TOPOLOGICAL MODELS OF COLUMNAR VAGUENESS

Abstract. The main aim of this paper is to show that the concept of vagueness based on an S4 modal operator **C** of clearness (**C**A to be read as "It is clear that A", "It is definitely the case that A", or similarly) leads to a concept of columnar higher-order vagueness. More precisely, the class of topological models of the operator **C** can be identified with the class of weakly scattered topological spaces such that the resulting concept of higher-order vagueness is columnar. A philosophically particularly interesting class of weakly scattered spaces, recently introduced by Rumfitt to cope with the Sorites paradox and other problems of vague concepts, is the class of polar topological spaces. More generally, all topological spaces support a concept of stably columnar vagueness that is only slightly weaker than proper columnar vagueness. Further, for all topological spaces, the boundaries of sets that satisfy the McKinsey axiom are columnar.

Finally, it is proved that higher-order vagueness is (stably) columnar higher-order vagueness not only for S4-operators but also for operators that instead satisfy the axioms of Williamson's "logic of clarity" (which is not S4 but characterized by Brouwer's axiom (B)). Thus, (stably) columnar vagueness may be said to be almost ubiquitous in the existing modal accounts of vagueness.

Key words: Columnar Higher-Order Vagueness, Topology, Weakly Scattered Spaces, Polar Spaces, McKinsey Axiom S4.1, McKinsey Algebra.

1. Introduction. The aim of this paper is to provide a topological clarification of the formal properties of the concept of higher-order vagueness (cf. Bobzien 2015,

Bobzien 2013, Keefe 2015). Higher-order vagueness of some kind or other shows up in many contemporary accounts of vagueness. According to Bobzien (2015, p.63) one may distinguish between two kinds of higher-order vagueness, hierarchical higher-order vagueness and columnar higher-order vagueness:

> Hierarchical higher-order vagueness is characterized by a hierarchy of consecutively higher orders of borderline cases of a vague predicate (that include clear (definite, determinate) borderline cases, and (ii) whose extensions do not overlap… Columnar higher-order vagueness differs from hierarchical higher-order vagueness in that, extensionally, it contains just one kind of borderline cases, and each borderline case is radically higher order, or radically borderline, i.e. borderline borderline …, ad infinitum (Bobzien (2015, 63)).

The general result of this paper is that for most accounts based on a precise modally defined concept of "clearness", "clarity" or something similar, the resulting concepts of vagueness turn out to be "almost" columnar (in a sense to be rendered precise in the following). In particular, vagueness amounts to columnar vagueness for S4 concepts, which, for familiar reasons, may be characterized as topological concepts of vagueness (cf. McKinsey and Tarski (1944)). Somewhat surprisingly perhaps, also for Williamson's logic of clarity that is not S4, higher-order vagueness turns out to be (stably) columnar. The details are as follows.

The concept of vagueness may be defined in a framework of a modal logic based on an operator **C** such that **C**A is interpreted as "It is clear that A", "It is definitely the case that A", or similarly. Then, with the aid of **C** an operator **U** is defined such that **U**A is as unclearness in the sense that **U**A is to be read as "It is not clear that

A, and it is not clear that not A". More formally, in terms of **C**, the operator **U** is defined as

$$(1.1) \qquad \mathbf{U}A = \text{NOT } \mathbf{C}A \text{ \& NOT } (\mathbf{C}(\text{NOT} A)))$$

Abbreviating the n-th iteration **U** •… • **U** of **U** by $\mathbf{U}^n$ ($n \geq 1$), the basic claim of a columnar account of vagueness is that **U** satisfies the equation

$$(1.2) \qquad \mathbf{U}^n A = \mathbf{U}A \quad , \quad n \geq 1.$$

In the following, **C** is assumed to be an S4 modal operator, i.e., **C** • **C** = **C**. This assumption is far from unanimously accepted. For a recent discussion of this issue, see Bobzien (2015) and Keefe (2015). Actually, the discussion whether **C** has to be assumed an S4-operator or not may be less important than it appears at first view since also for Williamson's „logic of clarity" based on an operator of clarity C – which does not satisfy the S4 principle - the „unclarity operator" U (corresponding to **U**) can be shown to be stably columnar for „fixed margin models" (cf. Williamson (1994)) in the sense that

$$(1.2)^* \qquad U^n A = U^2 A \quad , \quad n \geq 2.$$

Actually, this exactly corresponds to the behavior of **C**: If the logic of **C** is S4, only stable columnarity in the sense of (1.2)* is true for **U** and not the slightly stronger columnarity (1.2). Rather, as Bobzien rightly emphasizes, in order to ensure the validity of (1.2), S4 has to be strengthened to S4.1. As is well known S4.1 is characterized as an extension of S4 that satisfies the McKinsey axiom:

(1.3) **C** NOT **C** NOT $\Rightarrow$ NOT **C** NOT **C**

McKinsey and Tarski (1944) proved that the modal system S4 is the logic of all topological spaces in the sense that a proposition is valid in S4 if and only if its topological interpretation is valid in all topological spaces. McKinsey and Tarski thus laid the foundation for the new discipline of "spatial logic", which would become an extremely fruitful symbiosis of modal logic and set-theoretical topology. For a contemporary survey, see van Benthem and Bezhenishvili (2007). McKinsey and Tarski's result has been generalized in many ways, establishing a correspondence between certain classes of topological spaces on the one hand and certain extensions S4.X of standard S4 logic on the other. For the purposes of this paper, the following special case is relevant: Let WSC denote the class of weakly scattered topological spaces (to be defined precisely in the next section). Then, the extension of S4 corresponding to this class of spaces is S4.1 (cf. van Benthem, Bezhanishvili (2007), Bezhenashvili, Mines, Morandi (2003), Bezhenashvili, Esakia, Gabelaia (2004), Gabelaia (2001)).

In the following, we will exploit the correspondence between S4.1 and WSC to elucidate the formal properties of columnar vagueness. From a topological perspective, the clearness operator **C** corresponds to the topological operator int of the interior, and its operator **U** of unclearness corresponds to the topological boundary operator bd defined by

(1.4) bd(A) := Cint(A) & Cint(CA) = cl(A) $\cap$ cl(CA) (C set-theoretical complement)

In terms of the topological concept of boundary, the basic claim (1.2) of the columnar account of vagueness reads

(1.5) $$bd(bd(A)) = bd(A)$$

From now on, we will use topological terminology throughout; i.e., instead of using the modal operators **C** and **U**, we will use the corresponding topological operators int and bd. The only reference to a logically inspired language will be that the expression "columnar boundary" is used to refer to a topological boundary operator bd that satisfies $bd^2 = bd$.

The class of topological spaces for which the boundary bd(a) of a is columnar for all a is the class of weakly scattered spaces (cf. Gabelaia (2001), Bezhenashvili, Mines, Morandi (2003), Bezhenashvili, Esakia, Gabelaia (2003)).

Weakly scattered topological spaces, however, are a rather elusive species of topological spaces. This is evidenced by the fact that familiar topological spaces such as Euclidean spaces and their derivatives are anything but weakly scattered. Thus, if one intends to use topological means for the explication of concepts such as vagueness, and borderlineness, it would be highly desirable to have a reservoir of "concrete" weakly scattered spaces that are "naturally related" to matters of vagueness. Here, the polar spaces of Rumfitt come to the rescue. It will be shown that polar spaces, recently introduced by Rumfitt in his book *The Boundary Stones of Thought. An Essay in the Philosophy of Logic* (Rumfitt (2015)) to cope with the Sorites paradox, provide a class of intuitively appealing weakly scattered spaces with especially nice properties.

However, the concept of columnar boundaries makes sense not only for weakly scattered spaces but also for topological spaces in general. It will be shown that for all topological spaces, “many” subsets have columnar boundaries.[1] Issues concerning the columnarity of boundaries appear in general topological spaces and are to be considered general features of topological models of vagueness.

2. Weakly Scattered Spaces as Models of Columnar Vagueness. Let us begin by recalling the basic topological concepts that will be used throughout this paper. A topological space is a pair (X, OX), where X is a non-empty set and OX is a family of subsets of X containing X and Ø that is closed under finite intersections and arbitrary unions of its elements. The space (X, OX) is an Alexandroff space if the arbitrary intersection of elements of OX belongs to OX (cf. Alexandroff (1937)). The subsets of X belonging to OX are called open sets of the space X; the family OX of open subsets of X is also called a topology on X. Set-theoretical complements of opens are called closed sets. The ***interior*** int(A) of a set $A \subseteq X$ is the largest open set contained in A, whereas the ***closure*** cl(A) of A is the least closed set containing A. Clearly, int and cl are interdefinable, i.e., int = C cl C, and, correspondingly, cl = C int C, with C denoting the set-theoretical complement in X. For a topological space, the interior int and the closure cl can be conceived as operators on the set of subsets of X. They satisfy the so-called Kuratowski axioms:

(2.1) (Kuratowski Axioms). Let (X, OX) be a topological space. Then, the operators cl and int satisfy the following axioms ($A, B \subseteq X$):

(1)	$cl(A \cup B) = cl(A) \cup cl(B)$	(1)*	$int(A \cap B) = int(A) \cap int(B)$.
(2)	$cl(cl(A)) = cl(A)$.	(2)*	$int(int(A)) = int(A)$.
(3)	$A \subseteq int(A)$.	(3)*	$int(A) \subseteq A$.
(4)	$cl(Ø) = Ø$.	(4)*	$int(X) = X$.

[1] What this means exactly will be explained in detail later.

Clearly, (1)-(4) is equivalent to (1)*-(4)*. In the following, we will use these axioms without explicitly mentioning them. ♦

(2.2) Definition. Let (X, OX) be a topological space.

(i) A subset $A \subseteq X$ is dense iff $cl(A) = X$. A subset $B \subseteq X$ is nowhere dense iff $int(cl(B)) = Ø$.

(ii) A point $x \in X$ is isolated iff the singleton $\{x\} \in OX$. The set of isolated points of X is denoted by ISO(X).

(iii) (X, OX) is weakly scattered iff cl(ISO(X)) is dense in X, i.e., cl(ISO(X)) = X.♦

For a class K of topological spaces (X, OX), let L(K) denote the set of formulas of the modal propositional calculus that are valid for all members of K interpreting the modal operators $\Box$ and $\Diamond$ as interior operator int and closure operator cl, respectively. Due to the fundamental result of McKinsey and Tarski, L(K) is a normal extension of S4. The set L(K) of formulas is called the modal logic of the class K of topological spaces.

For instance, if T denotes the class of all topological spaces and ALEX the class of Alexandroff spaces, then L(T) = L(ALEX) = S4. Many results of this type have been obtained for special classes of topological spaces (cf., for instance, van Benthem, Bezhanishvili (2007), Bezhenashvili, Mines, Morandi (2003), Bezhenashvili, Esakia, Gabelaia (2003), or Gabelaia (2001)). For this paper, the following theorem is fundamental:

(2.3) Theorem. Let WSC be the class of weakly scattered topological spaces. Then, the modal logic L(WSC) corresponding to WSC is S4.1, defined as the extension of

S4 by the McKinsey axiom

(MK) $\Box\Diamond A \Rightarrow \Diamond\Box A$.♦

(2.4) Examples and Counter-examples.

(i) Denote the power set of X by PX. Then (X, PX) is a topological space called the discrete topological space on X. Clearly, (X, PX) is weakly scattered and Alexandroff.

(ii) For any set X, the structure (X, {Ø, X}) is a topological space called the indiscrete topology on X. If X has more than one element, (X, {Ø, X}) is not weakly scattered, since no point is isolated. Every non-empty set $A \subseteq X$ is dense with respect to the indiscrete topology.

(iii) Let X be the set {0, 1} of two elements and OX = {Ø, {1}, X}. The topological space (X, OX) is called the Sierpinski space. The only isolated point is the point {1} and clearly cl({1}) = X. Thus, the Sierpinski space is weakly scattered. For it, the set {1} is dense, and the set {o} is closed and nowhere dense. As a finite topological space, the Sierpinski space is Alexandroff.

(iv) The real line (**R**, O**R**) endowed with the standard Euclidean topology is not weakly scattered. (**R**, O**R**) has no isolated point at all. The set **Q** of rational numbers is dense. The set **Z** of integers is nowhere dense. The space (**R**, O**R**) is not Alexandroff, since the arbitrary intersection of open intervals may not be open.♦

These examples show that the concepts of weakly scattered spaces and Alexandroff spaces are consistent and non-trivial: weakly scattered spaces exist, but not all topological spaces are weakly scattered. In the next section, we will introduce several "realistic" examples of weakly scattered spaces that are "naturally related" to issues of vagueness.

3. Polar Spaces are Weakly Scattered. The aim of this section is to show that there are plenty of spaces relevant for issues of vagueness that are weakly scattered. For these spaces, the concept of boundary turns out to be columnar in the sense that bd(A) = bd(bd(A). This may be taken as an argument that columnar vagueness is indeed a "natural" phenomenon.

As mentioned, polar spaces were recently introduced by Rumfitt (Rumfitt (2015)). In topology, polar spaces and related classes of spaces have been discussed for some time (although not under this name, of course) (cf. van Benthem, Bezhanishvili (2007), Bezhenashvili, Mines, Morandi (2003), Bezhenashvili, Esakia, Gabelaia (2003), Gabelaia (2001)).

Although Rumfitt explicitly defines the topology of polar spaces, he does not address the specifics of this topology. This also holds for the numerous reviews of Rumfitt's book, which all take note of its topological argumentation only in passing. The topological structure of polar spaces is rather specific and deserves to be made explicit.

Rumfitt's approach may be succinctly described as follows: Given a set X, a subset P of X is selected. The elements of P are to be interpreted as prototypical or paradigmatic elements of X. For instance, if X is a set of colored objects, the elements of the subset P ⊆ X are to be considered "prototypically" or "paradigmatically" colored in some sense. For instance, an element of P is a "typically" blue object or a "typically" red object. In the following, the elements of

P are called poles of the space X. The selection of poles is assumed to satisfy the following requirements:

<u>(3.1) Definition (Pole Distribution for X)</u>. Let X be a non-empty set, $P \subseteq X$. A function $X \xrightarrow{\quad m \quad} 2^P$ is a pole distribution iff it satisfies the following requirements:

(1) For all $x \in X$, the sets $m(x) \subseteq P$ are not empty.

(2) For $p \in P$, one has $m(p) = \{p\}$.

A pole distribution is denoted by (X, m, P).♦

Rumfitt proved the following (cf. Rumfitt (2015, 243, Fn. 15):

<u>(3.2) Proposition</u>. A pole distribution (X, m, P) defines a topology (X, OX) by an interior kernel operator $PX \xrightarrow{\quad int \quad} PX$ defined by

$$y \in int(A) := y \in A \text{ and } \forall p \in P\ (p \in m(y) \Rightarrow p \in A) \qquad (A \subseteq X)$$

The topological space (X, OX) is called a polar topological space. The subset $P \subseteq X$ is called the set of poles.

<u>Proof</u>: To check that int satisfies Kuratowski axioms (1)*-(4)* of (2.1), see Rumfitt (2015, 243-244). ♦

Rumfitt does not investigate polar topologies in any detail. However, the details of the polar topology are important to show that polar spaces are (weakly) scattered spaces. Let us start with the following basic result:

<u>(3.3) Proposition</u>. The topology (X, OX) defined by the polar distribution (X, m, P) is an Alexandroff topology; i.e., <u>arbitrary</u> intersections of open sets are open.

Proof. Let A_i be an arbitrary family of open sets of X. To show that (X, OX) is Alexandroff, one must show that $\cap A_i = \mathrm{int}(\cap A_i)$. For trivial reasons one has int $\cap A_i \subseteq (\cap A_i)$. In order to show $\cap A_i \subseteq \mathrm{int}\,(\cap A_i)$ one argues as follows: By definition of the operator int, we obtain

$y \in \cap A_i$ iff for all $i(y \in A_i)$

iff $\forall i\ (y \in A_i$ and $\forall p(p \in P$ and $p \in m(x) \Rightarrow p \in A_i\,)$ (since A_i is open)

iff $(y \in \cap A_i$ and $\forall p(p \in P$ and $p \in m(x) \Rightarrow p \in \cap A_i)$

iff $y \in \mathrm{int}(\cap A_i)$.♦

Proposition (3.4). A polar topology (X, OX) is weakly scattered. More precisely, the set of isolated points ISO(X) of X is just the set P of poles. This set is dense in X, i.e., cl(P) = cl(ISO(X) = X.

Proof. First, we show that the singletons of poles are open in the polar topology, i.e., $\{p\} \in OX$. Then, it is proved that cl(P) = X. Let $q \in P$. By definition of int for $\{q\}$, one has

$$y \in \mathrm{int}(q) \text{ iff } y \in \{q\} \text{ and } \forall p(p \in P \text{ and } p \in m(y) \Rightarrow p \in A).$$

By definition of a pole distribution m for a pole q, there is only one pole in m(q), namely, q itself. Hence, $\{q\}$ is open for $q \in P$. Slightly more difficult is the calculation of cl(q): By definition, the closure operator cl of (X, OX) is given by (cf. Rumfitt (2015, p. 244)

$$x \in \mathrm{cl}(A) := x \in A \text{ or } \exists p \in P(p \in m(x) \text{ and } p \in A)$$

For $A = \{q\}$, $q \in P$, this yields $\mathrm{cl}(q) := \{x;\ q \in m(x)\}$. To show that the set P of poles is just the set ISO(X) of the isolated elements of X and is dense in X, one first observes that X is Alexandroff; therefore, one has $cl(P) = \cup_{p \in P} cl(p)$. Now, by definition of m for all x, the set m(x) is not empty. Hence, every x is an element of at least one set cl(p). Therefore, cl(P) is dense in X; i.e., X is weakly scattered.[2] ♦

The main example of a polar space discussed by Rumfitt is the well-known color circle X. In it, prototypical shades of colors such as red, orange, and yellow serve as poles P (cf. Rumfitt (2015, 235ff). This space plays a central role in philosophical discussions of vagueness. Thus, the fact that vagueness emerges as columnar vagueness for this space may be taken as an argument for the adequacy and naturalness of the concept of columnar vagueness. Moreover, the polar topology is a non-trivial topology, as the following proposition shows:

<u>(3.5.) Proposition.</u> Let (X, OX) be the color circle endowed with the polar topology generated by familiar poles such as red, orange, and yellow. For the color circle endowed with the polar topology, neither Brouwer's axiom (B) nor the converse MK* of McKinsey's axiom is valid[3]:

(MK)* $\quad CA \subseteq int(C(int(CA))$ (B) $\quad cl(int(A)) \subseteq int(cl(A))$

<u>Proof</u>. Let (X, OX) be a polar topology on X defined by a polar distribution (X, m, P), $p \in P$ such that $intcl(p)) \neq \{p\}$. That is to say, $\{x; m(x) = \{p\}\} \neq \{p\}$. Take $A = X - \{p\}$. Then, Brouwer's axiom requires that $\{p\} \subseteq int(C(int(C\ A)) = int(X - \{p\})$. Clearly, however, $\{p\} \not\subseteq int(X - \{p\})$. Hence, the polar space (X, OX) does not

[2] Polar spaces can easily be shown to be scattered, not just weakly scattered; however, we do not need this result.

[3] Axioms (MK)* and (B) have a certain relevance for epistemological matters: (MK)* is characteristic for the system S4.2, which corresponds to the class of extremally disconnected spaces that have proven useful for modeling the concept of belief. Brouwer's axiom (B) is a characteristic axiom of Williamson's logic of clarity (cf. Williamson 1994), see section 5.

satisfy (B).

The converse (MK)* of the McKinsey axiom is disproved by considering A = {p}, p ∈ P. Then, one obtains cl(int(p)) ={x; p ∈ m(x)} and intcl(p)) = {x; {p} = m(x)}. Clearly, = {x; p ∈ m(x)} ⊈ intcl(p)) = {x; {p} = m(x)}. Thus, (MK)* does not hold.♦

Obviously, the polar topology of the color circle does not coincide with the usual Euclidean topology of this space: in the Euclidean topology the color circle clearly lacks isolated points and is not weakly scattered, while in the polar topology it has plenty of isolated points (= poles) such that the set of isolated points is dense rendering thereby the color circle a (weakly) scattered. This difference between the two topologizations should be considered an advantage. The standard Euclidean circle is an overly strong structure that produces many artifacts that do not correspond to any empirical experience. For instance, what does it mean that certain color experiences x, y, z, and w are such that x and y have the same distance from each other as have z and w? In contrast, the polar topology is a much more modest structure that generates fewer structural artifacts.

Polar spaces are not, however, the only area where columnar boundaries show up. In the next section, we show that columnar boundaries are virtually ubiquitous in topological models.

4. The Boolean Algebra of McKinsey Sets of a Topological Space. In this section, we show that every topological space (X, OX) comes along with a large class of subsets A ⊆ X, the boundaries bd(A) of which are columnar, i.e., satisfy bd(bd(A))

= bd(A). Moreover, the class of these subsets has the structure of a Boolean algebra (with respect to the set-theoretical operations inherited from the set X).

(4.1) Definition. Let (X, OX) be a topological space. A set $A \subseteq X$ is called a McKinsey set of (X, OX) if and only if bd(bd(A)) = bd(A). The set of McKinsey sets of (X, OX) is denoted by MKX.

If (X, OX) is weakly scattered, all subsets of X are McKinsey sets. In general, however, this is not the case, as is shown by the following elementary example:

(4.2) Example. Let ($\mathbf{R}$, O$\mathbf{R}$) be the real line with the standard Euclidean topology. The boundary bd($\mathbf{Q}$) of the set $\mathbf{Q}$ of rational numbers is not columnar, i.e., bd($\mathbf{Q}$) ≠ bd(bd($\mathbf{Q}$)).

Proof. As is easily calculated, bd($\mathbf{Q}$) = $\mathbf{R}$ and bd(bd($\mathbf{Q}$)) = bd($\mathbf{R}$) = Ø.♦

In the rest of this section, we show that this example should be considered exceptional in the sense that there are many topologically better behaved sets that show a less complicated behavior with respect to boundaries.

(4.3) Lemma. Let (X, OX) be a topological space. Then, $OX \cup CX \subseteq MKX$, i.e., if A is an open set or a closed set bd(A) = bd(bd(A)).

Proof. Assume A to be open. By definition, bd(A) = cl(A) ∩ cl(CA) = cl(A) ∩ CA, since A is open. For bd(bd(A)), one obtains bd(bd(A)) = bd((cl(A) ∩ CA)) = (cl(A) ∩ CA) ∩ cl(C(cl(A) ∩ CA)). Obviously, cl(C(cl(A) ∩ CA) = cl(C(cl(A) ∪ CA)) = cl(C(cl(A)) ∪ clA)) = cl(X) = X. Hence, bd(bd(A)) = bd(A).

The boundary of a closed set B equals the boundary of its open complement CB. Thus, the boundaries of open sets as well as the boundareis of closed sets are columnar.♦

(4.4) Corollary. Let (X, OX) be a topological space. All $A \subseteq X$ are stably columnar in the sense that bd(bd(bd(A)) = bd(bd(A)).

Proof. By definition, bd(A) is closed. Applying (4.3) to bd(A), one immediately obtains bd(bd(bd(A))) = bd(bd(A)). ♦

(4.4) may be informally stated as follows: For all topological spaces (X, OX) and all $A \subseteq X$, the boundary bd(A) is stably columnar in the sense that bd(bd(A)) = $bd^n(A)$, $n \geq 2$; i.e., after the second iteration of the boundary operator, it becomes stable. Thus, instead of being a column consisting of equal layers bd(a), bd(bd(a)), bd(bd(bd(a))), … in the general case, the column of higher-order boundaries of a set A starts with a "pedestal" bd(A) on which equal layers bd(bd(A)), bd(bd(bd(A)) are put ad infinitum. The expression "pedestal" is adequate insofar as in general the "pedestal" bd(A) is "broader" than the layers bdbd(A), …, i.e., bd(A) $\supseteq$ bd(bd(A)).

(4.5) Lemma. For all (X, OX), one has bd(A) = bd(bd(A)) iff int(bd(A)) = Ø.

Proof. Assume bdbd(A) = bd(A). Then one calculates

$$\mathrm{bd}(A) = \mathrm{bd}(\mathrm{bd}(A))$$

$$\Leftrightarrow \mathrm{bd}(A) = \mathrm{bd}(A)) \cap \mathrm{cl}(C\mathrm{bd}(A))$$

$$\Leftrightarrow \mathrm{bd}(A) = \mathrm{bd}(A) \cap \mathrm{cl}(C\mathrm{bd}(A)) \Leftrightarrow \mathrm{bd}(A) \subseteq \mathrm{cl}(C\mathrm{bd}(A))$$

$$\Leftrightarrow \mathrm{bd}(A) \subseteq C\mathrm{int}(\mathrm{bd}(A)) \Leftrightarrow \mathrm{int}(\mathrm{bd}(A)) \cap \mathrm{bd}(A) = \emptyset$$

$$\Leftrightarrow \mathrm{int}(\mathrm{bd}(A)) = \emptyset.\blacklozenge$$

The class of sets with columnar boundaries comprises many more than just the open sets and the closed sets of a topological space. Rather, the sets with columnar boundaries have the structure of a Boolean algebra. The proof of this assertion requires the following technical lemma:

(4.6) Lemma. For every topological space (X, OX) and $A \subseteq X$, one has $\mathrm{int}(\mathrm{bd}(A)) = \emptyset$ iff A satisfies the topological McKinsey axiom:

$$\mathrm{int}(\mathrm{cl}(A)) \subseteq \mathrm{cl}(\mathrm{int}(A)) \Leftrightarrow \mathrm{int}(\mathrm{bd}(A)) = \emptyset.$$

Proof. $\mathrm{int}(\mathrm{cl}(A)) \subseteq \mathrm{cl}(\mathrm{int}(A)) \Leftrightarrow \mathrm{int}(\mathrm{cl}(A)) \cap C\mathrm{cl}(\mathrm{int}(A)) = \emptyset$

$$\Leftrightarrow \mathrm{int}(\mathrm{cl}(A)) \cap CC\mathrm{int}C(\mathrm{int}(A)) = \emptyset$$

$$\Leftrightarrow \mathrm{int}(\mathrm{cl}(A)) \cap C(\mathrm{int}(A)) = \emptyset$$

$$\Leftrightarrow \mathrm{int}(\mathrm{cl}(A)) \cap C(\mathrm{int}(A)) = \emptyset \Leftrightarrow \mathrm{int}(\mathrm{bd}(A)) = \emptyset. \blacklozenge$$

Recall that a set $A \subseteq X$ is nowhere dense in X iff $\mathrm{intcl}(A) = \emptyset$. A closed set A is nowhere dense iff the complement CA is dense in X. As is well known, the finite intersections of dense open sets are again dense. Equivalently, the finite unions of closed nowhere dense sets are still nowhere dense. The following equivalent formulation using these concepts of (4.6) will be useful:

(4.7) Proposition. For every topological space (X, OX), one has

$$\mathrm{bd}(\mathrm{bd}(A)) = \mathrm{bd}(A) \text{ iff } \mathrm{bd}(A) \text{ is nowhere dense.} \blacklozenge$$

(4.7) can be used to prove that the set of McKinsey sets has a very satisfying "logical" structure:

(4.8) Theorem. For every topological space (X, OX), the set MKX of McKinsey subsets of X is a Boolean algebra.

Proof. We already know that $A \in MKX$ iff $CA \in MKX$. Assume A and B to be columnar. Then, $A \cup B$ is also shown to be columnar by the following calculation:

$$\begin{aligned} bd(A \cup B) &= cl(A \cup B) \cap cl(C(A \cup B)) = (cl(A) \cup cl(B)) \cap (cl(CA) \cap cl(CB)) \\ &\subseteq (cl(A) \cup cl(B)) \cap (cl(CA) \cap cl(CB)) \\ &\subseteq (cl(A) \cap cl(CA)) \cup (cl(B) \cap cl(CB)) = bd(A) \cup bd(B). \end{aligned}$$

Since bd(A) and bd(B) are closed and nowhere dense sets, their union $bd(A) \cup bd(B)$ is closed and nowhere dense as well, i.e., $int(bd(A) \cup bd(B)) = \varnothing$. Thus, $intbd(A \cup B) = \varnothing$. Similarly, $bd(A \cap B)$ is shown to be nowhere dense by the following calculation using de Morgan's laws for the set-theoretical operations $\cap$ and $\cup$:

$$bd(A \cap B) = bd(C(A \cap B)) = bd(CA \cup CB)) \subseteq bd(CA) \cup bd(CB) = bd(A) \cup bd(B)$$

Hence, if bd(A) and bd(B) are nowhere dense, then $bd(A \cap B)$ is also nowhere dense. In sum, the set MKX of McKinsey sets is a Boolean algebra.♦

(4.9) Corollary: Let MKX be the McKinsey algebra of (X, OX). For $A \in MKX$, the boundary bd(A) is columnar: $bd^n(A) = bd(A)$, $n \geq 1$.♦

In sum, if one casts the concept of vagueness in a framework of an S4 operator **C** the corresponding boundary operator is always stably columnar ($bd^3 = bd^2$) and quite often even columnar ($bd^2 = bd$). One may conjecture that this is due to the fact that $\mathbf{C}$ satisfies $\mathbf{C} \bullet \mathbf{C} = \mathbf{C}$, i.e., a principle analogous to the famous "KK-principle" to the knowledge operator K. This is, however, not the case. As will be shown in the next section, also for Williamson's logic of clarity based on an operator C that does not satisfy the KK-principle, the corresponding boundary operator is stably columnar.

5. Columnar Vagueness in the Williamson's Logic of clarity. In the appendix of his well-known treatise *Vagueness* (Williamson 1994) the author gives a succinct sketch of a modal "logic of clarity" based on an sentential operator $\mathcal{C}$ such that $\mathcal{C}$A is to be read as "It is clearly the case that A". For fixed margin models (the only type of models of the logic of clarity we will consider in the following) $\mathcal{C}$ defines a modal logic KTB.

The details are as follows: We start with a similarity (W, ~), where W is a set (of possible worlds) and ~ a binary similarity relation that is reflexive and symmetric but not necessarily transitive. For $x \in W$ define $co(x) := \{y; x \sim y\}$ as the similarity neighborhood of x. Then it is easily checked that for $A \subseteq W$ Williamson's clarity operator $\mathcal{C}$ can be defined as

$$\mathcal{C}A := \{x; co(x) \subseteq A\}$$

Since the relation ~ is assumed to be reflexive, clearly $\mathcal{C}A \subseteq A$. More precisely, one can show that the modal logic based on $\mathcal{C}$ is just KTB (cf. Williamson (1994, 271)).[4] Moreover, one observes that in general $\mathcal{C}$ does not satisfy the "KK-

[4] In detail (W, ~) is equivalent to Williamson's fixed margin model <W, d, 1, []>.

principle" $\mathcal{C} \bullet \mathcal{C} = \mathcal{C}$.[5] Thus, one may conjecture that the concept of vagueness associated with the boundary operator bd correlated to $\mathcal{C}$ does not satisfy the axiom of (stable) columnarity. As will be shown in the following, this conjecture turns out to be wrong: also the boundary defined by Williamson's "clarity" operator $\mathcal{C}$ is stably columnar. In the appendix of *Vagueness* (Williamson 1994) does not deal with issues concerning the boundary operator defined by $\mathcal{C}$. Fortunately, Breysse and De Glas have elaborated (without being aware of it) Williamson's "logic of clarity" based on $\mathcal{C}$ – in particular they dealt with the problem of the boundary operator corresponding to $\mathcal{C}$, see their "A New Approach to the Concepts of Boundary and Contact: Toward an Alternative to Mereotopology" (Breysse and De Glas 2007).

Their approach is essentially equivalent to Williamson's approach of "fixed margin models being based on a similarity structure (W, ~) as defined above: Given (W, ~) Breysse and De Glas define operators $PW \xrightarrow{h} PW$ and $PW \xrightarrow{s} PW$ such that h is a (non-topological) interior kernel operator and s = **C**•h•**C** a (non-topological) closure operator. More precisely, h is just Williamson's $\mathcal{C}$ and s satisfies

(5.1) $s(A) = \{x; co(x) \cap A \neq \emptyset\}$

Then the operators h and s are used to define various concepts of boundary to be discussed in more detail in the following. In order not to confuse the reader with a mixture of different terminologies and denotations, in the rest of this section we will use that of Breysse and De Glas, i.e., in the rest of this section

[5] If the clarity operator $\mathcal{C}$ is to satisfy also S4 this amounts to the requirement that it is even S5.

Williamson's operator $\mathscr{C}$ is denoted by h.

(5.2) Proposition. Let (W, ~) be a similarity structure. Then operators h and s defined above for all A $\subseteq$ W have the following properties:

(i) $h(A) \subseteq A$, $A \subseteq s(A)$.

(ii) $h(A \cap B) = h(A) \cap h(B)$.

(iii) $sh(A) \subseteq A$.

(iv) hs is a closure operator, i.e., hs satisfies the first three Kuratowski axioms.♦

As is shown by Breysse and De Glas, the operators h and s can be used to defined define a very useful concept of boundary as follows:

(5.3) Definition. Let (W, ~) be a similarity relation with operators h and s defined as above. Then (in strict analogy to topology) a boundary operator bd can be defined for A $\subseteq$ W by

$$bd(A) := hs(A) \cap hs(CA).\blacklozenge$$

(5.4) Theorem. The boundary operator bd defined by (5.2) is stably columnar, i.e.:

$$bd(bd(bd(A))) = bd(bd(A)).$$

Proof: The proof is carried out by using (5.1)(i) – (iv) and some other familiar properties of closure operators. By definition

$$\begin{aligned} bd(bd(bd(A))) &= hs(bd(bd(A)) \cap hs(Cbd(bd(A)) \\ &= bd(bd(A)) \cap Csh(bd(bd(A)) \end{aligned}$$

We prove that Csh(bd(bd(A)) = W and are done.

$$\begin{aligned} Csh(bd(bd(A)) = W &\Leftrightarrow sh(bd(bd(A)) = \varnothing \Leftrightarrow h(bd(bd(A)) = \varnothing \\ &\Leftrightarrow h(hs(bd(A) \cap hs(Cbd(A)) = \varnothing \\ &\Leftrightarrow h(bd(A) \cap hCh(bd(A)) = \varnothing \\ &\Leftrightarrow hbd(A)) \cap h^2 Ch(bd(A)) = \varnothing \end{aligned}$$

Clearly hbd(A)) $\cap$ h^2 Ch(bd(A)) $\subseteq$ hbd(A)) $\cap$ Ch(bd(A)) = Ø.♦

In sum, both the S4 "logic of clearness" as well as the non-S4 "logic of clarity" give rise to stably columnar boundary operators (in the latter case if one restricts one's attention to fixed margin models. As is pointed by Williamson, the logic of variable margin models is KT, i.e., the Brouwer axiom (B) (equivalent to the fact that hs is a closure operator) is no longer valid (cf. Williamson (1994, 272)).

6. Concluding remarks. If one relies on topological, i.e., S4 models of the modal operators **C** and **U**, columnar vagueness crops up almost everywhere, be it in its strict version (for all subsets of weakly scattered spaces, or for McKinsey sets of arbitrary spaces) or in a weaker form (as stably columnar vagueness for all subsets of all topological spaces). An analogous result holds for fixed margin models (W, ~) that lead to a KTB logic of the modal operator C. Thus, if one rejects columnar vagueness for one philosophical reason or other, one must reject S4 or KTB models of vagueness altogether. On the other hand, if one accepts an S4 approach for the clearness operator **C** or a KTB approach for the clarity operator C one has to buy into columnar vagueness for the resulting concept of boundary. If one wants to avoid columnar vagueness one has to be content with a concept of vagueness that only satisfies KT.

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