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TOPOLOGICAL REPRESENTATIONS OF MEREOLOGICAL SYSTEMS

Abstract. The aim of this paper is to show that topology is a useful conceptual device for studying mereology. If M is a reasonable mereological system there is a topological space $pt(M)$ such that the class $O(pt(M))$ of open subsets of $pt(M)$ represents the class of the mereological individuals in a 1–1-fashion by the open sets of a topological space $pt(M)$ in such a way that the mereological relations and operations such as parthood, overlapping, fusion, etc. are faithfully represented by the corresponding relations of inclusion, intersection, union etc in the topological realm. Moreover, the topological model may be used to import concepts and relations into mereology which originally were defined only for topology.

1 Introduction

The aim of this paper is to show that topology is a useful conceptual device for studying mereology. More precisely, I want to show the following. If M is a reasonable mereological system there is a topological space $pt(M)$ such that the class $O(pt(M))$ of open subsets of $pt(M)$ represents M. That is to say, there is a representing map $r : M \longrightarrow O(pt(M))$ which maps the mereological individuals, i.e., the elements of M, in a 1–1-fashion onto the open sets of a topological space $pt(M)$ in such a way that the mereological relations and operations such as parthood, overlapping, fusion, etc. are faithfully represented by the corresponding relations of inclusion, intersection, union etc in the topological realm. Hence, at least sometimes, it is possible replacing the mereological system without loss by its topological model. Moreover, the topological model may be used to import concepts and relations into mereology which originally were defined only for topology. Hence, the theory of the representing topological domain may be pulled back, so to speak, to the represented mereological domain.

For special mereological systems this kind of topological representability has been known for some time. Already Whitehead pointed out that Euclidean geometry may be cast in a mereological framework by taking as its basic primitive concept not the set theoretical notion of point but rather the mereological concept of region (cf. Whitehead 1929). That is to say, the basic building blocks of space are taken to be extended regions instead of points. From spatial regions and their (mereological) relations all other geometric entities such as points, lines, and surfaces have to be reconstructed. Whiteheads own account of this reconstruction is rather sketchy, in a more rigorous fashion it has been partially carried out by later authors (cf. Grzegorczyk 1960, Clarke 1985, Gerla and Tortora 1992, Gerla and Tortora 1996, Roeper 1997).

The first step in this process is the construction of an underlying set of points the topology of which may be used to represent topologically the Whiteheadian mereological system of spatial regions. The basic fact which renders this possible is the lattice structure of the mereological system of regions. Hence, the theory of lattices may be considered as the bridge between merology and topology. More precisely, the following holds: if a mereological system M allows for unrestricted fusion, it already has the structure of a complete lattice[1]. This is sufficient for the existence of a reasonable topological representation of M.

The outline of the paper is as follows: in section 2 some basic concepts and axioms of mereology are recalled. In particular, the concept of fusion is elucidated. It is shown that mereological systems with unrestricted fusion have the structure of complete lattices. Some set theoretical and topological examples of mereological systems are dealt with in section 3. In section 4 we interpret Stones representation theorem as a representation theorem for Boolean mereological systems. Topological representations for general mereological systems are discussed in section 5. Finally, in section 6 the various pro and cons of different topological representations for systems of Whiteheadian regions are dealt with. We conclude with some general remarks on the relation between topology and mereology in section 7.

For convenience, mereology is cast in the framework of set theory, i.e., mereological systems are considered as relational systems, i.e. as sets

[1]According to Lewis 1991, unrestricted fusion should be considered as a minimal requirement for any mereological system.

endowed with a certain mereological structure. More precisely, I take a mereological system M to be a (special) partial ordering having a maximal element ("the universe") and a minimal element ("the null individual")[2]. The order relation may denoted by "pt", "$\leq$" or "$\subseteq$" depending on the context. Hence, a mereological system may be denoted by (M, pt) etc. When there is no danger of confusion a mereological system is denoted simply by M.

2 Basic Mereological Concepts

Mereology, more than most other theories has been cursed by a jungle of different notations. For the following, I adopt as far as possible the terminology of Simons 1987. Sometimes, I diverge using some common set theoretical or mathematical notation which has a direct mereological interpretation. Let us fix the minimal requirements mereological systems have to satisfy in the following definition:

(2.1) Definition
Let M be a set and $<\subseteq M \times M$ a binary relation. The relational system $(M, <)$ is called mereological system iff it is a partial order, i.e. iff the relation $<$ satisfies the following requirements:

(A1, Parthood)	$x < y \Rightarrow not(y < x)$,	*(Irreflexivity)*
(A2, Parthood)	$x < y$ and $y < z \Rightarrow x < z$	*(Transitivity)*

The relation $x < y$ is to be interpreted as the relation of "proper parthood", i.e., as "x is a proper part of y". The relation $x \leq y := x < y \wedge x = y$ is called the relation of parthood. Obviously, $\leq$ and $<$ are interdefinable. Hence, a mereological system may be denoted by $(M, \leq)$ or, equally well by $(M, <)$.

As is well-known, in mereology one has a choice of primitives. Usually one begins with "part" (or "proper part") and defines the other notions such as "fusion" of "overlapping" in terms of that notion.

(2.2) Definitional Circle of Mereological Concepts
Let $(M, \leq)$ be a mereological system. The relations of "overlapping"

[2]While most mereologists acknowledge the existence of a maximal individual, the null individual is generally met with suspicion (cf. Simons 1987, 13). In the following the existence of a null individual is assumed throughout. Id like to emphasize that I consider the null individual only as a device for neatening the (lattice theoretical) algebra.

("⊙") and "fusion" ("V") are defined as follows:

(2.2.1) *Two individuals x and y overlap iff there is a non-null individual z which is part of both of them: $x \odot y := \exists z(z \leq x$ and $z \leq y)$*

(2.2.2) *Let $M' \subseteq M$. The fusion VM is an individual of M that has all elements of M' as parts and no part that does not overlap with at least one of the elements of M'. The fusion $V\{x, y\}$ is written as $x \vee y$.*

(2.2.3) *If one takes fusion as the primitive concept parthood is defined by $x \leq y := x \vee y = y$.*

(2.2.4) *If one takes overlapping as the primitive concept parthood may be defined by $x \leq y$ iff $\forall z(z \odot x \Rightarrow z \odot y)$.*

Accordingly, the basic axioms (A1,<) and (A2,<) of irreflexibiliy and transitivity of the proper parthood relation have to be translated in the corresponding axioms for the concepts of overlapping and fusion, respectively. For instance, the analogues of (A1,<) and (A2,<) for the concept of fusion run as follows:

(A1, v) $\quad x \vee y = y \wedge x \neq y \Rightarrow x \vee y \neq x$

(A2, v) $\quad (x \vee y) \vee z) = x \vee (y \vee z)$

Analogously, the corresponding axioms (A1,⊙) and (A2,⊙) for overlapping may be defined. The axioms (A1) and (A2) are a far cry from fully characterizing a reasonable notion of parthood (cf. Simons 1987). Rather, as Simons has pointed out it is a non-trivial task of mereology is to provide a variety of mereological axioms which may be used to chart the area of acceptable concepts of parthood. I don't want to deal with this topic in greater detail being content to mention the following three axioms:

(2.3) Weak Supplementation Principle (WSP)
(A3) $\quad x < y \Rightarrow \exists z(z < y \wedge not(z \odot x))$

(2.4) The Proper Part Principle (PPP)
(A4) $\quad \exists z(z < x \wedge \forall z((z < x \Rightarrow z < y) \Rightarrow x \leq y))$

From (PPP) one easily gets the following extentionality principle: if x and y have proper parts and the proper parts of x are the proper parts of y and vice versa x and y are identical.

Mereologists are divided over the question of how large the range of

fusion or composition should be. Does every finite family of individuals have a (unique) fusion, or even any collection of individuals? In this paper I take the radical stance that any collection of individuals has a unique fusion. Since the mereological system are assumed to be sets this may be expressed as follows:

(2.5) Axiom of Unrestricted Fusion
Let M be a mereological system. If M' is a subset of M there exist a unique individual VM' such that all x of M' are parts of VM' and any part of VM' overlaps with at least one of the elements of M'.

As we will see in a moment mereological systems M which satisfy (2.5) have the structure of a complete lattice. Since the class of mereological individuals is assumed to be a set, we may talk about the set PM of subsets of M. The axiom of unrestricted fusion asserts that for any subset $N \subseteq M$ there is a unique element VN of M called the fusion of N. Hence, the operation of fusion may be considered as a map $V : PM \longrightarrow M$. A reasonable fusion map will have to satisfy the following structural properties:

(2.6) Structural Properties of Fusion
Let M be a mereological system, and $V : PM \longrightarrow M$ an operator. For $B \in PPM$ define $VB := \{VN : N \in B\} \in PM$. Then the fusion operator $V : PM \longrightarrow M$ induces an operator which maps PPM to PM. This operator is also denoted by $V : PPM \longrightarrow PM$. V is assumed to satisfy the following properties:

(1) $\quad V : PM \longrightarrow M$ *is surjective.*

(2) $\quad V(VB) = V(UB)$

Condition (2) is called the condition of full associativity (Erné 1982, 99). It ensures that the fusion of fusions coincides with the fusion of set theoretical unions.

(2.7) Definition and Lemma (Erné 1982, 5.6. Satz)
Let M be a mereological system with a fully associative fusion operator V. The operator V gives rise to an operator $\vee : M \times M \longrightarrow M$ defined by $x \vee y := V(\{x, y\})$. This operator is idempotent, commutative and associative. Moreover, $(M, \vee)$ is a complete join semi-lattice.

Starting with a general fusion operator V of a mereological system M one actually has two choices for defining an order on M. Fortunately, they coincide:

(2.8) Proposition
Let $V : PM \longrightarrow M$ be a fully associative fusion operator. Then it defines partial orders $\leq$ and pt (for parthood) on M in the following ways:

(1) $x \leq y := V(x, y) = y$, i.e., $x \leq y := x \vee y = y$.

(2) $x \; pt \; y := \exists z$ and $y \vee z = x$.

The relations pt and $\leq$ coincide.

The relation *pt* is the familiar parthood relation related to fusion by (2.2.3). That is to say, if in mereology we start with fusion as the primitive notion, the derived parthood relation is just *pt* (cf. Lewis 1991). Hence, due to (2.8) our fusion-based approach is fully in line with the more familiar conceptualizations of mereology. The essential point to note is the following: if we restrict our attention to mereological systems (M, V) with complete fusion V, the operator V defines in a natural way a complete partial order $\leq$ (or *pt*). Hence, we may consider mereological systems as complete lattices, not only as semilattices. Hence, mereology understood as the theory of mereological systems satisfying the axiom of unrestricted fusion may be pursued in the framework of the theory of complete lattices. This, as will become clear in the following, also opens the gate for topological considerations. Moreover, one may flesh out the concept of fusion in several interesting ways which as far as I know have hitherto not been pursued in mereology. For instance, one may stipulate that fusion is *distributive* in the following sense:

(2.9) Definition
Let (M, V) be a mereological system with at least finite fusion V. The system is distributive iff the following two conditions are satisfied:

$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ *and* $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

An even stronger condition concerning arbitrary fusion will turn out to be topologically important:

(2.10) Definition
Let (M, V) be a mereological system with universal fusion conceived of as a complete distributive lattice in the sense of (2.7). The fusion is a Heyting fusion iff $x \wedge V y_i = V\{x \wedge y_i\}$

As is well known, for the lattice of open sets $O(X)$ of a topological space $(X, O(X))$ the set theoretical union is a Heyting fusion in the sense of (2.9). As it will turn out in the following, the requirement that the fusion of a mereological system has a Heyting fusion suffices that it may be represented by a topological space.

3 Topological Examples of Mereological Systems

The mereological systems that most easily come to mind are the set theoretical ones. For any set X the power set $P(X)$ of X defines a mereological system $(P(X), \subseteq)$ by taking set theoretical inclusion as the parthood relation. This recipe, however, only yields a very special class of mereological systems, to wit, the so called Boolean mereological systems which have the structure of a Boolean algebra. A good strategy to find other examples is considering appropriate subclasses of $P(X)$. A rich source for mereological systems is provided by the various systems arising from topological considerations.

Assume that X is a topological space, i.e., a set provided with a topological structure $O(X) \subseteq P(X)$[3]. Then $(O(X), \subseteq)$ is a mereological system. Generally, $O(X)$ is not a Boolean algebra but only a Heyting algebra.[4] Hence, topological structures $O(X)$ provide a more general class of mereological systems than the class of set theoretical examples such as $(P(X), \subseteq)$. Actually, the mereological systems $(O(X), \subseteq)$ do not satisfy some of the mereological principles which are considered by some authors as the most natural mereological principles such as the proper parthood principle (PPP) or the weak and strong supplementation principles (WSP) and (SSP). This amounts to a sort of clash, so to speak, of mereological considerations on the one hand, and topological ones on the other: From a topological point of view, the class $O(X)$ of open sets of X is a quite natural and well-behaved entity. On the other hand, and this

[3] As a reference for the topology needed in this paper one may use any standard text book or, as a very concise summary, the "Topological Toolkit" of Davey and Priestley 1990. the ical $(O(X), \subseteq)$ is not the only mereological system arising in topology. Other, topologically defined mereological systems are $(C(X), \subseteq)$, $(O^*(X), \subseteq)$ and $(C^*(X), \subseteq)$ the sets of closed, regular open, and regular closed subsets of X, respectively. $O^*(X)$ is a complete Boolean lattice. Its role in mereology will be discussed in section 6.

[4] If the topology $O(X)$ is the trivial discrete topology, the mereological system $(O(X), \subseteq)$ is a Boolean system. Hence, "Boolean" may be considered as a trivial special case of "Heyting".

may be somewhat surprising, mereologically, $(O(X), \subseteq)$ might not behave well since it does not satisfy the (PPP). This is shown by the following example:

(3.1) Example
Let $(O(X), \subseteq)$ be the topological mereological system defined as follows: $X = \{a, b, c\}, O(X) = \{\emptyset, \{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.
The system $(O(X), \subseteq)$ does not satisfy (PPP).

Proof
As is easily checked the individuals $\{a, c\}, \{b, c\} \in O(X)$ have the same proper parts, namely $\{c\}$, nevertheless they do not coincide. Hence, $(O(X), \subseteq)$ does not satisfy (PPP).

(3.1) shows that the axioms of general topology are too weak to ensure that the resulting mereological systems $(O(X), \subseteq)$ satisfy even the most natural mereological requirements. For (PPP), this shortcoming may be easily overcome: there is a genuinely topological axiom which ensures that mereological systems $(O(X), \subseteq)$ satisfy (PPP).

(3.2) Definition and Lemma
(i) *A topological space X satisfies the separation axiom T_1 iff for any a, b, $a \neq b$, there exist $U(a), U(b) \in O(X)$ containing a and b respectively, such that $b \notin U(a)$ and $a \notin U(b)$. If X satisfies T_1 it is called a T_1-space.*
(ii) *If X is a T_1-space the mereological system $(O(X), \subseteq)$ satisfies (PPP).*

Proof
Assume that X is a T_1-space. Let U, V be elements of $O(X)$, and let us further assume that for them the premisses of (PPP) hold, i.e., there is a nonempty open subset W of U and any open subset of U is a subset of V. Note that $\emptyset \neq W \subseteq U \cap V$. Hence there is an element $y \in W \subseteq U \cap V$. We have to show that U is a subset of V.

Suppose this is not the case. Then there is an $x \in U$ with $x \notin V$. Due to (T_1) there is an open neighborhood Z of x which does not contain y. Hence Z is a proper part of U. Hence it is a subset of V. This implies $x \in V$. This is a contradiction.

The separation axiom T_1 is only one of a whole family of separation axioms. It is natural question whether other axioms of this family may have mereological counterparts as well. This is indeed the case (cf. (5.5).

Hence, in the case of (PPP) and T_1, mereological and topological considerations run quite parallel. The axiom (WSP) shows, however, that these two perspectives do not match always that smoothly. The following example shows that the mereological axiom (WSP) may not be satisfied even by topologically quite well-behaved systems:

(3.3) Example
Let J be the Euclidean space endowed with the standard topology $O(J)$. The mereological system $(O(J), \subseteq)$ does not satisfy (WSP).

Proof
In order to show that $O(J)$ does not satisfy (WSP) one constructs a proper open subset S of J that lacks a supplement in the sense of (WSP). Let Q denote the set of points of J with rational coordinates. Q is countable and dense in J, i.e., the closure of Q is J. Hence, it may be described as a sequence of points $(A_i)_{i \in N}$. Let e and e_i $(i \in N)$ be positive real numbers such that $\Sigma e_i \leq e$. Let S_i $(i \in N)$ be open spheres with centers A_i and volume smaller than e_i. Then the sum of the volumes of the S_i is smaller than e. Denote the fusion of the S_i by S. Since S is the join of elements of $O(J)$, and $O(J)$ is closed with respect to arbitrary joins it also belongs to $O(J)$. The volume of S is e, i.e., much smaller than the volume of the whole space J. Hence, S and J are different. More precisely $S \subset J$. Since $Q \subset S$, S is dense in J. Hence, there cannot be an open subset $T \subseteq J$ disjoint from S. Thus, (WSP) does not hold for $(O(J), \subseteq)$.[5]

(3.3) shows a certain disharmony between mereology and topology: what may appears to be "well-behaved" from a topological point of view may not be "well-behaved" from a mereological point of view, and vice versa. This topic will be pursued on a representational level in section 6.

4 Topological Representations of Boolean Mereological Systems

Now let us come to the central topic of this paper, namely, the topological representation of mereological systems. Let us start with a special but nevertheless quite paradigmatic case, to wit, the representation of (complete) Boolean mereological systems.

The proof of a topological representation theorem for these systems has been a major mathematical achievement, namely, the proof of the path-trailing Stone representation theorem (cf. Stone 1936). Of course, Stone

[5] The relevance of this fact will be discussed from a different angle in section 6.

was not interested in mereology.[6] Nevertheless, his result can naturally be interpreted as a fundamental achievement concerning the topological representations of a special class of mereological systems, to wit, Boolean systems.

Since the proof of Stone's representation theorem may be considered as paradigmatic for all subsequent of more general and more refined representation theorems let us briefly recall the basic lines of one of its simpler versions.

Let L be a lattice and denote the set of a prime ideals of a lattice L by $I_p(L)$. Then there is a natural map $r : L \longrightarrow P(I_p(L))$ defined as follows: For $a \in L$ define $r(a)$ as $r(a) := \{I \in I_p(L); a \notin I\}$. As is easily seen, the family $\{r(a); a \in L\}_{a \in L}$ is closed under finite intersections and finite unions, but possibly not under arbitrary unions. Hence, a topology $O(I_p(L))$ on $I_p(L)$ is defined by

$$O(I_p(L)) := \{V \subseteq I_p(L); V \text{ is a union of members of } \{r(a); a \in L\}_{a \in L}\}$$

In other words, the map $r : L \longrightarrow P(I_p(L))$ may be conceptualized as a map $r : L \longrightarrow O(I_p(L))$. Under suitable circumstances specified by Stone's theorem the representation r is an isomorphism. That is to say, r yields a faithful topological representation of the lattice L.

From a mereological point of view, however, Stone's representation theorem leaves something to desire. Actually, mereologists were not interested in abstract Boolean mereological systems, but rather with very specific ones, namely, Whiteheadian systems W of spatial regions. According to the Whiteheadian theory of space, spatial regions are the basic building blocks of physical (Euclidean) space. Although these systems may be conceptualized as Boolean lattices W the representing topological space $O(I_p(W))$ of the Stone representation theorem is not the Euclidean space the Whiteheadian account of space intended to reconstruct from the system of spatial regions. Far from it: whilst physical space is connected the Stone space of a Boolean algebra is extremely disconnected. Hence, an abstract Boolean system of regions does not suffice to reconstruct the corresponding topological space.

Abstract Boolean systems are lacking an essential feature of Euclidean space. Some further structure, beyond the mereological one, is needed, or so it seems. Already Whitehead proposed to take as this extra structure a connection or touching relation between regions. This relation describes an apparently non-mereological relation between spatial regions. In the 1-dimensional case such a relation is exhibited by the following situation:

[6] For Stone's motivational background see Piazza 1995.

take X to be the real line R and consider two open intervals $A = (a, b)$ and $B = (b, c), a \leq b \leq c$. Although A and B do not overlap, their relation should be distinguished from, say, the relation between A and B', the latter defined as $B' := (a + d, c + d)$, $d > 0$. Intuitively A and B may be said to touch each other while A and B' are strictly apart. Topologically, this relation may be described by the relation that the closures of A and B intersect non-trivially, i.e., $cl(A) \cap cl(B) \neq \emptyset$. As it seems, this cannot be done in purely mereological terms. Stone's representation does not represent this touching or connection relation: from a mereological point of view, both A and B, and A and B' are just non-overlapping mereological individuals. Hence, as it seems, pure mereology does not suffice to achieve what the adherents of a Whiteheadian account of space expected from it, namely, to offer a complete reconstruction of physical space from a purely mereological base.[7]

Instead of a crude topological representation $r : W \longrightarrow O(X(W))$ (here $(X(W)$ is a set of "points" somehow constructed from the system of spatial regions W, not necessarily $I_p(W)$) one therefore would need a refined topological representation $r : (W, C) \longrightarrow (O(X(W)), C_O)$ which somehow takes into account the connection relation C representing it by a topologically defined relation C_O. This track has been pursued by several authors, see Grzegorczyk 1960, Clarke 1985 or Roeper 1997.[8]

5 Topological Representations of General Mereological Systems

In section 3 we considered the topological systems $(O(X), \subseteq)$ as special case of mereological systems. The existence of topological representations for general mereological systems will show that the $(O(X), \subseteq)$ are much more common than one might have thought. More precisely, mereological systems $(M, \leq)$ turn out to have (more or less) faithful topological models $(O(pt(M)), \subseteq)$. This is not quite obvious since topological structures are special set-theoretical structures which seem to be dependent on the notion of point or element, not available in mereology.

[7] Actually, this is not true. It is possible to define a connection relation in purely mereological,i.e., lattice theoretical terms (cf. Mormann 1998.

[8] For several reasons, the presently available versions of a Whiteheadian theory of space are not fully satisfying. First, they introduce apparently non-mereological primitives such as the connection relation. Second, and this is a more important shortcoming, they are ignore some important mathematical tools which render the theory of continuous lattices the appropriate framework for dealing with Whitehead's problem (cf. Banaschewski and Hoffmann 1981). For instance, in the framework of continuous lattices the "Main Theorem" of Roeper's is obtained in a quite direct and natural way (cf. Mormann 1998).

The theories of pointless topology has shown that the dependence on points in topology is not as deep as one might have expected. Large parts of topology may be done, at least in principle, without recourse to points (cf. Johnstone 1983), Mac Lane and Moerdijk 1992). For many issues it suffices to deal with the lattice $O(X)$ of open sets, ignoring the underlying set X. It is even possible, to reconstruct the set X from the lattice $O(X)$. The recipe for this construction is practically the same as the one Stone employed in the proof of his representation theorem. One topologizes the set of prime ideals $pt(O(X))$ of $O(X)$ in an appropriate way thereby getting a topological space $(pt(O(X)), O(pt(O(X))$. Under mild restrictions, $O(pt(O(X))$ is isomorphic to $O(X)$. This yields a topological representation of $O(X)$.

The details are as follows: Let $2 := \{0, 1\}$ denote the 2-point set. 2 may be considered as a complete lattice, taking 0 as the bottom and 1 as the top element and the obvious lattice theoretical operations. Hence we may consider the following set of mappings $a : O(X) \longrightarrow 2$:

(5.1) Definition
Let L be a complete lattice.

(1) *Denote by $pt(L)$ the set of mappings $a : O(X) \longrightarrow 2$ which preserve infinite joins and finite meets. The set $pt(L)$ is called the set of (generalized) points of L.*

(2) *L is said to have enough points iff the points of L separate the elements of L, i.e., for any two elements $x, y \in L$ there is an $a \in L$ such that $a(x) \neq a(y)$.*

The rationale of (5.1) is the following: Suppose L is the lattice $O(X)$ of open sets of some topological space X. If $O(X)$ is well-behaved, there is a natural bijection $i : X \longrightarrow pt(O(X))$ defined by $i(x)(U) := 1$, if $x \in U$, and $i(x)(U) = 0$, otherwise. Hence, we may call the maps $a : pt(O(X)) \longrightarrow 2$ points of X, or even points of $O(X)$, even if $O(X)$ happens to lack an underlying set X. The point of this definition of points of $O(X)$ is that it only depends on the lattice properties of $O(X)$ and not on X itself.

(5.2) Definition and Lemma

(1) *A topological space X is sober iff for any $P \in O(X)$ such that (i) $P \neq X$, (ii) $U \cap V \subseteq P \Rightarrow U \subseteq P$ or $V \subseteq P$ there is a unique point $x \in X$ with $P = X - cl(\{x\})$.*

(2) *The mapping $r : X \longrightarrow pt(O(X))$ is a bijection iff X is sober.*

Sober spaces abound. For instance, Hausdorff spaces are sober. Hence, there is a large supply of spaces for which the representation $r : X \longrightarrow pt(O(X))$ is a bijection. For $U \in O(X)$ define $r(U)$ as $\{a; a(U) = 1\}$. Hence, the map r induces a map $r : O(X) \longrightarrow P(pt(O(X)))$. As is easily seen the sets $r(U)$ define a topology $O(pt(O(X))$ on $pt(O(X))$. With respect to the topologies $O(X)$ and $O(pt(O(X))r : X \longrightarrow pt(O(X))$ is a homeomorphism.

The definition of $pt(O(X))$ does not depend on the fact that the elements of $O(X)$ be sets of points. The only feature of $O(X)$ employed in the construction of $pt(O(X))$ is that $O(X)$ is a complete lattice. More precisely, the elements of $pt(O(X))$ are just the prime elements of $O(X)$. Hence, this construction may be carried out for any complete lattice L. Consequently, the elements of $pt(L)$ are called the points of L although L may not have any "real" points at all. Then we may define the topological representation of L as follows:

(5.3) Definition and Lemma
Let L be a complete (semi)lattice, and $pt(L)$ the set of points of L. Define the mapping $r : L \longrightarrow P(pt(L))$ by $r(x) := \{a; a(x) = 1\}$. The sets $\{r(x); x \in L\}$ define a topology $O(pt(L))$ on $pt(L)$. Hence, r may be conceptualized as a map $r : L \longrightarrow= (pt(L))$, and r may be called the topological representation of L.

Proof:
One has to show that the class of sets $\{r(x); x \in L\} \subseteq P(pt(L))$ is closed with respect to arbitrary union and finite intersection. This immediately follows the fact that the elements of $pt(L)$ preserver arbitrary joins and finite meets.

(5.4) Corollary (Topological Representation of mereological systems)
If (M, V) is a mereological system with complete fusion V there is a canonical representation $r : M \longrightarrow O(pt(M))$. The topology $O(pt(M))$ on $pt(M)$ is defined by $O(pt(M)) := \{r(x); x \in M\}$. If the fusion of (M, V) is a Heyting fusion and M, conceived as a lattice, has enough points in the sense of (5.1) the topological representation $r : M \longrightarrow O(pt(M))$ is a lattice isomorphism. Moreover, $O(pt(M))$ is sober.

To assess the strength of this representation theorem one has to get information under what conditions the space $pt(M)$ is sufficiently non-trivial to make the representation r useful. These conditions should be genuine mereological conditions, i.e. requirements formulated in purely mereological terms. We are content to give one paradigmatic example which shows how mereological counterparts of topological concepts may be constructed. This paradigmatic example is dealing with the mereological analogue of the familiar Hausdorff separation axiom T_2:

(5.5) Definition (Hausdorff Separation Axiom)
Let X be a topological space. The topology $O(X)$ is Hausdorff (or satisfies the axiom T_2) iff for all different $a, b \in X$, there exist disjoint open sets $U(a), U(b) \in O(X)$ containing a and b respectively.

If we want to define a mereological analogue of (5.5) first we have to say what is the analogue of "points" in mereological systems:

(5.6) Definition (Prime individuals)
Let M be a mereological system. An individual $p \in M$ with $p \neq 1$ is a prime individual iff $a \wedge b \leq p \Rightarrow a \leq p$ or $b \leq p$. M is a Hausdorff mereological system iff all different prime individuals p and p' have parts q and q', respectively, such that $q \subseteq p$, $q' \not\subseteq p'$, $q' \subseteq p$, and $q \not\subseteq p$.

Boolean mereological systems are Hausdorff systems. The following proposition confirms the intuition that prime individuals may serve as substitutes for points:

(5.7) Proposition
Let M be a complete Hausdorff mereological system with Heyting fusion. Then the topological representation $r : M \longrightarrow O(pt(M))$ is an isomorphism.

Proof:
First note that the prime individuals of M correspond in a 1-1-fashion to the elements of $pt(M)$: if p is a prime individual a homomorphism $\phi_p : M \longrightarrow 2$ is defined by $\phi_p(m) = 0$ iff $m \leq p$. As is easily seen, ϕ_p is an element of $pt(M)$, i.e., it preserves finite meets and infinite joints. In the other direction, $\phi \in pt(M)$ defines a prime individual p_ϕ by $p_\phi := V\{m; \phi(m) = 0\}$. Since M is assumed to be Hausdorff this correspondence is 1-1 and the topology $O(pt(M))$ induced on $pt(M)$ by the representation $r : M \longrightarrow O(pt(M))$ is Hausdorff.

Topological representability is not restricted to mereological systems with unrestricted fusion. For more general systems one may obtain a topological representation via completion: since for any mereological system $(M, <)$ the axioms (A1) and (A2) are assumed to be valid, any mereological system is at least a partial order. Hence we may embed $(M, <)$ into a complete lattice $(M^*, \vee, \wedge)$ in a canonical way $e : M \longrightarrow M^*$. There are several possibilities to do that. Maybe the most elementary and most economic one is the Dedekind-Mac Neille completion, (cf. Davey and Priestley 1990 Davey/Priestley 1990) which is isomorphic to M if M happens to be already complete. Via this embedding one may obtain a topological representation $M \xrightarrow{e} M^* \xrightarrow{r} O(pt(M^*))$. Summarizing we may say that all mereological systems have topological representations of one kind or another. Of course, these representations may widely vary with respect to the degree of how faithful they represent the mereological structures. Indeed, there are lattices L with $pt(L) = \emptyset$. For them, a topological representation is rather pointless.

6 Do Mereology and Topology Really Go Well Together?

In this section I'd like to resume the topic of a possible clash of mereological and topological considerations already briefly addressed in section 3. Recently, For-rest contended that the mereology and Whiteheadian theory of space are incompatible (cf. Forrest 1996b). More precisely he claims that the mereological axiom of countable fusion (CF) clashes with fairly plausible geometric convictions. To put it bluntly, according to him, countable fusion is incompatible with the rather plausible thesis (W) that space is Whiteheadian. Graham Oppy argues against Forrest that the inconsistency of (W) and (CF) only arises because Forrest's Whiteheadian account of space is formulated in terms which poach on the preserves of mereology (Oppy 1997).

In this section I want to show that both Forrest and Oppy are mistaken. Against Forrest I argue that his incompatibility thesis stands in need of qualification. More precisely, I argue that Forrests incompatibility result may be considered as an artifact of his representational framework. Against Oppy I contend that Forrest's account of a Whiteheadian theory of space can and should be maintained. Moreover, Oppy's liberalization of the concept of Whiteheadian regions amounts to the unconditional surrender of the Whiteheadian account, since it opens the door for points as primitive spatial entities. It will be shown that one can get everything Oppy is after without any alterations of the original Whiteheadian account.

Let us start with Forrest's incompatibility thesis. The proof of the incompatibility thesis crucially depends on two assumptions, namely, the volume assumption (V) and the assumption of regular representatibility (RR). (V) states that the volume of a fusion of regions does not exceed the sum of the volumes. (RR) states that Whiteheadian regions are to be represented by regular open (or regular closed) sets.[9] If either (V) or (RR) is dropped Forrests proof founders. Moreover, there are good reasons not to subscribe to the conjunction (RR) & (V).

First let us fix some notation. The natural numbers are denoted by N. Let J denote the 3-dimensional vector-space over the real numbers R. J is assumed to be endowed with the standard topology and the standard Lebesgue measure m defined on the Borel subsets of J. According to Forrest's reconstruction Whiteheadian regions have to satisfy the following requirements (among others):

(6.1.) Basic Assumption of a Whiteheadian Account of Space

(a) *Regions may be represented by sets of points in such a way that each representing set contains a sphere, that is, all the points distance less than some positive z from some point Z.*

(b) *Among the regions there are spherical regions of arbitrarily small diameter.*

(c) *The representations of regions as sets of points preserve volumes. That is, the volume of a region equals the Lebesgue measure of the corresponding set of points.*

The class of Whiteheadian regions of space is denoted by W. W is assumed to be a mereological system in the following sense: If the region X is part of the region Y this is denoted by $XptY$. Parthood is assumed to be transitive. Other mereological concepts are defined as usual: the regions X and Y overlap iff they have a common part, X and Y are said to be disjoint iff they do not overlap. The region Y is a fusion of the regions X_i iff all X_i are parts of Y and no part of Y is disjoint from each of the X_i. The axiom of countable fusion (CF) states that for any countably many regions a unique fusion exists (Lewis 1991). Forrest intends to show that this assumption is incompatible with a Whiteheadian account of space.

A representation of regions is a 1–1-mapping of regions to sets sending

[9] While Forrest explicitly argues for (V), the motive for (RR) remains implicit. In a footnote he asserts that for various reasons Whiteheadian regions should best be represented by regular open or regular closed sets (Forrest 1996b). The peculiar consequences of (RR), however, are not discussed at all.

proper parts to proper subsets (Forrest1996, 128, Footnote 4). This is rendered explicit as follows:

(6.2) Definition

Let W be the the class of Whiteheadian regions. A WF-representation of W into $P(J)$ is a 1–1-mapping $r : W \longrightarrow P(J)$ such that (6.1) (a) – (c) are satisfied. Moreover, for all regions X, Y of W the representing map r is assumed to satisfy the following conditions:

(1) *For any $B \in J$, arbitrarily small regions R exist such that $B \in r(R)$.*

(2) $X \leq Y \Rightarrow r(X) \subseteq r(Y)$.

These assumptions do not uniquely determine the class of sets which are to represent Whiteheadian regions. Not just any contrived subset of J should be allowed to represent a region. Rather, Whiteheadian regions should be represented by nice or well-formed subsets of J. Hence, a fully fledged Whiteheadian account of space should restrict the range of the representation $r : W \longrightarrow P(J)$ to a suitable subset $N(J) \subseteq P(J)$ such that the elements of $N(J)$ are nice subsets of J. In other words, a WF-representation is a mapping $r : W \longrightarrow N(J)$. What is to be understood by this informal notion may be explicated by topological notions. In the following I'd like to concentrate on two kinds of WF-representations:

(6.3) Definition

Let $r : W \longrightarrow P(J)$ be a WF-representation.

(1) *r is an open representation if it maps regions onto the open sets of J. An open representation is be denoted by $r : W \longrightarrow O(J)$.*

(2) *r is a regular open representation if it maps Whiteheadian regions onto the regular open sets of J. Hence, a regular open representation may be denoted by $r : W \longrightarrow O^*(J)$.*

Obviously, any regular open representation gives rise to an open representation, just forget about regularity. This is rendered precise as follows: let $i : O^*(J) \longrightarrow O(J)$ denote the canonical inclusion. Then a regular open representation $r^* : W \longrightarrow O^*(J)$ can be "prolonged" to an open representation $i \circ r^* : W \longrightarrow O^*(J) \longrightarrow O(J)$. Hence, open representations may be considered as more basic than regular open representations.

Forrest's proof of the incompatibility of countable fusion (CF) and

Whiteheadian account of space (W) is based on the construction of a region S represented by an open subset $r(S) \subseteq J$ having a small volume and a thick topological boundary. According to him, this leads to contradiction. As I want to show this conclusion is valid only if one assumes (V) and (RR).

Slightly simplified, Forrest's construction runs as follows: Let Q denote the set of points of J with rational coordinates. Q is countable and dense in J, i.e., $cl(Q) = J$. Hence, it may be described as a sequence of points $(A_i)_{i \in N}$. Let e and e_i $(i \in N)$ be positive real numbers such that $Se_i \leq e$. Let S_i $(i \in N)$ be regions mapped by r onto open spheres $r(S_i)$ with centers A_i and volume smaller than e_i. Then the sum of the volumes of the S_i is smaller than e. Let us assume that the fusion of the S_i exist and denote it by S.

Now the crucial question is: what is the representing set of the region S? The answer depends on what kind of representation we rely. If r is a regular open representation: $W \longrightarrow O^*(J)$ the region S must be represented by a regular open subset of J which contains the $r(S_i)$. As is easily seen the only regular open set which satisfies this requirement is J itself. Hence, for regular representations we get $r(S) = J$. The set $r(S)$ is the closure of the set theoretical union $\bigcup r(S_i)$. The volume of this set is smaller than e (which may be chosen arbitrarily small). Hence, S may rightly be called a region with a "thick" boundary, because the volume $m(r(S)) = m(J)$ which is much larger than e. Consequently, for a regular representation r the existence of the countable fusion S of the S_i contradicts the volume assumption (V), since we have: $m(r(S)) = m(J) > e > \Sigma e_i > \Sigma m(r(S_i))$. Now we can state Forrest's result as follows:

(6.4) Proposition
The conjunction (RR) & (V) & (W) is incompatible with (CF).

This incompatibility is, after all, not a very happy state of affairs. We may attempt to circumvent it by dropping either (W), (RR) or (V) thereby hopefully retaining (CF). Oppy's proposal to drop (or modify) (W) will be dealt with in the next section. Let us first consider the option of giving up (RR), i.e., we no longer require the representing sets of Whiteheadian regions to be regular open. Rather, we assume that r is an open representation only. In this case the set-theoretical union of the $r(Si)$ represents the fusion S, i.e., $r(S) = \bigcup r(S_i)$. Since $r(S)$ is a countable union of open sets, it is measurable. With respect to the volume we get $v(S) = m(\bigcup r(S_i)) \leq \Sigma m(r(S_i)) = \Sigma v(S_i) = e$. That is, for a non-regular open representation the volume assumption (V) is *not* violated, and Forrest's incompatibility thesis no longer holds. Hence we get:

(6.5) Proposition
The conjunction (W) & (V) is compatible with (CF).

Of course, we have a price to pay. The class of open sets $O(J)$ which represent Whiteheadian regions is larger than $O^*(J)$ and may contain members we do not like in the office of representing honest Whiteheadian regions. Hence, admittedly, $O(J)$ may not be the optimal candidate for the representation of Whiteheadian regions. It should be noted, however, that $O(J)$ need not be the last word. [10]

Now let us consider the move of giving up (V) and retaining (RR). In this case the axiom of countable fusion (CF) may hold, i.e., the fusion S of the S_i exists although its volume may exceed the sum of the volumes of the S_i. Hence we get:

(6.6) Proposition
The conjunction (W) & (RR) is compatible with (CF).

At first look, giving up (V) (and keeping (RR)) may be considered as a rather desperate move. It is, however, more plausible than Forrest wants to make us believe. In fact, (V) and (RR) do no harmonize very well with each other. Even for finitely many regions X_i, $i = 1, ..., n$, the regular fusion FX_i is often represented by a regular open set $r(FX_i)$ which is strictly larger than the set theoretical union of the sets $r(X_i)$ representing the X_i, i.e., one may have $\bigcup r(X_i) \subset r(FX_i)$ and $\bigcup r(X_i) \neq r(FX_i)$. Of course, for finitely many X_i the Lebesgue measure of the difference $r(FX_i)/\bigcup r(X_i)$ is zero. Nevertheless regular representations inevitably violate what may be dubbed the principle of extension (E) according

[10] With the aid of the Lebesgue measure m the class $O(J)$ may be restricted to a smaller better behaved class $O_L(J)$ of open sets without violating the volume assumption (V). Let $X \in P(J)$ be a Lebesgue measurable set and $x \in J$. For $r > 0$ denote the open ball around x of radius r by $B(x,r)$. Then we may define the density $\delta(X,x)$ of X at x by $\delta(X,x) := lim_{r \longrightarrow 0} m(X \cap B(x,r))/r^3$ whenever the limit exists. The measure theoretic closure $cl_L(X)$ of X is defined as the set of all points x which satisfy $d(X,x) \neq 0$. As is easily seen cl_L is a closure operator on $O(J)$. Hence we can define the class $O_L(J)$ of Lebesgue-regular open sets X as the class of open sets which satisfy $int_L(cl_L(X)) = X$. Replacing $O(J)$ by $O_L(J)$, ugly open sets, e.g., sets with low-dimensional holes, may be excluded from the class of region-representing sets. Moreover, since the volume $m(cl_L(X)/X)$ of the Lebesgue-boundary $cl_L(X)/X$ is known to be 0, Lebesgue-regular open mereological representations $W \longrightarrow O_L(J)$ satisfy (V).

to which the extension of a fusion should not exceed the fusion of the extensions. (E) may be considered as a weak version of (V). For adherents of (RR) who reject (E) anyway, it should not be overly difficult also to give up (V). The reward is high, namely, the restoration of mereology's innocence.

Trouble with countable fusion (CF) only arises, when (RR) is considered as a sacrosanct principle. This brings us in conflict with the volume assumption (V). Moreover, even if we abandon (CF) the conjunction (V) & (RR) appears to be problematic, since (V) and (RR) do not sit comfortably with each other due to the violation of the extension principle (E). Hence, Forrest's verdict on mereology should not be considered as the last word. For the present, (CF), i.e., mereology is to be considered as innocent.

Let us now come to Oppy's criticism of Forrest's contentions. Against Forrest he argues that "countable fusion [has] not yet [been] proven guilty: it may be the Whiteheadian account of space whatdunnit" (Oppy 1997, 253). In other words, Oppy proposes to give up (W) in order to retain (CF), i.e., mereology. In this section I want to show that Oppy's attempt of shifting the blame on the Whiteheadian account of space is ill-founded. For this purpose I have to rely on some facts of the mathematical theory of pointless topology (e.g. Mac Lane and Moerdijk 1992, Vickers 1989).

Oppy argues that a Whiteheadian has to subscribe to a concept of regions according to which the mereological difference between a region and a countable fusion of regions should again be a region. If it turns out that this is not the case the concerned concept of region should be replaced by another one which satisfies this requirement. He claims that Forrest's notion of a Whiteheadian region does not meet this condition.

Oppy's argument is based on the meanwhile notorious countable fusion S. The details are as follows. Let us assume that regions are (represented by) open subsets of the topological space J. Now consider the mereological difference S of the regions J and S. According to Oppy, the mereological difference of the regions J and S is just the set theoretical difference of their representing point sets. The set J/S is, of course, not open. Hence, it does not represent a region in the sense of Forrest. Thus, Oppy concludes, the Whiteheadian should liberalize his account of regions admitting regions that are not (representable by) open sets of J. As I will show this argument is not sound for several reasons. First, Oppy opts for the wrong concept of mereological difference. Secondly, his account amounts to a complete surrender of the Whiteheadian account. Thirdly, his liberalization of the concept of Whiteheadian region is unnecessary.

Let us first consider the concept of mereological difference. Oppy starts with an open representation of Whiteheadian regions, i.e., regions are to be (represented by) elements of $O(J)$. Since $O(J)$ is a *complete Heyting algebra* it has a well-behaved notion of difference. The difference between open sets X and Y is not the (Boolean) set theoretical difference $X/_mY$. Rather, the mereological difference $X/_mY$ of open sets X, Y is defined as the largest open subset contained in X and having empty intersection with Y, i.e., $X/_mY := int(X/CY)$, CY being the set theoretic complement of Y in J.

According to Oppy, the class of Whiteheadian regions should be closed with respect to the difference operation, i.e., if X and Y are regions the mereological difference $X/_mY$ should again be a region. If we identify, as Oppy does, the mereological difference $X/_mY$ with X/Y this requirement has desastrous consequences for a Whiteheadian account of space. Even ordinary points come out as Whiteheadian regions: In his 1-dimensional account this result may be sketched as follows. Start with the (non-trivial) difference of two open intervals. This is a half-open interval. Since the differences of regions are to be considered as a region, half-open intervals become regions. Appropriate intersections of half-open intervals help closed intervals to be recognized as regions. Finally, intersections of appropriate closed intervals generate points. This is the unconditional surrender of the Whiteheadian account of space. Hence, for the Whiteheadian, Oppy's generalized account of regions ends in desaster. If we stick to the correct notion of mereological difference, namely $X/_mY$, these fatal consequences do not arise.

Although Oppy's proposal is doomed to fail, his argumentation points to a task that cannot be easily dismissed. How we can differentiate in a Whiteheadian framework between the fusion S and the space J? As is easily seen the mereological difference $J/_mS$ between J and S is $\emptyset$. Hence, the perspectives for a Whiteheadian distinction between these entities look bleak, to say the least. However, we should not be discouraged. The fact that J and S are different does not imply, as Oppy assumes, that their difference can be directly expressed as a difference in terms of regions, i.e., we cannot expect that for different regions X and Y there exists a (generalized) region making that difference. As I want to show now, Oppy's proposed modification of Forrest's Whiteheadian account is unnecessary. Sticking faithfully to the original Whiteheadian account of space according to which regions are to be represented solely by open sets we get everything we want, namely, a completely satisfying account why and in what sense J and S are different.

The theory of pointless topology offers a ready-made Whiteheadian account of space (Mac Lane and Moerdijk 1992, Vickers 1989) which may be used to distinguish between between S and J in a Whiteheadian style. One need not invoke any spatial entities that are not regions such as points or Oppy's generalized non-open regions. Very briefly, this works as follows: Let K be a topological space, i.e., a set of points endowed with a topological structure given by the Heyting algebra $O(K)$ of open sets. Then one may recover the set K and its topological structure from $O(K)$ alone *provided* the topological structure of K is *sober*. Since J (and its subspaces) are well-known to be Hausdorff, they are sober. Hence, we may recover the difference of J and S from the difference of the Heyting algebras $O(J)$ and $O(S)$. In order to distinguish between between J and S we need not introduce, as Oppy does, a new kind of regions beyond the original ones represented by open sets. That train of thought is fully in line with the Whiteheadian account. For the Whiteheadian, the realm of spatial entities is not restricted to regions only. Rather, regions are to be regarded as the fundamental spatial entities from which the other spatial entities such as points are to be constructed. Hence, a Whiteheadian should be happy to be able to reconstruct the set of points J from the class of regions $O(J)$.

After having shown that Forrest's and Oppy's verdicts of guilty on mereology and the Whiteheadian account of space are not sound, a last remark on the relevance of their intuitive *Gedankenexperiments on colouring or filling space* may be in order. In the case of Forrest, these *Gedankenexperiments* attempt to show that oddities arise if one does not accept the incompatibility of (CF) and (W), in the case of Oppy they intend to show that one has to admit regions that have parts not representable by open sets. Forrest's and Oppy's considerations are based on the seemingly counterintuitive fact that a large region of space may be "filled" or "covered" by a region of arbitrarily small volume. A vivid illustration offers the 2-dimensional case: We may use an arbitrarily small amount of colour to paint black an arbitrarily large canvas. When we have already accepted that arbitrarily small amounts of colour are available, this fact shouldn't appear as quite so counter-intuitive. Provided the colour is cleverly put on the appropriate places, it seems plausible that the blackness of the canvas is essentially in the eyes of the beholder. The relevance of such intuitive *Gedankenexperiments* may be limited, however. As for empirical theories, we cannot expect that ontological theories have to be intuitively plausible. Thus, we should bite the bullet of apparent intuitive oddities and stick to precise formal argumentation.

Even if my pleading for the ontological innocence of mereology and Whiteheadian account of space should be considered as successful it should not be considered as the main result of this note. Rather, I'd like to have rendered plausible that the problems which beset Forrest's and Oppy's accounts point to a new and fascinating field of formal ontology located at the interface of mereology, topology and measure theory, namely, a representational theory of mereological structures.

7 Concluding Remarks

Topological representability of mereological systems forges a close link between both the areas of mereology and topology. Actually, this has been a desideratum for quite a long time (cf. Menger 1940). Although Menger made this proposal already in 1940, his proposal has been seriously pursued only much later. The establishment of a substantial relationship with topology would be profitable in particular for mereology. Until today, mereology is a rather isolated "philosophical" discipline, replete of terminology but not overly rich in substantial theorems. On the other hand, topology is one of the fundamental disciplines of modern mathematics. Bringing mereology into contact with the full-flegded and highly non-trivial theories of topological and lattice structures may help mereology to leave its present state of theoretical immaturity.

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References

Banaschewski, B. and R.-E. Hoffmann (Eds.) (1981). *Continuous Lattices*, Volume 871 of *Lecture Notes in Mathematics*, Berlin / Heidelberg / New York. Springer.

Biacino, L. and G. Gerla (1991). Connection structures. *Notre Dame Journal of Formal Logic 32*, 242–247.

Biacino, L. and G. Gerla (1996). Connection structures: Grzegorczyk's and Whitehead's definitions of point. *Notre Dame Journal of Formal Logic 37*, 431–439.

Clarke, B. L. (1981). A calculus of individuals based on "connection". *Notre Dame Journal of Formal Logic 22*, 204–218.

Clarke, B. L. (1985). Individuals and points. *Notre Dame Journal of Formal Logic 26*, 61–75.

Davey, B. A. and H. A. Priestley (1990). *Introduction to Lattices and Order.* Cambridge: Cambridge University Press.

Erné, M. (1982). *Einführung in die Ordnungstheorie.* Mannheim: Bibliographisches Institut.

Forrest, P. (1996a). From ontology to topology in the theory of regions. *The Monist 79*, 34–50.

Forrest, P. (1996b). How innocent is mereology? *Analysis 56*(3), 127–131.

Gerla, G. and R. Tortora (1992). La relazione di connessione in A. N. Whitehead: Aspetti matematici. *Epistemologia 15*, 351–364.

Gerla, G. and R. Tortora (1996). Dissezioni e intersezioni di regioni in A. N. Whitehead: Aspetti matematici. *Epistemologia 19*, 289–308.

Grzegorczyk, A. (1960). Axiomatizability of geometry without points. *Synthese 12*, 228–235.

Johnstone, P. T. (1983). The point of pointless topology. *Bulletin (New Series) of the American Mathematical Society 8*, 41–53.

Lewis, D. (1991). *Parts of Classes*. Oxford: Basil Blackwell.

Mac Lane, S. and I. Moerdijk (1992). *Sheaves in Geometry and Logic, A First Introduction to Topos Theory*. New York: Springer.

Menger, K. (1940). Topology without points. *Rice Institute Pamphlets 27*, 80–107.

Mormann, T. (1998). Continuous lattices and whiteheadian theory of space.

Oppy, G. (1997). Countable fusion not yet proven guilty: it may be the Whiteheadian account of space whatdunnit. *Analysis 57*(4), 249–253.

Piazza, M. (1995). "one must always topologize": Il teorema di stone, la 'topologia influente' e l'epistemologia matematica. *Rivista di storia della scienza (ser. II) 3*, 1–24.

Roeper, P. (1997). Region-based topology. *Journal of Philosophical Logic 26*, 251–309.

Simons, P. (1987). *Parts*. Oxford: Oxford University Press.

Stone, M. (1936). The theory of representations for Boolean algebras. *Trans. Amer. Math. Soc. 40*, 37–111.

Vickers, S. (1989). *Topology via Logic*. Cambridge: Cambridge University Press.

Whitehead, A. N. (1929). *Process and Reality. An Essay in Cosmology*. New York: Macmillan.

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TOOLS

Predicate Based Logical Relations between Events

Abstract. It is well-known that predicates play a central role in the constitution of events and facts. In the present paper, some ideas stemming from grammar theories are used to define and to back up logical relations between philosophical entities based on predicate forming operations. We suggest formal tools for such operations and give a sketch how the formal results allow to solve several puzzles. At the end, a new kind of categorial grammer shows the linguistic relevance of the method.

1 The Problem

Usually, events are introduced as relata of causal relations, that is, as entities belonging to reality, to empirical experience in a sense. There are other empirical relations between events: temporal and spatial relations, for instance. We are interested in *logical* relations between events. The matter of discussion is not that there are such relations. The question is how we can grasp these relations. The following sections do not give a complete theory of logical relations between events, but are intended to develop a tool for analysing such relations.

Sebastian leisurely strolls through the streets of Bologna at 2 a.m., and he meets Max during his stroll. Intuitively, this sentence is not only about Sebastian, it is also about a certain stroll. Strolls, like murders, explosions and widowings are *events* which may occur, recur, have parts