subset.♦

This theorem allows us to complete the results (2.3) and (2.6) as follows:

<u>(2.7) Theorem.</u> If the target set T is Borel, then Alice wins if and only if T is uncountable,

or, equivalently, Bob wins if and only if T is countable.♦

What about winning strategies for target sets that are not Borel? Before this question can be considered as meaningful, one has to deal with the problem if there exists non-Borel sets at all. In a sense, non-Borel sets are rather strange creatures that are seldom found in ordinary mathematical practice. Indeed, nobody has ever found a non-Borel set without relying on the axiom of choice. As far as I know, it is unknown whether the Borel assumption in (2.7) is necessary or not. The discussion of the Banach-Mazur game in the following provides some evidence that the Borel assumption (or something similar) is probably necessary to ensure that the game is a determined game, i.e. a game for which either Alice or Bob have a winning strategy.

Now let us discuss a slightly more complex topological game that the very first infinite topological game *überhaupt* invented by the Polish mathematicians Mazur and Banach around 1935 (cf. Telgársky 1987). There are various versions of this game in the literature, the one presented here has the advantage that it has especially perspicuous winning strategies and can be interpreted as a Newtonian supertask in a particularly simple way.

Choose a subset $T \subseteq [0, 1]$ as the target set. The Banach-Mazur game is played as follows: Alice as the first player chooses a closed interval $I_1 \subseteq [0, 1]$; then Bob chooses a closed interval $I_2 \subseteq I_1$; then Alice chooses a closed interval $I_3 \subseteq I_2$, and so on. It is further assumed that the length of the intervals I_{2n+1} chosen by Alice (or Bob) is smaller than $½^n$. In sum, the players construct a nested sequence of intervals I_n

(2.8) $$[0, 1] \supseteq I_1 \supseteq I_2 \supseteq I_3 \supseteq .... \supseteq I_{2n} \supseteq I_{2n+1} \supseteq ... \supseteq \{\omega\}$$

such that the intersection $\cap I_n$ contains exactly one point ω. Alice wins if and only $\omega \in T$, otherwise Bob wins. Since a closed interval of the real line is fully determined by its endpoints, Alice's choice of an interval I_{2n+1} can be described as the choice of two points a_{2n+1}, a^*_{2n+1} with $a_{2n+1} < a^*_{2n+1}$ and $I_{2n+1} = [a_{2n+1}, a^*_{2n+1}]$. In a similar vein, Bob's choices of intervals amount to the choices of two points b_{2n} and b^*_{2n} with $b_{2n} < b^*_{2n}$. Then the successive moves in a Banach-Mazur game can be described by the following diagram:

... ω ...

(2.9) 0--------------a_1 ------ b_2 -----a_3--------------.---------a^*_3------ b^*_2 ---- a^*_1 -----------1

In order to interpret the Banach-Mazur game as a supertask, consider the following equivalent geometrical description of this game: Assume that Alice and Bob are driving together in a car. At the start, Alice is the driver, and Bob is the codriver. In her first move, Alice drives from (0,1) to (a_1, a^*_1). There she stops, and hands over the steering wheel to Bob. Then Bob drives from (a_1, a^*_1) to (b_2, b^*_2) where he stops. There, they switch again their roles and Alice drives from (b_2, b^*_2) to (a_3, a^*_3), and so on:

(2.10) (0, 1)——-——(a_1, a^*_1)——-——(b_2, b^*_2)————(a_3, a^*_3)———— ....(ω, ω)

In sum, running the game amounts to alternately driving in direction southeast starting at (0, 1) and finally arriving after infinitely many stages at a point (ω, ω) of the diagonal of the unit square $I \times I$ - recall that according to the rules $\lim a_n = \lim a^*_n = \lim b_n = \lim b^*_n = \omega$. Alice wins if ω is a point of the target set T, otherwise, Bob wins.

Let us assume that Alice and Bob always move on straight paths directly from their starting points to their endpoints. Then all the distances that Alice and Bob are crossing are finite. More precisely, the total distance L that they have passed through together after all stages of the race from (0, 1) to (ω, ω) is calculated to be $\frac{1}{2}\sqrt{2} \leq L \leq 1$, the exact value of L depending on the individual choices of the players.

This game is rendered a collective supertask as follows. Assume that for the task of driving from (a_n, a^*_n) to (b_{n+1}, b^*_{n+1}) Alice needs the time $t/2^n$; similarly Bob for carrying out his drivings. Assuming that the car has two steering wheels that such that always only one is operative when the car is moving, the change from Alice's being the driver to Bob's being the driver does not consume any time. Alice just stops driving and then Bob starts his turn. Hence for carrying out the whole trip from (0, 1) to (ω, ω) Alice and Bob spend the time $2 \sum t/2^n = 2t$.

Now let us tackle the problem of characterizing those targets for which Alice and Bob have winning strategies, respectively. For this task we need to introduce some more topology. We formulate it for general topological spaces X but will apply it only for the special case of X = [0,1].

<u>(2.11) Definition.</u> Let X be a topological space.

(i) A set $A \subseteq X$ is nowhere dense if every non-empty open set U contains a non-empty open set $V \subseteq U$ such that $A \cap U = \emptyset$.

(ii) A set $B \subseteq X$ is called meager (or of the first category) if it is the countable union of nowhere sets, i.e. $B = \cup_{n \in \mathbf{N}} A_n$, A_n meager. All other sets are called sets of the second category.

(iii) A set $D \subseteq X$ has the Baire property if there are sets P, $G \subseteq X$ with P meager and G open such $D = G \Delta P$, with $G \Delta P$ the symmetric difference := $(\complement G \cap P) \cup (G \cap \complement M)$.♦

Before we state the winning conditions for the Banach-Mazur game, some remarks on how this battery of definitions works for the special case we are really interested in may be in order.

Clearly, for $a \in [0, 1]$ the singletons {a} are nowhere dense, hence countable subsets of [0,1] such as the rational numbers **Q** between 0 and 1 are meager. It should be noted, however, that not all meager sets are countable. For instance the Cantor set is meager in **I** but uncountable. On the other hand, as is well known (Theorem of Baire) the set of irrational numbers **IR** is not meager in **I** but of the second category. Further, all Borel sets have the Baire property. Intuitively, a set with the Baire property is "almost" an open set, since it differs from an open set only by a meager set. As is well-known the set BAIRE([0,1]) of subsets of [0,1] having the Baire property is a σ-complete Boolean algebra that comprises the Borel algebra BOREL(X) of X (cf. Jech 1978, Chapter 7, 39.23)).

After these preparatory remarks one can characterize the targets for which Alice or Bob have a winning strategy in the Banach-Mazur game:

<u>(2.12) Theorem (Banach-Mazur, Oxtoby).</u> Let T be the target set of the Banach-Mazur game on the unit interval [0, 1]. The the following holds:

(1) Alice has a winning strategy if and only if T is of the second category.

(2) Bob has a winning strategy if and only if T is of first category.♦

<u>(2.13) Theorem (Banach-Mazur, Oxtoby).</u> If the target set T has the property of Baire, then Alice or Bob have a winning strategy.

Proof. If T has the Baire property, one can find G, P such that T = G Δ P, where G is open and P is of first category (see 2.11). If G is empty, then T is meager and Bob has a winning strategy by (2.13). If G is not empty, then G is of the second category and Alice will win.♦

(2.13) offers a neat and complete answer for target sets T that are Baire. What happens, if the target set is not Baire? In order that this question makes sense, first one has to ensure that there are subsets of [0, 1] that do not have the Baire property. This is indeed the case, at least if one assumes the axiom of choice (AC) is valid. More precisely, one can show that the following class of sets is non-empty:

(2.14) Definition. Let X be a topological space. An uncountable subset T ⊆ X is a Bernstein set if it does not have a perfect subset.♦

Relying on the unrestricted axiom of choice (AC) one can show the existence of Bernstein sets for which the Banach-Mazur game is undetermined:

(2.15) Theorem (Oxtoby 1980, pp. 29-30). If the target set T ⊆ [0, 1] of the Banach-Mazur game is a Bernstein set, then the game is undetermined, i.e. neither Alice nor Bob have a winning strategy.♦

In other words, constructing a Bernstein set may be conceived as a hypertask that goes beyond the capacities even of ideal mathematical agents like Alice and Bob.

3. Newtonian Games. In this section we show that the topological games discussed in the previous section give rise to Newtonian supertasks (cf. Pérez Laraudogoitia 2004). Defining supertasks baldly as infinite sequences of actions carried out one after another in a finite time by some agent allows for rather strange activities to be characterized as supertasks. One might wish to have a more precise characterization of the context or the environment in which these tasks are carried out. To add some more flesh to the skeletal description of supertasks given up to now it seems useful to distinguish between physically possible supertasks and physically not possible ones (cf. Pérez Laraudogoitia 2004). One way to do this, is to define physically possible supertasks as those supertasks that are possible in a continuous Newtonian world (cf. Davies 2001). This might be a way to distinguish between purely mathematical supertasks and physical ones that are closer to

our physical reality. As has been mentioned above, hypertasks licensed by the general axiom of choice (AC) may safely considered to be possible only in a mathematical universe far remote from any physically possible world. This is not to deny, of course, that worlds in which Newtonian supertasks are possible, still are highly idealized worlds.

In order to exhibit the Newtonian character of the infinite topological games between Alice and Bob that we discussed in the last section the basic ingredient is the argument that the moves of the players can be described as “staccato runs” (cf. Grünbaum 1967, Pérez Laraudogoitia 2006). These runs have been shown to be physically possible in continuous Newtonian spacetimes. For explaining the concept of “staccato run” let us consider once more Achilles’ original race as depicted in the diagram:

a_0 a_1 a_2 a_3 a_4 … a_∞

(3.1) 0----------------------------------1/2-----------------------3/4--------------7/8---------------1

b_0 b_1 b_2 b_3 b_4 …b_∞

Grünbaum refers to the sequence of spatial intervals (a_n a_{n+1}) as the “Z-sequence”, to the sequence of their partial sums of their lengths as the “Z-series”, and to the points a_i as Z-points. After these preparations we can state Grünbaum’s definition of the “staccato run” as follows:

> The *staccato motion* is the traversal of the Z-sequence in unit time by a runner who runs discontinuously as follows: He takes ¼ of a unit of time to traverse the first Z-interval of length ½ and rests for an equal amount of time; then he takes 1/8 of a unit of time to traverse the second Z-interval of lenght ¼ and rests for an equal amount of time, and so on. (Grünbaum 1968, 79)

In order to see that a run of Baker’s game may be considered as a collective Newtonian supertask, let us look once again on its characterizing diagram:

….⇒ a b ⇐…

(3.2) 0-------------------- a_1 ----------- a_2 --------.------.----- b_2 ---------- b_1 -----------1

Let us choose as Z-points for Alice the a_i and as Z-points for B the b_i. Then the first move of Alice is running from 0 to a_1 in time t/4, say. There she stops. When she has arrived at a_1 Bob starts running from 1 to b_1 in time t/4, meanwhile Alice rests at a_1. After Bob has reached b_1 Alice starts running from a_1 to a_2 in time t/8. At a_2 she rests for t/8, while Bob is performing his race toward b_2 in time t/8, and so on. In other words, Alice and Bob are performing two staccato runs, since for being a staccato run in Grünbaum’s sense with respect to the sequences of Z-points a_i and b_i, it is sufficient that these sequences converge to a and b, respectively. As is easily calculated, the entire collective supertask consisting of alternating staccato runs is performed in time t.

For exhibiting the Newtonian character of the Banach-Mazur game, we rely on its "Southeastern race" model (2.11). Similarly as in Baker's game, in the BM game Alice and Bob perform alternating staccato runs in Grünbaum's sense: In her first turn Alice drives in a staccato run from (0,1) to (a_1, a^*_1) in a finite time, say, t/2, carrying Bob with her. Immediately after she has finished this task, Bob transfers her, in his staccato drive, from (a_1, a^*_1) to (a_2, a^*_2). This transfer is carried out in time t/2 as well. In the next part of her staccato run, Alice drives with Bob from (a_2, a^*_2) to (a_3, a^*_3) in time t/4, then Bob takes over, moving with Alice from (a_3, a^*_3) to (a_4, a^*_4) in time t/4, and so on. In sum, the collective supertask of travelling from (0,1) to (ω, ω) is carried out in the time 2t.

The admissibility of staccato runs allows us to conceive a quite general class of games as Newtonian games.

<u>(3.3) Definition.</u> Let T $\subseteq$ [0, 1]. The game G(T) is defined as follows: Alice and Bob alternatingly choose $n_i \in \{0, 1\}$ with Alice beginning with n_0. Alice wins if and only if the infinite sum $\sum_{i=0 \dots \infty} n_i 2^{-i-1}$ is an element of the target set T.[2]♦

In a similar vein as the Banach-Mazur game, this game may be conceived of as the superposition of two staccato runs on the unit interval [0, 1] performed by Alice and Bob, respectively. In the first stage Alice drives together with Bob from 0 to $_\infty$ n_0 1/2. There Bob takes the wheel and drives with Alice to $n_0 1/2 + n_1 1/4$. Then Alice takes over again driving from $n_0 1/2 + n_1 1/4$ to $n_0 1/2 + n_1 1/4 + n_2 1/8$, and so on. Together they reach the finish at $\sum_{i=0 \dots \infty} n_i 2^{-i-1}$. Of course, if $n_i = 0$, they have a break. Thus, if in accordance with Grünbaum, Pérez Laraudogoitia and others one considers staccato runs as admissible in Newtonian mechanics (cf. Grünbaum 1967, Pérez Laraudogoitia 2006), "superpositions" of staccato runs as that performed by Alice and Bob in Baker's game and in the Banach-Mazur game appear to be admissible as well. Hence we may conclude that these games are <u>Newtonian</u> supertasks that are possible in continuous Newtonian worlds. The most interesting consequence of this fact is that among these supertasks there are <u>undetermined</u> Newtonian supertasks as is exemplified by Banach-Mazur games with Bernstein targets. In the next section this issue will be treated in more detail by interpreting experiments as games between science and nature.

[2] Equivalently this game may be formulated as an infinite topological game on the Cantor space $2^{\mathbf{N}}$:= {f: **N**------>{0, 1}} much in the same way as the Banach-Mazur game.

4. Experiments as Games. In this section I propose to conceive infinite topological games as a model for experiments. More precisely, I propose to conceive an experiment as a game that a scientist (Alice) plays against nature (Bob). What is at stake in the game is that the trajectory of a certain physical system that is submitted to certain conditions ends up in a certain state that has been specified in advance by the scientist as belonging to a target set T. If a run of the game produces an element of T, the experiment is said to confirm the scientist's expectations, otherwise not. To have a concrete example, think of a particle collider such as the Large Hadron Collider (LHC) in Geneve designed to collide opposing particle beams. In order to circulate the particle beams correctly an immensely complicated machinery of magnets and other devices has to work properly. For instance, more than one thousand dipole magnets are used to keep the beams on their circular path, and hundreds of supercooled quadrupole magnets are used to keep the beams focused. The particle beams may have to circulated several hours until they have eventually have reached a sufficiently high level of energy to be evaluated at certain intersection points. In very abstract terms such a collider experiment may described in game-theoretical terms as follows. The set up of the experiment is given by the topological space X, the target set T, and the operations that the two players Science (Alice) and Nature (Bob) are allowed to perform.

The experiment starts with Alice's first move by which the initial conditions of the experiment are set. Bob's first move responds to the initial state put forward by Alice. It should be noted that Alice does not know how exactly Bob will react. This assumption takes into account the fact that one cannot realistically expect that nature will answer in a unique way. Even if it would, we could never know this due to our approximative knowledge of empirical reality. In other words, Bob has a certain margin of possible answers. After Bob has carried out his move, Alice responds in a way that she considers as appropriate to achieve her aim – in the case of the (LHC), say, to keep the particle beam on track so that the run of the experiment may produce a confirmation of her prediction, namely, that she is able to determine the result of the "collaboration" between her and Bob such that the final outcome belongs to the target set T selected in advance.

Alice seeks to obtain a reliable result, i.e. she seeks to figure out a strategy that always produces a T-result in every run of the experiment. In other words, the experiment should be stable or determinate. In the ideal case of a well-designed experiment an ideal scientist will have a winning strategy againtst Nature, i.e. he will be able to repeat the same result in every run of the experiment. This suggests that we conceive experiments as games against nature. So it might be surprising that even for a perfect experimentator such as Alice with infinite information and infinite resources a god (or perhaps better a demon) is able to set up games, i.e. experiments in Newtonian mechanics, that are not determined, i.e. for which she has no winning strategy and the results of which she cannot predict.

The lack of stability exhibited by infinite games is a consequence of the excessive complexity of target sets permitted by the unrestricted axiom of choice (AC). In order that an experiment is empirically meaningful, some kind of stability must be assumed, however. At least in principle it must be possible to repeat the "same experiment". In a game-theoretical language this is to say that at least in principle all runs of the same game should have the same outcome. This means, it must be possible to reproduce an infinite sequence of types of states of a system arbitrarily often. The tokens of these types are not the same, since strictly speaking they cannot be reproduced. In other words, an empirically meaningful experiment should be determined, i.e. exhibit a certain stability, to be meaningful at all.

So it might be surprising that even for a perfect experimentator with infinite information such as Alice a god (or a demon) is able to set up games, i.e. experiments in Newtonian mechanics, that are not determined, i.e. for which she has no winning strategy and the results of which she cannot predict. One should note, that in the case of undeterminacy neither Alice nor Nature have a winning strategy, since, trivially, if Alice always lost, this could be a valuable experiment too, since Alice could just switch from the target set T to its complement CT.

This indeterminacy arising from the complexity of the target set might be compared with the indeterminacy that shows up in certain supertasks extensively studied by Pérez Laraudogoitia and others. Take supertask ST1:

```
                ...pn ...  p3        p2                    p1        ⇐                 p0
(4.1)
            0------------1/8---------1/4--------------------1/2----------------------------------1
```

From a game-theoretical perspective, this supertask is trivial: the scientist starts the game by carrying out his first move, namely, by accelerating p_0 to the left, and takes care that no other external forces interfer. Nature responds by following the rules of Newtonian mechanics that ensure that the particles p_i, $i \geq 1$, inherit the impuls from p_0 without loss, then come to rest when they collide with their immediate neighbors p_{i+1} such that the games terminates in finite time when all particles at rest shifted to the left. More interesting appears to be the reverse ST1* of ST1 being a supertask of order type $\omega^* = \ldots, n, n-1, \ldots, 2, 1$ which turns put to be indeterministic. At the beginning the scientist is confronted with the following configuration of countable infinitely many particles p_i, $i \in \mathbf{N}$, all at rest and located in

(4.2)

$$\begin{array}{ccccc} \ldots p_n \ldots p_3 & p_2 & p_1 & & p_0 \end{array}$$

0-------1/8-----------1/4-----------------1/2---------------------------1

all of which are assumed to be at rest. The point is that even if the system is shielded from all external forces for some time t, the configuration of the p_i might undergo a selfexcitation due to the fact that ST1* is the reverse of ST1 so that ST1* is possible if and only if ST1 is possible. Since in ST1 the energy of the system disappears without a trace, in the reverse process ST1* it might reappear without previous announcement, so to speak. Nevertheless one cannot be sure that such a phenomenon will happen. It is simply undetermined whether the system will selfexcitate or not during time t. Thus ST1* is indeterministic.

The indeterministic character of ST1* differs essentially from the undeterminacy exhibited by Banach-Mazur games with Bernstein targets, however. ST1* is not an infinite game properly understood. Infinite games have a well-defined first move. In contrast, ST1* has no definite beginning, since the sequence of particles p_i has no first member on the left. Thus, in the case of selfexcitation, there is no first particle that moves.

As far as I can see, Pérez Laraudogoitia never provides an example of an indeterministic supertask of type ω. Rather, the indeterministic supertasks he and other authors present are all of order type $\omega^* = \ldots, n, n-1, \ldots, 2, 1$. Clearly, these supertasks are not infinite games, since they lack a proper beginning.

In other words, the gate through which indeterminism enters the Newtonian realm is Newton's second law that allows for mechanical processes and their reverse counterparts. The lack of stability exhibited by infinite games is of a different kind, namely, the excessive complexity of target sets permitted by the unrestricted axiom of choice (AC). In order that an experiment is empirically meaningful, some kind of stability must be assumed, however. At least in principle it must be possible to repeat the "same experiment". In a game-theoretical language this is to say that at least in principle all runs of the same game should have the same outcome. This means, it must be possible to reproduce an infinite sequence of types of states of a system arbitrarily often. The tokens of these types are not the same, since strictly speaking they cannot be reproduced. In other words, an empirically meaningful experiment should be determined, i.e. exhibit a certain stability, to be meaningful at all.

5. The Axiom of Determinacy and the Axiom of Choice. The Axiom of Choice may be identified as the culprit for the undeterminacy of many infinite games. For instance, the "construction" of Bernstein sets for Banach-Mazur games requires the unrestricted axiom of choice AC. Constructing a Bernstein set may characterized as a "hypertask" by some higher order god or demon whose intellectual power outsmarts even that of "ideal mathematicians such as Alice and Bob, who only have the capacity of performing countable supertasks. Even for very clever "ideal mathematical agents" such as Alice and Bob, whocan carry a countable many mathematical operations in a finite time a hypertask may pose problems they cannot cope with.

The existence of undetermined games might appear counter-intuitive. One might insist on the intuition that for every 2-person game with complete information for both players and no tie there must exist a strategy either for the first or the second player. Losing such a game might be attributed to a contingent lack of information or cleverness or both of one of the players which, by definition, does not occur in the idealized situation that we are dealing with – one player must have a winning strategy.

To tackle this problem, it might be expedient to cut down the problem to its bare bones, leaving aside all physical considerations:

(5.1) Definition . Let $\mathbf{N}^N$:= {f: N-----f---->N} be the set of all mappings of the natural numbers **N** into **N**.[3] As usual, an element f of $\mathbf{N}^N$ is identified with an infinite sequence (x_o, x_1, x_2 , ...), $x_i \in \mathbf{N}$. With each subset $T \subseteq \mathbf{N}^N$ we associate the following infinite game G(T): Alice and Bob successively choose natural numbers:

$$\begin{array}{lllllll} \text{Alice:} & a_0 & a_1 & a_2 & a_3 & \ldots\ldots & (a_i \in \mathbf{N}) \\ \text{Bob:} & b_0 & b_1 & b_2 & b_3 & \ldots\ldots & (b_i \in \mathbf{N}) \end{array}$$

[3] Actually N^N can be endowed with a canonical topological structure. Endowed with this structure it is is a called the Baire space (cf. Jech 1978, p 36/37). It can be shown to be homeomorphic to the space of irrational numbers of the unit interval [0, 1]. In a similar way as (3.2) the game (5.1) can be reformulated as a topological game like the Banacn-Mazur game.

Alice wins if the resulting sequence (a_0, b_0, a_1, b_1, a_2 , b_2, ...) is in the target set T, otherwise Bob wins.[4]♦

A strategy for a player is a set of rules which tell the player exactly how to move, depending on what has happened earlier in the game. The player uses the strategy s if each of his moves in the game obeys the rules of s. The strategy s is a winning strategy if the player wins every play in which he uses the strategy s. A formal definition of a winning strategy for topological games in terms of trees may be found in Kechris (1995, XX) or in Oxtoby (1980, 27- 28). A winning strategy guaranties that the player who uses it wins every run of the game, regardless what his adversary does. Thus a game for which one of the players has such a strategy may be called a stable or determined game, since the result of the game is determined in principle, i.e. if we ignore the possibility that one of players does not follow the winning strategy for lack of skill or negligence. Undetermined games show that even under highly idealized circumstances undeterminacy cannot be excluded. If undeterminacy is considered as a fault, one should subscribe to a set theoretical framework from which it is excluded. The most common way to do this is the introduction of the so called axiom of determinacy (AD):

<u>(5.2) Axiom of Determinateness (AD) (Jech 2008 (1973), 12.3).</u> For every target set T the game G(T) is determined, i.e., either Alice as a winning strategy or Bob has a winning strategy.♦

Recall that an ordering < of a set X is a well-ordering if and only if every non-empty subset Y ⊆ X has a first element with respect to the ordering <. As is well-known the axiom of choice (AC) is equivalent to the so-called *Well-Ordering Principle* according to which every set can be well-ordered. Evidently, the set of natural numbers **N** with its usual ordering is a well-ordering. On the other hand, the rational numbers **Q** and the real numbers **R** are not well-ordered with respect to their usual order structures. The well-ordering principle asserts that the task of well-ordering them can be carried out, i.e. that they can be endowed with different order structure that are well-orderings. As is well-known the axiom of choice (AC) is equivalent to the so-called *Well-Ordering Principle* according to which every set can be well-ordered. For the rational numbers **Q** an explicit well-ordering can be

[4] Actually the set N^N is a topological space, to wit, the Baire space, which has a highly interesting topological structure. More precisely, it may be shown to be homeomorphic with the space of irrational numbers (cf. Jech 1978).

given by Cantor's diagonal method, relying only on (CAC). On the other hand, no explicit well-ordering of **R** is known.Now we can state the following theorem:

<u>(5.2) Theorem (Jech 2008 (1973), Theorem 12.14).</u> If the set of all real numbers **R** can be well-ordered , then there exists a game G(T) that is not determined.♦

To put it bluntly, determinateness and unrestricted choice contradict each other. There is a huge literature on the issue of how to assess the respective merits of (AD) and (AC) (cf. Jech 2008 (1973), Kechris (1995) which need not concern us in the present context. Be it sufficient to state (AD) is compatible with the very existence of ideal mathematical agents like Alice and Bob:

<u>(5.3) Proposition (Jech 2008 (1973), Lemma 12.15).</u> The axiom of determinateness (AD) implies the countable axiom of choice (CAC) according to which every countable family F of non-empty set of real numbers has a choice function.♦

The unrestricted axiom of choice (AC), contradicting the axiom of determinacy (AD), leads to undetermined Newtonian supertasks such as Banach-Mazur games with Bernstein targets. If undeterminateness is considered as a fault, one should subscribe to a set theory from which it is excluded. For this one has to pay a price, however. There are many areas in mathematics for which some version of the axiom of choice (stronger than (CAC)) is considered as almost indispensable. The interesting point is that, having the problem of the physical feasibility of supertasks in view, also physicists might have a saying on the issue which set theoretical axioms are to be considered as the "correct" ones. Of course, as long as we stick to outdated physical theories such as Newtonian mechanics, arguments from physical theories in favor or against some set-theoretical axiom may not be very compelling. Things might change if one could rely on other, more substantial physical theories in questions of supertasks.

A way how this could possibly achieved has been sketched in (Pérez Laraudogoitia, Bridger, and Alper 2002). In this paper the authors briefly mention the possibility of characterizing particle systems in two different ways as Newtonian, to wit, as "locally Newtonian" and as "globally Newtonian". A system of particles together with the forces acting on them, is said to be locally Newtonian if the motion of each particle can be described by Newton's laws, it is globally Newtonian if the system as a whole can be described in Newtonian terms. This requires to set up a convenient phase-space for the system, in which the temporal evolution of the system is described in terms of the system's trajectory that has to obey the laws of global Newtonian mechanics. If the number of the particles of the system is infinite the phase space has infinitely many dimensions. Although this may cause certain mathematical difficulties (cf. (Pérez Laraudogoitia, Bridger, and Alper 2002)), at least *grosso modo* the theory of globally Newtonian supertasks seems to follow the lines of the more elementary theory of locally Newtonian systems. The mathematical reason is that the resulting phase manifolds for global systems still belong to the class of so called "Polish spaces" (cf. Jech 1978, Kechris 1995). As is well known Polish spaces behave topologically essentially in the same way as the unit interval [0, 1]. Thus it is to be expected that supertasks in the more general arena of Polish spaces behave more or less in the same way as supertasks that take place in the apparently quite narrow framework of the unit interval. At least, this is the case for infinite games such as the Banach-Mazur game (cf. Oxtoby 1980, Kechris 1995).

In general, then, it might be promising to study supertasks that concern physical theories other than Newtonian mechanics. If the phase spaces of these theories are Polish spaces - as will be usually the case - questions of determinacy and indeterminacy may be studied essentially in the same manner as for supertasks in the arena of the unit interval [0, 1], but possibly stronger requirements for their physical feasibility of these supertasks can be formulated. This might be useful for the aim of singling out "really" physically possible supertasks.

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