On Partial and Paraconsistent Logics *

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Abstract

In this paper we consider the theory of predicate logics in which the principle of Bivalence or the principle of Non-Contradiction or both fail. Such logics are partial or paraconsistent or both. We consider sequent calculi for these logics and prove Model Existence. For $\mathbf{L}_4$, the most general logic under consideration, we also prove a version of the Craig-Lyndon Interpolation Theorem. The paper shows that many techniques used for classical predicate logic generalise to partial and paraconsistent logics once the right set-up is chosen. Our logic $\mathbf{L}_4$ has a semantics that also underlies Belnap's [4] and is related to the logic of bilattices. $\mathbf{L}_4$ is in focus most of the time, but it is also shown how results obtained for $\mathbf{L}_4$ can be transferred to several variants.

1 Introduction

The principle of Bivalence states that a sentence is either true or false; the principle of Non-Contradiction says that no sentence is both true and false. These two principles have been part and parcel of all standard formulations of logic since the subject began with Aristotle. But they need not be accepted and in fact if one of them is rejected (or if both are) we get a straightforward generalisation of classical logic. Allowing the possibility that a sentence is

neither true nor false yields a partial logic (see e.g Cleave [9], Blamey [5], Langholm [21]) and allowing sentences to be both true and false leads to a paraconsistent logic. The possibility of having partiality and paraconsistency at the same time is exemplified in Belnap [4]. Partial and paraconsistent logics have applications in database theory (see the motivation given in [4]), in treatments of the Liar paradox (see e.g. Woodruff [30], Visser [29]), in knowledge representation (Thijsse [28], Jaspars [19]), in logic programming (Fitting [13], Bochman [6]), and in natural language semantics (see Barwise & Perry [3], Muskens [23, 24]). Apart from such applications there is another motivation to study them which derives from an interest in classical logic: how much of the latter's metatheory rests on Bivalence and Non-Contradiction and how much remains if these principles are removed?

In this paper we shall consider predicate logics in which Bivalence or Non-Contradiction or both fail. Our prime example will be the logic $\mathbf{L_4}$, a partial and paraconsistent predicate logic based on [4]. For this logic we shall give a simple Gentzen sequent calculus and prove Model Existence (with useful corollaries such as Completeness, Compactness and the Löwenheim-Skolem theorem) and Craig Interpolation.

There will be logics other than $\mathbf{L_4}$ which we shall also consider briefly. Most of our methods are in fact independent from the way in which certain basic choices for setting up partial or paraconsistent logics are resolved. One such basic choice concerns the notion of *consequence*. In classical logic a set of premises Γ entails a set of conclusions Δ iff in each model in which all $\gamma \in \Gamma$ are true some $\delta \in \Delta$ is true. The same definition can be used for a partial or paraconsistent logic and in this case one obtains a notion $\models^{tr}$, *transmission of truth*. In classical systems this is indistinguishable from *transmission of non-falsity:* We can define $\Gamma \models^{nf} \Delta$ to hold iff some $\delta \in \Delta$ is not false in each model in which no $\gamma \in \Gamma$ is false, or, conversely, iff some $\gamma \in \Gamma$ is false whenever all $\delta \in \Delta$ are. A basic fact about the logics under consideration is that transmission of truth and transmission of non-falsity in general are not equivalent. For example, if we have logical constants (zero-place connectives) $\mathbf{t}$ and $\mathbf{b}$, with $\mathbf{t}$ denoting the proposition that is always true and never false and $\mathbf{b}$ the proposition that is always both true and false, we have $\mathbf{t} \models^{tr} \mathbf{b}$ but not $\mathbf{t} \models^{nf} \mathbf{b}$. Note that we needed the nonclassical connective $\mathbf{b}$ for this example; it is well-known that $\models^{tr}$ and $\models^{nf}$ are identical on formulae with only the classical $\mathbf{t}$, $\neg$, $\wedge$, $\vee$, $\forall$, and $\exists$ (given the interpretations for these connectives considered below).

Which notion is the "right" notion of validity, $\models^{tr}$ or $\models^{nf}$? In $\mathbf{L_4}$ neither

of these notions is taken, but the choice is resolved by requiring transmission of truth as well as non-falsity. The relation of entailment is defined by letting $\Gamma \models \Delta$ iff $\Gamma \models^{tr} \Delta$ and $\Gamma \models^{nf} \Delta$. Thus, while $\models^{tr}$ and $\models^{nf}$ are duals in an obvious sense, the notion $\models$ will be its own dual. This, we feel, is a strong argument in its favour. The choice for a 'double-barrelled' notion of consequence was also taken in [4, 5, 24], but in the literature we find instantiations of the other possibilities as well. In [9, 21, 28, 19, 6], for example, $\models^{tr}$ is taken to be the basic notion of consequence, as it is in Hähnle [17] and Baaz et al. [2]. In Holden [18], on the other hand, we find its dual $\models^{nf}$. The results in this paper generalise over such variations in a simple way. As will be explained in some detail below, our basic notion will be $\models$; but extra *structural elements* $\not\vdash$ and $\not\dashv$ can be present in our Gentzen sequents. Addition of $\not\vdash$ (but not $\not\dashv$) leads to the notion $\models^{tr}$, while adding $\not\dashv$ (but not $\not\vdash$) leads to $\models^{nf}$. The presence of $\not\vdash$ means that transmission of falsity from conclusions to premises need not obtain while the presence of $\not\dashv$ signals that there need be no transmission of truth.

The double-barrelled notion of consequence distinguishes $\mathbf{L_4}$ from the approach taken in the tradition of *many-valued* logics (see e.g. Schröter [26], Rousseau [25], Carnielli [7, 8], [2], [17] and Zach [31]). Such logics are standardly associated with a set N of truth values and a set $D \subseteq N$ of *designated* truth values. A sentence φ follows from a set of sentences Γ in this approach if and only if φ evaluates to an element of D in every model in which each $\gamma \in \Gamma$ evaluates to an element of D. $\models^{tr}$ and $\models^{nf}$ easily fit within this scheme, as will be seen below, but $\models$ is an animal of a different kind. Properties of $\models$ can not always be reduced to properties of $\models^{tr}$ and $\models^{nf}$.

It is well-known that partial and paraconsistent logics can usually be embedded into classical logic. Such embeddings (see Feferman [12], Gilmore [15], [21, 24]) give useful abstract information about the embedded logic, but for more concrete information direct methods are necessary. For instance, [24] observes that for $\mathbf{L_4}$, the compactness theorem, the Löwenheim-Skolem theorem and the recursive axiomatisability of $\models$ all follow from a simple embedding into predicate logic and the corresponding theorems there. But this method of translation does not give a concrete axiomatisation and cannot be used to obtain Interpolation.

Apart from their technical use, embeddings of partial and paraconsistent logics into the classical system give some intuitive guidance. The existence of such embeddings strongly suggests that many proofs for the classical theory will generalise to cases where Bivalence or Non-Contradiction are not as-

3

sumed to hold. One purpose of this paper, next to simply providing concrete syntactic characterisations of the consequence relation for various useful logics and studying properties of this consequence relation, is to show that this is indeed the case. The reader, therefore, should not be disappointed if our proofs turn out to be generalisations of similar proofs for the classical theory. Fascinating as partial and paraconsistent logics are, many of their properties can be studied with the same arsenal of methods that is used for the classical case.

The proof system in this paper will stay close to the sequent format introduced in Langholm [22]. For reasons that will be discussed below, Langholm's sequents are set up as 'quadrants', with *four* structural positions instead of the usual two (left and right). We found that this format helped to formulate Gentzen rules in a very concise way.

Our axiomatisation of the $\mathbf{L_4}$ consequence relation with the help of Langholm's 'quadrants' and the two structural elements $\not\vdash$ and $\not\dashv$ may seem strange at a first encounter. Is not a Gentzen calculus which depends on such unusual devices simply a 'hack'? One way to test the quality of a calculus is to see whether it admits of Interpolation and indeed we shall find that a version of the Craig-Lyndon theorem can be proved in a very straightforward way. The result here should well be distinguished from the result in Langholm [21], where Interpolation is proved for a partial, but not paraconsistent, logic based on $\models^{tr}$. On the one hand, we have not been able to extend our interpolation result for $\models$ to three-valued logics. On the other, [21] remarks that although $\models$ "is perhaps a more worthy counterpart to the classical consequence relation," an interpolation theorem for this notion "does not seem to be as easily obtained as the interpolation theorem for $\models_3$" (the partial but not paraconsistent version of our $\models^{tr}$). Langholm tends to emphasize the difference between partial logic and classical logic, arguing that the resemblances that people have noted between the two extend only to concepts (such as $\models^{tr}$ and $\models^{nf}$) that concern only the truth or only the falsity behaviour of sentences, while "the picture becomes considerably more complex when questions concerning the interaction between the two are brought into focus." If the results in this paper are right, such conclusions are at least not warranted for four-valued logics.

The set-up of the rest of the paper will be as follows. In the next section we recall what happens when the classical connection between truth and falsity is given up: under reasonable assumptions we then arrive at the bilattice *FOUR* which was introduced in Belnap [4]. Section 3 describes the
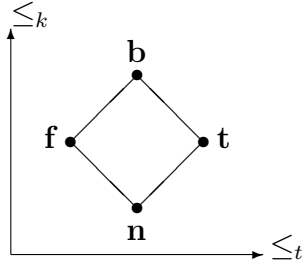
4

Figure 1: The bilattice *FOUR*

truth definition for $\mathbf{L_4}$ and shows the functional completeness of its basic set of connectives. Section 4 discusses semantic consequence and introduces sequents, and section 5 gives a sequent calculus. The completeness proof for this calculus is given in section 6, via Model Existence. Section 7 discusses ways to base our logic on three instead of four values, and the logic's interpolation theorem is proved in section 8. A last section gives conclusions.

2   The elements of *FOUR*

Let us introduce the basic notions that lead to the logic $\mathbf{L_4}$. If we give up both Bivalence and Non-Contradiction, i.e. if we sever the classical relation between truth and falsity completely, we arrive at Belnap's four values *true and not false*, *false and not true*, *both true and false* and *neither true nor false* (see [4]). These we shall abbreviate as $\mathbf{t}$, $\mathbf{f}$, $\mathbf{b}$ and $\mathbf{n}$ respectively. The first two of these values correspond to the classical possibilities; the third represents contradicting information and the fourth no information at all. If we order $\mathbf{4} = \{\mathbf{t}, \mathbf{f}, \mathbf{b}, \mathbf{n}\}$ according to the information content of its elements, we arrive at the lattice ordering $\leq_k$ depicted in figure 1. If we order the same elements with respect to their degrees of truth and non-falsity, we get the lattice ordering $\leq_t$. The structure $\langle \mathbf{4}, \leq_k \rangle$ was called an *approximation lattice* in [4], while $\langle \mathbf{4}, \leq_t \rangle$ was called a *logical lattice*.

Given that formulas φ and ψ take their values in $\mathbf{4}$, how can we compute values for $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$? This can be answered in a very simple way by separating conditions for truth and conditions for falsity (see e.g. Dunn [11]):

   i. $\neg\varphi$ is true if and only if φ is false,

5

$\neg\varphi$ is false if and only if φ is true;

  ii. $\varphi \wedge \psi$ is true if and only if φ is true and ψ is true,
     $\varphi \wedge \psi$ is false if and only if φ is false or ψ is false;

 iii. $\varphi \vee \psi$ is true if and only if φ is true or ψ is true,
     $\varphi \vee \psi$ is false if and only if φ is false and ψ is false.

So, for example, if φ receives the value **n** (neither true nor false) and ψ gets the value **t** (true and not false), then $\varphi \wedge \psi$ is evaluated as **n**: $\varphi \wedge \psi$ is not true since φ is not true and it is not false since neither φ nor ψ is false. Reasoning similarly in all other cases we arrive at the following tables.

$\wedge$	t	f	n	b
t	t	f	n	b
f	f	f	f	f
n	n	f	n	f
b	b	f	f	b

$\vee$	t	f	n	b
t	t	t	t	t
f	t	f	n	b
n	t	n	n	t
b	t	b	t	b

$\neg$	
t	f
f	t
n	n
b	b

It is easily seen that $\wedge$ is meet and $\vee$ is join in the lattice $\langle \mathbf{4}, \leq_t \rangle$. Note that the Strong Kleene truth tables are obtained if we restrict $\wedge$, $\vee$ and $\neg$ to $\{\mathbf{t}, \mathbf{f}, \mathbf{n}\}$ or to $\{\mathbf{t}, \mathbf{f}, \mathbf{b}\}$. In fact restricting values to $\{\mathbf{t}, \mathbf{f}, \mathbf{n}\}$ corresponds to accepting Non-Contradiction but not Bivalence, while restricting to $\{\mathbf{t}, \mathbf{f}, \mathbf{b}\}$ corresponds to accepting Bivalence but leaving open the possibility of para-consistency. In section 7 we shall show how the results from this paper can easily be adapted to logics that are either partial or paraconsistent, but not both.

The structure $FOUR = \langle \mathbf{4}, \leq_t, \leq_k, \neg \rangle$ is a prime example of what Ginzburg [16] has called a *bilattice*. For the general notion of a bilattice see [16] or one of Fitting's papers on the subject (e.g. Fitting [13]). Here we shall content ourselves with considering predicate logics in which formulas can have their values only in $\mathbf{4}$.

$\otimes$	t	f	n	b
t	t	n	n	t
f	n	f	n	f
n	n	n	n	n
b	t	f	n	b

$\oplus$	t	f	n	b
t	t	b	t	b
f	b	f	f	b
n	t	f	n	b
b	b	b	b	b

$-$	
t	t
f	f
n	b
b	n

The truth functions $\wedge$, $\vee$ and $\neg$ are *classical* in the sense that they always yield a value in $\{\mathbf{t}, \mathbf{f}\}$ when given arguments from $\{\mathbf{t}, \mathbf{f}\}$. The zero-place functions $\mathbf{t}$ and $\mathbf{f}$ are also classical in this sense of course. Composition of classical functions can only give new classical functions and so it is clear that $\{\wedge, \vee, \neg, \mathbf{t}, \mathbf{f}\}$ cannot be functionally complete on $\mathbf{4}$. In fact many new interesting operators can be investigated once the classical connections between truth and falsity have been severed. E.g. one can introduce connectives $\otimes$ and $\oplus$, with truth tables as above, into the logical language. These connectives (called 'consensus' and 'gullibility' in the literature on bilattices) correspond to meet and join in the approximation lattice $\langle \mathbf{4}, \leq_k \rangle$, while $-$, *conflation*, does on $\langle \mathbf{4}, \leq_k \rangle$ what negation does on $\langle \mathbf{4}, \leq_t \rangle$. As was shown in [24] the set $\{\otimes, \wedge, \neg, -\}$ is in fact functionally complete.

Since $\mathbf{L_4}$ is based on the bilattice *FOUR* there is a clear relation between this logic and what Arieli & Avron [1] have called *bilattice logics*. But there is also an important difference between $\mathbf{L_4}$ and the system presented in [1], as the latter's consequence relation is a certain generalisation of $\models^{tr}$ adequate for arbitrary bilattices. Our preferred notion of consequence is $\models$, which directly reflects $\leq_t$. (Another difference is our restriction to *FOUR* of course.) In section 5 below we shall give more information about the difference between our set-up and Arieli & Avron's.

3 Satisfaction and Functional Completeness

Having described the basic domain of truth values for $\mathbf{L_4}$ we may proceed with defining the syntax and semantics of the logic. The syntax is defined in the usual way with the help of function and relation symbols in some countable language $\mathcal{L}$, a countable set of variables, and the logical operators $\{\mathbf{n}, \approx, \neg, -, \wedge, \forall\}$. Of the latter, we have met $\mathbf{n}$, $\neg$, $-$ and $\wedge$ already; $\approx$ is identity and $\forall$ is universal quantification. The usual definitions of *free* and *bound* variables, *sentences* etc. obtain. *Constants* are zero-place function symbols. We write $[t_1/x_1, \ldots, t_n/x_n]\varphi$ for the simultaneous substitution of t_1 for x_1 and $\ldots$ and t_n for x_n in φ. The function $[t_1/x_1, \ldots, t_n/x_n]$ is called a *substitution*. φ is a *substitution instance* of ψ if $\varphi = \sigma\psi$ for some substitution σ. A *model* is a pair $\langle \mathcal{D}, \mathcal{I} \rangle$ where $\mathcal{D} \neq \emptyset$ and $\mathcal{I}$ is a function with domain $\mathcal{L}$ such that $\mathcal{I}(f)$ is an n-ary function on $\mathcal{D}$ if $f \in \mathcal{L}$ is an n-ary function symbol and $\mathcal{I}(R)$ is a *pair* of n-ary relations on $\mathcal{D}$ if $R \in \mathcal{L}$ is an n-ary relation symbol. We denote the first element of this pair as $\mathcal{I}^+(R)$, the second

7

element as $\mathcal{I}^-(R)$. *Assignments* and the notation $a_{d_1}^{x_1} \cdots_{d_n}^{x_n}$ for assignments are defined as usual. The value of a term t in a model $\mathcal{M}$ under an assignment a is written as $[\![t]\!]^{\mathcal{M},a}$, or $[\![t]\!]^{\mathcal{M}}$ if t is closed.

Definition 1 We define the three-place relations $\mathcal{M} \models \varphi[a]$ (formula φ is *true* in model $\mathcal{M}$ under assignment a) and $\mathcal{M} =\!\!| \varphi[a]$ (φ is *false* in $\mathcal{M}$ under a) as follows.

1. $\mathcal{M} \not\models \mathbf{n}[a]$,
   $\mathcal{M} \not=\!\!| \mathbf{n}[a]$;

2. $\mathcal{M} \models Rt_1 \ldots t_n[a] \Leftrightarrow \langle [\![t_1]\!]^{\mathcal{M},a}, \ldots, [\![t_n]\!]^{\mathcal{M},a} \rangle \in \mathcal{I}^+(R)$,
   $\mathcal{M} =\!\!| Rt_1 \ldots t_n[a] \Leftrightarrow \langle [\![t_1]\!]^{\mathcal{M},a}, \ldots, [\![t_n]\!]^{\mathcal{M},a} \rangle \in \mathcal{I}^-(R)$;

3. $\mathcal{M} \models t_1 \approx t_2[a] \Leftrightarrow [\![t_1]\!]^{\mathcal{M},a} = [\![t_2]\!]^{\mathcal{M},a}$,
   $\mathcal{M} =\!\!| t_1 \approx t_2[a] \Leftrightarrow [\![t_1]\!]^{\mathcal{M},a} \neq [\![t_2]\!]^{\mathcal{M},a}$;

4. $\mathcal{M} \models \neg\varphi[a] \Leftrightarrow \mathcal{M} =\!\!| \varphi[a]$,
   $\mathcal{M} =\!\!| \neg\varphi[a] \Leftrightarrow \mathcal{M} \models \varphi[a]$;

5. $\mathcal{M} \models -\varphi[a] \Leftrightarrow \mathcal{M} \not=\!\!| \varphi[a]$,
   $\mathcal{M} =\!\!| -\varphi[a] \Leftrightarrow \mathcal{M} \not\models \varphi[a]$;

6. $\mathcal{M} \models \varphi \wedge \psi[a] \Leftrightarrow \mathcal{M} \models \varphi[a] \ \& \ \mathcal{M} \models \psi[a]$,
   $\mathcal{M} =\!\!| \varphi \wedge \psi[a] \Leftrightarrow \mathcal{M} =\!\!| \varphi[a]$ or $\mathcal{M} =\!\!| \psi[a]$;

7. $\mathcal{M} \models \forall x \, \varphi[a] \Leftrightarrow \mathcal{M} \models \varphi[a_d^x]$ for all $d \in \mathcal{D}$,
   $\mathcal{M} =\!\!| \forall x \, \varphi[a] \Leftrightarrow \mathcal{M} =\!\!| \varphi[a_d^x]$ for some $d \in \mathcal{D}$.

We write $\mathcal{M} \models \varphi$ ($\mathcal{M} =\!\!| \varphi$) if φ is a sentence and $\mathcal{M} \models \varphi[a]$ ($\mathcal{M} =\!\!| \varphi[a]$) for some a.

Definition 1 uses the format of assigning truth conditions and falsity conditions separately, as discussed in the previous section. Alternatively, we can let formulas take their values directly in **4** by letting

$$
\begin{aligned}
[\![\varphi]\!]^{\mathcal{M},a} = \mathbf{t} \quad &\text{iff} \quad \mathcal{M} \models \varphi[a] \text{ and } \mathcal{M} \not=\!\!| \varphi[a], \\
[\![\varphi]\!]^{\mathcal{M},a} = \mathbf{f} \quad &\text{iff} \quad \mathcal{M} \not\models \varphi[a] \text{ and } \mathcal{M} =\!\!| \varphi[a], \\
[\![\varphi]\!]^{\mathcal{M},a} = \mathbf{n} \quad &\text{iff} \quad \mathcal{M} \not\models \varphi[a] \text{ and } \mathcal{M} \not=\!\!| \varphi[a], \\
[\![\varphi]\!]^{\mathcal{M},a} = \mathbf{b} \quad &\text{iff} \quad \mathcal{M} \models \varphi[a] \text{ and } \mathcal{M} =\!\!| \varphi[a] \ .
\end{aligned}
$$

Again we suppress superscripts where this may be done. It is easily verified that the connectives $\mathbf{n}$, $\neg$, $-$ and $\wedge$ have a semantics as discussed in the previous section under this interpretation. The semantics of $\forall$ is just what one would expect and bears the usual relation to that of $\wedge$. Note that

$$[\![\forall x \varphi]\!]^{\mathcal{M},a} = \bigwedge_{d \in D} [\![\varphi]\!]^{\mathcal{M},a_d^x} \ ,$$

where $\bigwedge$ denotes arbitrary meet in $\langle \mathbf{4}, \leq_t \rangle$.

This leaves it for us to motivate the semantics of $\approx$, for which we need a short digression. One common way (see [4]) to motivate logics in which truth and non-falsity are not the same concept is to point out the existence of situations in which there is some form of distributed but fallible knowledge. Suppose we have a database which can be updated by more than one employee. Then it may occur that Tim enters that p, while Tom enters $\neg p$. If the reasoning system that comes with the database is based on classical logic this means that all future questions posed to the system will be answered with 'yes'. A partial or paraconsistent logic can avoid this, for, as we shall see shortly, $p, \neg p \models q$ does not hold in such systems.

But this motivation does not preclude the possibility that the reasoning system decides for *some* sentences that they must take their values in $\{\mathbf{t}, \mathbf{f}\}$. For example, whatever information there is in the database system, it makes little sense for the computer to have doubts about statements it can decide itself, such as, say, $27 + 45 \approx 73$. The fact that *some* knowledge is distributed does not mean that *all* knowledge must be treated as such by the reasoning system.

For arbitrary formulas φ it is possible to state that the formula is true and not false by stating $\mathbf{t} \to \varphi$ (where $\mathbf{t}$ abbreviates $\neg(-\mathbf{n} \wedge \mathbf{n})$ and $\to$ is as below). We may imagine that an automated system which has expertise in a certain field simply asserts $\mathbf{t} \to \varphi$ for certain φ and overrules all employee attempts of entering potentially conflicting information. For identity statements it seems that bivalence is even the only possibility, provided that we wish to preserve two properties: (a) self-identity and (b) replacement of equals by equals. No respectable notion of identity can do without these. Suppose that some statement $t_1 \approx t_2$ could be both true and false. Then $\neg t_1 \approx t_2$ would also be both true and false. Given the definition of $\models$, self-identity, the property that $\models t_1 \approx t_1$, requires that $t_1 \approx t_1$ is true and not false and hence that $\neg t_1 \approx t_1$ is false and not true. Note that $t_1 \approx t_2, \neg t_1 \approx t_2 \models \neg t_1 \approx t_1$ is an instance of replacing equals by equals. But now we have a valid sequent with

two premises which are both true (and false) but a conclusion which is not true. Contradiction. The assumption that $t_1 \approx t_2$ could be neither true nor false is dealt with in a similar way. In that case $t_1 \approx t_2, \neg t_1 \approx t_2 \models \neg t_1 \approx t_1$ has a false conclusion but no false premises. Again this is a contradiction. It follows that $t_1 \approx t_2$ must be bivalent and hence that the semantics as it is given is the only reasonable one.

We can introduce more connectives by means of abbreviation.

Definition 2 (Abbreviations) Write

$$\varphi \otimes \psi \quad \text{for} \quad (\varphi \wedge \psi) \vee ((\varphi \vee \psi) \wedge \mathbf{n})$$
$$\varphi \oplus \phi \quad \text{for} \quad (\varphi \wedge \psi) \vee ((\varphi \vee \psi) \wedge -\mathbf{n})$$
$$\varphi \to \psi \quad \text{for} \quad (\neg \varphi \vee -\psi) \wedge (\neg - \varphi \vee \psi)$$

It is not difficult to check that $\otimes$, and $\oplus$ denote meet and join in the approximation lattice. The connective $\to$ is related to $\leq_t$, for we have that

$$[\![\varphi \to \psi]\!]^{\mathcal{M},a} = \mathbf{t} \quad \text{iff} \quad [\![\varphi]\!]^{\mathcal{M},a} \leq_t [\![\psi]\!]^{\mathcal{M},a}$$
$$[\![\varphi \to \psi]\!]^{\mathcal{M},a} = \mathbf{f} \quad \text{iff} \quad [\![\varphi]\!]^{\mathcal{M},a} \not\leq_t [\![\psi]\!]^{\mathcal{M},a} \ .$$

Suitable definitions of $\mathbf{f}$, $\mathbf{b}$, $\vee$, and $\exists$ are left to the reader.

Theorem 1 (Functional Completeness) *Every truth function is expressed by a formula.*

Proof. Directly from the functional completeness of $\{\otimes, \wedge, \neg, -\}$, shown in [24], and the definability of $\otimes$. □

4 Consequence

When we study the consequence relation $\models$ it immediately becomes apparent that the usual rules for negation are no longer valid: $\Gamma, \neg \varphi \models \Delta$ does not follow from $\Gamma \models \varphi, \Delta$ (for example, we have $p \models p$, but not $p, \neg p \models$) and $\Gamma, \varphi \models \Delta$ does not entail $\Gamma \models \neg \varphi, \Delta$ (since $\not\models p, \neg p$). This means that such rules can no longer appear in a syntactic characterisation of the consequence relation and that we must find something weaker. One solution is to give mixed rules for negation and other connectives as it is done e.g. in [9]. For example, we can split the left rule for $\wedge$ in two as follows.

$$\frac{\Gamma, \neg \varphi \vdash \Delta \quad \Gamma, \neg \psi \vdash \Delta}{\Gamma, \neg(\varphi \wedge \psi) \vdash \Delta} \qquad \qquad \frac{\Gamma, \varphi, \psi \vdash \Delta}{\Gamma, \varphi \wedge \psi \vdash \Delta}$$

We may also split the right rule:

$$\frac{\Gamma \vdash \varphi, \Delta \quad \Gamma \vdash \psi, \Delta}{\Gamma \vdash \varphi \wedge \psi, \Delta} \qquad\qquad \frac{\Gamma \vdash \neg\varphi, \neg\psi, \Delta}{\Gamma \vdash \neg(\varphi \wedge \psi), \Delta}$$

And other rules may be split in a similar way.

While it is possible to arrive at a sound and complete characterisation of consequence in $\mathbf{L_4}$ in this manner, it may be thought less than nice that some of these rules are of a mixed character and combine two connectives. These rules do not conform to the so-called *subformula property*, as neither $\neg\varphi$ nor $\neg\psi$ is a subformula of $\neg(\varphi \wedge \psi)$. Notice, moreover, the similarity between the combined left rule for negation and conjunction and the right rule for conjunction without negation. A further similarity obtains between the left rule for unnegated conjunction and the combined right rule. We would do better if we could let such similar rules be instantiations of a single one.

In order to obtain such a more compact characterisation we follow [22] in taking sequents to have *four* structural positions instead of the usual two and in letting these positions be arranged in a so-called *quadrant*. [22] also considers the various directions in which transmissions of truth and falsity may go and obtains sequents such as the following.

$$\frac{\Gamma \mid \Delta}{\Pi \mid \Sigma}\rightarrow \qquad\qquad \overleftarrow{\frac{\Gamma \mid \Delta}{\Pi \mid \Sigma}} \qquad\qquad \overleftarrow{\frac{\Gamma \mid \Delta}{\Pi \mid \Sigma}}\rightarrow$$

Here the two 'north' positions correspond to the two positions in a normal Gentzen sequent $\Gamma \vdash \Delta$ and the two other positions are added for a convenient treatment of negation: having φ in a 'southern' position will be equivalent to having $\neg\varphi$ in the corresponding 'northern' position and vice versa. The idea of using sequents with multiple components dates back to [26, 25], but Langholm's set-up is different from these approaches, as will become apparent below. The 'biconsequence relations' of [6], on the other hand, are very similar to Langholm's quadrants.

We linearise notation by attaching two *signs* i and j to formulae. i can be n *(north)* or s *(south)*, j can be e *(east)* or w *(west)*. Instead of the rightmost sequent displayed above we write

$$\{\varphi^{n,w} \mid \varphi \in \Gamma\} \cup \{\varphi^{n,e} \mid \varphi \in \Delta\} \cup \{\varphi^{s,w} \mid \varphi \in \Pi\} \cup \{\varphi^{s,e} \mid \varphi \in \Sigma\}.$$

While [22] considers the graphical representations shown above as different species of sequents, we let them be manifestations of a single variety. In order

to distinguish the different kinds, we introduce the two *structural elements* $\not\vdash$ and $\not\dashv$ mentioned in the introduction. With the help of these we can define our basic data structure.

Definition 3 A *sequent* is a set of signed sentences and structural elements.

The usual notation for sequents will be employed. In particular, we write (as we did before) Γ, ϑ instead of $\Gamma \cup \{\vartheta\}$ whenever ϑ is a signed sentence or a structural element. We do not require sequents to be finite. The leftmost representation above corresponds to

$$\{\varphi^{n,w} \mid \varphi \in \Gamma\} \cup \{\varphi^{n,e} \mid \varphi \in \Delta\} \cup \{\varphi^{s,w} \mid \varphi \in \Pi\} \cup \{\varphi^{s,e} \mid \varphi \in \Sigma\} \cup \{\not\dashv\} \ .$$

The idea is that the direction from right to left in a sequent Γ is not considered if $\not\dashv \in \Gamma$ and that the direction from left to right is not considered if $\not\vdash \in \Gamma$. The situation that $\{\not\vdash, \not\dashv\} \subseteq \Gamma$ is a limiting case; Γ will then be an axiom.

   We say that a signed sentence $\varphi^{i,j}$ is a *north* sentence if $i = n$, otherwise it is a *south* sentence. Similarly, $\varphi^{i,j}$ is a *west* sentence if $j = w$ and an *east* sentence if $j = e$. A model $\mathcal{M}$ *accepts* a north sentence φ if $\mathcal{M} \models \varphi$; it accepts a south sentence φ if $\mathcal{M} =\!\!\!\mid \varphi$. $\mathcal{M}$ *rejects* a north sentence φ if $\mathcal{M} =\!\!\!\mid \varphi$; it rejects a south sentence φ if $\mathcal{M} \models \varphi$.

Definition 4 $\mathcal{M}$ *refutes*$^{\rightharpoonup}$ Γ if $\not\vdash \notin \Gamma$ and $\mathcal{M}$ accepts all west sentences but no east sentence in Γ; $\mathcal{M}$ *refutes*$^{\leftharpoonup}$ Γ if $\not\dashv \notin \Gamma$ and $\mathcal{M}$ rejects all east sentences but no west sentence in Γ; and $\mathcal{M}$ *refutes* Γ if it refute$^{\rightharpoonup}$ or refute$^{\leftharpoonup}$ Γ. A sequent Γ is *valid* if no $\mathcal{M}$ refutes Γ.

The notions of consequence considered in the introduction clearly are specialisations of the notion of a valid sequent, as we have that

$$\begin{aligned}
\Pi \models \Sigma \quad &\text{iff} \quad \{\varphi^{n,w} \mid \varphi \in \Pi\} \cup \{\varphi^{n,e} \mid \varphi \in \Sigma\} \text{ is valid,} \\
\Pi \models^{tr} \Sigma \quad &\text{iff} \quad \{\varphi^{n,w} \mid \varphi \in \Pi\} \cup \{\varphi^{n,e} \mid \varphi \in \Sigma\} \cup \{\not\dashv\} \text{ is valid,} \\
\Pi \models^{nf} \Sigma \quad &\text{iff} \quad \{\varphi^{n,w} \mid \varphi \in \Pi\} \cup \{\varphi^{n,e} \mid \varphi \in \Sigma\} \cup \{\not\vdash\} \text{ is valid.}
\end{aligned}$$

Remark 1 This place is as good as any to emphasise the fact that in general there is no unique way to associate quadrant positions with truth values in our system. It is true that for the notions refute$^{\rightharpoonup}$ and refute$^{\leftharpoonup}$ the following pictures emerge.

true	not true		not false	false
false	not false		not true	true

(See also the tableau system of D'Agostino [10] which is based on the values *true, false, non-true* and *non-false*.) But the general notion of refutation is a combination of refute$^{\rightharpoonup}$ and refute$^{\leftharpoonup}$ and there is no similar picture corresponding to it. For this reason our main notion of logical consequence $\models$ is not the kind of consequence relation that is studied in the tradition of many-valued logics, where an argument is valid whenever the conclusion gets a designated truth value if all premises get a designated truth value. The relations $\models^{tr}$ and $\models^{nf}$, on the other hand, do fall within this realm. $\models^{tr}$ is the relation we get when **t** and **b** are designated, while $\models^{nf}$ is the relation we obtain when **t** and **n** are.

Definition 5 Let Γ be a sequent. We define the *dual* of Γ, *dual*(Γ), to be the sequent which results from Γ by simultaneously replacing every superscript n in Γ by s, every s by n, every w by e, every e by w, $\nrightarrow$ by $\nleftarrow$ and $\nleftarrow$ by $\nrightarrow$.

Lemma 2 $\mathcal{M}$ *refutes*$^{\rightharpoonup}$ Γ *iff* $\mathcal{M}$ *refutes*$^{\leftharpoonup}$ *dual*(Γ).

Proof. Immediate from the definitions. □

5 A Sequent Calculus

We turn to the proof theory of our system and provide the notion of validity defined in the previous section with a corresponding notion of provability.

Definition 6 A sequent is *provable* if it follows in the usual way from the following sequent rules. (Here and elsewhere we shall let $-n = s, -s = n, -e = w, -w = e$.)

(R) $\dfrac{}{\Gamma, \varphi^{i,w}, \varphi^{i,e}}$, if φ is atomic

($\nrightarrow, \nleftarrow$) $\dfrac{}{\Gamma, \nrightarrow, \nleftarrow}$

(**n**^{w}) $\dfrac{\Gamma, \nrightarrow}{\Gamma, \mathbf{n}^{i,w}}$ (**n**^{e}) $\dfrac{\Gamma, \nleftarrow}{\Gamma, \mathbf{n}^{i,e}}$

($\neg$) $\dfrac{\Gamma, \varphi^{i,j}}{\Gamma, \neg\varphi^{-i,j}}$

($-$) $\dfrac{\Gamma, \varphi^{i,j}}{\Gamma, -\varphi^{-i,-j}}$

$(\wedge_{sw}^{ne})$ $\dfrac{\Gamma, \varphi^{i,j} \quad \Gamma, \psi^{i,j}}{\Gamma, (\varphi \wedge \psi)^{i,j}}$, where $\langle i,j \rangle \in \{\langle n,e \rangle, \langle s,w \rangle\}$

$(\wedge_{se}^{nw})$ $\dfrac{\Gamma, \varphi^{i,j}, \psi^{i,j}}{\Gamma, (\varphi \wedge \psi)^{i,j}}$, where $\langle i,j \rangle \in \{\langle n,w \rangle, \langle s,e \rangle\}$

$(\forall_{sw}^{ne})$ $\dfrac{\Gamma, [c/x]\varphi^{i,j}}{\Gamma, \forall x \varphi^{i,j}}$, where c is not in Γ or φ and $\langle i,j \rangle \in \{\langle n,e \rangle, \langle s,w \rangle\}$

$(\forall_{se}^{nw})$ $\dfrac{\Gamma, [t/x]\varphi^{i,j}}{\Gamma, \forall x \varphi^{i,j}}$, where $\langle i,j \rangle \in \{\langle n,w \rangle, \langle s,e \rangle\}$

(id) $\dfrac{}{\Gamma, t \approx t^{i,j}}$, where $\langle i,j \rangle \in \{\langle n,e \rangle, \langle s,w \rangle\}$

(L) $\dfrac{\Gamma, [t_2/x]\varphi^{i',j'}}{\Gamma, t_1 \approx t_2{}^{i,j}, [t_1/x]\varphi^{i',j'}}$, $\begin{array}{l} \text{where } \langle i,j \rangle \in \{\langle n,w \rangle, \langle s,e \rangle\}, \\ i' \in \{n,s\}, \ j' \in \{e,w\} \text{ and } \varphi \text{ is} \\ \text{atomic} \end{array}$

It is clear that this cut free calculus obeys the modularity constraint that only one logical operator is dealt with in each rule. We also have a version of the *subformula property* as it is not difficult to show that, for each sentence φ occurring somewhere in a proof Π, φ is a substitution instance of some ψ occurring as a subformula of a sentence in the last sequent of Π.

Just as in the case of validity we can specialise our notion of provability and, letting Π and Σ vary over sets of sentences, write

$$\begin{array}{lll} \Pi \vdash \Sigma & \text{iff} & \{\varphi^{n,w} \mid \varphi \in \Pi\} \cup \{\varphi^{n,e} \mid \varphi \in \Sigma\} \text{ is provable,} \\ \Pi \vdash^{tr} \Sigma & \text{iff} & \{\varphi^{n,w} \mid \varphi \in \Pi\} \cup \{\varphi^{n,e} \mid \varphi \in \Sigma\} \cup \{\nvdash\} \text{ is provable,} \\ \Pi \vdash^{nf} \Sigma & \text{iff} & \{\varphi^{n,w} \mid \varphi \in \Pi\} \cup \{\varphi^{n,e} \mid \varphi \in \Sigma\} \cup \{\nvdash\} \text{ is provable.} \end{array}$$

Example 1 The following proof shows that $-(\varphi \wedge \psi) \vdash -\varphi \wedge -\psi$.

$$\dfrac{\dfrac{\dfrac{}{\varphi^{s,e}, \psi^{s,e}, \varphi^{s,w}} \ (R)}{\dfrac{\varphi^{s,e}, \psi^{s,e}, -\varphi^{n,e}}{\dfrac{\varphi^{s,e}, \psi^{s,e}, -\varphi \wedge -\psi^{n,e}}{\dfrac{\varphi \wedge \psi^{s,e}, -\varphi \wedge -\psi^{n,e}}{-(\varphi \wedge \psi)^{n,w}, -\varphi \wedge -\psi^{n,e}} \ (-)} \ (\wedge_{se}^{nw})} \ (-)} \quad \dfrac{\dfrac{}{\varphi^{s,e}, \psi^{s,e}, \psi^{s,w}} \ (R)}{\varphi^{s,e}, \psi^{s,e}, -\psi^{n,e}} \ (-)}{\ } \ (\wedge_{sw}^{ne})$$

Remark 2 Consider a calculus with rules as above except that $(\nvdash, \nvdash)$, $(\mathbf{n}^w)$, and $(\mathbf{n}^e)$ are replaced by the single

$$\dfrac{}{\Gamma, \mathbf{n}^{i,w}} \ .$$

14

Call a sequent *tr-provable* if it follows from this calculus. It is not difficult to show that, if Γ does not contain $\not\mathrel{\leftarrow}$, Γ is tr-provable iff $\Gamma, \not\mathrel{\rightarrow}$ is provable. This gives an alternative characterisation of $\vdash^{tr}$. An alternative characterisation of $\vdash^{nf}$ is obtained by proceeding dually. For Γ not containing structural elements, we have that Γ is provable iff Γ is provable in the system resulting from the present one with $(\not\mathrel{\leftarrow}, \not\mathrel{\rightarrow})$, $(\mathbf{n}^w)$, and $(\mathbf{n}^e)$ replaced by

$$\frac{}{\Gamma, \mathbf{n}^{i,w}, \mathbf{n}^{i',e}} \ .$$

We conclude that the structural elements are not strictly necessary for the set-up. But see Remark 3 below.

It is easy to check that the following are derived rules of our calculus.

$(\mathbf{f}^{nw}_{se})$ $\quad \dfrac{}{\Gamma, \mathbf{f}^{i,j}}$, where $\langle i,j \rangle \in \{\langle n,w \rangle, \langle s,e \rangle\}$

$(\vee^{nw}_{se})$ $\quad \dfrac{\Gamma, \varphi^{i,j} \quad \Gamma, \psi^{i,j}}{\Gamma, (\varphi \vee \psi)^{i,j}}$, where $\langle i,j \rangle \in \{\langle n,w \rangle, \langle s,e \rangle\}$

$(\vee^{ne}_{sw})$ $\quad \dfrac{\Gamma, \varphi^{i,j}, \psi^{i,j}}{\Gamma, (\varphi \vee \psi)^{i,j}}$, where $\langle i,j \rangle \in \{\langle n,e \rangle, \langle s,w \rangle\}$

$(\rightarrow^{ne}_{sw})$ $\quad \dfrac{\Gamma, \varphi^{n,w}, \psi^{n,e} \quad \Gamma, \varphi^{s,e}, \psi^{s,w}}{\Gamma, (\varphi \rightarrow \psi)^{i,j}}$, where $\langle i,j \rangle \in \{\langle n,e \rangle, \langle s,w \rangle\}$

$(\rightarrow^{nw}_{se})$ $\quad \dfrac{\Gamma, \varphi^{n,e}, \psi^{s,e} \quad \Gamma, \varphi^{n,e}, \varphi^{s,w} \quad \Gamma, \psi^{n,w}, \psi^{s,e} \quad \Gamma, \psi^{n,w}, \varphi^{s,w}}{\Gamma, (\varphi \rightarrow \psi)^{i,j}}$,
$\quad\quad$ where $\langle i,j \rangle \in \{\langle n,w \rangle, \langle s,e \rangle\}$

$(\exists^{nw}_{se})$ $\quad \dfrac{\Gamma, [c/x]\varphi^{i,j}}{\Gamma, \exists x \varphi^{i,j}}$, where c is not in Γ or φ and $\langle i,j \rangle \in \{\langle n,w \rangle, \langle s,e \rangle\}$

$(\exists^{ne}_{sw})$ $\quad \dfrac{\Gamma, [t/x]\varphi^{i,j}}{\Gamma, \exists x \varphi^{i,j}}$, where $\langle i,j \rangle \in \{\langle n,e \rangle, \langle s,w \rangle\}$

An inspection of the rules shows that if we restrict ourselves to sentences in which $\mathbf{n}$ does not occur our various notions of provability collapse, i.e. for Π and Σ in which no signed sentence contains $\mathbf{n}$ we have that $\Pi \vdash \Sigma$ iff $\Pi \vdash^{tr} \Sigma$ iff $\Pi \vdash^{nf} \Sigma$, since no application of a rule can create $\not\mathrel{\leftarrow}$ or $\not\mathrel{\rightarrow}$. But as soon as $\mathbf{n}$ enters the picture it is important to keep track of the extra structural elements. Here are derived rules for $\mathbf{b}$, $\otimes$ and $\oplus$ in which $\not\mathrel{\leftarrow}$ and $\not\mathrel{\rightarrow}$ play an important role.

$$(\mathbf{b}^e) \quad \frac{\Gamma, \not\nearrow}{\Gamma, \mathbf{b}^{i,e}} \qquad\qquad (\mathbf{b}^w) \quad \frac{\Gamma, \not\nrightarrow}{\Gamma, \mathbf{b}^{i,w}}$$

$$(\otimes^e) \quad \frac{\Gamma, \varphi^{i,e}, \psi^{i,e} \quad \Gamma, \varphi^{i,e}, \not\nrightarrow \quad \Gamma, \psi^{i,e}, \not\nrightarrow}{\Gamma, (\varphi \otimes \psi)^{i,e}}, \text{ where } i \in \{n, s\}$$

$$(\otimes^w) \quad \frac{\Gamma, \varphi^{i,w}, \psi^{i,w} \quad \Gamma, \varphi^{i,w}, \not\nearrow \quad \Gamma, \psi^{i,w}, \not\nearrow}{\Gamma, (\varphi \otimes \psi)^{i,w}}, \text{ where } i \in \{n, s\}$$

$$(\oplus^e) \quad \frac{\Gamma, \varphi^{i,e}, \psi^{i,e} \quad \Gamma, \varphi^{i,e}, \not\nearrow \quad \Gamma, \psi^{i,e}, \not\nearrow}{\Gamma, (\varphi \oplus \psi)^{i,e}}, \text{ where } i \in \{n, s\}$$

$$(\oplus^w) \quad \frac{\Gamma, \varphi^{i,w}, \psi^{i,w} \quad \Gamma, \varphi^{i,w}, \not\nrightarrow \quad \Gamma, \psi^{i,w}, \not\nrightarrow}{\Gamma, (\varphi \oplus \psi)^{i,e}}, \text{ where } i \in \{n, s\}$$

Remark 3 Note that the use of our structural elements here makes it possible to formulate these rules without any violation of the subformula property.

Remark 4 Arieli & Avron [1] offer the following sequent rules for $\otimes$ and $\oplus$.

$$[\otimes \Rightarrow] \quad \frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \otimes \psi \Rightarrow \Delta} \qquad\qquad [\Rightarrow \otimes] \quad \frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \otimes \psi}$$

$$[\neg\otimes \Rightarrow] \quad \frac{\Gamma, \neg\varphi, \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \otimes \psi) \Rightarrow \Delta} \qquad [\Rightarrow \neg\otimes] \quad \frac{\Gamma \Rightarrow \Delta, \neg\varphi \quad \Gamma \Rightarrow \Delta, \neg\psi}{\Gamma \Rightarrow \Delta, \neg(\varphi \otimes \psi)}$$

$$[\oplus \Rightarrow] \quad \frac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \oplus \psi \Rightarrow \Delta} \qquad [\Rightarrow \oplus] \quad \frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \oplus \psi}$$

$$[\neg\oplus \Rightarrow] \quad \frac{\Gamma, \neg\varphi \Rightarrow \Delta \quad \Gamma, \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \oplus \psi) \Rightarrow \Delta} \quad [\Rightarrow \neg\oplus] \quad \frac{\Gamma \Rightarrow \Delta, \neg\varphi, \neg\psi}{\Gamma \Rightarrow \Delta, \neg(\varphi \oplus \psi)}$$

It is not difficult to show that these rules are derivable in our calculus plus an additional Thinning rule, provided we interpret $\Rightarrow$ as $\vdash^{tr}$. In view of the soundness result below, this also means that they are sound with respect to $\models^{tr}$. However, soundness with respect to $\models$ does not obtain. Consider $[\otimes \Rightarrow]$ for example. Since $\mathbf{f}, \mathbf{t} \Rightarrow \mathbf{f}$, it follows from this rule that $\mathbf{f} \otimes \mathbf{t} \Rightarrow \mathbf{f}$ is derivable. But clearly $\mathbf{f} \otimes \mathbf{t} \not\models \mathbf{f}$, as $\mathbf{f} \otimes \mathbf{t}$ evaluates as $\mathbf{n}$.

We now prove the soundness of our calculus after stating a useful lemma.

Lemma 3 $\Gamma_1, \ldots, \Gamma_n / \Gamma$ *is a sequent rule iff* $dual(\Gamma_1), \ldots, dual(\Gamma_n)/dual(\Gamma)$ *is a sequent rule.*

Proof. By a simple inspection of the sequent rules. $\quad\square$

Theorem 4 (Soundness) *If a sequent is provable then it is valid.*

Proof. It can be shown for any sequent rule that some condition of the rule is refutable$^\rightharpoonup$ if the conclusion of the rule is refutable$^\rightharpoonup$. We prove this statement for $(\forall_{sw}^{ne})$, leaving the other cases to the reader. Suppose that $\mathcal{M} = \langle \mathcal{D}, \mathcal{I} \rangle$ and that $\mathcal{M}$ refutes$^\rightharpoonup$ $\Gamma, \forall x \varphi^{n,e}$. Then $\mathcal{M} \not\models \forall x \varphi[a]$ for some a, so that there is some $d \in \mathcal{D}$ such that $\mathcal{M} \not\models \varphi[a_d^x]$. Let $\mathcal{I}'$ be the interpretation function which is just like $\mathcal{I}$ with the possible exception that $\mathcal{I}'(c) = d$ and let $\mathcal{M}' = \langle \mathcal{D}, \mathcal{I}' \rangle$. Then $\mathcal{M}' \not\models [c/x]\varphi$ and, since c does not occur in $\Gamma \cup \{\varphi\}$, $\mathcal{M}'$ refutes$^\rightharpoonup$ $\Gamma, [c/x]\varphi^{n,e}$. For the case that $\langle i, j \rangle = \langle s, w \rangle$, repeat this argument, but uniformly replace $\not\models$ with $=\mid$.

Other cases are proved in a similar vein and this settles that, for any sequent rule, some condition of the rule is refutable$^\rightharpoonup$ if the conclusion is refutable$^\rightharpoonup$. For the dual case, suppose that the conclusion Γ of a rule $\Gamma_1, \ldots, \Gamma_n / \Gamma$ is refutable$^\leftharpoonup$. Then, by Lemma 2, $dual(\Gamma)$ is refutable$^\rightharpoonup$. By Lemma 3, $dual(\Gamma_1), \ldots, dual(\Gamma_n) / dual(\Gamma)$ is a sequent rule, whence, by the previous reasoning, $dual(\Gamma_k)$ is refutable$^\rightharpoonup$ for some k. A second application of Lemma 2 gives that Γ_k is refutable$^\leftharpoonup$.

We conclude that the conclusion of any rule is valid if all its conditions are valid and the theorem follows by an induction on the complexity of proofs. $\square$

6 Elementary Model Theory

The purpose of this section is to prove a Model Existence theorem for our logic. From this some useful corollaries in the form of a Compactness theorem, a Löwenheim-Skolem theorem and a Completeness theorem will follow.

Definition 7 A sequent Γ is called a *Hintikka sequent* iff

1. $\{\varphi^{i,w}, \varphi^{i,e}\} \not\subseteq \Gamma$, if φ is atomic;

2. $\{\not\vdash, \not\dashv\} \not\subseteq \Gamma$;

3. $\mathbf{n}^{i,w} \in \Gamma \Rightarrow \not\vdash \in \Gamma$,
   $\mathbf{n}^{i,e} \in \Gamma \Rightarrow \not\dashv \in \Gamma$;

4. $\neg \varphi^{i,j} \in \Gamma \Rightarrow \varphi^{-i,j} \in \Gamma$;

5. $-\varphi^{i,j} \in \Gamma \Rightarrow \varphi^{-i,-j} \in \Gamma$;

6. $\varphi \wedge \psi^{i,j} \in \Gamma \Rightarrow \varphi^{i,j} \in \Gamma$ or $\psi^{i,j} \in \Gamma$, if $\langle i,j \rangle \in \{\langle n,e \rangle, \langle s,w \rangle\}$,
   $\varphi \wedge \psi^{i,j} \in \Gamma \Rightarrow \{\varphi^{i,j}, \psi^{i,j}\} \subseteq \Gamma$, if $\langle i,j \rangle \in \{\langle n,w \rangle, \langle s,e \rangle\}$;

7. $\forall x \varphi^{i,j} \in \Gamma \Rightarrow [t/x]\varphi^{i,j} \in \Gamma$, for all closed terms t, if $\langle i,j \rangle \in \{\langle n,w \rangle, \langle s,e \rangle\}$,
   $\forall x \varphi^{i,j} \in \Gamma \Rightarrow [c/x]\varphi^{i,j} \in \Gamma$, for some constant c, if $\langle i,j \rangle \in \{\langle n,e \rangle, \langle s,w \rangle\}$;

8. $t \approx t^{i,j} \notin \Gamma$, if $\langle i,j \rangle \in \{\langle n,e \rangle, \langle s,w \rangle\}$,
   $\{t_1 \approx t_2{}^{i,j}, [t_1/x]\varphi^{i',j'}\} \subseteq \Gamma \Rightarrow [t_2/x]\varphi^{i',j'} \in \Gamma$, if $\langle i,j \rangle \in \{\langle n,w \rangle, \langle s,e \rangle\}$,
   $i' \in \{n,s\}$, $j' \in \{e,w\}$ and φ is atomic.

Lemma 5 *If Γ is a Hintikka sequent then $dual(\Gamma)$ is a Hintikka sequent.*

Proof. By inspection. □

Lemma 6 (Hintikka Lemma) *Each Hintikka sequent is refutable by a countable model.*

Proof. Let Γ be a Hintikka sequent. We first consider the case that $\nrightarrow \notin \Gamma$ and construct a model $\mathcal{M}$ which accepts all west sentences but no east sentence in Γ. Define the relation $\sim$ between closed terms by setting

$$t_1 \sim t_2 \Leftrightarrow (t_1 \approx t_2^{n,w} \in \Gamma \text{ or } t_1 \approx t_2^{s,e} \in \Gamma)$$

It is easily verified that $\sim$ is an equivalence relation. For each term t, let $\tilde{t}$ be the equivalence class $\{t' \mid t' \sim t\}$ and let $\mathcal{D}$ be the set $\{\tilde{t} \mid t \text{ is a closed term}\}$. Define, for each n-ary function symbol $f \in \mathcal{L}$ and each n-ary relation symbol $R \in \mathcal{L}$

$$\mathcal{I}(f)(\tilde{t_1}, \ldots, \tilde{t_n}) = \widetilde{ft_1 \ldots t_n},$$
$$\mathcal{I}(R) = \langle \{\langle \tilde{t_1}, \ldots, \tilde{t_n} \rangle \mid Rt_1 \ldots t_n{}^{n,w} \in \Gamma\}, \{\langle \tilde{t_1}, \ldots, \tilde{t_n} \rangle \mid Rt_1 \ldots t_n{}^{s,w} \in \Gamma\} \rangle.$$

The last clause of Definition 7 ensures that this definition does not depend on the choice of $t_1, \ldots, t_n$. Now let $\mathcal{M} = \langle \mathcal{D}, \mathcal{I} \rangle$. Clearly, $\mathcal{M}$ is a countable model. An induction on term complexity shows that, for each t, $[\![t]\!]^{\mathcal{M}} = \tilde{t}$. Another induction on the number of connectives occurring in a sentence establishes that, for each φ

A. $\varphi^{n,e} \in \Gamma \Rightarrow \mathcal{M} \not\models \varphi$

B. $\varphi^{s,w} \in \Gamma \Rightarrow \mathcal{M} \models \varphi$

C. $\varphi^{n,w} \in \Gamma \Rightarrow \mathcal{M} \models \varphi$

D. $\varphi^{s,e} \in \Gamma \Rightarrow \mathcal{M} \not\models \varphi$

We work out the $\forall$ case of the induction.

A. Assume that $\forall x \varphi^{n,e} \in \Gamma$. Then, by the definition of a Hintikka sequent, $[c/x]\varphi^{n,e} \in \Gamma$ for some constant c. By the induction hypothesis, $\mathcal{M} \not\models [c/x]\varphi$, so that $\mathcal{M} \not\models \varphi[a_{\bar{c}}^x]$, where a is arbitrary, follows by the usual Substitution Lemma. From this we have that $\mathcal{M} \not\models \forall x \varphi$.

B. $\forall x \varphi^{s,w} \in \Gamma \Rightarrow [c/x]\varphi^{s,w} \in \Gamma$ for some $c \Rightarrow \mathcal{M} =\!\mid [c/x]\varphi \Rightarrow \mathcal{M} =\!\mid \varphi[a_{\bar{c}}^x] \Rightarrow \mathcal{M} =\!\mid \forall x \varphi$.

C. $\forall x \varphi^{n,w} \in \Gamma \Rightarrow [t/x]\varphi^{n,w} \in \Gamma$ for all closed terms $t \Rightarrow \mathcal{M} \models [t/x]\varphi$ for all $t \Rightarrow \mathcal{M} \models \varphi[a_{\bar{t}}^x]$ for all $t \Rightarrow \mathcal{M} \models \varphi[a_d^x]$ for all $d \in \mathcal{D} \Rightarrow \mathcal{M} \models \forall x \varphi$.

D. $\forall x \varphi^{s,e} \in \Gamma \Rightarrow [t/x]\varphi^{s,e} \in \Gamma$ for all closed $t \Rightarrow \mathcal{M} \not\models [t/x]\varphi$ for all $t \Rightarrow \mathcal{M} \not\models \varphi[a_{\bar{t}}^x]$ for all $t \Rightarrow \mathcal{M} \not\models \varphi[a_d^x]$ for all $d \in \mathcal{D} \Rightarrow \mathcal{M} \not\models \forall x \varphi$.

The other cases of this induction are similar and are left to the reader. It follows that $\mathcal{M}$ refutes$^{\rightharpoonup} \Gamma$.

Now consider the case that $\not\vdash\, \notin \Gamma$. Since $dual(\Gamma)$ is a Hintikka sequent by Lemma 5 and since $\not\rightarrow\, \notin dual(\Gamma)$ we have that there is a countable $\mathcal{M}$ which refutes$^{\rightharpoonup} dual(\Gamma)$ and hence, by Lemma 2 refutes$^{\leftharpoonup} \Gamma$. Since $\not\rightarrow$ and $\not\vdash$ cannot both be elements of Γ, we have established the theorem. $\square$

Definition 8 Let $\mathcal{P}$ be a set of sequents in the language $\mathcal{L}$. $\mathcal{P}$ is a *provability property* with respect to $\mathcal{L}$ iff

1. If $\{\Gamma_1, \ldots, \Gamma_n\} \subseteq \mathcal{P}$ and $\Gamma_1, \ldots, \Gamma_n / \Gamma$ is a sequent rule, then $\Gamma \in \mathcal{P}$,

2. If $\Gamma \in \mathcal{P}$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \in \mathcal{P}$, for each Γ' in $\mathcal{L}$.

Theorem 7 (Model Existence) *Let $\mathcal{L}$ be a language and let C be a countably infinite set of constants such that $\mathcal{L} \cap C = \emptyset$. Assume that $\mathcal{P}$ is a provability property with respect to $\mathcal{L} \cup C$ and that Γ is a sequent in the language $\mathcal{L}$. If $\Gamma \notin \mathcal{P}$ then Γ is refutable by a countable model.*

Proof. Let $\mathcal{P}$ and Γ be as described. We construct a Hintikka sequent Γ^* such that $\Gamma \subseteq \Gamma^*$. Let $\vartheta_1, \ldots, \vartheta_n, \ldots$ be an enumeration of all signed sentences in $\mathcal{L} \cup C$ plus the structural elements. Write $\iota(\vartheta)$ for the index that the signed

19

sentence or structural element ϑ obtains in this enumeration. Define

$$\Gamma_0 \;=\; \Gamma$$

$$\Gamma_{n+1} \;=\; \begin{cases} \Gamma_n, & \text{if } \Gamma_n \cup \{\vartheta_n\} \in \mathcal{P} \\ \Gamma_n \cup \{\vartheta_n\}, & \text{if } \Gamma_n \cup \{\vartheta_n\} \notin \mathcal{P} \text{ and } \vartheta_n \text{ is not of} \\ & \text{the form } \forall x \varphi^{n,e} \text{ or } \forall x \varphi^{s,w} \\ \Gamma_n \cup \{\vartheta_n, [c/x]\varphi^{i,j}\}, & \text{if } \Gamma_n \cup \{\vartheta_n\} \notin \mathcal{P}, \ \langle i,j \rangle \in \\ & \{\langle n,e \rangle, \langle s,w \rangle\} \text{ and } \vartheta_n \text{ is of the} \\ & \text{form } \forall x \varphi^{i,j}, \text{ where } c \text{ is the first} \\ & \text{constant in } C \text{ which does not oc-} \\ & \text{cur in } \Gamma_n \cup \{\vartheta_n\} \end{cases}$$

This is well-defined since each Γ_n contains only a finite number of constants from C. That $\Gamma_n \notin \mathcal{P}$ for each n follows by a simple induction which uses the definition of a provability property and the fact that $(\forall^{ne}_{sw})$ is a sequent rule. Define $\Gamma^* = \bigcup_n \Gamma_n$. We prove that, for all finite sets $\{\vartheta_{k_1}, \ldots, \vartheta_{k_n}\}$ and for all $k \geq max\{k_1, \ldots, k_n\}$

$$\{\vartheta_{k_1}, \ldots, \vartheta_{k_n}\} \subseteq \Gamma^* \Leftrightarrow \Gamma_k \cup \{\vartheta_{k_1}, \ldots, \vartheta_{k_n}\} \notin \mathcal{P} \tag{1}$$

In order to show that this holds, let $k \geq max\{k_1, \ldots, k_n\}$ and suppose that $\{\vartheta_{k_1}, \ldots, \vartheta_{k_n}\} \subseteq \Gamma^*$. Then there is some ℓ such that $\{\vartheta_{k_1}, \ldots, \vartheta_{k_n}\} \subseteq \Gamma_\ell$. Let $m = max\{k, \ell\}$. We have that $\Gamma_k \cup \{\vartheta_{k_1}, \ldots, \vartheta_{k_n}\} \subseteq \Gamma_m$. Since $\Gamma_m \notin \mathcal{P}$ and $\mathcal{P}$ is closed under supersets it follows that $\Gamma_k \cup \{\vartheta_{k_1}, \ldots, \vartheta_{k_n}\} \notin \mathcal{P}$. For the reverse direction, suppose that $\Gamma_k \cup \{\vartheta_{k_1}, \ldots, \vartheta_{k_n}\} \notin \mathcal{P}$. Then, since $\mathcal{P}$ is closed under supersets, $\Gamma_{k_i} \cup \{\vartheta_{k_i}\} \notin \mathcal{P}$, for each of the k_i. By the construction of Γ^* each $\vartheta_{k_i} \in \Gamma^*$ and $\{\vartheta_{k_1}, \ldots, \vartheta_{k_n}\} \subseteq \Gamma^*$.

With the help of (1) we verify that Γ^* is a Hintikka sequent. Here we check only a few conditions of Definition 7.

- In order to check the first part of condition 6 of Definition 7, let $\langle i,j \rangle \in \{\langle n,e \rangle, \langle s,w \rangle\}$ and suppose $\varphi \wedge \psi^{i,j} \in \Gamma^*$. Let k be the maximum of $\iota(\varphi \wedge \psi^{i,j})$, $\iota(\varphi^{i,j})$ and $\iota(\psi^{i,j})$. (1) entails that $\Gamma_k \cup \{\varphi \wedge \psi^{i,j}\} \notin \mathcal{P}$. Since $\mathcal{P}$ is closed under sequent rules, it follows with $(\wedge^{ne}_{sw})$ that $\Gamma_k \cup \{\varphi^{i,j}\} \notin \mathcal{P}$ or $\Gamma_k \cup \{\psi^{i,j}\} \notin \mathcal{P}$. By (1) this implies that $\varphi^{i,j} \in \Gamma^*$ or $\psi^{i,j} \in \Gamma^*$.

- We verify that the seventh condition of Definition 7 holds for Γ^*. First suppose that $\forall x \varphi^{i,j} \in \Gamma^*$, that $\langle i,j \rangle \in \{\langle n,e \rangle, \langle s,w \rangle\}$ and that t is an arbitrary closed term. Let $k = max\{\iota(\forall x \varphi^{i,j}), \iota([t/x]\varphi^{i,j})\}$. (1) gives that $\Gamma_k \cup \{\forall x \varphi^{i,j}\} \notin \mathcal{P}$ and by the closure of $\mathcal{P}$ under sequent rules

20

we find that $\Gamma_k \cup \{[t/x]\varphi^{i,j}\} \notin \mathcal{P}$. This in its turn, by (1), has as a consequence that $[t/x]\varphi^{i,j} \in \Gamma^*$. The construction of Γ^* ensures that $[c/x]\varphi^{i,j} \in \Gamma^*$ for some c if $\forall x \varphi^{n,e} \in \Gamma^*$ if $\langle i, j \rangle \in \{\langle n, e \rangle, \langle s, w \rangle\}$, so that the second part of clause 7 of Definition 7 is satisfied.

Checking the other conditions in the definition of a Hintikka sequent gives rise to considerations that are very similar to those already encountered and is left to the reader.

Since $\Gamma \subseteq \Gamma^*$ and Γ^* is refutable by a countable model, Γ is refutable by that model. $\square$

In the following corollaries Γ will always be a sequent in some language $\mathcal{L}$ while Δ ranges over sequents in $\mathcal{L} \cup C$, where $\mathcal{L}$ and C are as in the formulation of Theorem 7.

Corollary 8 (Compactness) *If Γ is valid then there is some finite $\Gamma_0 \subseteq \Gamma$ which is valid.*

Proof. The set $\{\Delta \mid$ some finite $\Delta_0 \subseteq \Delta$ is valid$\}$ is easily seen to be a provability property. It follows by Theorem 7 that Γ is refutable if no finite $\Gamma_0 \subseteq \Gamma$ is valid. By Contraposition we find that some finite $\Gamma_0 \subseteq \Gamma$ must be valid if Γ is valid. $\square$

Corollary 9 (Löwenheim–Skolem) *If Γ is not valid then Γ is refutable by a countable model.*

Proof. $\{\Delta \mid \Delta$ is valid$\}$ is a provability property. $\square$

Corollary 10 (Completeness) *If Γ is valid then Γ is provable.*

Proof. The set $\{\Delta \mid \Delta$ is provable$\}$ is a provability property. It follows that Γ is refutable if Γ is not provable. $\square$

7   Three Values

We have obtained our results for a logic that was both partial and paraconsistent. What if we do not want to allow paraconsistency or do not want partiality?

The solution is simple. For a logic without paraconsistency we must remove $-$ from our syntax and in order to make up for this loss we must add $\mathbf{f}$ and $\rightarrow$ as primitive connectives. Next we additionally require in the definition of a *model* that, for any relation symbol R, $\mathcal{I}^+(R) \cap \mathcal{I}^-(R) = \emptyset$ (*"no gluts"*). This removes the possibility of paraconsistency. The truth definition should give the new primitives the semantics they previously obtained by expansion of their definitions:

- $\mathcal{M} \not\models \mathbf{f}$
  $\mathcal{M} =\!\mid \mathbf{f}$

- $\mathcal{M} \models \varphi \rightarrow \psi[a] \Leftrightarrow (\mathcal{M} \models \varphi[a] \Rightarrow \mathcal{M} \models \psi[a])\ \&\ (\mathcal{M} =\!\mid \psi[a] \Rightarrow \mathcal{M} =\!\mid \varphi[a])$
  $\mathcal{M} =\!\mid \varphi \rightarrow \psi[a] \Leftrightarrow (\mathcal{M} \models \varphi[a]\ \&\ \mathcal{M} \not\models \psi[a])$ or $(\mathcal{M} =\!\mid \psi[a]\ \&\ \mathcal{M} \not=\!\mid \varphi[a])$

That the set $\{\mathbf{f}, \mathbf{n}, \neg, \wedge, \rightarrow\}$ is functionally complete for the new set-up is easily seen to hold on the basis of a minor variant of the proof of Theorem 1. Apart from $(\mathbf{f}^{ne}_{sw})$, $(\rightarrow^{ne}_{sw})$ and $(\rightarrow^{nw}_{se})$ two extra rules are added to the sequent calculus to counterbalance our restriction of the class of models (the rule for $-$ will be omitted of course).

$$(3^w) \quad \frac{\Gamma, \not\!\!\nearrow}{\Gamma, \varphi^{n,w}, \varphi^{s,w}} \qquad (3^e) \quad \frac{\Gamma, \not\!\!\nwarrow}{\Gamma, \varphi^{n,e}, \varphi^{s,e}}$$

The five new rules bring five new conditions in Definition 7 with them.

- $\mathbf{f}^{i,j} \notin \Gamma$, if $\langle i, j \rangle \in \{\langle n, w \rangle, \langle s, e \rangle\}$

- $\{\varphi^{n,w}, \varphi^{s,w}\} \subseteq \Gamma \Rightarrow \not\!\!\nearrow \in \Gamma$

- $\{\varphi^{n,e}, \varphi^{s,e}\} \subseteq \Gamma \Rightarrow \not\!\!\nwarrow \in \Gamma$

- $\varphi \rightarrow \psi^{i,j} \in \Gamma \Rightarrow \{\varphi^{n,w}, \psi^{n,e}\} \subseteq \Gamma$ or $\{\varphi^{s,e}, \psi^{s,w}\} \subseteq \Gamma$, if $\langle i, j \rangle \in \{\langle n, e \rangle, \langle s, w \rangle\}$

- $\varphi \rightarrow \psi^{i,j} \in \Gamma \Rightarrow \{\varphi^{n,e}, \psi^{s,e}\} \subseteq \Gamma$ or $\{\varphi^{n,e}, \varphi^{s,w}\} \subseteq \Gamma$ or $\{\psi^{n,w}, \psi^{s,e}\} \subseteq \Gamma$ or $\{\psi^{n,w}, \varphi^{s,w}\} \subseteq \Gamma$, if $\langle i, j \rangle \in \{\langle n, w \rangle, \langle s, e \rangle\}$

It must then be checked in the proof of the Hintikka Lemma that the model which is constructed satisfies our new requirement that $\mathcal{I}^+(R) \cap \mathcal{I}^-(R) = \emptyset$ for each R. But in fact this is trivial from the construction.

The reader will have no difficulty in seeing that, under these new conditions, all our previous proofs will go through.

In order to get logics that are not partial (but may be paraconsistent), we proceed dually. First, we require $\mathbf{n}$ and $-$ not to be in the language, but introduce $\mathbf{b}$, $\mathbf{f}$ and $\rightarrow$. We put the additional constraint on models that $\mathcal{D}^n \subseteq \mathcal{I}^+(R) \cup \mathcal{I}^-(R)$ for all n-ary relation symbols R ("no gaps"). The extra rules which need to be added are $(\mathbf{b}^e)$, $(\mathbf{b}^w)$, $(\mathbf{f}^{ne}_{sw})$, $(\rightarrow^{ne}_{sw})$ and $(\rightarrow^{nw}_{se})$ plus the following.

$$(3'^w) \qquad \frac{\Gamma, \not\leftarrow}{\Gamma, \varphi^{n,w}, \varphi^{s,w}} \qquad (3'^e) \qquad \frac{\Gamma, \not\leftarrow}{\Gamma, \varphi^{n,e}, \varphi^{s,e}}$$

Again we must add conditions corresponding to these rules to Definition 7 and we must check that this causes the new requirement on models to be satisfied by the model constructed in the Hintikka Lemma.

This shows that we can easily trade our four-valued logics for three-valued ones. The choice which notion of logical consequence should be used, $\models$, $\models^{tr}$, or $\models^{nf}$, is independent from the choice which truth-values should be accepted.

To obtain classical logic add the *no gluts* and the *no gaps* requirements on models, remove $\mathbf{n}$, and $-$, but add $\mathbf{f}$ and $\rightarrow$. A sequent calculus is obtained by adopting extra rules $(\mathbf{f}^{ne}_{sw})$, $(\rightarrow^{ne}_{sw})$, $(\rightarrow^{nw}_{se})$, (3^w), (3^e), $(3'^w)$ and $(3'^e)$.

8 Interpolation in $\mathbf{L_4}$

The purpose of this section is to prove a Craig-Lyndon interpolation theorem for $\mathbf{L_4}$. The theorem will be restricted to languages without function symbols other than individual constants, as a consideration of complex terms leads to certain complications orthogonal to the main concerns of this paper. The Lyndon part of the theorem can be formulated in a way that is slightly subtler than it is possible in classical logic. Due to the fact that we have *two* negations ($\neg$ and $-$) in our set-up, there will also be two notions of positive and negative occurrences of formulas. It will turn out that the theorem holds for both these notions. The proof of the theorem was inspired by the proof in Kleene [20] for the classical case, but also bears resemblance to the Maehara method discussed in Takeuti [27]. Proofs of interpolation for $\vdash^{tr}$ and $\vdash^{nf}$ have been known before (see [21]), but since it is unclear how interpolation for $\vdash$ could be obtained from these, we give a direct proof here.

We *colour* signed sentences in proofs in order to be able to keep track of them. A coloured signed sentence is a signed sentence which carries an additional subscript r (red) or b (blue) and a colouring of a sequent proof Π is obtained by colouring the signed sentences in the sequents of Π in such a way that a signed sentence shares its colour with its ancestors in the proof (because of our representation of sequents as sets this may require that some signed sentences $\varphi^{i,j}$ now get two representations, $\varphi_r^{i,j}$ and $\varphi_b^{i,j}$). It is clear that, given any initial colouring of the end sequent of any proof Π, Π itself can be coloured.

If Γ is a sequent in which the signed sentences are coloured, we write Γ^b (Γ^r) for the sequent which results from Γ by removing all signed sentences coloured red (blue) in Γ (note that this leaves the structural elements in place) and removing all subscripts from the remaining signed sentences.

A relation symbol R *occurs $\neg$-positively ($\neg$-negatively)* in a sentence φ if it occurs within the scope of an even (odd) number of negation symbols $\neg$. Similarly R *occurs $-$-positively ($-$-negatively)* in φ if it occurs within the scope of an even (odd) number of $-$s. These notions of positive and negative occurrence within an unsigned sentence can be extended to signed coloured sentences by counting certain combinations of signs and colours as extra negation symbols in the following way.

- R occurs $\neg$-positively ($\neg$-negatively) in $\varphi_k^{i,j}$ iff R occurs $\neg$-positively ($\neg$-negatively) in φ and $\langle i, j, k \rangle \in \{\langle n, w, r \rangle, \langle n, e, b \rangle, \langle s, w, b \rangle, \langle s, e, r \rangle\}$.

- R occurs $\neg$-negatively ($\neg$-positively) in $\varphi_k^{i,j}$ iff R occurs $\neg$-positively ($\neg$-negatively) in φ and $\langle i, j, k \rangle \in \{\langle s, w, r \rangle, \langle s, e, b \rangle, \langle n, w, b \rangle, \langle n, e, r \rangle\}$.

- R occurs $-$-positively ($-$-negatively) in $\varphi_k^{i,j}$ iff R occurs $-$-positively ($-$-negatively) in φ and $\langle i, j, k \rangle \in \{\langle n, w, r \rangle, \langle n, e, b \rangle, \langle s, w, r \rangle, \langle s, e, b \rangle\}$.

- R occurs $-$-negatively ($-$-positively) in $\varphi_k^{i,j}$ iff R occurs $-$-positively ($-$-negatively) in φ and $\langle i, j, k \rangle \in \{\langle n, w, b \rangle, \langle n, e, r \rangle, \langle s, w, b \rangle, \langle s, e, r \rangle\}$.

The main interpolation lemma can now be formulated and proved as follows.

Lemma 11 *Let Δ be a coloured provable sequent in a language without function symbols other than individual constants. Then there is a sentence χ such that*

1. $\Delta^r, \chi^{n,e}$ and $\Delta^b, \chi^{n,w}$ are provable, and

24

2. *each individual constant occurring in χ also occurs in Δ^r and in Δ^b.*

3. *each relation symbol which occurs $\neg$-positively ($\neg$-negatively, $-$-positively, $-$-negatively) in χ also occurs $\neg$-positively ($\neg$-negatively, $-$-positively, $-$-negatively) in both Δ^r and Δ^b.*

χ *is called an* interpolant *of Δ and we write $\Delta : \chi$.*

Proof. Let Π be a coloured proof for Δ. The argument will proceed by induction on the complexity of Π. Consider the last rule ρ used in Π and suppose that ρ has premises $\Delta_1, \ldots, \Delta_n$. Induction gives us $\chi_1, \ldots, \chi_n$ such that $\Delta_1 : \chi_1, \ldots, \Delta_n : \chi_n$. In the following statements, which exhaust all possibilities for ρ, the abbreviation

$$\frac{\Delta_1 : \chi_1, \ldots, \Delta_n : \chi_n}{\Delta : \chi}$$

means 'χ is an interpolant for Δ if each χ_i is an interpolant for Δ_i ($1 \leq i \leq n$).'

$(\mathbf{R})^{int}$

(a) $\dfrac{}{\Gamma, \varphi_r^{i,w}, \varphi_r^{i,e} : \mathbf{f}}$,

(b) $\dfrac{}{\Gamma, \varphi_b^{i,w}, \varphi_b^{i,e} : \mathbf{t}}$,

(c) $\dfrac{}{\Gamma, \varphi_r^{n,w}, \varphi_b^{n,e} : \varphi}$,

(d) $\dfrac{}{\Gamma, \varphi_b^{n,w}, \varphi_r^{n,e} : \neg - \varphi}$,

(e) $\dfrac{}{\Gamma, \varphi_r^{s,w}, \varphi_b^{s,e} : \neg\varphi}$,

(f) $\dfrac{}{\Gamma, \varphi_b^{s,w}, \varphi_r^{s,e} : -\varphi}$,

(φ atomic)

$(\nrightarrow, \nleftarrow)^{int}$ $\quad \dfrac{}{\Gamma, \nrightarrow, \nleftarrow : \mathbf{t}}$

$(\mathbf{n}^w)^{int}$

(a) $\dfrac{\Gamma, \nrightarrow : \chi}{\Gamma, \mathbf{n}_r^{i,w} : \chi \wedge \mathbf{n}}$

(b) $\dfrac{\Gamma, \nrightarrow : \chi}{\Gamma, \mathbf{n}_b^{i,w} : \chi \vee \mathbf{b}}$

$(\mathbf{n}^e)^{int}$

(a) $\dfrac{\Gamma, \nleftarrow : \chi}{\Gamma, \mathbf{n}_r^{i,e} : \chi \wedge \mathbf{b}}$

(b) $\dfrac{\Gamma, \nleftarrow : \chi}{\Gamma, \mathbf{n}_b^{i,e} : \chi \vee \mathbf{n}}$

$(\neg)^{int}$ $\quad \dfrac{\Gamma, \varphi_k^{i,j} : \chi}{\Gamma, \neg\varphi_k^{-i,j} : \chi}$

$(-)^{int}$ $\quad \dfrac{\Gamma, \varphi_k^{i,j} : \chi}{\Gamma, -\varphi_k^{-i,-j} : \chi}$

$(\wedge_{sw}^{ne})^{int}$

(a) $\dfrac{\Gamma, \varphi_r^{i,j} : \chi_1 \quad \Gamma, \psi_r^{i,j} : \chi_2}{\Gamma, (\varphi \wedge \psi)_r^{i,j} : \chi_1 \vee \chi_2}$

(b) $\dfrac{\Gamma, \varphi_b^{i,j} : \chi_1 \quad \Gamma, \psi_b^{i,j} : \chi_2}{\Gamma, (\varphi \wedge \psi)_b^{i,j} : \chi_1 \wedge \chi_2}$

25

$$(\langle i,j \rangle \in \{\langle n,e \rangle, \langle s,w \rangle\})$$

$(\wedge_{se}^{nw})^{int}$ $\quad \dfrac{\Gamma, \varphi_k^{i,j}, \psi_k^{i,j} : \chi}{\Gamma, (\varphi \wedge \psi)_k^{i,j} : \chi},$ $\qquad (\langle i,j \rangle \in \{\langle n,w \rangle, \langle s,e \rangle\})$

$(\forall_{sw}^{ne})^{int}$ $\quad \dfrac{\Gamma, [c/x]\varphi_k^{i,j} : \chi}{\Gamma, \forall x \varphi_k^{i,j} : \chi},$ $\qquad (k \in \{r,b\},\ c \text{ not in } \Gamma, \varphi \text{ and } \langle i,j \rangle \in \{\langle n,e \rangle, \langle s,w \rangle\})$

$(\forall_{se}^{nw})^{int}$ $\quad$ (a) $\dfrac{\Gamma, [c/x]\varphi_r^{i,j} : \chi}{\Gamma, \forall x \varphi_r^{i,j} : \chi}$, if c occurs in Γ^r or φ

$\qquad\qquad$ (b) $\dfrac{\Gamma, [c/x]\varphi_r^{i,j} : \chi}{\Gamma, \forall x \varphi_r^{i,j} : \forall x [x/c]\chi}$, if c does not occur in Γ^r or φ

$\qquad\qquad$ (c) $\dfrac{\Gamma, [c/x]\varphi_b^{i,j} : \chi}{\Gamma, \forall x \varphi_b^{i,j} : \chi}$, if c occurs in Γ^b or φ

$\qquad\qquad$ (d) $\dfrac{\Gamma, [c/x]\varphi_b^{i,j} : \chi}{\Gamma, \forall x \varphi_b^{i,j} : \exists x [x/c]\chi}$, if c does not occur in Γ^b or φ

$$(\langle i,j \rangle \in \{\langle n,w \rangle, \langle s,e \rangle\})$$

$(\text{id})^{int}$ $\quad$ (a) $\dfrac{}{\Gamma, d \approx d_r^{i,j} : \mathbf{f}}$, $\quad$ (b) $\dfrac{}{\Gamma, d \approx d_b^{i,j} : \mathbf{t}},$ $\qquad (\langle i,j \rangle \in \{\langle n,e \rangle, \langle s,w \rangle\})$

$(\text{L})^{int}$ $\quad$ (a) $\dfrac{\Gamma, [c/x]\varphi_k^{i',j'} : \chi}{\Gamma, d \approx c_{k'}^{i,j}, [d/x]\varphi_k^{i',j'} : \chi}$,
$\qquad\qquad$ if $k = k'$ ($k, k' \in \{r,b\}$) or x is not free in φ;

$\qquad\qquad$ (b) $\dfrac{\Gamma, [c/x]\varphi_r^{i',j'} : \chi}{\Gamma, d \approx c_b^{i,j}, [d/x]\varphi_r^{i',j'} : \neg d \approx c \vee \chi}$,
$\qquad\qquad$ if x is free in φ and c occurs in d, φ, or Γ^r;

$\qquad\qquad$ (c) $\dfrac{\Gamma, [c/x]\varphi_r^{i',j'} : \chi}{\Gamma, d \approx c_b^{i,j}, [d/x]\varphi_r^{i',j'} : \forall y(\neg d \approx y \vee [y/c]\chi)}$,
$\qquad\qquad$ if x is free in φ and c does not occur in d, φ, or Γ^r;

$\qquad\qquad$ (d) $\dfrac{\Gamma, [c/x]\varphi_b^{i',j'} : \chi}{\Gamma, d \approx c_r^{i,j}, [d/x]\varphi_b^{i',j'} : d \approx c \wedge \chi}$,
$\qquad\qquad$ if x is free in φ and c occurs in d, φ, or Γ^b;

$\qquad\qquad$ (e) $\dfrac{\Gamma, [c/x]\varphi_b^{i',j'} : \chi}{\Gamma, d \approx c_r^{i,j}, [d/x]\varphi_b^{i',j'} : \exists y(d \approx y \wedge [y/c]\chi)}$,
$\qquad\qquad$ if x is free in φ and c does not occur in d, φ, or Γ^b;

$$(\langle i,j \rangle \in \{\langle n,w \rangle, \langle s,e \rangle\},\ i' \in \{n,s\},\ j' \in \{e,w\} \text{ and } \varphi \text{ is atomic})$$

This ends the long list of possible cases for ρ. We shall prove one character-istic case, the (c) case of $(\mathrm{L})^{int}$, leaving the others to an interested reader. Suppose that $\Gamma, [c/x]\varphi_r^{i',j'} : \chi$, where x is free in φ and c does not occur in d, φ, or Γ^r. Then by definition $\Gamma^r, [c/x]\varphi^{i',j'}, \chi^{n,e}$ and $\Gamma^b, \chi^{n,w}$ are provable. But a proof of $\Gamma^r, [c/x]\varphi^{i',j'}, \chi^{n,e}$ can be extended as follows.

$$
\cfrac{
\cfrac{
\cfrac{
\cfrac{
\Gamma^r, [c/x]\varphi^{i',j'}, \chi^{n,e}
}{
\Gamma^r, [d/x]\varphi^{i',j'}, d \approx c^{s,e}, \chi^{n,e}
} \ (\mathrm{L})
}{
\Gamma^r, [d/x]\varphi^{i',j'}, \neg d \approx c^{n,e}, \chi^{n,e}
} \ (\neg)
}{
\Gamma^r, [d/x]\varphi^{i',j'}, \neg d \approx c \vee \chi^{n,e}
} \ (\vee_{sw}^{ne})
}{
\Gamma^r, [d/x]\varphi^{i',j'}, \forall y(\neg d \approx y \vee [y/c]\chi)^{n,e}
} \ (\forall_{sw}^{ne})
$$

The last step was possible because c does not occur in d, φ, or Γ^r. The proof of $\Gamma^b, \chi^{n,w}$ can be extended as follows.

$$
\cfrac{
\cfrac{
\Gamma^b, \chi^{n,w} \quad
\cfrac{
\cfrac{
\overline{\Gamma^b, c \approx c^{s,w}} \ (\mathrm{id})
}{
\Gamma^b, \neg c \approx c^{n,w}
} \ (\neg)
}{
\Gamma^b, \neg c \approx c \vee \chi^{n,w}
}
}{
\cfrac{
\Gamma^b, d \approx c^{i,j}, \neg d \approx c \vee \chi^{n,w}
}{
\Gamma^b, d \approx c^{i,j}, \forall y(\neg d \approx y \vee [y/c]\chi)^{n,w}
} \ (\forall_{se}^{nw})
} \ (\vee_{se}^{nw})
} \ (\mathrm{L})
$$

Since x occurs free in φ, d will occur in $[d/x]\varphi$ as well as in $d \approx c$. Moreover, since $\Gamma, [c/x]\varphi_r^{i',j'} : \chi$, each individual constant in $[y/c]\chi$ occurs in Γ^b and also either in Γ^r or in φ. This means that each individual constant in $\forall y(\neg d \approx y \vee [y/c]\chi)$ occurs both in $\Gamma^b, d \approx c^{i,j}$ and in $\Gamma^r, [d/x]\varphi^{i',j'}$. If R is a non-logical relation constant occurring $\neg$-positively ($\neg$-negatively, etc.) in $\forall y(\neg d \approx y \vee [y/c]\chi)$, then R occurs $\neg$-positively ($\neg$-negatively, etc.) in χ. Using the fact that $\Gamma, [c/x]\varphi_r^{i',j'} : \chi$ we easily find that R occurs $\neg$-positively ($\neg$-negatively, $-$-positively, $-$-negatively) both in $\Gamma^b, d \approx c^{i,j}$ and in $\Gamma^r, [d/x]\varphi^{i',j'}$. We conclude that $\forall y(\neg d \approx y \vee [y/c]\chi)$ is the required interpolant.

This concludes the construction of an interpolant for Δ. $\square$

Theorem 12 (Craig-Lyndon Interpolation Theorem) *Let Γ and Δ be sets of sentences in a language without function symbols other than individual constants. If $\Gamma \vdash \Delta$ there is a sentence χ such that*

1. $\Gamma \vdash \chi$ and $\chi \vdash \Delta$,

2. each individual constant which occurs in χ also occurs in Γ and in Δ.

3. each relation symbol which occurs $\neg$-positively ($\neg$-negatively, $-$-positively, $-$-negatively) in χ also occurs $\neg$-positively ($\neg$-negatively, $-$-positively, $-$-negatively) in both Γ and Δ.

Here $\vdash$ can uniformly be replaced with $\vdash^{tr}$ or with $\vdash^{nf}$.

Proof. The $\vdash$ case follows by applying the previous lemma to the coloured sequent $\Theta = \{\varphi_r^{n,w} \mid \varphi \in \Gamma\} \cup \{\varphi_b^{n,e} \mid \varphi \in \Delta\}$. For the $\vdash^{tr}$ and $\vdash^{nf}$ cases consider $\Theta \cup \{\nvdash\}$ and $\Theta \cup \{\nvDash\}$ respectively. $\square$

A set of sentences Π is *inconsistent* if $\Pi \vdash$, i.e. if $\{\varphi^{n,w} \mid \varphi \in \Pi\}$ is provable.

Corollary 13 (Robinson Joint Consistency Theorem) *Suppose Π_1 and Π_2 are sets of sentences in a language without function symbols and that $\Pi_1 \cup \Pi_2$ is inconsistent. Then there is a sentence χ, all whose non-logical symbols also occur in $\Pi_1 \cup \Pi_2$, such that $\Pi_1 \vdash \chi$ and $\Pi_2, \chi \vdash$.*

Proof. Directly from Lemma 11. $\square$

9 Conclusion

We have generalised the sequent calculus for predicate logic to systems for a range of partial and paraconsistent logics. Our methods work for systems based on one-directional notions of logical consequence ($\models^{tr}$ and $\models^{nf}$), but also for the bidirectional notion $\models$ based on $\leq_t$, which to us seems more attractive from an esthetic point of view. The bidirectional notion differs from the one-directional notions only if connectives are considered that can only be defined in terms of **n**, but among these are the important $\oplus$ and $\otimes$. Other interesting connectives which are definable with **n**, but not without this connective, are discussed in [5]. That the techniques we have used stay close to the techniques usually employed for predicate logic (e.g. in Fitting [14]) comes as a surprise in view of the remarks in the otherwise excellent [21, 22]. The fact that we have been able to prove an interpolation theorem for $\mathbf{L_4}$ gives some support to the idea that a reasonable sequent formalisation for this logic was found.

References

[1] O. Arieli and A. Avron. Reasoning with Logical Bilattices. *Journal of Logic, Language and Information*, 5:25–63, 1996.

[2] M. Baaz, C.G. Fermüller, and R. Zach. Elimination of Cuts in First-order Finite-valued Logics. *Journal of Information Processing and Cybernetics*, 29:333–355, 1994.

[3] J. Barwise and J. Perry. *Situations and Attitudes*. MIT Press, Cambridge, Massachusetts, 1983.

[4] N.D. Belnap Jr. A Useful Four-Valued logic. In J.M. Dunn and G. Epstein, editors, *Modern Uses of Multiple-Valued Logic*, pages 8–37. Reidel, Dordrecht, 1977.

[5] S. Blamey. Partial Logic. In D.M. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic*, volume III, pages 1–70. Reidel, Dordrecht, 1986.

[6] A. Bochman. A Logical Foundation for Logic Programming i: Biconsequence Relations and Nonmonotonic Completion. *The Journal of Logic Programming*, 35:151–170, 1998.

[7] W.A. Carnielli. Systematization of Finite Many-valued Logics through the Method of Tableaux. *Journal of Symbolic Logic*, 52:473–493, 1987.

[8] W.A. Carnielli. On Sequents and Tableaux for Many-valued Logics. *The Journal of Non-Classical Logic*, 8:59–78, 1991.

[9] J.P. Cleave. The Notion of Logical Consequence in the Logic of Inexact Predicates. *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, 20:307–324, 1974.

[10] M. D'Agostino. Investigations into the Complexity of some Propositional Calculi. Technical report, Oxford University Computing Laboratory, 1990.

[11] J.M. Dunn. Intuitive Semantics for First-Degree Entailments and 'Coupled Trees'. *Philosophical Studies*, 29:149–168, 1976.

[12] S. Feferman. Toward Useful Type Free Theories I. *Journal of Symbolic Logic*, 49:75–111, 1984.

[13] M. Fitting. Bilattices and the Semantics of Logic Programming. *Journal of Logic Programming*, 11:91–116, 1991.

[14] M. Fitting. *First-Order Logic and Automated Theorem Proving*. Springer, New York, 1996.

[15] P.C. Gilmore. The Consistency of Partial Set Theory without Extensionality. In *Axiomatic Set Theory. Proceedings of Symposia in Pure Mathematics 13, Part II*, Providence, 1974. AMS.

[16] M.L. Ginzburg. Multivalued Logics: A Uniform Approach to Reasoning in AI. *Computer Intelligence*, 4:256–316, 1988.

[17] R. Hähnle. *Automated Deduction in Multiple-valued Logics*. Oxford University Press, Oxford, 1993.

[18] M. Holden. Weak Logic Theory. *Theoretical Computer Science*, 79:295–321, 1991.

[19] J. Jaspars. *Calculi for Constructive Communication*. PhD thesis, Tilburg University, 1994.

[20] S.C. Kleene. *Introduction to Metamathematics*. Van Nostrand, 1952.

[21] T. Langholm. *Partiality, Truth and Persistence*. CSLI Lecture Notes, Stanford, 1988.

[22] T. Langholm. How Different is Partial Logic? In P. Doherty, editor, *Partiality, Modality, and Nonmonotonicity*, pages 3–43. CSLI, Stanford, 1996.

[23] R.A. Muskens. Going Partial in Montague Grammar. In R. Bartsch, J.F.A.K. van Benthem, and P. van Emde Boas, editors, *Semantics and Contextual Expression. Proceedings of the Sixth Amsterdam Colloquium*. Foris, Dordrecht, 1989.

[24] R.A. Muskens. *Meaning and Partiality*. CSLI, Stanford, 1995.

[25] G. Rousseau. Sequents in Many-valued Logic I. *Fundamenta Mathematicae*, 60:23–33, 1967.

[26] K. Schröter. Methoden zur Axiomatisierung Beliebiger Aussagen- und Prädikatenkalküle. *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, 1:241–251, 1955.

[27] G. Takeuti. *Proof Theory*. North-Holland, Amsterdam, 1975.

[28] E. Thijsse. *Partial Logic and Knowledge Representation*. PhD thesis, Tilburg University, 1992.

[29] A. Visser. Four Valued Semantics and the Liar. *Journal of Philosophical Logic*, 13:181–212, 1984.

[30] P.W. Woodruff. Paradox, Truth and Logic, Part I: Paradox and Truth. *Journal of Philosophical Logic*, 13:213–232, 1984.

[31] R. Zach. Proof Theory of Finite-valued Logics. Technical Report TUW-E185.2-Z.1-93, Institut für Computersprachen, Technische Universität Wien, 1993.