Updating Classical Mereology [1]

Thomas Mormann

University of the Basque Country UPV/EHU

ylxmomot@sf.ehu.es

ABSTRACT. In this paper it is argued that classical Boolean mereology cannot deal with problems concerning structured wholes and structured parts. Elementary examples of structured mereological systems are provided by Boolean algebras, groups, similarity structures and topological structures. In general mereological systems turn to be non-Boolean. Classical Boolean mereology is to be considered only as a very special case. A truly general mereology as a general theory of parthood has to take into account the various kinds of structures present in the mereological wholes under consideration.

This idea can be rendered precise in the framework of category theory. More precisely, it can be shown that every category comes along with its own specific mereology. Depending on the category's structure the category-relative mereology more or less deviates from classical Boolean mereology.

1 Classical Mereology

Classical mereology is the theory of classical mereological systems. I take a classical mereological system to be a complete Boolean algebra. In this way, formally, classical mereology may be considered as an elementary part of the theory of Boolean algebras, since philosophers usually ignore the more advanced parts of the theory of Boolean algebras, and leave it to the mathematicians.[2]

Standard examples of classical mereological systems in this sense are the power sets of sets. Take for instance the set $X = \{a, b, c\}$. Then the resulting mereological system can be depicted by the following familiar lattice diagram in Fig.1:

In other words, the *elements* of the Boolean algebra PX are the parts of X. Not all complete Boolean algebras are power sets PX, however. A more general class of classical Boolean systems is provided by the class of Boolean algebras

[1] The author is with the Department of Logic and Philosophy of Science, University of the Basque Country UPV/EHU, Donostia-San Sebastian, Spain. Email:ylxmomot@sf.ehu.es

[2] Many mereologists prefer to exclude the bottom element 0 of a Boolean algebra. According to them, there is no "empty part". Then they conceive a classical mereological system as a Boolean algebra minus 0. Others have qualms even with the existence of a top element 1, often called the universe that comprises all parts of mereological system, still others only admit certain kinds of fusions, in particular they consider fusions of infinitely many mereological individuals as unpalatable. In this paper I don't want to discuss problems of this kind. Since this paper is intended to be a contribution to the area of formal methodology I will stick to the mathematically more convenient presentation of the theory. Hence from now on I consider a classical mereological system to be a complete Boolean algebra with bottom 0 and top 1.

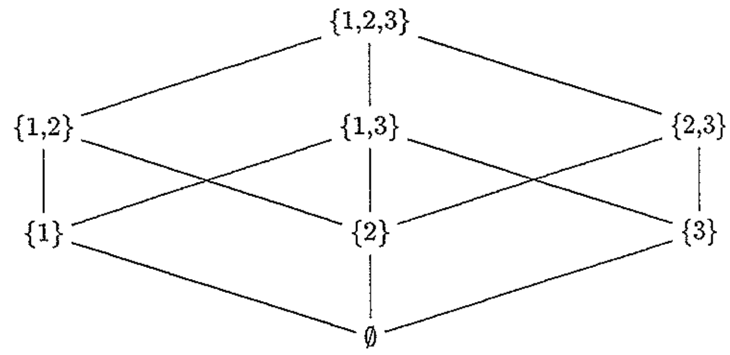

Figure 1:

O^*X of regular open subsets of topological spaces X.[3] Notwithstanding this fact, power sets PX may still be considered as the most important class of classical (complete) mereological systems, in particular since David Lewis in *Parts of Classes* vigorously advocated the thesis that sets and their subsets are to be conceived as mereological systems. More precisely, Lewis argued that mereology and set theory are intimately related in that the former is to be conceived as the elementary "innocent" part of the latter (cf. [7, chap.1]). Conceiving classical Boolean mereology as the elementary part of set theory he self-confidently asserted: "I myself take mereology to be perfectly understood, unproblematic, and certain." (ibidem, 75). For philosophers like him, mereology is just a tool that by itself is philosophically not very interesting. This is a rather bold claim. In the following I'd like to argue that it is wrong. Mereology is more complex and more interesting than partisans of classical mereology might have imagined. Asserting that mereology is to be understood formally as the theory of Boolean algebras is to claim that the mereological concepts of part and whole are fully captured by the conceptual framework of Boolean algebras. In other words, everything concerning parts and wholes can be expressed in terms of complete Boolean algebras. This is a rather strong claim. In this paper I'll argue that it is not only strong but untenable.

Instead of casting mereology in the narrow framework of Boolean algebras I propose to conceive the notions of part, whole, and their relatives as *context dependent*. It is not plausible to assume that "part" and "whole" mean always the same, in particular it is not plausible that the parts of a whole always form a complete Boolean algebra. Take, for instance, the body of a living being. Then one may consider its heart as a part of the whole body. It seems, however, strange to say that the "body minus the heart" is just another part of the body that serves as the "complement" of the "heart-part". But this is required from the perspective of Boolean mereology. Thus, classical mereology is not very good in dealing with structured wholes and structured parts. But most wholes to be met in the world are structured wholes in some way or other.

Traditional mereology bluntly assumes that "structured wholes" are of no concern for mereology in its genuine sense. Mereology is assumed thus abstract

[3] Actually, due to Stone's representation theorem, all complete Boolean algebras can be represented in this way.

that it can safely ignore any non-mereological aspects of the systems it is dealing with. Thereby the ken of mereology is severely restricted, and not much is left for it. The problem of the relation between structure and mereology not only concerns "difficult" entities such as "organic wholes", it pops up also for artifacts. If the carburetor of my car is part of the car, is the "car minus the carburetor" another "complementary" part of the car? From the point of view of a car mechanic the latter "part" of the car hardly makes sense. Or, take another more formal example. Take Boolean algebras and consider them as candidates of mereological investigations asking the question

What are the parts of a Boolean algebra?

In analogy to Lewis's claim that the parts of a set are its subsets it does not seem too far-fetched to contend that the parts of a Boolean algebra are its Boolean subalgebras. As we shall see in a moment, this opens the gate to a wealth of non-classical, i.e. non-Boolean mereologies, since, as it is well-known, the *Boolean subalgebras* of a Boolean algebra in general do *not* form a Boolean mereological system. Thus, the mereology of Boolean algebras does not fit into the framework of classical Boolean mereology. This is already shown by the algebra of Boolean subalgebras of the power set $P(\{a, b, c\})$ of a set with three elements a, b, and c. It looks as follows:

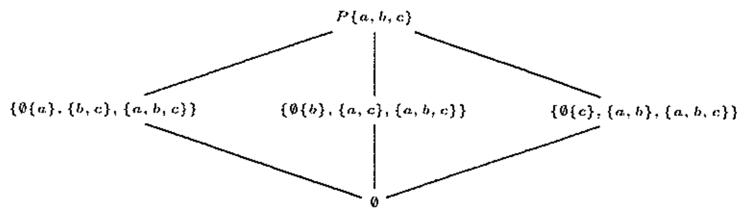

Figure 2:

This lattice, often called the "diamond" (cf. [3, p.132]), is clearly not a Boolean lattice, it is not even a distributive lattice. Thus, conceiving a Boolean algebra B as a structured whole whose structured parts are its Boolean subalgebras leads us outside the ken of classical Boolean mereology. This phenomenon is, of course, not restricted to Boolean algebras. Virtually all non-trivially structured entities lead to a non-Boolean structural mereologies. To put it bluntly, Boolean mereological systems are not the rule but rather the exception. In general, mereological systems are non-Boolean systems. I take the failure of classical Boolean mereology to cope with problems of structure as a good reason to consider the project of revising and updating classical mereology.

The outline of this paper is as follows: In Section 2 we will deal with an elementary example of a structural mereology, to wit, the structural mereology of groups. The mereology of groups, it will be argued, may be considered as a paradigmatic example of a structural mereology that considerably differs from standard Boolean mereology. In Section 3 we outline the general format of structural mereologies in the framework of the mathematical theory of categories. The upshot will be that every category **C** comes along with its own mereology. Depending on the kind of **C**, this **C**-mereology turns out to be more or less similar to the classical Boolean mereology of sets.[4] As is to be expected the classical Boolean mereology of sets can be identified with the mereology of the category **SET** of sets. As another example of a structural mereology, which is rather similar but still different from set-theoretical mereology, in Section 4 we consider the mereology of similarity structures, i.e., sets endowed with a reflexive and symmetric similarity relation. A similarity structure may be considered as a rather simple kind of a spatial structure that allows to introduce the notion of neighborhood. Another example of a structural mereology related to spatial structures is discussed in Section 5, namely the structural mereology of spaces dealing with topologically well behaved parts of topological spaces. In Section 6 it is argued that the general category-theoretical perspective on mereology may also shed new light on a classical mereological problem that has been considered already by Plato and Aristotle, namely, the problem whether the whole is "more" than its parts. The paper concludes with some general remarks on the prospects of a generalized mereology.

2 Structural Mereology I: Groups

The objects of the world are rarely blobs lacking any structure, rather they are structured in some way or other. One may even doubt that the concept of an object without any structure makes sense at all. Hence let us take as our starting point the general assumption that the objects of the world we are dealing with are structured objects, "structured" to be understood in a broad sense that need not be specified for the moment. Then, given a structured whole W, it is reasonable to ask for its *structured* parts, not just for its parts.

In the first section we already mentioned the case of Boolean algebras and their parts. If one considers the Boolean subalgebras of a Boolean algebra as its parts, the resulting mereological system of subalgebras is in general *not* a Boolean algebra. For didactical reasons, instead of Boolean algebras, I propose to consider the even more elementary case of groups and their structural parts. Recall that a group G is a set endowed with an associative multiplication $G \times G \xrightarrow{m} G$ such that there is a unique neutral element $e \in G$ satisfying $m(a, e) = m(a, e) = e$, and for all $a \in G$ there is an $a^* \in G$ with $m(a, a^*) = m(a^*, a) = e$. The element a^* is called the inverse of a, and the neutral element e is often called the unit of G. For $a, b \in G$ the product $m(a, b)$ is often denoted by $a \cdot b$ or simply by ab. A group is called abelian or commutative if $m(a, b) = m(b, a)$. A subgroup H of G is a subset of G that is a group under the multiplication of G, i.e., it contains e, and with $a, b \in H$ also a^{-1} and $m(a, b)$ are elements of H.

Groups abound in mathematics, physics, and elsewhere. Let us just mention the group of integers $\mathbf{Z}$, the groups $\mathbf{Z}_n$, of natural numbers modulo n, $n \in \mathbf{N}$, the real numbers $\mathbf{R}$, and symmetry groups such as the Lie groups $O(m), SU(m)$ endowed with their standard (matrix) multiplications.

There are several candidates for the office of the structural parts of a group G, but certainly the most straight-forward choice is to take the subgroups of G

[4] For more detailed accounts of this category-relative mereology (although not under this name) from a mathematical perspective the reader may consult the presentations to be found in Lawvere and Schanuel's Conceptual Mathematics [5](elementary), and Lawvere and Rosebrugh's *Set Theory* for Mathematicians [4] (more advanced).

as its structural parts.[5] In line with our experiences with parts of sets one may naively conjecture that the more elements a group has the more complicated its subgroup structure tends to be. This is not always true, however. Take the groups $\mathbf{Z_p}$ of integers modulo p, $p \in \mathbf{N}$ and prime. Since there are arbitrarily large prime numbers p there are groups of arbitrary large order whose only parts are the trivial group E having only one element and $\mathbf{Z}_p$ itself. Mereologically more interesting is the group $\mathbf{Z}$ of integers (with standard addition + as group operation) that has infinitely many subgroups. They are all of the form $n\mathbf{Z} := \{\cdots, -2n, -n, 0, n, 2n, \cdots\}$ for $n \in \mathbf{N}$.

In order to keep matters as simple as possible, in the rest of this paper we will consider only finite groups having therefore finitely many subgroups. To deal with these groups it is expedient to characterize them by generators and relations. More precisely, we will deal with the following groups ($z^k = z \times \cdots \times z$ (k times)):

$$\begin{aligned} \mathbf{Z}_m &:= \{e, z, z^2, , z^{m-1}; z^m = e, m \in \mathbf{N}\} \\ K &:= \{e, x, y, xy; x^2 = y^2 = e \text{ and } xy = yx\} \\ &= Z_2 \times Z_2 \text{ (Klein group)} \\ S_3 &:= \{e, x, y, y^2, xy, xy^2; x^2 = y^3 = e \text{ and } xy = y^2 x\} \end{aligned}$$

In analogy to the Boolean lattice Fig.1, which describes the parts of the set $\{1, 2, 3\}$, the structural parts of these groups may be conspicuously exhibited by lattices $PART(G)$. For $G = Z_3, Z_4, Z_6, K,$ and S_3 the lattices $PART(G)$ can be depicted diagrammatically as follows:

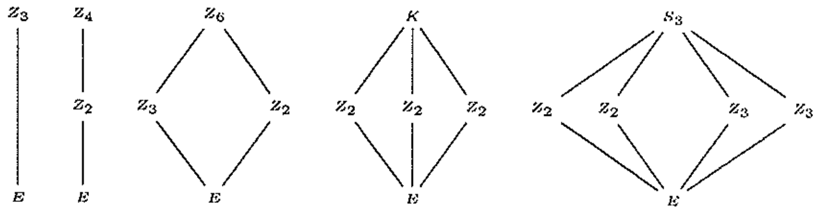

Figure 3:

More explicitly, these lattices are to be read as follows: If, according to the recipe given above, the group $\mathbf{Z}_4$ can be considered as the set $\{e, z, z^2, z^3\}$ its subgroup $\mathbf{Z}_2$ is to be conceived of as the set $\{e, z^2\}$ endowed with the canonical multiplication inherited from $\mathbf{Z}_4$. Similarly, if $\mathbf{Z}_6$ is given by set $\{e, z, z^2, z^3, z^4, z^5\}$ the subgroup $\mathbf{Z}_2$ is given by $\{e, z^3\}$ and the subgroup $\mathbf{Z}_3$ is given by $\{e, z^2, z^4\}$ endowed with the multiplication inherited from $\mathbf{Z}_6$. A bit more interesting is the case of the Klein group K. Its three subgroups are given by $\{e, x\}$, $\{e, y\}$, and $\{e, xy\}$, respectively. Following these lines the reader may calculate $PART(S_3)$ for himself.

As is directly observed only $PART(\mathbf{Z}_3)$ and $PART(\mathbf{Z}_6)$ are Boolean.[6] Thus the structural mereology of groups in general is not Boolean. Already these

[5] Another plausible choice would be to consider only *normal* subgroups of G as parts of G. From this mereological perspective simple groups (= groups having no normal subgroups) have a trivial mereological structure.

[6] As is easily shown that lattice $PART(\mathbf{Z}_m)$ of structural parts of $\mathbf{Z}_m$ is Boolean iff the natural number m is square-free.

elementary examples show that in general the structural mereology of groups is not Boolean. Different types of subgroup lattices $PART(G)$ may be distinguished and related to the types of the groups G. For a comprehensive account of this perspective on group theory see [10]. For instance, groups of prime order such as $\mathbf{Z}_3$ have a trivial subgroup lattice, since there only part is the trivial group E. The group $\mathbf{Z}_4$ is a bit more complex having a single proper part $\mathbf{Z}_2$. Evidently the parthood structure of $\mathbf{Z}_4$ is non-Boolean since its only non-trivial proper part $\mathbf{Z}_2$ has no complementary counter-part such that the fusion of it and $\mathbf{Z}_2$ would yield the whole $\mathbf{Z}_4$.

More interesting are the cases of the Klein group K that may be described as the Cartesian product $\mathbf{Z}_2 \times \mathbf{Z}_2$ of two copies of the group $\mathbf{Z}_2$ and the symmetric group $\mathbf{S}_3$ that can be characterized as the group of permutations of three elements 1, 2, and 3. Naively one might have guessed that the subgroup $\mathbf{Z}_2$ appears exactly twice in K and once in $\mathbf{S}_3$ due to the fact that K has four elements and $\mathbf{S}_3$ has six elements. The lattices $PART(K)$ and $PART(\mathbf{S}_3)$ refute this guess. The lattice $PART(K)$ is the "diamond" $\mathbf{M}_3$ in which the structural part $\mathbf{Z}_2$ of K appears three times, and $PART(\mathbf{S}_3)$ is even more complicated, exhibiting three copies of $\mathbf{Z}_2$ and one of $\mathbf{Z}_3$. Nevertheless, this lattice structure is not to be interpreted extensionally in the sense that this K is construed of three copies of $\mathbf{Z}_2$, since the group K has only four elements. Rather, one and the same group $\mathbf{Z}_2$ appears three times in the part structure of the Klein group K. Still more complicated is the case of the symmetric group $\mathbf{S}_3$.

In *Against Structural Universals* [6] David Lewis pointed out that some intricate metaphysical problems are lurking here. Does it really make sense to say that, say, one and the same part $\mathbf{Z}_2$ appears three times in one and the same whole, or that the same parts can form different mereological wholes, depending on how they are assembled? Lewis flatly denied that this is possible. He contended that a composition of this kind cannot claim to be mereological at all. According to him the talk that one and the same part appears repeatedly in one and the same whole is simply unintelligible.[7] I don't want to discuss Lewis's examples in this paper, rather I am content to show that his objections can be defused for groups and similar structures.

The appropriate general framework for dealing with this kind of questions concerning structural mereology has turned out to be the mathematical theory of categories. A category **C** may be described as a local mathematical universe of discourse dealing with a certain kind of objects, called the **C**-objects, and their relations called **C**-morphisms. Thereby one obtains, say, the category **TOP** of topological spaces, the category **GROUP** of groups, and countless categories in mathematics, physics, computer science, and other areas (cf. [4, 5, 8]).

Before we deal with mereological problems on the general level of categories, I'd like to deal with the mereology of a concrete and easily accessible specific category, to wit, the category of groups. Then the general case may be more easily grasped.

As has become evident in the evolution of modern mathematics, groups and other structures do not live in isolation. Rather, group theory does not only study groups in themselves, an essential part of group theory is the study of

[7] Lewis did not deal with groups but with "structural universals" such as "methane" or "butane" that allegedly were composed of more primitive universals such as "hydrogene" and "carbon" as is indicated by their chemical formulas CH_4 and C_4H_8, respectively.

relations between groups. The most important relations are homomorphisms (or structure-preserving mappings) $H \xrightarrow{f} G$ defined as follows:

Definition 1 *Let H and K be groups. A homomorphism h with domain H and codomain K is a set-theoretical map $H \xrightarrow{h} K$ satisfying the conditions:*

$$f(ab) = f(a)f(b) \quad (1a)$$
$$f(e) = e \quad (1b)$$
$$f(a^*) = f(a)^* \quad (1c)$$

The set $h(H) := \{h(x); x \in H\}$ is a subset of G and called the image of H in G. As is easily shown the image $h(H)$ of H is a subgroup of G. Between any two groups H and K there always exists the trivial homomorphism $H \xrightarrow{t} K$ mapping all elements a of H to $t(a) = e$. Given two groups H and K it is usually difficult to find a non-trivial homomorphism $H \xrightarrow{f} K$. It may even happen that none exists. For instance, there is no non-trivial homomorphism between groups whose order is prime to each other, e.g. $\mathbf{Z}_2$ and $\mathbf{Z}_3$. For the following we need to distinguish between different types of homomorphisms:

Definition 2 *A group homomorphism $H \xrightarrow{h} K$ is a monomorphism if and only if $h(x) = h(y)$ entails $x = y$. A group homomorphism $H \xrightarrow{h} K$ is an epimorphism if and only if for every $y \in K$ there is at least one $x \in H$ such that $h(x) = y$. A homomorphism is an isomorphism if and only if it is a monomorphism and an epimorphism.*

Definition 3 *Let $H \xrightarrow{h} K$ and $H' \xrightarrow{h'} K$ be two monomorphisms with the same target K. The monomorphisms h and h' are called equivalent if and only if there is an isomorphism $H \xrightarrow{s} H'$ such that $h = h' \cdot s$, i.e., for all $a \in H$ one has $h(a) = h'(s(a))$.*

It is easily proved that Definition 3 defines indeed an equivalence relation between monomorphisms. The equivalence class of a monomorphism $H \xrightarrow{h} G$ may be denoted by $[h]$. In order not to overburden notion we will often blur the distinction between h and $[h]$, i.e., we will talk of a monomorphism $H \xrightarrow{h} G$ even if we really mean the equivalence class $[h]$ of h. Now we are ready to formulate the central notion of this section, to wit, the concept "part of a group":

Definition 4 *Let G be a group. A part of G is an equivalence class of monomorphisms $H \xrightarrow{h} G$ according to the equivalence relation Definition 3. If $H \xrightarrow{h} G$ represents a part of G, then H is called the type of h (or $[h]$). If it is not necessary to refer explicitly to h, we may simply call H a part-type of G.*

Assume that $H \xrightarrow{h} G$ and $H' \xrightarrow{h'} G$ are equivalent monomorphisms, i.e., define the same part of G. Then the images $h(H)$ and $h'(H')$ coincide. Since $h(H) = h'(H')$ is a subgroup of G one can identify a part of G with a subgroup of G, to wit the image $h(H)$ or $h'(H')$. Since we aim at a general clarification of the concept of part, which is not confined to group theory, it is nevertheless expedient to stick the clumsier terminology of parts as (equivalence classes of) monomorphisms $H \xrightarrow{h} G$, even if for groups a more elegant terminology is available.

If $H \xrightarrow{h} G$ is a part of G it is natural to express this by saying that h is a way how the part-type H is involved in G. Or, still differently, we may say a part of G is a part-type-in-a-way. Distinguishing between parts and part-types allows us to speak meaningfully that one part may appear in different ways. This will be shown by the following elementary examples. More precisely, we will show that one and the same group H – conceived as a part-type of an other group G - may give rise to different parts of G, i.e., there may exist non-equivalent monomorphisms $H \xrightarrow{h} G$ and $H \xrightarrow{g} G$. Take, for instance the Klein group K, and denote the generator of the group $\mathbf{Z}_2$ by z. Then three non-equivalent monomorphism $Z_2 \xrightarrow{h_i} K, i = 1, 2, 3$, are defined by

$$
\begin{aligned}
h_1(z) &= x \\
h_2(z) &= y \\
h_3(z) &= xy
\end{aligned}
$$

According to Definition 3, these homomorphisms define *different* parts of K all of which are of the same type $\mathbf{Z}_2$. It goes without saying that this phenomenon is not restricted to the Klein group but occurs for many groups. The group of permutations S_3 provides a particularly interesting example since it reveals that in the case of groups a structured whole is not necessarily determined by its part-types alone. It might happen that two different structured wholes have the same part types. This is quite in line with our mereological intuitions in that often the same parts may be assembled in different ways so that they form different wholes. Take the groups $\mathbf{Z}_6$ and S_3. Although they both have six elements they are non-isomorphic, since the former is commutative and the latter not. A classical theorem of group theory tells us that they can only have parts of type $\mathbf{Z}_2$ or $\mathbf{Z}_3$. Indeed, the only parts of Z_6 are given by the monomorphisms $Z_2 \xrightarrow{h} Z_6$ and $Z_3 \xrightarrow{k} Z_6$ defined by $h(z) = x^3$ and $k(z) = x^2$. On the other hand, S_3 has a more complicated parthood structure since there are three non-equivalent ways $\mathbf{Z}_2 \xrightarrow{g_i} S_3, i = 1, 2, 3$ for $\mathbf{Z}_2$ to become a subgroup of S_3:

$$
\begin{aligned}
g_1(w) &= x \\
g_2(w) &= xy \\
g_3(w) &= xy^2
\end{aligned}
$$

On the other hand there is only one way $\mathbf{Z}_3 \longrightarrow S_3$ of how $\mathbf{Z}_3$ be a part of S_3, namely by mapping its generator z onto y. In sum, it is possible that the very same components $\mathbf{Z}_2$ and $\mathbf{Z}_3$ can be assembled in two different ways. This is in no way paradoxical if one bases one's considerations on the sophisticated definition of parthood, which takes into account the way, how the components are assembled, to wit, the monomorphisms by which the "abstract" groups $\mathbf{Z}_2$ and $\mathbf{Z}_3$ are embedded into Z_6 and S_3. Thereby it is revealed that there is only one way that renders $\mathbf{Z}_2$ and $\mathbf{Z}_3$ parts of Z_6, while in the case of S_3 the group $\mathbf{Z}_2$ gives rise to three different parts, while for $\mathbf{Z}_3$ there is still only one way to become part of S_3.

Pace Lewis, then, according to Definition 4, for groups the talk of "a part many times over" *does* make sense. One and the same group H may be part of a larger group G in many different ways, namely, provided there are different, non-equivalent monomorphisms that embed H in G. There is nothing mysterious about it.

In order to convince a Lewisian skeptic it may be more expedient, not to rely on groups and their structural parts but to show that the same kind of argument goes through for Lewis's preferred kind of mereological systems, to wit, set-theoretical ones. In this case, instead of group-theoretical monomorphisms we simply deal with set-theoretical monomorphisms, i.e. 1-1-set-theoretical functions $Y \xrightarrow{m} X$. Following Definition 3 a part of the set X is defined to be an equivalence class of set theoretical monomorphisms $A \xrightarrow{m} X$. Then m defines a subset of X, namely $m(A) \subseteq X$. An equivalent monomorphism $A' \xrightarrow{m'} X$ yields the same subset $m(A) = m'(A')$. Hence, every part of X defines a unique subset of X in the ordinary sense. On the other hand, if A is a subset of X in the ordinary sense, then the inclusion of A in X yields a canonical monomorphism $A \xrightarrow{i} X$. The equivalence class of this monomorphism defines a unique part of X in the sense of Definition 4. Hence, the notions of subsets and parts of X coincide. Denoting the class of subsets of X by PX we get that PX is a lattice with respect to the order relation $\leq$ that is just the familiar set-theoretical inclusion. More precisely, PX is a Boolean lattice with bottom element $\emptyset$ and top element X.

The essential difference between the lattices PX of subsets of X and the lattices $PART(G)$ of group parts is that the PX are *Boolean* lattices while the subgroup lattices $PART(G)$ in general are not Boolean.[8] The lack of Booleaness for the lattices $PART(G)$ reflects the extra structure present in groups G but not present in arbitrary sets X. If G is a group, not all subsets $A \subseteq G$ are G-parts but only those that are compatible with the group structure, i.e., those that are subgroups. In other words, only a selected subclass of subsets of G qualifies as group parts. Formally, then, the generalization of traditional mereology to a wealth of structural mereologies amounts to giving up the requirement that mereological lattices have to be Boolean lattices. Instead, we subscribe to a more liberal account that allows for other types of lattices as well.

Definition 4 is, however, only the beginning of a full-fledged theory of structural mereology for groups. Up to now, we have only defined the notions of structural parts and part type but have said nothing why the class $PART(G)$ of structural parts G is actually a lattice. For groups, this is intuitively more or less

[8] Among the parthood lattices displayed above, only that of $\mathbf{Z}_3$ and Z_6 are Boolean. In general, lattices of subgroups are not even distributive as is shown by the examples K and S_3. For a comprehensive treatment of subgroup lattices see [10].

clear since the subgroups of G are subsets of the set of elements of G endowed with a group multiplication inherited from that of G. Thereby it is easily seen that the set $PART(G)$ of structural parts inherits a lattice structure from the power set PG of G.

Nevertheless, in view of the general category-theoretical account of structural mereology to be developed in the next section, it is desirable to show that this lattice structure can also be obtained from the new definition of part as laid down in Definition 4. That is to say, we'd like to formulate the basic mereological notion such as overlapping, disjointness, fusion in terms of monomorphisms. This can indeed be done, as is shown by the following definition, which provides the base for a full-fledged structural mereology of groups:

Definition 5 *Let* $H \xrightarrow{h} G$ *and* $H' \xrightarrow{h'} G$ *represent two parts of* G. *If there is a monomorphism* $H \xrightarrow{p} H'$ *such that* $h = h' \cdot p$ *this is denoted by* $h \leq h'$, *or, by an abuse of language, simply by* $H \leq H'$. *Then, again committing an abuse of language,* H *is called a smaller part of* G *than* H'.

The definition of $\leq$ is reasonable in the sense that it depends only on the equivalence classes of h and h', not on the representing monomorphisms h and h'. Then it is easily seen that $\leq$ is an order structure on the parts of G, i.e., $\leq$ is reflexive, transitive, and anti-symmetric. Even more can be proved:

Proposition 1 *Let* G *be a group and denote the set of parts of* G *as defined in Definition 5 by* $PART(G)$. *Then* $PART(G)$ *endowed with* $\leq$ *has the structure of a lattice, i.e.,* $(PART(G), \leq)$ *is an order structure with minimal element* E, *maximal element* G *and every finite subset of* $PART(G)$ *has an infimum and a supremum with respect to* $\leq$. *As usual, the infimum* $inf(H, K)$ *is denoted by* $H \wedge K$, *and the supremum* $sup(H, K)$ *is denoted by* $H \vee K$.

This lattice structure on $PART(G)$ enables us to speak of inclusion, composition, overlapping, and disjointness of G-parts much in the same way as for the parts of classical mereological systems such as the system PX of subsets of a set X. For instance, $H \leq H'$ is to be interpreted as that H is included in H', and $H \wedge H' \neq E$ means that H and H' overlap nontrivially. It may be expedient to spell out in some detail what is the infimum $H \wedge K$ and the supremum $H \vee K$, respectively. By definition H and K – as parts of G – are defined by some monomorphisms $H \xrightarrow{h} G$ and $K \xrightarrow{g} G$, respectively. Then $h(H) \cap g(K) \subseteq G$ is a subgroup of G, independent of the representing monomorphisms h and g. Hence the inclusion $h(H) \cap g(K) \xrightarrow{i} G$ defines a well defined part of G in the sense of Definition 4. This part is denoted by $H \wedge K$. Suppressing the various monomorphisms involved one may simply say that $H \wedge K$ is just the intersection of H and K. Thus, the construction of the infimum $H \wedge K$ of the G-parts H and K is not essentially different from the construction of the infimum of two sets X and Y in set theory, which is just the intersection $X \cap Y$.

Things become more interesting when we consider the supremum of $H \vee K$. In this case $H \vee K$ is not the set-theoretical union of H and K, since this is usually not a subgroup of G. Rather, $H \vee K$ may be described as the subgroup of G generated by H and K. More precisely, the following holds: Let H and K – as parts of G – be represented by the monomorphisms $H \xrightarrow{h} G$ and $K \xrightarrow{g} G$, respectively. Then there is a smallest subgroup of G that contains $h(G)$ and $g(K)$. This subgroup may be denoted by $H \vee K \subseteq G$ and defines a part of G in the sense of Definition 5. In other words, just as in the case of standard mereology, for G-parts one can form the infimum (intersection, overlapping) and the supremum (fusion, composition) in such a way, that these operations render the set $PART(G)$ of G-parts a lattice. Although the composition $H \vee K$ of H and K is not just the set-theoretical union but something different. Nevertheless, the "non-mereological" composition $H \vee K$ makes perfect sense, since it is a well-defined subgroup of G. *Pace* Lewis, then, group theory is an example of a domain which provides an honest notion of a "sui generis composition".

It should be clear that our approach not only works for groups but for many other structures as well. The details will be described precisely in the next section. What is going on may be informally described as follows. According to Fig.2 a group G is a "structured set" in that it is a "set plus group structure". Other types of structured sets may be defined analogously. Usually it is not difficult to define appropriate structure-preserving homomorphisms between these structures. Thereby for each type of structured sets one may set up a specific notion of parthood and composition mimicking the definitions Definition 2–5 . Thereby we obtain for each structure S a structure-specific mereology encapsulated in the notion structure specific lattice $PART(S)$ of structured parts of S. This program will be carried out in detail in the next section in terms of category theory. The category-theoretical generalization of mereology reveals that the essential structure of generalized or relativised mereology that does not depend on the specific features of group theory or set theory. Moreover, in the category theoretical framework it can be shown that the traditional Boolean structure of set-theoretical mereology is only a special case of the general approach of structural mereology. In other words, standard Boolean mereology, which Lewis took as the only feasible one, and other structural mereologies are on an equal footing. All of them are special cases of a general theory of parthood and composition.

3 A Category-theoretical Framework for General Mereology

The mereologically interesting point is that every category **C** comes along with its own specific **C**-mereology that deals with the **C**-parts of its **C**-objects. Instead of describing how this works in precise abstract terms, let us be content to state that the structural mereologies of Boolean algebras and groups we just mentioned correspond exactly to the category-theoretical mereologies of the categories **BOOLE** of Boolean algebras, and the category **GROUP** of groups. As it should be, the mereology of sets favored by Lewis, is nothing but the mereology of the category **SET** of sets. The examples of the categories **BOOLE** and **GROUP** show that most categories have mereologies that are not classical. From a general category-theoretical point of view, then, there is no reason for mereologists to restrict their attention to Boolean mereology that is only a small facet of the whole range of mereological possibilities. Mereology in general is structural mereology, and it is just a "Boolean prejudice" to ignore non-Boolean mereological systems.

After having established that one can set up a structural mereology for groups, encapsulated in the concept a group-part, let us now consider how this recipe can be generalized to all kind of structures. This is done by showing that nothing depends on the specific features of groups in the refined definition of parthood as given in Definition 5. Rather, conceiving the theory of groups and group homomorphisms as one category among many, for every category **C** one may set up its own specific "**C**-mereology".

Definition 6 *A category C is given by the following ingredients:*

1. *A collection of things* $A, B, D, \cdots$ *called* **C**-*objects.*
2. *A collection of things* $f, g, h, \cdots$ *called* **C**-*morphisms.*
3. *An operation that assigns to each* **C**-*morphism* f *a* **C**-*object* $dom(f)$ *(the domain of* f*), and an other operation that assigns to* f *a* **C**-*object* $cod(f)$ *(the codomain or target of* f*). Thus, morphisms may be displayed as* $A \xrightarrow{f} B$ *whereby it is assumed that* A *is the* $dom(f)$ *and* B *is* $cod(f)$.
4. *An operation assigning to each pair* (f, g) *of* **C**-*morphisms with* $dom(g) = cod(f)$ *a* **C**-*morphism* $(g \cdot f)$ *with* $dom(g \cdot f) = dom(f)$ *and* $cod(g \cdot f) = cod(g)$ *such that the following law of associativity is satisfied. Given the configuration* $A \xrightarrow{f} B \xrightarrow{g} D \xrightarrow{h} E$ *one has* $h \cdot (g \cdot f) = (h \cdot g) \cdot f$.
5. *An assignment to each* **C**-*object* A *a* **C**-*morphism* $A - id_A \rightarrow A$ *such that for any* **C**-*morphisms* $A \xrightarrow{f} B$ *and* $B \xrightarrow{g} A$ *one has* $id_A \cdot g = g$ *and* $f \cdot id_A = f$.

Intuitively, the category-theoretical notion of a morphism intends to capture the essential features of the idea of a set-theoretical function. More precisely the requirements Definition 63–5 generalize the essential aspects of the concatenation of set-theoretical functions. Thus, it is an easy exercise to show that there is a category **SET** whose objects are sets, and whose morphisms are set-theoretical maps such that $g \cdot f$ is the familiar set-theoretical concatenation of set-theoretical functions. **SET** is not, however, the only category. Categories abound in mathematics and elsewhere. Large lists of categories occurring in mathematics, computer science and other sciences can be found in any textbook on category theory (cf. [4, 5, 8]). The category **GROUP** of groups is defined as having as objects groups and as morphisms homomorphisms in the sense of group theory. Analogously, categories of manifolds, vector spaces, rings, fields and other structures may be defined. It should be noted, however, that there are categories of a quite different kind than **GROUP** or **SET** whose morphisms are not set-theoretical mappings at all. For instance, a lattice or any other ordered structure may be conceived as a category. All of them have specific notions of parthood and composition.

In order to set up a general category-theoretical analogue of Proposition 1 of group-theoretical parthood, we need purely category-theoretical characterizations of the concepts of monomorphism, epimorphism and isomorphism that do not presuppose the notions of set, structure, and elementhood. This is achieved by the following definition:

Definition 7 *Let* **C** *be a category.*

1. *A* **C**-*morphism* $B \xrightarrow{m} D$ *is a* **C**-*monomorphism if and only if for all* **C**-*morphisms* $A \xrightarrow{f,g} B$ *the identity* $m \cdot f = m \cdot g$ *entails* $f = g$.
2. *A morphism* $D \xrightarrow{e} E$ *is a C-epimorphism if and only if for all morphisms* $E \xrightarrow{s,t} F$ *the identity* $s \cdot e = t \cdot e$ *entails* $s = t$.
3. *A* **C**-*morphism* $X \xrightarrow{h} Z$ *is a* **C**-*isomorphism if and only if it is a* **C**-*monomorphism and a C-epimorphism.*

Informally, monomorphisms are morphisms that can be cancelled from the left, and epimorphisms can be cancelled from the right. As is easily checked set-theoretical and group-theoretical homomorphisms are iso/epi/mono/morphisms in the ordinary sense if and only if they are iso/epi/mono/morphisms in the sense of 7. Thus, 7 is the "correct" category-theoretical generalization of the original, more restricted versions of these notions. The point is that 7 does not refer to set-theoretical notions such as set and element. Observing that the equivalence relation 4 is already formulated in purely category-theoretical terms the desired category-theoretical generalization of the concept of parthood is at hands:

Definition 8 *Let X be an object of a category* **C**, *and* $Z \xrightarrow{f} X$ *a* **C**-*monomorphism. A monomorphism* $Z' \xrightarrow{f'} X$ *is equivalent to* $Z \xrightarrow{f} X$ *if and only if there is a* **C**-*isomorphism* $Z \xrightarrow{j} Z'$ *such that* $f' \cdot j = f$. *Then a* **C**-*part*[9] *of X is defined as an equivalence class of* **C**-*monomorphisms* $Z \xrightarrow{f} X$.

Analogously as for the category **GROUP** for any category **C** one may define the relation $\leq$ between **C**-parts of X. Then, under some mild conditions on **C**, for every **C**-object X one obtains an order structure, or, under somewhat stronger conditions, even a *lattice* of its **C**-parts (cf. [4]). In this way, every category comes along with its own ready-made notions of **C**-mereology.

Thus, 8 and the ensuing theory of **C**-parts and **C**-composition answers Lewis's question of what is the general notion of composition and parthood: The general theory of parthood and composition is the theory of the category-relative concepts of **C**-parthood and **C**-composition, **C** an arbitrary category. It should be noted that this theory is not a philosopher's fancy invention but rather is an established mathematical enterprise built up over the last decades.

Before we leave this sketch of a general theory of parthood let us prove that for the category **SET** of sets the definition 8 yields what it should, namely, the standard mereological theory of parthood and composition. This is seen as follows. Let X be a set and $A \subseteq X$ a subset. Then the inclusion of A in X yields a canonical monomorphism $A \xrightarrow{i} X$. The equivalence class of this monomorphism defines a **SET**-part of the **SET**-object X in the sense of 8. On the other hand, every monomorphism $B \xrightarrow{m} X$ defines a subset of X, namely $m(B) \subseteq X$. It is clear that equivalent monomorphism $B \xrightarrow{m} X$ and $B' \xrightarrow{m'} X$

[9]In category theory, a **C**-part of a **C**-object X is often called a subobject of X. The family of subobjects is denoted by $Sub(X)$.

yield the same subset $m(B) = m'(B')$. Thus, every **SET**-part of X in the sense of 7 defines a unique subset of X and vice versa. Therefore the set PX of subsets of X and the set of **SET**-parts of X coincide. This evidences that the definition 8 is the correct generalization of the familiar mereological notion of a subset to arbitrary categories. This means that our definition of structural parthood is indeed continuous with the original mereological definition.

4 Structural Mereology II: Similarity Structures

The concept of similarity enjoys a mixed reputation in philosophy. On the one hand, authors such as Goodman and Quine considered similarity or resemblance as highly suspicious and finally philosophical useless. Their verdict was that "similarity was a quack ...". On the other hand, there have been philosophers such as Carnap who, at least for some time, considered the relation of similarity as a sufficient base for carrying out "the logical constitution of the world" (cf. [2]). Although the detractors of similarity have dominated the debate on the philosophical dignity of this concept there are some signs for a change of the tide. After all, similarity plays an important role in many realms of scientific and common sense argumentation.

The aim of this section is to show how mereology can be contextualized in such a way that it takes into account the concept of similarity. For this purpose we start from a rather weak concept of similarity inspired by Carnap's notion of "recollection of similarity" (cf. [2]). There similarity is conceived as a binary relation $\sim$ defined on some domain S of objects, and it is assumed that $\sim$ is reflexive and symmetric, i.e., every object x of S is similar to itself ($x \sim x$) and if x is similar to y then y is also similar to $x(x \sim y \Rightarrow y \sim x)$. It is not assumed that the similarity relation $\sim$ is transitive, i.e. from $x \sim y$ and $y \sim z$ one cannot infer that $x \sim z$. Apparently, this is a quite weak, almost trivial notion of similarity. Goodman and Quine have argued at length that this notion of similarity is too weak to be interesting. In the following I'd like to argue that their assessment was wrong.

To get started let us fix notation as follows: Let $(S, \sim)$ be a similarity structure, i.e. a set S endowed with a binary similarity relation $\sim$.[10] In the usual way, one may define various categories of similarity structures, depending on what kinds of similarity structures and similarity-preserving morphisms are admitted. In the following we will restrict our attention to one such category, denoted by **SIM**.[11] As every category **SIM** comes along with a ready-made specific **SIM**-mereology. Indeed, **SIM**-mereological systems are complete Heyting-algebras that can be conceived as generalizations of the classical Boolean systems PX that arise from **SET**-objects X. This is not too surprising since the category **SET** of sets may be conceived as a subcategory of **SIM**, namely, the subcategory of similarity structures $(X, \sim)$ endowed with a trivial similarity relation $\sim$ defined by $x \sim y := x = y$. The **SIM**-mereology provide an elementary and easily

[10] Similarity structures appear in the literature under many names: tolerance spaces, coherence spaces and others. These names emphasize the broadly spatial character of these structures.

[11] **SIM** has some interesting categorial properties. Among other things it can be shown that it is almost a topos, namely a quasi-topos.

accessible example of a structural mereology that, although similar to standard **SET**-mereology, differs in some aspects from it.

Definition 9 *The category* **SIM** *of similarity structures is defined as follows:*

SIM-*Objects:* *Similarity structures* $(S, \sim)$.

SIM-*Morphisms:* *Set-theoretical maps* $S \xrightarrow{f} T$ *that are similarity-preserving in the sense that* $x \sim y \Rightarrow f(x) \sim f(y)$

Since identity maps $S \xrightarrow{id} S$ are obviously **SIM**-morphisms and the concatenations of **SIM**-morphisms are **SIM**-morphisms again, 9 defines a category. The parts of a **SIM**-object $(S, \sim)$ are defined in the usual way as equivalence classes of **SIM**-monomorphism (see 8). Denote the set of all **SIM**-parts of $(S, \sim)$ by $PART(S, \sim)$. Then $PART(S, \sim)$ is endowed with an order structure $\leq$ as explained in the previous section. Then the following proposition can be proved:

Proposition 2 *Let* $(S, \sim)$ *be a similarity structure. Then the lattice* $PART(S, \sim)$ *of structural parts of* $(S, \sim)$ *is a complete Heyting algebra. If the similarity relation* $\sim$ *coincides with identity is trivial, then* $PART(S, \sim)$ *is just the power set* PS.

Similarity structures abound. For instance, a natural intuitive interpretation of similarity conceives similarity as some sort of nearness, i.e., two things are considered as similar, if they are near to each other in some sense to be specified. Given a element x of a similarity structure $(S, \sim)$ one may therefore consider the set $S(x) := y; x \sim y$ as a kind of neighborhood of x. Thereby a similarity structure $(S, \sim)$ obtains a kind of rudimentary spatial structure. A concrete example of such a spatial similarity structure can be obtained as follows: Let (X, d) be a metrical space, i.e., a set endowed with a non-negative real-valued function $X x X \xrightarrow{d} R$. Let $\varepsilon > 0$. Then X is rendered a similarity structure by stipulating $x \sim y := d(x, y) < \varepsilon$.

A quite different, more algebraic kind of similarity structure can be defined for Boolean algebras and similar structures. If $(B, \leq)$ is a Boolean algebra with bottom element 0, the set $B - -\{0\}$ is rendered a similarity structure by stipulating $x \sim y := infimum(x, y) \neq 0$. These two kinds of similarity structures in no way exhaust the possibilities. Similarity is a very flexible structure that allows to define similarity structures almost ad libitum.

5 Spatial Mereology

Let us conclude the series of examples of structural mereologies by an example that has met considerable attention in recent years, to with, the mereology of space. To be as specific as possible let us concentrate on the most prominent case, to wit, Euclidean space E. It should be clear, however, that the following considerations apply to a much larger variety of spaces. Conceiving E as a mereological system requires to give a reasonable answer to the mereologically fundamental question "What are the parts of Euclidean space?" The standard Euclidean answer is "The parts of Euclidean space E are sets of Euclidean points". This is, however, not a very convincing answer. It amounts to conceive E simply as a set of points, thereby completely ignoring its genuine spatial structure. In other words, the point of a genuinely spatial mereology is somehow to take into account the spatial structure of E. There are several ways to achieve this. One is to describe the Euclidean space as a topological space. Recall that a topological space (X, O^*X) is defined as a set X for which a set $O^*X \subseteq PE$ is singled out.[12] The elements of O^*X are called the open sets of the topological space X and form a complete Heyting algebra with bottom element $\emptyset$ and top element X. The Euclidean space E carries a canonical topological structure OE inherited from its canonical metrical structure. Then a reasonable answer, at least prima facie, to the fundamental mereological question "What are the parts of E?" would be "The parts of Euclidean space are the open subsets of E, i.e. the elements of OE." Considering only the elements of OE as spatial parts of E, and not just any contrived subset of E amounts to the requirement that genuine spatial parts have to be structurally "nice" or "natural", or in other words, the notion of parthood in the case of E has to take into account the spatial structure of E. This is done by denying that certain "wild" subsets of E deserve the predicate of "being a spatial part of E". For this maneuver, there is, of course, a price to pay. The resulting mereology, which takes as parts of E only the elements of OE, and not all elements of PX, is no longer Boolean, but solely Heyting. This entails that the elements of OE no longer have Boolean complements. All this can be cast in the category-theoretical framework of Section 3 introducing the category **TOP** whose objects are topological spaces and whose morphisms are continuous maps.

It should be noted that the class of open subsets O^*X of a topological space X is in no way the only reasonable choice for genuine spatial parts of X. Since even open sets may often look rather unwieldy and unnatural, one may prefer to restrict the ken of proper spatial parts of a space X still further. Accordingly, some authors have proposed to consider only regular open subsets of X as its spatial parts. This class, denoted by O^*X, is a proper subclass of O^*X. Indeed, the move from O^*X to O^*X has some advantages. For instance, O^*X is a complete Boolean algebra – and not only a Heyting algebra. On the other hand, new difficulties arise. For instance, while (under some mild restrictions) a topological space (X, O^*X) is determined (up to isomorphism) by the Heyting algebra O^*X of its parts, this is no longer true for O^*X. There may be topologically different spaces X and Y having isomorphic algebras O^*X and O^*Y. This can be interpreted as the fact that the spatial mereology of a space no longer fully determines it. Accepting only regular open subsets $A \in O^*X$ of X as spatial parts entails that some information on X is lost in that the Boolean lattice O^*X alone does not uniquely determine the point set X. Hence, the full topological structure of (X, O^*X) escapes a mereological description in terms of O^*X. In other words, topological mereology is inevitably non-Boolean mereology.

6 Platonic versus Aristotelian Mereology

The category-theoretical generalization of classical mereology has not only the virtue to offer a unifying framework for a large variety of structural mereologies.

[12]For a succinct account of the basic notions of topology the reader may consult [3, chap.10]

It may also be used to shed new light on a classical metaphysical problem that already occupied Plato and Aristotle, namely, the time-honored problem whether a whole is identical with its parts. The pertinent texts are *Socrates' Dream* in the *Theaetetus* and *Aristotle's Metaphysics Z 17*, respectively (cf. [9, chap.4, 60ff]). As is well known, Plato held that the whole is identical with its parts. Accordingly, he claimed that the syllable "SO" is identical to the letters "S" and "O". Then the difficulty arises that the letters "S" and "O" may not only be assembled to form the syllable "SO" but also to the syllable "OS". This may be considered as analogous to the group-theoretical fact that the groups $\mathbf{Z}_2$ and $\mathbf{Z}_3$ may be assembled in two different ways, namely, one way to yield the abelian cyclic group $\mathbf{Z}_6$ and another way to yield the non-abelian group $\mathbf{S}_3$.

In contrast to Plato, Aristotle maintained that the whole and its parts are different. The syllable, said Aristotle is not just the letters "S" and "O" but something else, too, since when the syllable is "dissolved, the whole, i.e. the syllable, no longer exists, but the elements of the syllable exist." Hence, Aristotle concluded, the syllable consists of the elements plus a further item, which is of a completely different type than the elements namely its substance. As Scaltsas points out, the classical dispute between a Platonic and an Aristotelian account to mereology finds a certain rehearsal in the dispute between Armstrong, Lewis and other contemporary philosophers on the possibility of non-mereological composition of structural universals (cf. [1, 6]). Structural mereology in the sense of category-relative mereological as outlined in this paper may not directly contribute to this problem, but at least it may widen the horizon of mereologists and thereby help indirectly to better understand some classical problems of traditional mereology. For reasons of space I cannot further elaborate this point, but it seems that the account of structural mereology presented in this paper favors an Aristotelian stance in mereological matters.

7 Concluding Remarks

In this paper I have argued that classical Boolean mereology needs to be contextualized or customized in the sense that it has to take into account the various kinds of structures that are present in the objects of our mereological investigations. This kind of customized mereology defies the confines of standard Boolean mereology. Elementary examples of structured mereological systems in this sense are provided by Boolean algebras, groups, similarity structures, topological structures, and many other structures as well. These examples show that in general, the lattices of structural parts of structured wholes are not Boolean. These considerations can be rendered precise in the framework of category theory according to which every category **C** comes along with its own specific **C**-mereology. Classical Boolean mereology turns out to be closely related to the category **SET** of lattices. In sum, then, I'd to contend that mereology is far from being "perfectly understood, and unproblematic" part of formal methodology, as the late David Lewis maintained. Rather, it is to be considered as an open field of research that offers a wealth of interesting philosophical, logical, and mathematical problems.

Bibliography

[1] Armstrong, D. M. *Universals and Scientific Realism: A Theory of Universals.* Cambridge: Cambridge University Press, 1980.

[2] Carnap, R. *Der logische Aufbau der Welt, Berlin, Weltkreis Verlag.* 1928. Translated by George, R. as *The Logical Structure of the World*, Berkeley and Los Angeles:University of California Press, 1967.

[3] Davey, B. A., Priestley, H. A. *Introduction to Lattices and Order.* Cambridge: Cambridge University Press, 1990.

[4] Lawvere, F., Rosebrugh, R. *Sets for Mathematicians.* Berlin and New York: Springer, 2003.

[5] Lawvere, F., Schanuel, S. *Conceptual Mathematics: A First Introduction to Categories.* Cambridge: Cambridge University Press, 1997.

[6] Lewis, D. Against structural universals. *Australasian Journal of Philosophy*, 64:25–46, 1986.

[7] Lewis, D. *Parts of Classes.* Oxford: Basil Blackwell, 1991.

[8] Mac Lane, S. *Categories for the Working Mathematician.* New York: Springer, 1998.

[9] Scaltsas, T. *Substances and Universals in Aristotle's Metaphysics.* Ithaca and London: Cornell University Press, 1994.

[10] Schmidt, R. Subgroup lattices of groups. In *Expositions in Mathematics*, vol. 14. Berlin: de Gruyter, 1994.