Embedding Classical Logic in S4

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Abstract

In this thesis, we will study the * embedding of classical first-order logic in first-order S4, which is based on the * translation, originally introduced in Fitting (1970). The initial, main part is dedicated to a detailed model-theoretic proof of the soundness of the embedding. This will follow the proof sketch in Fitting (1970). We will then outline a proof procedure for a proof-theoretic replication of the soundness result. Afterwards, a potential proof of faithfulness of the * embedding, read in terms of soundness and completeness, will be discussed. We will particularly highlight the many difficulties coming with it. In the final section, we will relate this discussion to the debate on notational variance in French (2019). We will do this by showing how a weaker version of French's notion of 'expressive equivalence' conforms to the model-theoretic soundness result. We will then conclude that the soundness result without completeness might contain rather little overall insight by relating it to the extensibility of classical logic to S4.

Contents

1 Introduction: embeddings of logics

[T]ruth may be replaced
with provable consistency
(Fitting, 1970, p. 529)

What does Fitting (1970) mean by this quote? What does he mean by *truth*? What does he refer to by *provable consistency*? How should we understand the *may be replaced with* relation? We will try to answer these questions and rationally reconstruct Fitting's quote in this thesis. As this requires viewing the quote in its context, we will have to have a close look at the main quest of Fitting (1970): an embedding of classical logic in S4, the * embedding.

1.1 Embeddings...

What exactly is an embedding of logics, in particular a modal embedding, such as Fitting's? The most popular modal embedding might be the embedding of (non-modal) intuitionistic logic in (modal) S4, tracing back to McKinsey and Tarski (1948) and Prawitz (1966) (the latter for the first-order case) (see also Fitting (1970, p. 529), Schütte (1968, pp. 33-39), and Troelstra and Schwichtenberg (2000, pp. 288-291)). We will call this the $^\circ$ embedding. It will serve us as an example to sketch a general recipe for embeddings. A more detailed discussion of the * embedding will be worked out in the course of this thesis.
The backbone of any embedding is a translation, i.e. a procedure which converts formulas of one logical language (here: intuitionistic logic, IL) into formulas of another one (in this case: S4). So, we define:

Definition 0 (The $^\circ$ translation)
Be A, B, C formulas, which do not contain any modal symbols, $\Box, \Diamond$.

(1) $$A \text{ is atomic} \Rightarrow A^\circ := \Box A,$$

(2) $$(\neg B)^\circ := \Box\neg B^\circ,$$

(3) $$(B \wedge C)^\circ := \Box(B^\circ \wedge C^\circ),$$

(4) $$(B \vee C)^\circ := \Box(B^\circ \vee C^\circ),$$

(5) $$(B \to C)^\circ := \Box(B^\circ \to C^\circ),$$

(6) $$(\forall x(A(x)))^\circ := \Box\forall x((A(x))^\circ),$$

(7) $$(\exists x(A(x)))^\circ := \Box\exists x((A(x))^\circ).$$

We could also say that applying the $\circ$ translation means boxing every sub-formula. This kind of translation is a rather intuitive procedure, which can be read just as vocabulary books, the way we know them from foreign language lessons, where there is a word of one language (above: IL) in the left column and a word of the other language (above: S4) in the right column (where columns are replaced by the definitional identity sign, :=).

Table 1: excerpt from a fictional vocabulary book

Deutsch	**English**
Übersetzung	translation
Einbettung	embedding
Abschlussarbeit	thesis

The application of the $\circ$ translation to concrete logics is called a $\circ$ embedding, in this case of IL in S4. As we will see later on, several properties of such an embedding have been considered of relevance in the literature. However, two of them are of particular interest: 'soundness' and 'completeness' (of an embedding). Informally speaking, the $\circ$ embedding is sound iff all intuitionistically true formulas are truths of S4 modulo $\circ$ translation. Expressed a bit more formally: $\forall$ (non-modal) $A :\vDash_{IL} A \Rightarrow \vDash_{S4} A^\circ$[1]. 'Completeness' is defined as the converse of 'soundness'. We will call the conjunction of both properties 'faithfulness'. We know that the $\circ$ embedding of intuitionistic logic in S4 satisfies this notion of faithfulness (Troelstra and Schwichtenberg, 2000, pp. 288-291).

[1] Soundness (and completeness) can be proven semantically or syntactically. Hence, we could also substitute $\vdash$ for $\vDash$ here.

1.2 ...and their philosophy

So, we should have acquired some rough grasp on embeddings at this point. Nonetheless, how does this relate to our initial quote?
Intuitionistic logic was originally developed as a formal framework to capture the notion of mental constructions (in the context of mathematics). This interpretation can be traced back to Brouwer (1907). Until today, many people would consider intuitionistic logic a good means to model constructability in mathematics, for instance by serving as the underlying logical language for constructive mathematics (cf. Moschovakis, 2018). Although there is less consensus in the debate centring around the quest for a logic of provability, S4 is still considered a good candidate to capture many properties of (absolute and informal) provability (cf. Verbrugge, 2017). Historically speaking, S4 was also the first proposal of a logic for modelling provability (cf. Gödel, 1986).
We do not want to get into this debate in more detail for the sake of this thesis. Nevertheless, if we accept intuitionistic logic as the 'logic of constructability' and S4 as the 'logic of provability' (or: 'provable consistency'), we are able to re-phrase the result of the $^\circ$ embedding as follows:

Constructability can be faithfully embedded in provable consistency.

We might be able to see some similarities to our initial quote here. However, the * embedding does not embed intuitionistic logic in S4 but classical logic, the underlying formal framework of which is widely considered to replicate many intuitions about (mathematical or natural language) truth (Shapiro and Kissel, 2018). If we want to re-phrase this in a catchy way again, we might want to call classical logic 'the logic of truth'.

Table 2: logics, their interpretation, and corresponding translations

intuitionistic logic	constructabiliy	$\circ$
classical logic	truth	$*$
S4	provability	$\circ$, $*$

Therefore, we can tell now what Fitting means by *truth*, namely the framework for modelling truth: classical logic. We can also make out what he means by *provable consistency*: its modelling framework, which is S4.
Nevertheless, we still have to clarify what is meant by the *may be replaced with* relation. This will require closer investigation of the * embedding. The quest for a notion of it will be the overarching question of this thesis.

1.3 The structure of this thesis

To accomplish finding an answer to this question, we will employ a three step battle plan.
Firstly, we will spell the model-theoretic proof sketch from Fitting (1970) in full, and prove the soundness of Fitting's * embedding. We will thereby fill any gaps if necessary.
Secondly, we will sketch a procedure to replicate this model-theoretic soundness result in a proof-theoretic setting. We will then continue the proof-theoretic discussion by giving a precise notion of 'faithfulness'. Working our way through some of the many potential challenges accompanying a completeness proof for the * embedding will lead us to formulate a suspicion that no proof might be possible at all. This thought will conclude the proof theory section.
In the last part, we will evaluate the preceding work. Here is also where we will try to give an answer to our initial question by arguing for an explication of the replaceability relation in terms of a weak version of notational variance, primarily based on French (2019).
Before each of the main sections, we will give the definitions and notions involved in the following part in a separate 'preliminaries' section. We will start off with such a section, where we will introduce the languages of classical logic and S4.[2]

[2]Before we begin, we want to point at the proof-theoretic (cf. Czermak, 1975) and model-theoretic (cf. Czermak, 1976) results about the * embedding by Johannes Czermak. Although we will not discuss them in the course of this thesis, many additional interesting results can be found there, some of which also relate to our work in this thesis.

2 Preliminaries 1: $\mathcal{L}_{S4}$ and $\mathcal{L}_C$

In the following, we will define the language of first-order S4 without identity, $\mathcal{L}_{S4}$, and some related concepts, which will be of importance later on. We will not explicitly state the definitions for its classical first-order counterpart, $\mathcal{L}_C$. However, these will be the same as for $\mathcal{L}_{S4}$, with the only difference being that all clauses referring to modal symbols, $\Box, \Diamond$, have been deleted. These definitions have partly been taken from or inspired by Ebbinghaus et al. (2007) and, particularly, Gratzl (2018a).

We will employ a natural deduction system taken from Leitgeb (2018) and Gratzl (2018b), combined with the terminology of Gratzl (2018a) (and standard set theory), as our meta-language. We will use A, B as meta-variables for formulas, $\mathbb{V}, \mathbb{E}$ as meta-quantifiers, $\Rightarrow, \Leftrightarrow$ as meta-implication and meta-equivalence, and English words to represent the other meta-connectives.

2.1 The alphabet

Definition 1 (The alphabet of the language of first-order S4, $\mathcal{L}_{S4}$)
The alphabet of the language $\mathcal{L}_{S4}$ consists of:

1. $v_0, v_1, v_2, \ldots$ (variables),
2. $a_1, a_2, a_2, \ldots$ (individual constants, which can also be called 'parameters' (cf. Fitting, 1970, p. 530)),
3. $\neg, \land, \lor, \rightarrow$,
4. $\exists$, $\forall$,
5. $), ($,
6. Countably many predicates: $P_1^n, P_2^n, P_3^n, \ldots$ (where, for each n, $n \geq 1$),
7. $\Diamond, \Box$.

2.2 Terms and formulas

Definition 2 (Terms of $\mathcal{L}_{S4}$)

1. Each variable is a term of $\mathcal{L}_{S4}$.
2. Each constant is a term of $\mathcal{L}_{S4}$.
3. Otherwise, nothing is a term of $\mathcal{L}_{S4}$.

Definition 3 (Formulas of $\mathcal{L}_{S4}$)

1. Be $t_1, ..., t_n$ terms of $\mathcal{L}_{S4}$, $Pt_1, \ldots, t_n$ is a formula of $\mathcal{L}_{S4}$.
2. Be A a formula of $\mathcal{L}_{S4}$, $\neg A$ is a formula of $\mathcal{L}_{S4}$.
3. Be A, B formulas of $\mathcal{L}_{S4}$, $(A \wedge B)$, $(A \vee B)$, $(A \to B)$ be formulas by $\mathcal{L}_{S4}$.
4. Be A a formula of $\mathcal{L}_{S4}$ and be x a variable, $\forall x A$ and $\exists x A$ are formulas of $\mathcal{L}_{\mathcal{S}}$.
5. Be A a formula of $\mathcal{L}_{S4}$, $\Box A$ and $\Diamond A$ are formulas of $\mathcal{L}_{\mathcal{S}}$.

Remark 1 (Atomic formulas)
Formulas of the first type are called 'atomic formulas'.

Definition 4 (Free variables in a term t, fvt(t))

1. $\text{fvt}(x) := \{x\}$,
2. $\text{fvt}(c) := \varnothing$.

Definition 5 (Free variables in a formula A, free(A))

1. $\text{free}(Pt_1 \ldots t_n) := \text{fvt}(t_1) \cup \cdots \cup \text{fvt}(t_n)$,
2. $\text{free}(\neg A) := \text{free}(A)$,
3. $\text{free}((A * B)) := \text{free}(A) \cup \text{free}(B)$, for $* = \wedge, \vee, \to$,
4. $\text{free}(\exists x A) := \text{free}(A) - \{x\}$,

5. $\text{free}(\forall x A) := \text{free}(A) - \{x\}$,
6. $\text{free}(\Box A) := \text{free}(A)$,
7. $\text{free}(\Diamond A) := \text{free}(A)$.

Definition 6 (Closed formula)
A formula A is closed $:\Leftrightarrow$ free(A)= $\varnothing$.

Remark 2 (Closed formula convention)
In the following, the term 'formula' will always refer to closed formulas, unless stated otherwise.

Definition 7 (S4 assignment, α_{S4})
An S4 assignment is a function $\alpha_{S4} : \{v_n | n \in \mathbb{N}\} \to D$.

Definition 8 (The terms of a predicate, ter($P^n(x_1, ..., x_n)$))
The terms of an n-ary predicate ter($P^n(t_1, ..., t_n)$), where $t_1, ...t_n$ are terms, is a set, for which holds:

$\text{ter}(P^n(t_1, ..., t_n)) = \{e \,|\, \forall\alpha_{S4} : \forall t \in \{t_1, ..., t_n\} :$

1. t is a constant $\Rightarrow e = t$,
2. t is a variable and $t \in \text{free}(P^n(t_1, ..., t_n)) \Rightarrow \exists\alpha_{S4} : e = \alpha_{S4}(t)\}$.

Definition 9 (The terms of a formula A, ter(A))

1. $\text{ter}(P^n(t_1 \dots t_n))$, vide supra,
2. $\text{ter}(\neg A) := \text{ter}(A)$,
3. $\text{ter}((A * B)) := \text{ter}(A) \cup \text{ter}(B)$, for $* = \wedge, \vee, \rightarrow$,
4. $\text{ter}(\exists x A) := \text{ter}(A)$,
5. $\text{ter}(\forall x A) := \text{ter}(A)$,
6. $\text{ter}(\Box A) := \text{ter}(A)$,
7. $\text{ter}(\Diamond A) := \text{ter}(A)$.

3 Preliminaries 2: semantics

As we have just defined our languages, we will now define semantics for $\mathcal{L}_{S4}$. Similarly to our definitions of the languages, we will afterwards define validity for $\mathcal{L}_C$, exploiting the concepts of $\mathcal{L}_{S4}$. The notion of a truth set (cf. Smullyan, 1968) will play a crucial role here, as it does for Fitting (1970). Our definitions are taken from or inspired by Fitting (1970), Gratzl (2018a), Schütte (1968), Fitting (1969), and Kripke (1962).

3.1 S4: frames and models

Definition 10 (S4 frame, $\mathcal{F}_{S4}$)
An S4 frame $\mathcal{F}_{S4}$ is a pair $\langle W, R\rangle$, with the properties:

1. $W \neq \varnothing$,
2. $R \subseteq (W \times W)$ and for R holds:
   (a) $\forall w \in W : R(w, w)$ (reflexivity),
   (b) $\forall w, v, u \in W : (R(w, v) \text{ and } R(v, u)) \rightarrow R(w, u)$ (transitivity).

W is called 'the set of possible worlds', so to say the possible scenarios in which S4 formulas can be true (or false). R is a relation between these worlds.

Definition 11 (S4 model, $\mathcal{M}_{S4}$)
An S4 model, $\mathcal{M}_{S4}$, is a pair $\langle \mathcal{F}, V\rangle$, where

1. V is a function $V : W \rightarrow \wp(C)$,
2. C is the set of all constants,
3. $C \neq \varnothing$,
4. $\forall w_1, w_2 \in W : R(w_1, w_2) \Rightarrow V(w_1) \subseteq V(w_2)$.

Definition 12 (D)

$$D = \bigcup_{w \in W} V(w)$$

So, V is a function which assigns each world a set of individual constants and thus outputs the domain of that world. D on the other hand stands for the fused domain, of which each V is a subset. Def. 11.4 ensures that a subsequent world, i.e. a world to which a prior world has a relation, also contains all constants of the prior world.

Strictly speaking, we do not want constants, names of objects, inhabiting our worlds but the objects themselves since logics are normally applied in order to model objects (their properties, relations between them, etc.) and not their names. However, this equivocation of names and objects, as it is also done by Fitting (1970), does not pose a profound threat to our definitions, stand alone Fitting's results, which we will see in the next subsection. This discontent can easily be fixed by adding a function relative to a $w \in W$, the 'true domain function', $TrueV_w$, which assigns the set of all names of objects in a world w the objects themselves. Formally: $TrueV_w : V(w) \to \wp(O)$, where O is the set of all objects in question. Nevertheless, we will largely ignore this distinction in the following.

Definition 13 ($\hat{V}(w)$)
Be A a formula:
$\forall w \in W : \hat{V}(w) := \{A \,|\, \mathrm{ter}(A) \subseteq \wp(V(w))\}$.

This means that $\hat{V}(w)$ is the set of all formulas which contain solely terms (or constants, as the formulas are closed ones) which are in the domain of a world.

Example 1 ($\hat{V}(w)$)
Be $V(w_1) = \{a_1, a_2, a_3\}$ (so to say the 'inhabitants of w_1'). All formulas containing terms from $\{a_1, a_2, a_3\}$ are in $\hat{V}(w_1)$, for instance:

1. $P(a_1) \in \hat{V}(w_1)(\in \hat{D})$,
2. $\forall v_2 P(v_2, a_1, v_1) \in \hat{V}(w_1)$, for $\alpha_{S4,1}(v_1) = a_3$,
3. $P(a_1) \in \hat{V}(w_2)$ if $R(w_1, w_2)$ (cf. Def. 11.4),
4. $P(a_1) \in \hat{V}(w_3)$ if $R(w_1, w_2)$ and $R(w_2, w_3)$ (cf. Def. 11.4 and Def. 10.2b).

3.2 S4: satisfaction, validity, and truth

Definition 14 (Satisfaction in an S4 model)
$\mathbb{V}w \in W$:

1. $A \notin \hat{V}(w) \Rightarrow \mathcal{M}_{S4}, w \nvDash A$.

2. $A, B \in \hat{V}(w) \Rightarrow$

$$\mathcal{M}_{S4}, w \vDash \neg A \Leftrightarrow \mathcal{M}_{S4}, w \nvDash A, \tag{8}$$

$$\mathcal{M}_{S4}, w \vDash (A \wedge B) \Leftrightarrow \mathcal{M}_{S4}, w \vDash A \text{ and } \mathcal{M}_{S4}, w \vDash B, \tag{9}$$

$$\mathcal{M}_{S4}, w \vDash (A \vee B) \Leftrightarrow \mathcal{M}_{S4}, w \vDash A \text{ or } \mathcal{M}_{S4}, w \vDash B, \tag{10}$$

$$\mathcal{M}_{S4}, w \vDash (A \rightarrow B) \Leftrightarrow \mathcal{M}_{S4}, w \nvDash A \text{ or } \mathcal{M}_{S4}, w \vDash B, \tag{11}$$

$$\mathcal{M}_{S4}, w \vDash \forall x A x \Leftrightarrow \mathcal{M}_{S4}, w \vDash \mathbb{V}d \in V(w) : A(d), \tag{12}$$

$$\mathcal{M}_{S4}, w \vDash \exists x A x \Leftrightarrow \mathcal{M}_{S4}, w \vDash \mathbb{E}d \in V(w) : A(d), \tag{13}$$

$$\mathcal{M}_{S4}, w \vDash \Box A \Leftrightarrow \mathbb{V}w^* \in W, R(w, w^*) : \mathcal{M}_{S4}, w^* \vDash A, \tag{14}$$

$$\mathcal{M}_{S4}, w \vDash \Diamond A \Leftrightarrow \mathbb{E}w^* \in W, R(w, w^*) : \mathcal{M}_{S4}, w^* \vDash A. \tag{15}$$

To shorten our commentary for the proofs, we will sometimes abbreviate for instance 'Def. 14.2.7' as 'Def 14.7'.

Definition 15 (Validity in an S4 model)
A formula A is valid in an S4 model, $\langle \mathcal{F}, V \rangle$, if

$$\mathbb{V}w \in W, A \in \hat{V}(w) : \mathcal{M}_{S4}, w \vDash A.$$

Definition 16 (Logical truth)
Be A a formula:

$$\vDash_{FS} A \Leftrightarrow \mathbb{V}\mathcal{M}_{FS} : A \text{ is valid in } \mathcal{M}_{FS},$$

where FS is a formal system (for example S4).

Example 2 (Tertium non datur, boxed)
We want to showcase our semantics by a brief example:
$\overset{?}{\vDash}_{S4} \Box A \vee \Box\neg A$

(1) Not: $\vDash_{S4} \Box A \vee \Box\neg A$ | IP ass.

(2)
Not: $\forall \mathcal{M}_{S4} : \forall w \in W, A \in \hat{V}(w) :$
$\mathcal{M}_{S4}, w \vDash \Box A \vee \Box\neg A$ | 1., Def. 16, Def. 14

(3) Not: $(\mathcal{M}_{S4}, w \vDash \Box A$ or $\mathcal{M}_{S4}, w \vDash \Box\neg A)$ | 2., Def. 14.2.3

(4) Not: $\mathcal{M}_{S4}, w \vDash \Box A$ and Not: $\mathcal{M}_{S4}, w \vDash \Box\neg A)$ | 3.

(5)
$\exists v, R(w, v) :$ Not: $\mathcal{M}_{S4}, v \vDash A$ | 4., Def. 14.2.7

(6) $\mathcal{M}_{S4}, v' \vDash \neg A$ | 5., EE ass., Def. 14.2.1

(7)
$\exists u, R(w, u) :$ Not: Not: $\mathcal{M}_{S4}, u \vDash A$ | 4., Def. 14.2.1,7

(8) $\mathcal{M}_{S4}, u' \vDash A$ | 7., EE ass.

We have run into a dead end here, as we cannot derive a contradiction from 6. and 8.. Does this mean that the formula is invalid in some S4 models and it is thus not S4 true? If we want to show this, we must construct a counter model. Indeed, the derivation above already gives us one at hand:
$\mathcal{M}_{S4,\text{ counter}} = \langle W, R, V \rangle$, where

- $W = \{w, u, v\}$
- $R(w, u), R(w, v)$
- $\hat{V}(w) = \{\Box A \vee \Box\neg A\}; \hat{V}(u) = \{A\}; \hat{V}(v) = \{\neg A\}$

Why is this a counter model for our formula in question?

(1) Not: Not: $\mathcal{M}_{S4,\text{ counter}}, w \vDash \Box A \vee \Box\neg A$ | IP ass.

(2) $\mathcal{M}_{S4,\text{ counter}}, w \vDash \Box A$ or $\mathcal{M}_{S4,\text{ counter}}, w \vDash \Box\neg A$ | 1., Def. 14.2.3

(3) $\mathcal{M}_{S4,\text{ counter}}, w \vDash \Box A$ | PC ass. 1

(4)	$\mathcal{M}_{S4,\text{ counter}}, v \vDash A$	| 3., Def. 14.2.7, UE
(5)	Not: $\mathcal{M}_{S4,\text{ counter}}, v \vDash A$	| Def. $\hat{V}$(v), Def. 14.2.1
(6)	C or Not: C	| 4.,5., ECQ
(7)	$\mathcal{M}_{S4,\text{ counter}}, w \vDash \Box\neg A$	| PC ass. 2
(8)	Not: $\mathcal{M}_{S4,\text{ counter}}, u \vDash A$	| 7., Def. 14.2.7,1
(9)	$\mathcal{M}_{S4,\text{ counter}}, u \vDash A$	| Def. $\hat{V}(u)$
(10)	C or Not: C	| 8., 9., ECQ
(11)	C or Not: C	| 2.-10., PC
(12)	Not: $\mathcal{M}_{S4,\text{ counter}}, w \vDash \Box A \vee \Box\neg A$	| 1.-11., IP

This contradicts Def. 15 and, thus, Def. 16. So, we can conclude: $\nvDash_{S4} \Box A \vee \Box\neg A$.

Before we will begin with the main proof of this thesis, we have to define some additional concepts which we will require for the proof.

3.2.1 Trivial models

The first one of them is the model which contains exactly one world:

Definition 17 (Trivial model)
An S4 model is called trivial model, $\mathcal{M}_T$, if $|W| = 1$.

Corollary 1
Be A a formula:

$$\vDash_T A \Leftrightarrow \forall \mathcal{M}_T : A \text{ is valid in } \mathcal{M}_T.$$

follows from Def. 17 and Def. 16.

3.3 Truth sets

Another concept of interest is the classical truth set, based on Smullyan (1968). It is effectively a set containing all true (non-modal) formulas of classical logic. We will use this definition to define validity for classical logic.

Definition 18 (*Par*)
Par is a set for which holds:

1. $Par \subseteq C$, and

2. $Par \neq \varnothing$.

Definition 19 (Truth set, $\mathcal{T}_{Par}$)
Be A, B formulas, for which holds:

1. A, B do not contain any modal symbols, $\Box, \Diamond$, and

2. $\mathrm{ter}(A, B) \subseteq Par$.

A truth set with respect to Par, $\mathcal{T}_{Par}$, is a set of formulas, which do not contain any modal symbols, $\Box, \Diamond$, for which holds:

$$(\neg A) \in \mathcal{T}_{Par} \Leftrightarrow A \notin \mathcal{T}_{Par}, \tag{1}$$
$$(A \wedge B) \in \mathcal{T}_{Par} \Leftrightarrow A \in \mathcal{T}_{Par} \text{ and } B \in \mathcal{T}_{Par}, \tag{2}$$
$$(A \vee B) \in \mathcal{T}_{Par} \Leftrightarrow A \in \mathcal{T}_{Par} \text{ or } B \in \mathcal{T}_{Par}, \tag{3}$$
$$(A \rightarrow B) \in \mathcal{T}_{Par} \Leftrightarrow A \notin \mathcal{T}_{Par} \text{ or } B \in \mathcal{T}_{Par}, \tag{4}$$
$$(\forall x A(x)) \in \mathcal{T}_{Par} \Leftrightarrow \mathbb{V} c \in Par : A(c) \in \mathcal{T}_{Par}, \tag{5}$$
$$(\exists x A(x)) \in \mathcal{T}_{Par} \Leftrightarrow \mathbb{E} c \in Par : A(c) \in \mathcal{T}_{Par}. \tag{6}$$

3.3.1 Classical validity

Definition 20 (Classical validity)
Be A a formula, for which holds: A does not contain any modal symbols, $\Box, \Diamond$.
Be $\mathcal{T}_{Par}$ a truth set with respect to Par.

$$\vDash_C A :\Leftrightarrow \mathbb{V}\mathcal{T}_{Par}, \mathrm{ter}(A) \subseteq Par : A \in \mathcal{T}_{Par}.$$

4 The embedding 1: model-theoretically

By defining the necessary semantic concepts, we have now set the stage to rationally reconstruct and elaborate on the brief model-theoretic proof sketch in Fitting (1970, pp. 531-534). This is a combination of the model-theoretic concept of a complete sequence (cf. Cohen, 1966) and the basic structure of a Henkin style completeness proof for classical logic (cf. e.g. Ebbinghaus et al., 2007).

4.1 Theorem 1: a relation of classical and trivial validity

Theorem 1 (A relation of classical and trivial validity)
A be a formula, which does contain modal symbols, $\Box, \Diamond$.
$A^{\#} = A$, but all occurrences of modal symbols, $\Box, \Diamond$, have been deleted in $A^{\#}$.

$$\vDash_T A \Rightarrow \vDash_C A^{\#}$$

Proof

(1)	$\vDash_T A.$	| CP ass.
(2)	$\nvDash_C A^{\#}$	| IP ass.
(3)	$\text{Not}: \forall \mathcal{T}_{Par}, \text{ter}(A^{\#}) \subseteq Par : A^{\#} \in \mathcal{T}_{Par}.$	| 2, Def. 20.
(4)	$\exists \mathcal{T}_{Par}, \text{ter}(A^{\#}) \subseteq Par : A^{\#} \notin \mathcal{T}_{Par}.$	| 3.

We define a trivial model:

$$\mathcal{M}_T = \langle \mathcal{F}, V \rangle = \langle W, R, V \rangle, \text{ where}$$
$$W = \{\mathcal{T}_{Par}\},$$
$$R(\mathcal{T}_{Par}, \mathcal{T}_{Par}),$$
$$V(\mathcal{T}_{Par}) = Par,$$
$$\forall E, E \text{ is atomic} : E \in \mathcal{T}_{Par} \Leftrightarrow \mathcal{M}_T, \mathcal{T}_{Par} \vDash E.$$

(5)	$\mathcal{M}_T, \mathcal{T}_{Par} \vDash \Box B$	| CP ass.
(6)	$\forall w \in W, R(\mathcal{T}_{Par}, w) : \mathcal{M}_T, w \vDash B$	| 5., Def. 14.7

(7) Not: $\mathcal{M}_T, \mathcal{T}_{Par} \models B$ | IP ass.

(8) $\exists w \in W, R(\mathcal{T}_{Par}, w) : \text{Not: } \mathcal{M}_T, w \models B$

by 7., EI, Def. 17, and Def. 10.2a ($w \overset{!}{=} \mathcal{T}_{Par}$).

(9) Not: $\forall w \in W, R(\mathcal{T}_{Par}, w) : \mathcal{M}_T, w \models B$ | 8.

(10) $\mathcal{M}_T, \mathcal{T}_{Par} \models B$ | 6.,9., ECQ, 7.-9., IP

(11) $\mathcal{M}_T, \mathcal{T}_{Par} \models \Box B \Rightarrow$

$\mathcal{M}_T, \mathcal{T}_{Par} \models B$ | 5.-10., CP

(12) $\mathcal{M}_T, \mathcal{T}_{Par} \models B$ | CP ass.

(13) Not: $\mathcal{M}_T, \mathcal{T}_{Par} \models \Box B$ | IP ass.

(14) Not: $\forall w \in W, R(\mathcal{T}_{Par}, w) : \mathcal{M}_T, w \models B$ | 13., Def. 14.7

(15) $\forall w \in W, R(\mathcal{T}_{Par}, w) : \mathcal{M}_T, w \models B$

by 12. and the definition of our trivial model.

(16) $\mathcal{M}_T, \mathcal{T}_{Par} \models \Box B$ | 14.,15., ECQ, 13.-15., IP

(17) $\mathcal{M}_T, \mathcal{T}_{Par} \models B \Rightarrow$

$\mathcal{M}_T, \mathcal{T}_{Par} \models \Box B$ | 12.-16., CP

(18) $\mathcal{M}_T, \mathcal{T}_{Par} \models \Box B \Leftrightarrow$

$\mathcal{M}_T, \mathcal{T}_{Par} \models B$ | 11., 17., EQI

(19) $\mathcal{M}_T, \mathcal{T}_{Par} \models \Diamond B$ | CP ass.

(20) $\exists w \in W, R(\mathcal{T}_{Par}, w) : \mathcal{M}_T, w \models B$ | 19., Def. 14.8

(21) Not: $\mathcal{M}_T, \mathcal{T}_{Par} \models B$ | IP ass.

(22) $\mathcal{M}_T, w \models B$ | EE ass.

(23) C and Not: C | 21., 22., ECQ

(24) C and Not: C | 22.-23., EE

(25) $\mathcal{M}_T, \mathcal{T}_{Par} \models B$ | 21.-24., IP

(26) $\mathcal{M}_T, \mathcal{T}_{Par} \models \Diamond B \Rightarrow$

$\mathcal{M}_T, \mathcal{T}_{Par} \models B$ | 19.-25., CP

(27) $\mathcal{M}_T, \mathcal{T}_{Par} \models B$ | CP ass.

(28) Not: $\mathcal{M}_T, \mathcal{T}_{Par} \models \Diamond B$ | IP ass.

(29) Not: $\exists w \in W, R(\mathcal{T}_{Par}, w) : \mathcal{M}_T, w \models B$ | 28., Def. 14.8

(30) $\forall w \in W, R(\mathcal{T}_{Par}, w) :$ Not: $\mathcal{M}_T, w \models B$ | 29.

(31) Not: $\mathcal{M}_T, \mathcal{T}_{Par} \models B$ | 30., Def. 17

(32) $\mathcal{M}_T, \mathcal{T}_{Par} \models \Diamond B$ | 27., 31., ECQ, 28.-31., IP

(33) $\mathcal{M}_T, \mathcal{T}_{Par} \models B \Rightarrow$
$\mathcal{M}_T, \mathcal{T}_{Par} \models \Diamond B$ | 27.-32., CP

(34) $\mathcal{M}_T, \mathcal{T}_{Par} \models \Box B \Leftrightarrow$
$\mathcal{M}_T, \mathcal{T}_{Par} \models B$ | 26., 33., EQI

(35) $\mathcal{M}_T, \mathcal{T}_{Par} \models \Box B \Leftrightarrow$
$\mathcal{M}_T, \mathcal{T}_{Par} \models \Diamond B \Leftrightarrow$
$\mathcal{M}_T, \mathcal{T}_{Par} \models B.$

by 18., 34., and transitivity of $\Leftrightarrow$.

(36) $\forall B \in \hat{V}(\mathcal{T}_{Par}) : \mathcal{M}_T, \mathcal{T}_{Par} \models \Box B \Leftrightarrow$
$\mathcal{M}_T, \mathcal{T}_{Par} \models \Diamond B \Leftrightarrow \mathcal{M}_T, \mathcal{T}_{Par} \models B.$ | 35., UI

(37) $\forall B, B^{\#} \in \hat{V}(\mathcal{T}_{Par}) : \mathcal{M}_T, \mathcal{T}_{Par} \models B^{\#} \Leftrightarrow$
$\mathcal{M}_T, \mathcal{T}_{Par} \models B.$ | 36., Def. $B^{\#}$

(38) $\forall B^{\#} \in \hat{V}(\mathcal{T}_{Par}) :$
$\mathcal{M}_T, \mathcal{T}_{Par} \models B^{\#} \Leftrightarrow B^{\#} \in \mathcal{T}_{Par}.$

by the definition of $B^{\#}$, the definition our trivial model, and Def. 19.

(39) $A^{\#} \notin \mathcal{T}_{Par}$ | 4., EE ass., Def. 17

(40) $\mathcal{M}_T, \mathcal{T}_{Par} \not\models A^{\#}$ | 39., 38., EQE, MT

(41) $\mathcal{M}_T, \mathcal{T}_{Par} \not\models A$ | 37., EQE, 40., MT

(42) $\forall w \in W, A \in \hat{V}(w) :$
$\mathcal{M}_T, w \models A$ | 1., Corr. 1, Def. 15

(43)	$\mathcal{M}_T, \mathcal{T}_{Par} \vDash A$	| 42., UE
(44)	C and Not: C	| 41., 43., ECQ
(45)	$\vDash_C A^{\#}$.	| 2.-44., EEs, IP
(46)	$\vDash_T A \Rightarrow \vDash_C A^{\#}$.	| 1.-45., CP

■

This theorem says that if any formula is trivially S4 valid, its non-modal counterpart is classically valid.

4.2 The centrepiece: the * translation

Now we come to the beating heart of the embedding: Fitting's * translation, which bridges the non-modal classical formulas and the modal S4 formulas. It is fairly simple: we obtain the latter from the former by putting $\Box\Diamond$ in front of it.

Definition 21 (The * translation)
Be A, B, C formulas, which do not contain any modal symbols, $\Box, \Diamond$.

(1) $$A \text{ is atomic} \Rightarrow A^* := \Box \Diamond A,$$

(2) $$(\neg B)^* := \Box \Diamond \neg B^*,$$

(3) $$(B \wedge C)^* := \Box \Diamond (B^* \wedge C^*),$$

(4) $$(B \vee C)^* := \Box \Diamond (B^* \vee C^*),$$

(5) $$(B \to C)^* := \Box \Diamond (B^* \to C^*),$$

(6) $$(\forall x(A(x)))^* := \Box \Diamond \forall x((A(x))^*),$$

(7) $$(\exists x(A(x)))^* := \Box \Diamond \exists x((A(x))^*).$$

Corollary 2 (*, classical and tivial validity)
A be a formula, which does not contain modal symbols, $\Box, \Diamond$.

$$\vDash_T A^* \Rightarrow \vDash_C A.$$

This follows as A and A^* are identical, except for the modal operators, $\Box, \Diamond$, that have been added to A according to Def. 21. However, as all other aspects (variables, constants, predicates, logical operators, quantifiers) are identical, Theorem 1 can be applied here.

Definition 22 (A')

$$\forall A : A^* := \Box \Diamond A'.$$

4.3 Lemma 1: some properties of non-modal S4 formulas

Lemma 1 (Some properties of non-modal S4 formulas)
A be a formula, which does not contain modal symbols, $\Box, \Diamond$.
$\forall \mathcal{M}_{S4} : \forall w \in W : \forall w^* \in W, R(w, w^*) : \forall A \in \hat{V}(w) :$

$$\mathcal{M}_{S4}, w \vDash A^* \Rightarrow \forall w^* : \mathcal{M}_{S4}, w^* \vDash A^*, \tag{1}$$

$$\mathcal{M}_{S4}, w \nvDash A^* \Rightarrow \exists w^* : \mathcal{M}_{S4}, w^* \vDash (\neg A)^*, \tag{2}$$

$$\mathcal{M}_{S4}, w \vDash (\exists x(A(x)))^* \Rightarrow \exists w^* : \exists a \in V(w^*) : \mathcal{M}_{S4}, w^* \vDash (A(a))^*, \tag{3}$$

$$\mathcal{M}_{S4}, w \nvDash (\forall x(A(x)))^* \Rightarrow \exists w^* : \exists a \in V(w^*) : \mathcal{M}_{S4}, w^* \vDash (\neg A(a))^*. \tag{4}$$

We will prove this lemma along the lines of its four cases.

1.

(1)	A^* begins with an occurance of $\Box$.	| Def. 21.
	So, we define: $A^* := \Box B$	
(2)	$\mathcal{M}_{S4}, w \vDash A^*$	| CP ass.
(3)	$\mathcal{M}_{S4}, w \vDash \Box B$	| 1.,2.
(4)	$\forall w^* \mathcal{M}_{S4}, w^* \vDash \Box B$	| Def. 14.7
(5)	$\forall w^* \mathcal{M}_{S4}, w^* \vDash A^*$	| 1.
(6)	$\mathcal{M}_{S4}, w \vDash A^* \Rightarrow \forall w^* \mathcal{M}_{S4}, w^* \vDash A^*$	| CP

2.

(1)	$\mathcal{M}_{S4}, w \nvDash A^*$	| CP ass.

(2)	$\mathcal{M}_{S4}, w \nvDash \Box \Diamond A'$	| 1., Def. 22
(3)	Not: $\forall w^* : \mathcal{M}_{S4}, w^* \vDash \Diamond A'$	| 2., Def. 14.2.7
(4)	$\exists w^*$: Not: $\mathcal{M}_{S4}, w^* \vDash \Diamond A'$	| 3.
(5)	$\exists w^*$: Not: $\exists v, R(w^*, v) : \mathcal{M}_{S4}, v \vDash A'$	| 4., Def. 14.2.8
(6)	$\exists w^* : \forall v, R(w^*, v) : \mathcal{M}_{S4}, v \nvDash A'$	| 5.
(7)	$\exists w^* : \forall v, R(w^*, v) : \mathcal{M}_{S4}, v \vDash \neg A'$	| 6., Def. 14.2.1
(8)	$\exists w^* : \mathcal{M}_{S4}, w^* \vDash \Box \neg A'$	| 7., Def. 14.2.7
(9)	$\mathcal{M}_{S4}, v \vDash \Box \neg A'$	| CP ass.
(10)	Not: $\mathcal{M}_{S4}, v \vDash \Box \Diamond \neg \Box \Diamond A'$	| IP ass.
(11)	$\exists u, R(v, u)$: Not: $\mathcal{M}_{S4}, u \vDash \Diamond \neg \Box \Diamond A'$	| 10., Def. 14.2.7-8
(12)	Not: $\exists t, R(u', t) : \mathcal{M}_{S4}, t \vDash \neg \Box \Diamond A'$	| EE ass., Def. 14.2.8
(13)	$\forall t, R(u', t)$: Not: $\mathcal{M}_{S4}, t \vDash \neg \Box \Diamond A'$	| 12.
(14)	Not: Not: $\mathcal{M}_{S4}, u' \vDash \Box \Diamond A'$	

by 13., Def. 10.2a ($t = u'$), Def. 14.2.1.

(15)	$\mathcal{M}_{S4}, u' \vDash \Box \Diamond A'$	| 14.
(16)	$\forall s, R(u', s) \mathcal{M}_{S4}, s \vDash \Diamond A'$	| 15., Def. 14.2.7
(17)	$\mathcal{M}_{S4}, u' \vDash \Diamond A'$	| 16., Def. 10.2a ($s = u'$)
(18)	$\exists r, R(u', r) : \mathcal{M}_{S4}, r \vDash A'$	| 17., Def. 14.2.8
(19)	$\forall q, R(v, q) : \mathcal{M}_{S4}, q \vDash \neg A'$	| 9., Def. 14.2.7
(20)	$\mathcal{M}_{S4}, r' \vDash A'$	| EE ass.
(21)	$\mathcal{M}_{S4}, r' \vDash \neg A'$	

by 19., Def. 10.2b ($R(v, u'), R(u', r')$).

(22)	Not: $\mathcal{M}_{S4}, r' \vDash A'$	| 21., Def. 14.2.1
(23)	C and Not: C	| 20., 22., ECQ
(24)	C and Not: C	| 20.-23., EE, 12.-13., EE
(25)	$\mathcal{M}_{S4}, v \vDash \Box \neg A' \Rightarrow$	

$\mathcal{M}_{S4}, v \vDash \Box \Diamond \neg \Box \Diamond A'$ | 10.-23., IP, 9.-13., CP

(26) $\forall v : \mathcal{M}_{S4}, v \vDash \Box \neg A' \Rightarrow$

$\mathcal{M}_{S4}, v \vDash \Box \Diamond \neg \Box \Diamond A'$ | 9.-25., UI

(27) $\exists w^* : \mathcal{M}_{S4}, w^* \vDash \Box \Diamond \neg \Box \Diamond A'$ | 8., 26.

(28) $\exists w^* : \mathcal{M}_{S4}, w^* \vDash \Box \Diamond \neg A^*$ | 27., Def. 22

(29) $\exists w^* : \mathcal{M}_{S4}, w^* \vDash (\neg A)^*$ | 28., Def. 21.2

(30) $\mathcal{M}_{S4}, w \nvDash A^* \Rightarrow$

$\exists w^* : \mathcal{M}_{S4}, w^* \vDash (\neg A)^*$ | 1.-29., CP

For the propositional case, we could stop here. However, as we are interested in the full first-order version (without identity), we will take a look at the quantifier cases now.

3.

(1) $\mathcal{M}_{S4}, w \vDash (\exists x(A(x)))^*$ | CP ass.

(2) $\mathcal{M}_{S4}, w \vDash \exists x \Box \Diamond (A(x))^*$ | 1., Def. 21.7

(3) $\forall v, R(w,v) : \mathcal{M}_{S4}, v \vDash \exists x \Diamond (A(x))^*$ | 2., Def. 14.7

(4) $\mathcal{M}_{S4}, w \vDash \exists x \Diamond (A(x))^*$ | 3., Def. 10.2a ($w = v'$)

(5) $\exists w^* : \mathcal{M}_{S4}, w^* \vDash \exists x (A(x))^*$ | 4., Def. 14.8

(6)

$\exists w^* : \exists a \in V(w^*) : \mathcal{M}_{S4}, w^* \vDash (A(a))^*$ | 5., Def. 14.6, EE

(7) $\mathcal{M}_{S4}, w \vDash (\exists x(A(x)))^* \Rightarrow$

$\exists w^* : \exists a \in V(w^*) : \mathcal{M}_{S4}, w^* \vDash (A(a))^*$ | 1.-6., CP

4.

(1) $\mathcal{M}_{S4}, w \nvDash (\forall x(A(x)))^*$ | CP ass.

(2) $\mathcal{M}_{S4}, w \nvDash \Box \Diamond \forall x((A(x))^*)$ | 1., Def. 21.6

(3) Not: $\forall w^* : \mathcal{M}_{S4}, w^* \vDash \Diamond \forall x((A(x))^*)$ | 2., Def. 14.7

(4) $\exists w^*$: Not: $\mathcal{M}_{S4}, w^* \vDash \Diamond \forall x((A(x))^*)$ | 3., Def. 14.7

(5)
$\exists w^* :$ Not: $\exists v, R(w^*, v) : \mathcal{M}_{S4}, v \vDash \forall x((A(x))^*)$ | 4., Def. 14.8

(6)
$\exists w^* : \forall v, R(w^*, v) :$ Not: $\mathcal{M}_{S4}, v \vDash \forall x((A(x))^*)$ | 5.

(7) $\forall v, R(w^*, v) :$ Not: $\mathcal{M}_{S4}, v \vDash \forall x((A(x))^*)$ | EE ass.

(8) Not: $\mathcal{M}_{S4}, w^* \vDash \forall x((A(x))^*)$ | 7., UI ($v \overset{!}{=} w^*$)

(9) Not: $\forall a \in V(w^*) : \mathcal{M}_{S4}, w^* \vDash (A(a))^*$ | 8., Def. 14.5

(10) $\exists a \in V(w^*) : \mathcal{M}_{S4}, w^* \nvDash (A(a))^*$ | 9.

(11) $\mathcal{M}_{S4}, w^* \nvDash ((A(a'))^*$ | EE ass.

(12) $\mathcal{M}_{S4}, w^* \vDash (\neg(A(a'))^*$ | 11., Lemma 1.2

(13) $\exists a \in V(w^*) : \mathcal{M}_{S4}, w^* \vDash (\neg(A(a))^*$ | 12., EI

(14) $\exists a \in V(w^*) : \mathcal{M}_{S4}, w^* \vDash (\neg(A(a))^*$ | 11.-13., EE

(15) $\exists w^* : \exists a \in V(w^*) : \mathcal{M}_{S4}, w^* \vDash (\neg(A(a))^*$ | 14., EI

(16) $\exists w^* : \exists a \in V(w^*) : \mathcal{M}_{S4}, w^* \vDash (\neg(A(a))^*$ | 7.-15., EE

(17) $\mathcal{M}_{S4}, w \nvDash (\forall x(A(x)))^* \Rightarrow$
$\exists w^* : \exists a \in V(w^*) : \mathcal{M}_{S4}, w^* \vDash (\neg(A(a))^*$ | 1.-16., CP

■

4.4 Complete sequences

Cohen (1966) defines the concept of a complete sequence, which Fitting (1970) strongly relies on for his proof. It is rather difficult to grasp intuitively. However, we can basically figure it to ourselves as a chain of worlds which only contains non-modal formulas, and whose order depends on the type of these formulas. We present a precise definition in the following:

Definition 23 (Complete sequence, ϑ)
Be $\mathcal{M}_{S4}$ an S4 model, where $w \in W$.
A complete sequence ϑ in W beginning from w, $\vartheta = \{w_1; w_2; w_3; \dots\}$, is constructed in the following way:

- $A_1; A_2; A_3; \dots$ be an enumeration of all formulas which do not contain any modal symbols.
- $w_1 \stackrel{!}{=} w$.
- Suppose we have defined w_n. Consider A_n: Distinction of cases:

  1. $\mathbb{V} w_n^*, R(w_n, w_n^*) : A_n \notin \hat{V}(w_n^*)) \Rightarrow w_{n+1} = w_n$.
  2. $\mathbb{E} w_n^*, R(w_n, w_n^*) : A_n \in \hat{V}(w_n^*) \Rightarrow$

     2.1

     (2.1.1)
     $(\mathcal{M}_{S4}, w_n^* \vDash A_n^*$ and A_n is *not* of the form $\exists x A(x) \Rightarrow$
     $w_{n+1} = w_n^*)$,

     (2.1.2)
     $(\mathcal{M}_{S4}, w_n^* \vDash A_n^*$ and A_n is of the form $\exists x A(x) \Rightarrow$
     $\mathbb{E} w_n^{*^*}, R(w_n^*, w_n^{*^*}) : \mathbb{E} a \in V(w_n^{*^*}) : \mathcal{M}_{S4}, w_n^{*^*} \vDash (A(a))^*)$[3] and
     $w_{n+1} = w_n^{*^*})$,

     2.2

     (2.2.1)
     $(\mathcal{M}_{S4}, w_n^* \nvDash A_n^*$ and A_n is *not* of the form $\forall x A(x) \Rightarrow$
     $\mathbb{E} w_n^{*^*}, R(w_n^*, w_n^{*^*}) : \mathcal{M}_{S4}, w_n^{*^*} \vDash (\neg A_n)^*$ and
     $w_{n+1} = w_n^{*^*})$,

     (2.2.2)
     $(\mathcal{M}_{S4}, w_n^* \nvDash A_n^*$ and A_n is of the form $\forall x A(x) \Rightarrow$

[3] by lemma 1.3

$\exists w_n^{*^*}, R(w_n^*, w_n^{*^*}) : \exists a \in V(w_n^{*^*}) : \mathcal{M}_{S4}, w_n^{*^*} \models (\neg A(a))^{*4}$ and $w_{n+1} = w_n^{*^*}$).

Remark 3 (¬ and ϑ)

$$\exists w_n^*, R(w_n, w_n^*) : A_n \in \hat{V}(w_n^*) \text{ and } \mathcal{M}_{S4}, w_n^* \not\models A_n^* \Rightarrow$$

$$\exists w_{n+1}, R(w_n^*, w_{n+1}) : \mathcal{M}_{S4}, w_{n+1} \models (\neg A_n)^*$$

follows via Def. 23.

Proof:

2.2.1 immediate by EE, CP

2.2.2

(1)	$\mathcal{M}_{S4}, w'_{n+1} \models (\neg A(a))^*$	| EE, CP
(2)	$\mathcal{M}_{S4}, w'_{n+1} \models \Box \Diamond \neg \Box \Diamond A'(a)$	| 1., Def. 21.2 + 22
(3)	Not: $\mathcal{M}_{S4}, w'_{n+1} \models \Box \Diamond \neg \Box \Diamond \forall x \Box \Diamond A'(x)$	| IP ass.
(4)	Not: $\forall v, R(w'_{n+1}, v)$: $\mathcal{M}_{S4}, v \models \Diamond \neg \Box \Diamond \forall x \Box \Diamond A'(x)$	| 3., Def. 14.7
(5)	$\exists v, R(w'_{n+1}, v)$:Not: $\mathcal{M}_{S4}, v \models \Diamond \neg \Box \Diamond \forall x \Box \Diamond A'(x)$	| 4.
(6)	Not: $\exists u, R(v', u)$: Not: $\mathcal{M}_{S4}, v' \models \Box \Diamond \forall x \Box \Diamond A'(x)$	| EE ass., Def. 14.8,1
(7)	$\mathcal{M}_{S4}, v' \models \Diamond \neg \Box \Diamond A'(a)$	| 2., Def. 14.7, UE
(8)	$\exists u, R(v', u)$: Not: $\mathcal{M}_{S4}, u \models \Box \Diamond A'(a)$	| 7., Def. 14.8
(9)	Not: $\mathcal{M}_{S4}, u' \models \Box \Diamond A'(a)$	| EE ass.
(10)	$\mathcal{M}_{S4}, u' \models \Box \Diamond \forall x \Box \Diamond A'(x)$	| 6., UE

[4] by lemma 1.4

(11) $\exists t, R(u', t)$: Not:

$\mathcal{M}_{S4}, t \vDash \Diamond A'(a)$ | 9., Def. 14.7

(12) Not: $\mathcal{M}_{S4}, t' \vDash \Diamond A'(a)$ | EE ass.

(13) $\exists s, R(t', s)$:

$\mathcal{M}_{S4}, s \vDash \forall x \Box \Diamond A'(x)$

by 10., Def. 14.7, UE, and Def. 14.8.

(14) $\mathcal{M}_{S4}, s' \vDash \forall x \Box \Diamond A'(x)$ | 13., EE ass.

(15) $\forall a \in V(s') : \mathcal{M}_{S4}, s' \vDash \Box \Diamond A'(x)$ | 14., Def. 14.5

(16) $\mathcal{M}_{S4}, s' \vDash \Box \Diamond A'(a)$ | 15., UE

(17) $\forall r, R(s', r) : \mathcal{M}_{S4}, r \vDash \Diamond A'(a)$ | 16., Def. 14.7

(18) $\mathcal{M}_{S4}, s' \vDash \Diamond A'(a)$ | 17., UE, Def. 10.2a+b

For the last step: $R(t', s')$ and $R(s', r) \Rightarrow R(t', r)$ as well as $R(t', t')$.

(19) C and Not: C | 12.,18., ECQ

(20) C and Not: C

by 14.-19., 12.-19., 9.-19., 6.-19., EE.

(21) $\mathcal{M}_{S4}, w'_{n+1} \vDash \Box \Diamond \neg \Box \Diamond \forall x \Box \Diamond A'(x)$ | 3.-20., IP

(22) $\mathcal{M}_{S4}, w'_{n+1} \vDash (\neg \forall x A(x))^*$ | 21., Def. 21.6,2 + 22

(23) $\mathcal{M}_{S4}, w'_{n+1} \vDash (\neg A_n)^*$ | 22., 2.2.2

(24)

$\exists w_n^*, R(w_n, w_n^*) : A_n \in \hat{V}(w_n^*)$ and

$\mathcal{M}_{S4}, w_n^* \nvDash A_n^* \Rightarrow$

$\exists w_{n+1}, R(w_n^*, w_{n+1}) : w_{n+1} \vDash (\neg A_n)^*$ | 1.-24., EI, CP

■

4.4.1 $\bar{\vartheta}$ and other derived concepts

We will now take a look at some derived concepts, which will be of importance to bring the concept of the complete sequence in workable form. We will need this because the notion of a complete sequence combined with the idea of a truth set, as we have seen it earlier, will be major building-blocks of the model-theoretic proof.

Definition 24 ($\bar{\vartheta}$)

$$\bar{\vartheta} := \{A | \exists w_n \in \vartheta : \mathcal{M}_{S4}, w_n \models A^*\}.$$

So, $\bar{\vartheta}$ is the set of all formulas, whose * translation is satisfied in at least one world of the complete sequence ϑ.

Definition 25 (Relevance to $\bar{\vartheta}$)
A formula A is relevant to $\bar{\vartheta}$ if

$$\text{ter}(A) \subseteq Par_{\bar{\vartheta}} = \bigcup_{w_n \in \vartheta} V(w_n).$$

Expressed differently, we call a formula relevant to a $\bar{\vartheta}$ if all of its terms are in the domain of the complete sequence ϑ, called $Par_{\bar{\vartheta}}$.

The goal of the following lemmata is to show that $\bar{\vartheta}$ is a classical truth set with respect to $Par_{\bar{\vartheta}}$.

4.5 $\bar{\vartheta}$ as a truth set

We will now relate the $\bar{\vartheta}$ and the notion of a truth set to one another.

4.5.1 Lemma 2: tertium non datur in $\bar{\vartheta}$

Lemma 2 (Tertium non datur in $\bar{\vartheta}$)
Be A, B formulas, which do not contain any modal symbols, $\Box$, $\Diamond$.

$$\forall A, A \text{ is relevant to } \bar{\vartheta} : ((A \in \bar{\vartheta} \text{ or } \neg A \in \bar{\vartheta}) \text{ and } \{A; \neg A\} \notin \bar{\vartheta}).$$

Proof:
Will will prove each conjunct separately.

1. Proof of $\{A^+; \neg A^+\} \notin \bar{\vartheta}$:

(1)	$\{A^+; \neg A^+\} \in \bar{\vartheta}$	| IP ass.
(2)	$A^+ \in \bar{\vartheta}$ and $\neg A^+ \in \bar{\vartheta}$	| 1., notation
(3)	$\exists w_n \in \vartheta : \mathcal{M}_{S4}, w_n \models A^*$ and $\exists w_m \in \vartheta : \mathcal{M}_{S4}, w_m \models (\neg A)^*$	| 2., Def. 24
(4)	Either $R(w_n, w_m)$ or $R(w_m, w_n)$	| 3., Def. 23
(5)	$R(w_n, w_m)$	| PC ass. 1
(6)	$\mathcal{M}_{S4}, w'_m \models A^*$	| 3., 5., Lemma 1.1, UE
(7)	Not: Not: $\mathcal{M}_{S4}, w'_m \models \Box \Diamond A'$	| IP ass.
(8)	$\exists w_m \in \vartheta : \mathcal{M}_{S4}, w_m \models (\neg A)^*$	| 3.
(9)	$\mathcal{M}_{S4}, w'_m \models (\neg A)^*$	| 8., EE ass.
(10)	$\mathcal{M}_{S4}, w'_m \models \Box \Diamond \neg \Box \Diamond A'$	| 9., Def. 21.2, Def. 22
(11)	$\forall v, R(w'_m, v) : \mathcal{M}_{S4}, v \models \Diamond \neg \Box \Diamond A'$	| 10., Def. 14.7
(12)	$\mathcal{M}_{S4}, w'_m \models \Diamond \neg \Box \Diamond A'$	| 11., Def. 10.2a
(13)	$\exists u, R(w'_m, u)$: Not: $\mathcal{M}_{S4}, u \models \Box \Diamond A'$	| 12., Def. 14.8,1
(14)	Not: $\mathcal{M}_{S4}, u' \models \Box \Diamond A'$	| EE ass.
(15)	$\exists t, R(u', t)$: Not: $\mathcal{M}_{S4}, t \models \Diamond A'$	| 14., Def. 14.7
(16)	Not: $\mathcal{M}_{S4}, t' \models \Diamond A'$	| EE ass.
(17)	$\forall s, R(t', s)$: Not: $\mathcal{M}_{S4}, s \models A'$	| 16., Def. 14.8
(18)	$\mathcal{M}_{S4}, w'_m \models \Box \Diamond A'$	| 7.
(19)	$\forall r, R(w'_m, r) : \mathcal{M}_{S4}, r \models \Diamond A'$	| 18., Def. 14.7
(20)	$\mathcal{M}_{S4}, t' \models \Diamond A'$	| 19., Def. 10.2b

The last step holds as $R(w'_m, u')$ and $R(u', t')$, so $R(w'_m, t')$ by transitivity.

(21)	$\exists l, R(t', l) : \mathcal{M}_{S4}, l \vDash A'$	| 20., Def. 14.8
(22)	$\mathcal{M}_{S4}, l' \vDash A'$	| EE ass.
(23)	Not: $\mathcal{M}_{S4}, l' \vDash A'$	| 17., UE
(24)	C and Not: C	| 22.,23., ECQ
(25)	C and Not: C	| 22.-, 16.-, 14.-, 9.-24., EE
(26)	Not: $\mathcal{M}_{S4}, w'_m \vDash \Box \Diamond A'$	| 7.-25., IP
(27)	Not: $\mathcal{M}_{S4}, w'_m \vDash A^*$	| 26., Def. 22
(28)	C and Not: C	| 6., 27., ECQ
(29)	$R(w_m, w_n)$	| PC ass. 2
(30)	$\mathcal{M}_{S4}, w'_n \vDash (\neg A)^*$	| 3., 29., Lemma 1.1, UE
(31)	$\exists w_n \in \vartheta : \mathcal{M}_{S4}, w_n \vDash A^*$	| 3.
(32)	$\mathcal{M}_{S4}, w'_n \vDash A^*$	| EE ass.
(33)	Not: Not: $\mathcal{M}_{S4}, w'_n \vDash \Box \Diamond A'$	| IP ass.
(34)	$\mathcal{M}_{S4}, w'_n \vDash \Box \Diamond \neg \Box \Diamond A'$	| 30., Def. 21.2, Def. 22
(35)	$\forall v, R(w'_n, v) :$	
	$\mathcal{M}_{S4}, v \vDash \Diamond \neg \Box \Diamond A'$	| 34., Def. 14.7
(36)	$\mathcal{M}_{S4}, w'_n \vDash \Diamond \neg \Box \Diamond A'$	| 35., Def. 10.2a
(37)	$\exists u, R(w'_n, u) :$ Not:	
	$\mathcal{M}_{S4}, u \vDash \Box \Diamond A'$	| 36., Def. 14.8,1
(38)	Not: $\mathcal{M}_{S4}, u' \vDash \Box \Diamond A'$	| EE ass.
(39)	$\exists t, R(u', t) :$ Not:	
	$\mathcal{M}_{S4}, t \vDash \Diamond A'$	| 38., Def. 14.7
(40)	Not: $\mathcal{M}_{S4}, t' \vDash \Diamond A'$	| EE ass.
(41)	$\forall s, R(t', s) :$ Not:	
	$\mathcal{M}_{S4}, s \vDash A'$	| 40., Def. 14.8
(42)	$\mathcal{M}_{S4}, w'_n \vDash \Box \Diamond A'$	| 33.
(43)	$\forall r, R(w'_n, r) :$	

	$\mathcal{M}_{S4}, r \vDash \Diamond A'$	| 42., Def. 14.7
(44)	$\mathcal{M}_{S4}, t' \vDash \Diamond A'$	| 43., Def. 10.2b

The last step holds as $R(w_n', u')$ and $R(u', t')$, so $R(w_n', t')$ by transitivity.

(45)	$\exists l, R(t', l) : \mathcal{M}_{S4}, l \vDash A'$	| 44., Def. 14.8
(46)	$\mathcal{M}_{S4}, l' \vDash A'$	| EE ass.
(47)	Not: $\mathcal{M}_{S4}, l' \vDash A'$	| 41., UE
(48)	C and Not: C	| 46.,47., ECQ
(49)	C and Not: C	| 46.-, 40.-, 38.-48., EE
(50)	Not: $\mathcal{M}_{S4}, w_n' \vDash \Box \Diamond A'$	| 33.-49., IP
(51)	Not: $\mathcal{M}_{S4}, w_n' \vDash A^*$	| 50., Def. 22
(52)	C and Not: C	| 31., 51., ECQ
(53)	C and Not: C	| 31.-52., ECQ
(54)	C and Not: C	| 5.-53., PC
(55)	$\{A^+; \neg A^+\} \notin \bar{\vartheta}$	| 1.-53., IP

2. Proof of $A^+ \in \bar{\vartheta}$ or $\neg A^+ \in \bar{\vartheta}$:

(56)	$A^+ \notin \bar{\vartheta}$	| CP ass.

Let A^+ be A_n for some n.

(57)	Not: $\exists w_n \in \vartheta : \mathcal{M}_{S4}, w_n \vDash A_n^*$	| 56., Def. 24
(58)	$\forall w_n \in \vartheta : \mathcal{M}_{S4}, w_n \nvDash A_n^*$	| 57.
(59)	$\exists w_n^*, R(w_n, w_n^*) : \mathcal{M}_{S4}, w_n \nvDash A_n^*$	| 58., UE, Def. 23

Since A_n is relevant to $\bar{\vartheta}$, we can say by 58., Def. 25, EE, and EI that:

(60) $\exists w_n^*, R(w_n, w_n^*) :$
$A_n \in \hat{V}(w_n^*)$ and $\mathcal{M}_{S4}, w_n \nvDash A_n^*$

(61) $\exists w_n^*, R(w_n, w_n^*) : \exists w_{n+1}, R(w_n^*, w_{n+1}) :$
$\mathcal{M}_{S4}, w_{n+1} \vDash (\neg A_n)^*$ | 60., Remark 3, EE, EI

As $w_n \in \vartheta$ (cf. 58./59.), and since $R(w_n, w_n^*), R(w_n^*, w_{n+1})$ and thus $R(w_n, w_{n+1})$ (by Def. 10.2b), we know that:

(62) $\exists w_{n+1} \in \vartheta : w_n + 1 \vDash (\neg A_n)^*$ | 61., Def. 23

(63) $\neg A_n \in \bar{\vartheta}$ | 62., Def. 24

(64) $\neg A^+ \in \bar{\vartheta}$ | 63., $A^+ = A_n$

(65) $A^+ \notin \bar{\vartheta} \Rightarrow \neg A^+ \in \bar{\vartheta}$ | 56.-64., CP

(66) $A^+ \in \bar{\vartheta}$ or $\neg A^+ \in \bar{\vartheta}$ | 65.

Now, we join our conjuncts and obtain:

(67) $\forall A, A$ is relevant to $\bar{\vartheta}$:
$((A \in \bar{\vartheta}$ or $\neg A \in \bar{\vartheta})$ and $\{A; \neg A\} \notin \bar{\vartheta})$. | 66., 55., CON, UI

■

4.5.2 Lemma 3: the connectives and $\bar{\vartheta}$

Lemma 3 (The connectives and $\bar{\vartheta}$)
Be A, B formulas, which do not contain any modal symbols, $\Box, \Diamond$.
$\forall A, A$ is relevant to $\bar{\vartheta} : \forall B, B$ is relevant to $\bar{\vartheta}$:

(1) $(A \vee B) \in \bar{\vartheta} \Leftrightarrow A \in \bar{\vartheta}$ or $B \in \bar{\vartheta}$,

(2) $(A \wedge B) \in \bar{\vartheta} \Leftrightarrow A \in \bar{\vartheta}$ and $B \in \bar{\vartheta}$,

(3) $(A \rightarrow B) \in \bar{\vartheta} \Leftrightarrow A \notin \bar{\vartheta}$ or $B \in \bar{\vartheta}$.

Proof:
We will prove 1.-3. separately:

1. First, we will prove left to right:

(1)	$(A^+ \vee B^+) \in \bar{\vartheta}$	| CP ass.
(2)	Not: $(A^+ \in \bar{\vartheta}$ or $B^+ \in \bar{\vartheta})$	| IP ass.
(3)	Not: $A^+ \in \bar{\vartheta}$ and Not: $B^+ \in \bar{\vartheta}$	| 2.
(4)	$\neg A^+ \in \bar{\vartheta}$ and $\neg B^+ \in \bar{\vartheta}$	| 3., Lemma 2
(5)	$\exists w_n \in \vartheta :$ $\mathcal{M}_{S4}, w_n \vDash (\neg A^+)^*$ and $\exists w_m \in \vartheta :$ $\mathcal{M}_{S4}, w_m \vDash (\neg B^+)^*$	| 4., Def. 24
(6)	Either $R(w_n, w_m)$ or $R(w_m, w_n)$	| 5., Def. 23
(7)	$\mathcal{M}_{S4}, w'_n \vDash (\neg A^+)^*$ and $\mathcal{M}_{S4}, w'_m \vDash (\neg B^+)^*$	| EE ass.
(8)	$R(w_m, w_n)$	| PC ass. 1
(9)	$\mathcal{M}_{S4}, w'_n \vDash (\neg A^+)^*$ and $\mathcal{M}_{S4}, w'_n \vDash (\neg B^+)^*$	| 7., 8., Lemma 1.1
(10)	$\mathcal{M}_{S4}, w'_n \vDash \Box \Diamond \neg \Box \Diamond (A^+)'$	| 9., Def. 21.2 + 22
(11)	$\mathcal{M}_{S4}, w'_n \vDash \Box \Diamond \neg \Box \Diamond (B^+)'$	| 9., Def. 21.2 + 22
(12)	Not: $\mathcal{M}_{S4}, w'_n \vDash \Box \Diamond \neg \Box \Diamond$ $(\Box \Diamond (A^+)' \vee \Box \Diamond (B^+)')$	| IP ass.
(13)	$\exists v, R(w'_n, v) :$ Not: $\mathcal{M}_{S4}, v \vDash \Diamond \neg \Box \Diamond$ $(\Box \Diamond (A^+)' \vee \Box \Diamond (B^+)')$	| 12., Def. 14.7
(14)	Not: $\mathcal{M}_{S4}, v' \vDash \Diamond \neg \Box \Diamond$ $(\Box \Diamond (A^+)' \vee \Box \Diamond (B^+)')$	| EE ass.
(15)	$\mathcal{M}_{S4}, v' \vDash \Diamond \neg \Box \Diamond (A^+)'$	| 10., Def. 14.7, UE

(16) $\exists u, R(v', u)$: Not:

$\mathcal{M}_{S4}, u \vDash \Box \Diamond (A^+)'$ | 15., Def. 14.8,1

(17) Not:

$\mathcal{M}_{S4}, u' \vDash \Box \Diamond (A^+)'$ | EE ass.

(18) $\exists t, R(u', t)$: Not:

$\mathcal{M}_{S4}, t \vDash \Diamond (A^+)'$ | 17. Def. 14.7

(19) Not:

$\mathcal{M}_{S4}, t' \vDash \Diamond (A^+)'$ | EE ass.

(20) $\forall s, (v', s)$: Not: Not:

$\mathcal{M}_{S4}, s \vDash \Box \Diamond$

$(\Box \Diamond (A^+)' \vee \Box \Diamond (B^+)')$ | 14., Def. 14.8

(21) $\mathcal{M}_{S4}, u' \vDash \Box \Diamond$

$(\Box \Diamond (A^+)' \vee \Box \Diamond (B^+)')$ | 20., UE

(22) $\mathcal{M}_{S4}, t' \vDash \Diamond$

$(\Box \Diamond (A^+)' \vee \Box \Diamond (B^+)')$ | 21., Def. 14.7, UE

(23) $\exists r, R(t', r)$:

$\mathcal{M}_{S4}, r \vDash \Box \Diamond (A^+)' \vee \Box \Diamond (B^+)'$ | 22., Def. 14.8

(24) $\mathcal{M}_{S4}, r' \vDash \Box \Diamond (A^+)' \vee \Box \Diamond (B^+)'$ | EE ass.

(25) $\mathcal{M}_{S4}, r' \vDash \Box \Diamond (A^+)'$ or

$\mathcal{M}_{S4}, r' \vDash \Box \Diamond (B^+)'$ | 24., Def. 14.3

(26) $\mathcal{M}_{S4}, r' \vDash \Box \Diamond (A^+)'$ | 25., PC ass. 1

(27) $\mathcal{M}_{S4}, r' \vDash \Diamond (A^+)'$ | 26., Def. 14.7, Def. 10.2a

(28) $\exists l, R(r', l)$:

$\mathcal{M}_{S4}, l \vDash (A^+)'$ | 27., Def. 14.8

(29) $\mathcal{M}_{S4}, l' \vDash (A^+)'$ | 27., Def. 14.8

(30) $\forall j, R(t', j)$Not:

$\mathcal{M}_{S4}, j \vDash (A^+)'$ | 19., Def. 14.8

(31) Not:

	$\mathcal{M}_{S4}, l' \vDash (A^+)'$	| EE ass.
(32)	C and Not: C	| 29., 31., ECQ
(33)	C and Not: C	| 31., 33., EE
(34)	$\mathcal{M}_{S4}, r' \vDash \Box\Diamond(B^+)'$	| 25., PC ass. 2
(35)	$\mathcal{M}_{S4}, r' \vDash \Diamond\neg\Box\Diamond(B^+)'$	| 11., Def. 14.7, Def. 10.2b
(36)	$\exists i, R(r', i)$: Not:	
	$\mathcal{M}_{S4}, i \vDash \Box\Diamond(B^+)'$	| 35., Def. 14.8,1
(37)	Not:	
	$\mathcal{M}_{S4}, i' \vDash \Box\Diamond(B^+)'$	| EE ass.
(38)	$\mathcal{M}_{S4}, h' \vDash \Diamond(B^+)'$	| 34., Def. 14.7, Def. 10.2b
(39)	Not:	
	$\mathcal{M}_{S4}, h' \vDash \Diamond(B^+)'$	| 37., Def. 14.7, UE
(40)	C and Not: C	| 38., 39., ECQ
(41)	C and Not: C	| 37-40., EE
(42)	C and Not: C	| 26.-41., PC
(43)	C and Not: C	

by 24.-, 19.-, 17.-, 14.-42., EE.

(44)	$\mathcal{M}_{S4}, w_n' \vDash \Box\Diamond\neg\Box\Diamond$	
	$(\Box\Diamond(A^+)' \vee \Box\Diamond(B^+)')$	| 12.- 43., IP
(45)	$\mathcal{M}_{S4}, w_n' \vDash (\neg(A^+ \vee B^+))^*$	| 44., Def. 21.4 + 22
(46)	$\exists w_n \in \vartheta$:	
	$\mathcal{M}_{S4}, w_n \vDash (\neg(A^+ \vee B^+))^*$	| 45., EI, Def. 25
(47)	$\neg(A^+ \vee B^+) \in \bar{\vartheta}$	| 46., Def. 24
(48)	$A^+ \vee B^+ \notin \bar{\vartheta}$	| 47., Lemma 2
(49)	C and Not: C	| 1., 48., ECQ

The second half of the proof by cases which we have begun in 8. is identical to the first half, except that we substitute all instances of n by m and vice

versa (only the pedant to 8. ought to be unchanged).
Hence, we can conclude:

(91) C and Not: C | 8.-90., PC

(92) C and Not: C | 7.-91., EE

(93) $A^+ \in \bar{\vartheta}$ or $B^+ \in \bar{\vartheta}$ | 2.-92., IP

(94) $(A^+ \vee B^+) \in \bar{\vartheta} \Rightarrow$
$A^+ \in \bar{\vartheta}$ or $B^+ \in \bar{\vartheta}$ | 1.-93., CP

Now, we will prove the other direction of the equivalence relation. Since the proof idea is very similar to the other direction, we will shorten this:

(95) $A^+ \in \bar{\vartheta}$ or $B^+ \in \bar{\vartheta}$ | CP ass.

(96) $\mathcal{M}_{S4}, w'_n \vDash (A^+)^*$ or
$\mathcal{M}_{S4}, w'_n \vDash (B^+)^*$ | 35., Def. 24, UE, PC, Lemma 2.

(97) $\mathcal{M}_{S4}, w'_n \vDash \Box\Diamond(A^+)'$ or
$\mathcal{M}_{S4}, w'_n \vDash \Box\Diamond(B^+)'$ | 36., Def. 21.1 + 22

(98) Not:
$\mathcal{M}_{S4}, w'_n \vDash$
$\Box\Diamond(\Box\Diamond(A^+)' \vee \Box\Diamond(B^+)')$ | IP ass.

(99) Not:
$\mathcal{M}_{S4}, v' \vDash$
$\Diamond(\Box\Diamond(A^+)' \vee \Box\Diamond(B^+)')$ | 98., Def. 14.7, EE ass.

(100) $\mathcal{M}_{S4}, w'_n \vDash \Box\Diamond(A^+)'$ | PC ass. 1, Def. 22

(101) $\mathcal{M}_{S4}, u' \vDash (A^+)'$ | 100., Def. 14.7,8, UE, EE ass.

(102) Not:
$\mathcal{M}_{S4}, u' \vDash$
$\Box\Diamond(A^+)' \vee \Box\Diamond(B^+)'$ | 99., Def. 14.8, EE ass.

(103) $\mathcal{M}_{S4}, u' \vDash$
$\neg\Box\Diamond(A^+)' \wedge \neg\Box\Diamond(B^+)'$ | 102., Def. 14.1

(104) $\mathcal{M}_{S4}, u' \vDash \neg \Box \Diamond (A^+)'$ | 103., Def. 14.2

(105) $\mathcal{M}_{S4}, t' \vDash \neg \Diamond (A^+)'$ | 104., Def. 14.1,7, EE ass.

(106) $\mathcal{M}_{S4}, t' \vDash \Diamond (A^+)'$ | 100., Def. 14.8, Def. 10.2a

(107) C and Not: C | 105., 106., ECQ, EEs

The second half of the proof by cases which we started in 100 is identical to the first half except that we substitute A for all instances of B and vice versa.

(116) C and Not: C | 97., 100-115., PC

(117) $\mathcal{M}_{S4}, w_n' \vDash$
$\Box \Diamond (\Box \Diamond (A^+)' \vee \Box \Diamond (B^+)')$ | 98.-116., EE, IP

(118) $\mathcal{M}_{S4}, w_n' \vDash (A^+ \vee B^+)^*$ | 117., Def. 21.3, Def. 22

(119) $(A^+ \vee B^+) \in \bar{\vartheta}$ | 118., Def. 24

(120) $A^+ \in \bar{\vartheta}$ or $B^+ \in \bar{\vartheta} \Rightarrow$
$(A^+ \vee B^+) \in \bar{\vartheta}$ | 95.-119., CP

(121) $(A^+ \vee B^+) \in \bar{\vartheta} \Leftrightarrow$
$A^+ \in \bar{\vartheta}$ or $B^+ \in \bar{\vartheta}$ | 94., 120., EQI

2. We start again left to right:

(1) $(A^+ \wedge B^+) \in \bar{\vartheta}$ | CP ass.

(2) $\mathcal{M}_{S4}, w_n' \vDash (A^+ \wedge B^+)^*$ | 1., Def. 24, EE ass.

We have assumed $R(w_m, w_n)$, analogously to case 1. (steps 6 to 8). The $R(w_n, w_m)$ case can also be obtained by the same strategy as in 1. step 50-90.

(3) $\mathcal{M}_{S4}, w_n' \vDash$
$\Box \Diamond (\Box \Diamond (A^+)' \wedge \Box \Diamond (B^+)')$ | 2., Def. 21.3 + 22

(4) Not:

$\mathcal{M}_{S4}, w'_n \vDash \Box\Diamond(A^+)' \wedge \Box\Diamond(B^+)'$ | IP ass.

(5) Not:

$\mathcal{M}_{S4}, w'_n \vDash \Box\Diamond(A^+)'$ or

Not: $\mathcal{M}_{S4}, w'_n \vDash \Box\Diamond(B^+)'$ | 4., Def. 14.2

(6) Not:

$\mathcal{M}_{S4}, w'_n \vDash \Box\Diamond(A^+)'$ | PC ass. 1

(7) Not:

$\mathcal{M}_{S4}, v' \vDash \Diamond(A^+)'$ | 6., Def. 14.7, EE ass.

(8) $\mathcal{M}_{S4}, v' \vDash \Diamond(\Box\Diamond(A^+)' \wedge \Box\Diamond(B^+)')$ | 3., Def. 14.7, UE

(9) $\mathcal{M}_{S4}, u' \vDash \Box\Diamond(A^+)' \wedge \Box\Diamond(B^+)'$ | 8., Def. 14.8, EE ass.

(10) $\mathcal{M}_{S4}, u' \vDash \Diamond(A^+)'$ | 9., Def. 14.2,7, Def 9.2a

(11) $\mathcal{M}_{S4}, u' \nvDash \Diamond(A^+)'$ | 6., Def. 14.7, Def 9.2b

(12) C and Not: C | 10., 11., ECQ, EEs

The second half of the proof by cases which we started in 6 is identical to the first half except that we substitute A for all instances of B and vice versa.

(20) $\mathcal{M}_{S4}, w'_n \vDash \Box\Diamond(A^+)' \wedge \Box\Diamond(B^+)'$ | 5.-19. PC, 4.-19., IP

(21) $\mathcal{M}_{S4}, w'_n \vDash (A^+)^*$ and

$\mathcal{M}_{S4}, w'_n \vDash (B^+)^*$ | 20., Def. 21.3, Def. 22

(22) $A^+ \in \bar{\vartheta}$ and $B^+ \in \bar{\vartheta}$ | 21., Def. 24

(23) $(A^+ \wedge B^+) \in \bar{\vartheta} \Rightarrow$

$A^+ \in \bar{\vartheta}$ and $B^+ \in \bar{\vartheta}$ | 2.-22., EE, 1.-22., CP

Now, we proceed right to left:

(24) $A^+ \in \bar{\vartheta}$ and $A^+ \in \bar{\vartheta}$ | CP ass.

(25) $\mathcal{M}_{S4}, w'_n \vDash (A^+)^*$ and

$\mathcal{M}_{S4}, w'_n \vDash (B^+)^*$ | 24., Def. 24, EE ass.

(26) $\mathcal{M}_{S4}, w'_n \vDash \Box\Diamond(A^+)' \wedge \Box\Diamond(B^+)'$ | 25., Def. 21.3, Def. 22

(27)	$\mathcal{M}_{S4}, w_n' \vDash \Box \Diamond (A^+)'$	| 26., Def. 14.2
(28)	$\mathcal{M}_{S4}, w_n' \vDash \Box \Diamond (B^+)'$	| 26., Def. 14.2
(29)	Not: $\mathcal{M}_{S4}, w_n' \vDash \Box\Diamond$ $(\Box \Diamond (A^+)' \wedge \Box \Diamond (A^+)')$	| IP ass.
(30)	Not: $\mathcal{M}_{S4}, v' \vDash \Diamond$ $(\Box \Diamond (A^+)' \wedge \Box \Diamond (A^+)')$	| 29., Def. 14.7, EE ass.
(31)	$\mathcal{M}_{S4}, v' \vDash \Diamond(A^+)'$	| 30., Def. 14.7, UE
(32)	$\mathcal{M}_{S4}, u' \vDash (A^+)'$	| 31., Def. 14.8, EE ass.
(33)	Not: $\mathcal{M}_{S4}, u' \vDash$ $(\Box \Diamond (A^+)' \wedge \Box \Diamond (B^+)')$	| 32., Def. 14.8, UE
(34)	$\mathcal{M}_{S4}, u' \nvDash \Box \Diamond (A^+)'$ or $\mathcal{M}_{S4}, u' \nvDash \Box \Diamond (B^+)'$	| 33., Def. 14.2
(35)	$\mathcal{M}_{S4}, u' \nvDash \Box \Diamond (A^+)'$	| PC ass. 1
(36)	$\mathcal{M}_{S4}, u' \nvDash \Diamond(A^+)'$	| 35., Def. 14.7, EE ass.
(37)	$\mathcal{M}_{S4}, u' \vDash \Diamond(A^+)'$	| 27., Def. 14.7, Def. 10.2b
(38)	C and Not: C	| 36., 37., ECQ, EE
(39)	$\mathcal{M}_{S4}, u' \nvDash \Box \Diamond (B^+)'$	| PC ass. 2
(40)	$\mathcal{M}_{S4}, u' \nvDash \Diamond(B^+)'$	| 39., Def. 14.7, EE ass.
(41)	$\mathcal{M}_{S4}, u' \vDash \Diamond(B^+)'$	| 28., Def. 14.7, Def. 10.2b
(42)	C and Not: C	| 36., 37., ECQ, EE
(43)	C and Not: C	| 34.-42., PC
(44)	$\mathcal{M}_{S4}, w_n' \vDash \Box\Diamond$ $(\Box \Diamond (A^+)' \wedge \Box \Diamond (A^+)')$	| 29.-42., EEs, IP
(45)	$\mathcal{M}_{S4}, w_n' \vDash (A^+ \wedge A^+)^*)$	| 44., Def. 21.3 + 22
(46)	$A^+ \wedge A^+ \in \bar{\vartheta}$	| 45., EI, Def. 24

(47) $A^+ \in \bar{\vartheta}$ and $A^+ \in \bar{\vartheta} \Rightarrow$
$A^+ \wedge A^+ \in \bar{\vartheta}$ | 24.-46., CP, EE

(48) $A^+ \wedge A^+ \in \bar{\vartheta} \Leftrightarrow$
$A^+ \in \bar{\vartheta}$ and $A^+ \in \bar{\vartheta}$ | 23., 47., EQI

3. For the third case, we will start left to right again:

(1) $(A^+ \rightarrow B^+) \in \bar{\vartheta}$ | CP ass.

(2) Not: $(A^+ \notin \bar{\vartheta}$ or $B^* \in \bar{\vartheta})$ | IP ass.

(3) $A^+ \in \bar{\vartheta}$ and $B^* \notin \bar{\vartheta}$ | 2.

(4) $\mathcal{M}_{S4}, w'_n \vDash (A^+)^*$ and
Not: $\mathcal{M}_{S4}, w'_{n-1} \vDash (B^+)^*$ | 3., Def. 24, EE ass., UE

(5) $\mathcal{M}_{S4}, w'_n \vDash \Box\Diamond(A^+)'$ | 4., Def. 22.

(6) $\mathcal{M}_{S4}, w'_n \vDash (\neg B^+)^*$ | 4., Def. 23, Def. 25, Rem. 3

(7) $\mathcal{M}_{S4}, w'_n \vDash \Box\Diamond\neg\Box\Diamond(B^+)'$ | 6., Def. 21.2, Def. 22

(8) Not: $\mathcal{M}_{S4}, w'_n \vDash \Box\Diamond\neg\Box\Diamond$
$(\Box\Diamond(A^+)' \rightarrow \Box\Diamond(B^+)')$ | IP ass.

(9) Not: $\mathcal{M}_{S4}, w'_n \vDash \Diamond\neg\Box\Diamond$
$(\Box\Diamond(A^+)' \rightarrow \Box\Diamond(B^+)')$ | 8., Def. 14.7, Def. 10.2a

(10) $\mathcal{M}_{S4}, w'_n \vDash \Box\Diamond$
$(\Box\Diamond(A^+)' \rightarrow \Box\Diamond(B^+)')$ | 9., Def. 14.8,1, Def. 10.2a

(11) $\mathcal{M}_{S4}, w'_n \vDash \Diamond$
$(\Box\Diamond(A^+)' \rightarrow \Box\Diamond(B^+)')$ | 10., Def. 14.7, Def. 10.2a

(12) $\mathcal{M}_{S4}, v' \vDash$
$(\Box\Diamond(A^+)' \rightarrow \Box\Diamond(B^+)')$ | 11., Def. 14.8, EE ass.

(13) Not: $\mathcal{M}_{S4}, v' \vDash \Box\Diamond(A^+)'$ or
$\mathcal{M}_{S4}, v' \vDash \Box\Diamond(B^+)'$ | 12., Def. 14.4

(14) Not: $\mathcal{M}_{S4}, v' \vDash \Box\Diamond(A^+)'$ | PC ass. 1

(15) Not: $\mathcal{M}_{S4}, u' \vDash \Diamond(A^+)'$ | 14., Def. 14.7, EE ass.

(16)	$\mathcal{M}_{S4}, u' \vDash \Diamond(A^{+})'$	| 5., Def. 14.7, Def. 10.2b
(17)	C and Not: C	| 15., 16., ECQ, EE
(18)	$\mathcal{M}_{S4}, v' \vDash \Box\Diamond(B^{+})'$	| PC ass. 2
(19)	$\mathcal{M}_{S4}, v' \vDash \Diamond\neg\Box\Diamond(B^{+})'$	| 7., Def. 14.7, UE
(20)	Not: $\mathcal{M}_{S4}, s' \vDash \Diamond(B^{+})'$	| 20., Def. 14.1,8, EE ass.
(21)	$\mathcal{M}_{S4}, v' \vDash \Diamond(B^{+})'$	| 18., Def. 14.7
(22)	C and Not: C	| 20., 21., ECQ, EE
(23)	C and Not: C	| 13.-22., PC
(24)	$\mathcal{M}_{S4}, w'_n \vDash \Box\Diamond\neg\Box\Diamond$ $(\Box\Diamond(A^{+})' \rightarrow \Box\Diamond(B^{+})')$	| 8.-23., EE, IP
(25)	$\mathcal{M}_{S4}, w'_n \vDash$ $(\neg(A^{+} \rightarrow B^{+}))^{*}$	| 24., Def. 21.5 + 22
(26)	$(\neg(A^{+} \rightarrow B^{+})) \in \bar{\vartheta}$	| 25., EI, Def. 24
(27)	$(A^{+} \rightarrow B^{+}) \notin \bar{\vartheta}$	| 26., Lemma 2
(28)	C and Not: C	| 1., 27., ECQ
(29)	$A^{+} \notin \bar{\vartheta}$ or $B^{*} \in \bar{\vartheta}$	| 2.-27., EEs, IP
(30)	$(A^{+} \rightarrow B^{+}) \in \bar{\vartheta} \Rightarrow$ $A^{+} \notin \bar{\vartheta}$ or $B^{*} \in \bar{\vartheta}$	| 1.-28., CP

Right to left:

(31)	$A^{+} \notin \bar{\vartheta}$ or $B^{*} \in \bar{\vartheta}$	| CP ass.
(32)	Not: $\mathcal{M}_{S4}, w'_n \vDash (A^{+})^{*}$ or $\mathcal{M}_{S4}, w'_n \vDash (B^{+})^{*}$	| 31., Def. 24, EE ass.

We have assumed $R(w_m, w_n)$, analogously to case 1. (steps 6 to 8). The $R(w_n, w_m)$ case can also be obtained by the same strategy as in 1., steps 50-90.

(33)	$\mathcal{M}_{S4}, w'_n \vDash (\neg A^{+})^{*}$ or	

	$\mathcal{M}_{S4}, w'_n \models (B^+)^*$	| 32., Def. 23. + 25, Rem 3
(34)	$\mathcal{M}_{S4}, w'_n \models \Box\Diamond\neg\Box\Diamond(A^+)'$ or	
	$\mathcal{M}_{S4}, w'_n \models \Box\Diamond(B^+)'$	| 33., Def. 21.2 + 22
(35)	Not: $\mathcal{M}_{S4}, w'_n \models \Box\Diamond$	
	$(\Box\Diamond(A^+)' \to \Box\Diamond(B^+)')$	| IP ass.
(36)	Not: $\mathcal{M}_{S4}, v' \models \Diamond$	
	$(\Box\Diamond(A^+)' \to \Box\Diamond(B^+)')$	| 35., Def. 14.7, EE ass.
(37)	$\mathcal{M}_{S4}, w'_n \models \Box\Diamond\neg\Box\Diamond(A^+)'$	| PC ass. 1
(38)	$\mathcal{M}_{S4}, v' \models \Diamond\neg\Box\Diamond(A^+)'$	| 37., Def. 14.7, UE
(39)	Not: $\mathcal{M}_{S4}, u' \models \Box\Diamond(A^+)'$	| 38., Def. 14.8,1, EE ass.
(40)	Not: $\mathcal{M}_{S4}, u' \models$	
	$(\Box\Diamond(A^+)' \to \Box\Diamond(B^+)')$	| 36., Def. 14.8, UE
(41)	$\mathcal{M}_{S4}, u' \models \Box\Diamond(A^+)'$ and	
	Not: $\mathcal{M}_{S4}, u' \models \Box\Diamond(B^+)'$	| 40., Def. 14.4
(42)	$\mathcal{M}_{S4}, u' \models \Box\Diamond(A^+)'$	| 41.
(43)	C and Not: C	| 39., 41., ECQ, EE
(44)	$\mathcal{M}_{S4}, w'_n \models \Box\Diamond(B^+)'$	| PC ass. 2
(45)	$\mathcal{M}_{S4}, t' \models \Box\Diamond(A^+)'$ and	
	Not: $\mathcal{M}_{S4}, t' \models \Box\Diamond(B^+)'$	| 36., UE, Def. 14.4
(46)	Not: $\mathcal{M}_{S4}, t' \models \Box\Diamond(B^+)'$	| 45.
(47)	Not: $\mathcal{M}_{S4}, t' \models \Diamond(B^+)'$	| 46., Def. 14.7, EE ass.
(48)	$\mathcal{M}_{S4}, t' \models \Diamond(B^+)'$	| Def. 14.7., Def. 10.2b
(49)	C and Not: C	| 44., 48., ECQ
(50)	C and Not: C	| 34., 37.-49., EE, PC
(51)	$\mathcal{M}_{S4}, w'_n \models \Box\Diamond$	
	$(\Box\Diamond(A^+)' \to \Box\Diamond(B^+)')$	| 35.-50., EE, IP
(52)	$\mathcal{M}_{S4}, w'_n \models (A^+ \to B^+)^*$	| 51., Def. 21.5 + 22
(53)	$(A^+ \to B^+) \in \bar{\vartheta}$	| 52., EI, Def. 24

(54)	$A^+ \notin \bar{\vartheta}$ or $B^* \in \bar{\vartheta} \Rightarrow$	
	$(A^+ \rightarrow B^+) \in \bar{\vartheta}$	| 52., EI, Def. 24
(55)	$(A^+ \rightarrow B^+) \in \bar{\vartheta} \Leftrightarrow$	
	$A^+ \notin \bar{\vartheta}$ or $B^* \in \bar{\vartheta}$	| 30.54., EQI

Therefore, by CON and UI:

$$\mathbb{\forall} A, A \text{ is relevant to } \bar{\vartheta} : \mathbb{\forall} B, B \text{ is relevant to } \bar{\vartheta} :$$

(1.121.) $(A \lor B) \in \bar{\vartheta} \Leftrightarrow A \in \bar{\vartheta}$ or $B \in \bar{\vartheta}$,

(2.48.) $(A \land B) \in \bar{\vartheta} \Leftrightarrow A \in \bar{\vartheta}$ and $B \in \bar{\vartheta}$,

(3.55.) $(A \rightarrow B) \in \bar{\vartheta} \Leftrightarrow A \notin \bar{\vartheta}$ or $B \in \bar{\vartheta}$.

■

4.5.3 Lemma 4: the quantifiers and $\bar{\vartheta}$

Lemma 4 (the quantifiers and $\bar{\vartheta}$)
Be $A(x)$ a formula, which does not contain any modal symbols, $\Box, \Diamond$, and which is relevant to $\bar{\vartheta}$.

(1) $(\forall x(A(x))) \in \bar{\vartheta} \Leftrightarrow \mathbb{\forall} a \in Par_{\bar{\vartheta}} : A(a) \in \bar{\vartheta}$.

(2) $(\exists x(A(x))) \in \bar{\vartheta} \Leftrightarrow \mathbb{\exists} a \in Par_{\bar{\vartheta}} : A(a) \in \bar{\vartheta}$.

Proof:
We will prove 1. and 2. separately.

1. We will start with the left to right case:

(1)	$(\forall x(A(x))) \in \bar{\vartheta}$	| CP ass.
(2)	$\mathcal{M}_{S4}, w'_n \vDash (\forall x(A(x)))^*$	| 1., Def. 24
(3)	$\mathcal{M}_{S4}, w'_n \vDash \Box \Diamond \forall x(\Box \Diamond A'(x))$	| 2., Def. 21.6 + 22
(4)	Not: $\mathcal{M}_{S4}, w'_n \vDash \Box \Diamond A'(a')$	| IP ass.

(5)	Not: $\mathcal{M}_{S4}, v' \vDash \Diamond A'(a')$	| 4., Def. 14.7, EE ass.
(6)	$\mathcal{M}_{S4}, v' \vDash \Diamond \forall x(\Box \Diamond A'(x))$	| 3., Def. 14.7, UE
(7)	$\mathcal{M}_{S4}, u' \vDash \forall x(\Box \Diamond A'(x))$	| 6., Def. 14.8, EE ass.
(8)	$\mathcal{M}_{S4}, u' \vDash \Box \Diamond A'(a')$	| 7., Def. 14.5, UE
(9)	$\mathcal{M}_{S4}, u' \vDash \Diamond A'(a')$	| 8., Def. 14.7, Def. 10.2a
(10)	$\mathcal{M}_{S4}, t' \vDash A'(a')$	| 8., Def. 14.8, EE ass.
(11)	Not: $\mathcal{M}_{S4}, t' \vDash A'(a')$	| 5., Def. 14.8, Def. 10.2b
(12)	C and Not: C	| 10., 11., ECQ
(13)	$\mathcal{M}_{S4}, w'_n \vDash \Box \Diamond A'(a')$	| 4.-12., EEs, IP
(14)	$\mathcal{M}_{S4}, w'_n \vDash (A(a'))^*$	| 13., Def. 22
(15)	$A(a') \in \bar{\vartheta}$	| 14., Def. 24
(16)	$\mathbb{V}a \in Par_{\bar{\vartheta}} : A(a) \in \bar{\vartheta}$	| 15., Def. 25, UI
(17)	$(\forall x(A(x))) \in \bar{\vartheta} \Rightarrow$ $\mathbb{V}a \in Par_{\bar{\vartheta}} : A(a) \in \bar{\vartheta}$	| 1.-16., CP

Now, we will consider the right to left case in the following:

(18)	$\mathbb{V}a \in Par_{\bar{\vartheta}} : A(a) \in \bar{\vartheta}$	| CP ass.
(19)	$(\forall x(A(x))) \notin \bar{\vartheta}$	| IP ass.

Let $\forall x(A(x))$ be A_n for some n.

(20)	Not: $\mathcal{M}_{S4}, w'_n \vDash (A_n)^*$	| 19., Def. 24, EE ass.
(21)	Not: $\mathcal{M}_{S4}, w'_n \vDash (\forall x(A(x)))^*$	| 20.
(22)	$\mathbb{E}w'_{n+1}, R(w'_n, w'_{n+1}) : \mathbb{E}a \in V(w'_{n+1}) :$ $\mathcal{M}_{S4}, w'_{n+1} \vDash (\neg(A(a))^*$	| 21., Def. 23.2.2.2.
(23)	$\mathbb{E}a \in V(w'_{n+1}) : (\neg(A(a)) \in \bar{\vartheta}$	| 22., Def. 24
(24)	$(\neg(A(a')) \in \bar{\vartheta}$	| EE ass.
(25)	$(A(a')) \notin \bar{\vartheta}$	| 24., Lemma 2
(26)	$(A(a')) \in \bar{\vartheta}$	| 18., UE

(27)	C and Not: C	$\mid$ 25., 26., ECQ
(28)	$(\forall x(A(x))) \in \bar{\vartheta}$	$\mid$ 19.-27., EEs, IP
(29)	$\mathbb{V}a \in Par_{\bar{\vartheta}} : A(a) \in \bar{\vartheta} \Rightarrow$	
	$(\forall x(A(x))) \in \bar{\vartheta}$	$\mid$ 18.-28., CP
(30)	$(\forall x(A(x))) \in \bar{\vartheta} \Leftrightarrow$	
	$\mathbb{V}a \in Par_{\bar{\vartheta}} : A(a) \in \bar{\vartheta}$	$\mid$ 17., 29., EQI

2. Now, we will prove 2., starting left to right.

(1)	$(\exists x(A(x))) \in \bar{\vartheta}$	$\mid$ CP ass.

Let $\exists x(A(x))$ be A_n for some n.

(2)	Not: $\mathcal{M}_{S4}, w'_n \vDash A^*_n$	$\mid$ IP ass.
(3)	Not: $\mathcal{M}_{S4}, w'_n \vDash (\exists x(A(x)))^*$	$\mid$ 2.
(4)	$\mathbb{E}w'_{n+1} : \mathcal{M}_{S4}, w'_{n+1} \vDash (\neg\exists x(A(x)))^*$	$\mid$ 3., Def. 23.2.2.1
(5)	$(\neg\exists x(A(x))) \in \bar{\vartheta}$	$\mid$ 4., Def. 24
(6)	$(\exists x(A(x))) \notin \bar{\vartheta}$	$\mid$ 5., Lemma 2
(7)	$\mathcal{M}_{S4}, w'_n \vDash (\exists x(A(x)))^*$	$\mid$ 1., 6., ECQ, 2.-6., IP
(8)	$\mathbb{E}w'_{n+1} : \mathbb{E}a \in V(w'_{n+1}) :$	
	$\mathcal{M}_{S4}, w'_n \vDash (A(a))^*$	$\mid$ 7., Def. 23.2.1.2
(9)	$\mathbb{E}a \in V(w'_{n+1}) : (A(a)) \in \bar{\vartheta}$	$\mid$ 8., EE, EI, Def. 24
(10)	$\mathbb{E}a \in Par_{\bar{\vartheta}} : (A(a)) \in \bar{\vartheta}$	$\mid$ 9., Def. 23 + 25 + 13
(11)	$(\exists x(A(x))) \in \bar{\vartheta} \Rightarrow$	
	$\mathbb{E}a \in Par_{\bar{\vartheta}} : (A(a)) \in \bar{\vartheta}$	$\mid$ 1.-11., CP

Finally, we will prove the right to left case:

(12)	$\mathbb{E}a \in Par_{\bar{\vartheta}} : A(a) \in \bar{\vartheta}$	$\mid$ CP ass.
(13)	$\mathcal{M}_{S4}, w'_n \vDash (A(a'))^*$	$\mid$ 12., EE ass.s, Def. 24

(14)	$\mathcal{M}_{S4}, w'_n \vDash \Box\Diamond A'(a')$	| 13., Def. 22
(15)	Not: $\mathcal{M}_{S4}, w'_n \vDash \Box\Diamond \exists x(\Box\Diamond A'(x))$	| IP ass.
(16)	Not: $\mathcal{M}_{S4}, v' \vDash \Diamond\exists x(\Box\Diamond A'(x))$	| 15., Def. 14.7, EE ass.
(17)	Not: $\mathcal{M}_{S4}, v' \vDash \exists x(\Box\Diamond A'(x))$	| 16., Def. 14.8, Def. 10.2a
(18)	Not: $\mathcal{M}_{S4}, v' \vDash \Box\Diamond A'(a')$	| 17., Def. 14.6, EE ass.
(19)	Not: $\mathcal{M}_{S4}, u' \vDash \Diamond A'(a')$	| 18., Def. 14.7, EE ass.
(20)	$\mathcal{M}_{S4}, u' \vDash \Diamond A'(a')$	| 14., Def. 14.7, Def. 10.2b
(21)	C and Not: C	| 19., 20., ECQ
(22)	$\mathcal{M}_{S4}, w'_n \vDash \Box\Diamond \exists x(\Box\Diamond A'(x))$	| 15.-21., EEs, IP
(23)	$\mathcal{M}_{S4}, w'_n \vDash (\exists x(A(x))^*$	| 22., Def. 21.7 + 22
(24)	$(\exists x(A(x)) \in \bar{\vartheta}$	| 23., EI, Def. 24
(25)	$\mathbb{E}a \in Par_{\bar{\vartheta}} : A(a) \in \bar{\vartheta} \Rightarrow$	
	$(\exists x(A(x)) \in \bar{\vartheta}$	| 12.-24., EEs, IP
(26)	$(\exists x(A(x))) \in \bar{\vartheta} \Leftrightarrow$	
	$\mathbb{E}a \in Par_{\bar{\vartheta}} : (A(a)) \in \bar{\vartheta}$	| 11., 25., EQI

Hence:

$$A(x) \text{ is relevant to } \bar{\vartheta} \Rightarrow$$

(1.30.) $$(\forall x(A(x))) \in \bar{\vartheta} \Leftrightarrow \mathbb{V}a \in Par_{\bar{\vartheta}} : A(a) \in \bar{\vartheta}.$$

(2.26.) $$(\exists x(A(x))) \in \bar{\vartheta} \Leftrightarrow \mathbb{E}a \in Par_{\bar{\vartheta}} A(a) \in \bar{\vartheta}.$$

■

Observation 1 ($\bar{\vartheta}$ as a classical truth set)
Let us briefly recapitulate what we have just proven. All formulas which are used in Lemma 2-4 do not contain any modal symbols, $\Box, \Diamond$, and are relevant to $\bar{\vartheta}$. The latter, however, means by Def. 25 that the set of terms of these formulas is a subset of the set of all terms of all ϑ worlds. Consider Def. 19. We can see that both pre-conditions 1. and 2. (where $Par = Par_{\bar{\vartheta}}$) are satisfied by all of the lemmata. Furthermore, Lemma 2 satisfies Def. 19.1, Lemma 3 satisfies Def.

19.2-4, and Lemma 4 satisfies Def. 19.5+6, for $Par = Par_{\bar{\vartheta}}$ and $\bar{\vartheta} = \mathcal{T}_{Par}$. Hence, if follows from Lemma 2-4 that $\bar{\vartheta}$ is a truth set with respect to $Par_{\bar{\vartheta}}$.

4.6 Theorem 2: the semantic embedding

Theorem 2 (Soundness of the * embedding, semantically)
$\forall A, A$ does not contain any modal symbols, $\Box, \Diamond$:

$$\vDash_C A \Rightarrow \vDash_{S4} A^*.$$

Proof:

(1)	$\vDash_C A$	| CP ass.
(2)	Not: $\vDash_{S4} A^*$	| IP ass.
(3)	Not: $\forall \mathcal{M}_{S4} : \forall w \in W, A^* \in \hat{V}(w) :$ $\mathcal{M}_{S4}, w \vDash A^*$	| 2., Def. 16 +15
(4)	$\exists \mathcal{M}_{S4} : \exists w \in W, A^* \in \hat{V}(w) :$ $\mathcal{M}_{S4}, w \nvDash A^*$	| 3.
(5)	$\exists w^* : \mathcal{M}'_{S4}, w^* \vDash (\neg A)^*$	| 4., Lemma 1.2, EEs, UE, EI

Consider a complete sequence, ϑ, beginning from w^*, constructed according to Def. 23.

(6)	$(\neg A) \in \bar{\vartheta}$	| 5., Def. ϑ Def. 24
(7)	$A \notin \bar{\vartheta}$	| 6., Lemma 2
(8)	$\nvDash_C A$	

follows from Def. 23 + Def. 25 (the fact that $\text{ter}(A) \subset Par_{\bar{\vartheta}}$), Observation 1 + Lemma 2-4 (the fact that $\bar{\vartheta}$ is a truth set with respect to $Par_{\bar{\vartheta}}$), and Def. 20.

(9)	$\vDash_{S4} A^*$	| 1.-7., ECQ, EE, IP
(10)	$\vDash_C A \Rightarrow \vDash_{S4} A^*$	| 1.-8., CP

■

This theorem says that all non-modal formulas are classically valid only if their * translation is valid in S4.

By this we conclude our reconstruction of Melvin Fitting's proof of a model-theoretic embedding of classical logic into S4, using the * translation and the notion of complete sequences from Cohen (1966). Now, we will proceed to the proof-theoretic part of this thesis.

5 Preliminaries 3: syntax

According to Fitting (1970, p. 530), a proof-theoretic proof of his embedding is possible. However, Fitting does not provide such a proof. Now, it will be our task to fill this gap. However, we will not be able to provide a spelled-out proof in this thesis due to space constraints. We will nonetheless try to investigate two properties in particular, (proof-theoretic) soundness and faithfulness of the embedding (cf. Troelstra and Schwichtenberg, 2000, pp. 288-291). We want to argue that a proof-theoretic soundness proof will work well for the * embedding. Nonetheless, we will raise doubts about whether a proof for faithfulness can be provided at all. This concern will be related to the modest resources of S4 to deal with the $\Diamond$ operator, the limited distributivity of the underlying translation, and a problem, which could be referred to as 'meta-sklemisation', as we will see.

However, before we can start to prove, we will first proceed to set up the deductive systems involved.
Here, our definitions will proceed the other way around than in the semantics section. We will first introduce a Gentzen-style sequent calculus for classical logic and we will then add rules for the modal operators to obtain full S4.
The (standard) notion of a sequent (not to be mistaken with the concept of a complete sequence as we used it earlier) is taken from on Negri and von Plato (2001, pp. 13-15), where $\Rightarrow$ will serve as our symbol for the sequent arrow.

5.1 Derivations and proofs

We define, inspired by Gratzl (2018a):

Definition 26 (Derivation)
Let Φ, Ψ be sets of formulas of $\mathcal{L}_C$.
Derivability of Ψ from Φ, $\Phi \vdash_{\mathbf{G3c}} \Psi$, is defined as follows:

$$\Phi \vdash \Psi :\Leftrightarrow \exists A \in \Psi : \exists B_1, \ldots, B_n \in \Phi : B_1, \ldots, B_n \Rightarrow A \text{ can be derived in } \mathbf{G3c}.$$

Or, expressed differently:

$$\Phi \vdash \Psi \Leftrightarrow \bigwedge B \in \Phi \Rightarrow \bigvee A \in \Psi$$

is derivable in **G3c**.

Remark 4 (Proof)
We will write **G3c** $\vdash \Phi$[5] as an abbreviation for $\varnothing \vdash_{\mathbf{G3c}} \Phi$.

We can easily obtain the respective definition for S4 if we substitute 'S4' for 'C' and '**G3s**' for '**G3c**'.

5.2 G3c

What exactly is **G3c**? It is a classical Gentzen system, in which weakening and contraction have been implemented implicitly into the system (unlike for example G1 systems, where they are explicit rules) (cf. Negri and von Plato, 2001, p. 28). We have imported the following definitions from Troelstra and Schwichtenberg (2000, p. 77). We will use the notation and definitions for substitution from Gratzl (2018a, pp. 44-50), where $A[t/x]$ stands for the formula which is identical to A except that we have uniformly substituted the therm t for all free occurrences of the variable x. Technically speaking, we also need to change our languages slightly by adding the $\bot$ symbol to them, which will represent a contradictory formula. An example for the necessary changes our definitions need to undergo due to that modification can be found in Negri and von Plato (2001, pp. 61-67). We also need to add the following (rather unsurprising) case to our * translation: $\bot^* := \bot$.
Having this in mind, we can start defining:

Definition 27 (G3c)
Be A, B formulas and Γ, Δ sets of formulas.

The axioms:
Ax A is atomic $\Rightarrow A, \Gamma \Rightarrow \Delta, A$ L$\bot$ $\bot, \Gamma \Rightarrow \Delta$

[5]We can express this verbally as '**G3c** proves Φ' or 'Φ is provable in **G3c**'.

The logical rules:

$$\text{L}\wedge\ \frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \qquad \text{R}\wedge\ \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}$$

$$\text{L}\vee\ \frac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \qquad \text{R}\vee\ \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}$$

$$\text{L}\rightarrow\ \frac{\Gamma \Rightarrow \Delta, A \qquad B, \Gamma \Rightarrow \Delta}{A \rightarrow B, \Gamma \Rightarrow \Delta} \qquad \text{R}\rightarrow\ \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \rightarrow B}$$

$$\text{L}\forall\ \frac{\forall x A, A[t/x], \Gamma \Rightarrow \Delta}{\forall x A, \Gamma \Rightarrow \Delta} \qquad \text{R}\forall^{!}\ \frac{\Gamma \Rightarrow \Delta, A[y/x]}{\Gamma \Rightarrow \Delta, \forall x A}$$

$$\text{L}\exists^{!}\ \frac{A[y/x], \Gamma \Rightarrow \Delta}{\exists x A, \Gamma \Rightarrow \Delta} \qquad \text{R}\exists\ \frac{\Gamma \Rightarrow \Delta, A[t/x], \exists x A}{\Gamma \Rightarrow \Delta, \exists x A}$$

In the $^{!}$ cases, the additional constraint that $\forall D, D$ is a formula below the inference line: $y \notin \text{free}(D)$ must be respected.

5.3 G3s

Now, we can easily introduce a proof system for S4. Our presentation is based on Troelstra and Schwichtenberg (2000, p. 287).

Definition 28 (G3s)
G3s = **G3c** +

$$\text{L}\Box\ \frac{\Gamma, A, \Box A \Rightarrow \Delta}{\Gamma, \Box A \Rightarrow \Delta} \qquad \text{R}\Box\ \frac{\Box\Gamma \Rightarrow A, \Diamond\Delta}{\Gamma', \Box\Gamma \Rightarrow \Box A, \Delta}$$

$$\text{L}\Diamond\ \frac{\Box\Gamma, A \Rightarrow \Diamond\Delta}{\Gamma', \Box\Gamma, \Diamond A \Rightarrow \Diamond\Delta, \Delta'} \qquad \text{R}\Diamond\ \frac{\Gamma \Rightarrow A, \Diamond A, \Delta}{\Gamma \Rightarrow \Diamond A, \Delta}$$

In both systems, **G3c** and **G3s**, weakening, contraction, and cut are permissible but eliminable. This means that, although they can be proven within the systems, they are not necessary for any of the proofs in these systems (cf. Troelstra and Schwichtenberg, 2000, pp. 66-85, 284-288). We will state these redundant rules in the following:

$$\text{LW}\ \frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad \text{RW}\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow A, \Delta}$$

$$\text{LC}\ \frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad \text{RC}\ \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}$$

$$\text{Cut}\ \frac{\Gamma \Rightarrow \Delta, A \qquad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

Both systems also allow for permutations on any side of the sequent arrow.

As we have the proof systems we need at hand, we can now sketch a syntactic proof of the embedding theorem. We will only do this in a rather rough manner and leave a more detailed presentation to another paper. Some of the main ideas for the proof have been inspired by the discussion of the $^\circ$ embedding in Troelstra and Schwichtenberg (2000, pp. 288-291).

6 The embedding 2: proof-theoretically (sketch)

6.1 Soundness

The first question, in which we are interested, is whether the * embedding is sound, i.e. deducability-preserving.

Definition 29 (Soundness of an ^{e} embedding)
An ^{e} embedding, based on an ^{e} translation, which translates each set of formulas, Γ, Δ, of a language $\mathcal{L}_1$ into a set of formulas, Γ^e, Δ^e, of the language $\mathcal{L}_2$, is sound :⇔

$$\mathbf{PS1} \vdash \Gamma \Rightarrow \Delta \Rightarrow \mathbf{PS2} \vdash \Gamma^e \Rightarrow \Delta^e$$

holds for the corresponding proof systems **PS1** and **PS2**.

If we apply this to our concrete case, we obtain the following, which is the proof-theoretic counterpart to our model-theoretic Theorem 2.

Theorem 3 (Soundness of the * embedding, proof-theoretically)
The * embedding is sound ⇒

$$\mathbf{G3c} \vdash \Gamma \Rightarrow \Delta \Rightarrow \mathbf{G3s} \vdash \Gamma^* \Rightarrow \Delta^*.$$

Proof sketch
How shall we prove this? We will employ an induction on the length of a derivation in **G3c**. The induction start is the * translation of the axiom. The inductive hypothesis can be formulated as follows: 'If the * translation of a sequent with n steps can be derived from the translated axiom, such a sequent will also hold for the $n + 1^{th}$ step'. As the inductive step, we will now need to prove for each **G3c** rule that the * translation of its conclusion can be derived from the * translation of its premises, in **G3s**.

Example 3 (Two cases of the induction step)
We want to give exemplary derivations for two typical cases of the induction step, L→ and R→, in order to illustrate the procedure.
L→:

$$\dfrac{\dfrac{\dfrac{\Gamma^* \Rightarrow \Delta^*, A^* \qquad B^*, \Gamma^* \Rightarrow \Delta^*}{\Gamma^*, A^* \to B^* \Rightarrow \Delta^*}\,\text{L}{\to}}{\Diamond(A^* \to B^*), \Box\Diamond(A^* \to B^*), \Gamma^* \Rightarrow \Delta^*}\,\text{L}\Diamond}{\Box\Diamond(A^* \to B^*), \Gamma^* \Rightarrow \Delta^*}\,\text{L}\Box$$

R→:

$$\dfrac{\dfrac{\dfrac{\Gamma^*, A^* \Rightarrow B^*, \Delta^*}{\Gamma^* \Rightarrow A^* \to B^*, \Diamond(A^* \to B^*), \Delta^*}\,\text{R}{\to},\ \text{RW}}{\Gamma^* \Rightarrow \Diamond(A^* \to B^*), \Delta^*}\,\text{R}\Diamond}{\Gamma^* \Rightarrow \Delta^*, \Box\Diamond(A^* \to B^*)}\,\text{R}\Box$$

6.2 Faithfulness

Thus, we have seen that the * embedding is sound. However, is it also faithful?

6.2.1 A notion of faithfulness

There is some variance in regards to the notion of faithfulness in the literature. Some define it in terms of model stability over conservative model extensions, such as de Bruijn et al. (cf. 2011, p. 304) or Wagner (cf. 1998, p. 9), some in terms of consistency over language extensions, such as Schwarz (cf. 1996, p. 357) or de Brujin et al. (cf. 2008, p. 488), some in terms of soundness and completeness and proof-theoretic property preservation, such as Troelstra and Schwichtenberg (cf. 2000, pp. 289-291), and some in terms of mere soundness and completeness, such as Benzmüller et al. (cf. 2018, p. 9). The latter notion seems to be the most modest concept of faithfulness as it is a necessary condition in all of the other conceptions. Under this interpretation, it resembles the notion of adequacy of formal systems (cf. Ebbinghaus et al., 2007, p. 85). We will use this notion in the following. Hence, we define:

Definition 30 (Completeness of an ^{e} embedding)
An ^{e} embedding, based on an ^{e} translation, which translates each set of formulas, Γ, Δ, of a language $\mathcal{L}_1$ into a set of formulas, Γ^e, Δ^e, of the language $\mathcal{L}_2$, is complete :⇔

$$\mathbf{PS2} \vdash \Gamma^e \Rightarrow \Delta^e \Rightarrow \mathbf{PS1} \vdash \Gamma \Rightarrow \Delta$$

holds for the corresponding proof systems **PS1** and **PS2**.

Definition 31 (Faithfulness of an ^{e} embedding)
An ^{e} embedding, based on an ^{e} translation, which translates each set of formulas, Γ, Δ, of a language $\mathcal{L}_1$ into a set of formulas, Γ^e, Δ^e, of the language $\mathcal{L}_2$, is faithful :⇔

$$\mathbf{PS1} \vdash \Gamma \Rrightarrow \Delta \Leftrightarrow \mathbf{PS2} \vdash \Gamma^e \Rrightarrow \Delta^e$$

holds for the corresponding proof systems **PS1** and **PS2**.

This means, that completeness is the converse of soundness. The conjunction of both properties is what we call faithfulness. Consequently, in the case of the * translation, we get: if the * embedding is correct (faithful) then

$$\mathbf{G3s} \vdash \Gamma^* \Rrightarrow \Delta^* \Rightarrow (\Leftrightarrow)\ \mathbf{G3c} \vdash \Gamma \Rrightarrow \Delta.$$

6.2.2 Deliberations on faithfulness: $^\circ$ vs. *

If we substituted **G3i**, the intuitionistic version of a G3 system, for **G3c**, this result would surely hold (cf. Schütte, 1968, pp. 33-43). But is this also true for our case? We want to consider this by comparing the * embedding to the $^\circ$ embedding (see Def. 0) in the following.

Multi vs. single conclusion Indeed, it seems to be an easier endeavour to prove the classical case than proving the intuitionistic one at first glace: a lot of the difficulty accompanying the latter is related to **G3s** being a multi-conclusion system, i.e. $0 \leq |\Delta|$, whereas **G3i** only allows for at most single conclusion succedents, $0 \leq |\Delta| \leq 1$ (cf. Troelstra and Schwichtenberg, 2000, pp. 289-291). Since **G3c** is multi-conclusion permissive, this problem should not occur.

Boxing and unboxing Nonetheless, there is a different and possibly more grave problem: a proof of completeness of a modal embedding requires a back translation from the modal to the non-modal realm. In the intuitionistic case, where the $^\circ$ translation is employed, this is comparably straightforward. The $^\circ$ forth-translation consists in putting $\Box$ in front of every non-modal formula. In S4, boxing and unboxing formulas is made trivial by the conditions on the relation (cf. Def. 10.2). Expressed syntactically, and utilising an axiomatic, Hilbert-style formulation of the system, we can see this easily by taking a look at the two axioms normally added to K in order to obtain full S4 (cf. Troelstra and Schwichtenberg, 2000, p. 284):

(T) $\vdash_{S4} \Box A \rightarrow A$

(4) $\vdash_{S4} \Box A \rightarrow \Box \Box A$

Employing (T), every formula lead by a $\Box$ operator can directly be transformed into a non-modal formula. In the case of the * translation, however, a we will not achieve our goal as easily. The resources of S4 to deal with the $\Diamond$ operator, as it becomes introduced by the $*$ translation, in particular its elimination, are more limited. This, hence, complicates the proof.

Extracting the modal

Distributivity There is a second aspect to the task of proving faithfulness. Before we can employ the axioms or rules of S4 to transform modal into non-modal formulas, we need to extract the modal content from it. This means, we need to bring our S4 candidate formula for back-translation into the following form: $\Box C$ (or $\Box \Diamond C$ for the * case), where C does not contain any modal symbols $\Box, \Diamond$. Here again, $^\circ$ has the edge over *: as Fitting (1970, pp. 529-530) states correctly, the $^\circ$ translation is fully distributable over $\wedge, \vee, \exists$, whereas the * translation is only over $\wedge$, complicating the proof procedure. This will lead to more proof diversions in the * case.

Fuzz with the meta-quantifiers An additional problem for obtaining the desired form is a problem with the order of $\Box, \Diamond$ in the * translation. By taking a look at the corresponding semantic clauses in Def. 14.2.7-8, we can see that we cannot directly invert the order of the initial operators in the * translated formulas. We can see this as $\mathbb{V}A : \mathbb{E}B : C \nvdash_{LPL} \mathbb{E}B : \mathbb{V}A : C$ in our classical, first-order meta-language LPL (Leitgeb, 2018, p. 226). This again complicates matters since we cannot easily get the $\Diamond$s to the front of the modal formulas which we need to back-translate.

Additionally to extracting them, we need to eliminate the existential meta-quantifiers, i.e. (modulo Def. 14.2.7) the ◇s. This is a related problem to that of 'Skolemisation' (Troelstra and Schwichtenberg (cf. 2000, p. 246) and Bimbó (cf. 2015, pp. 288-289). We could frame this problem as skolemising on two levels, the object and the meta level. Due to its ◇-less formulation, such a concern cannot arise for the $^\circ$ case.

6.2.3 The unfaithfulness of *?

We have just seen that, although the multi-conclusion formulation of **G3c** might help to simplify matters slightly, the problem of extracting the modal operators and eliminating them by the means of S4 will be a notably greater challenge for the completeness proof of the * embedding than for its $^\circ$ counterpart. Giving the proof a hands on attempt may lead one to suspect that the proof is altogether impossible.

Suspicion 1 (Unfaithfulness)
We suspect that

$$\mathbf{G3s} \vdash \Gamma^* \Rightarrow \Delta^* \not\Rightarrow (\not\Leftarrow)\ \mathbf{G3c} \vdash \Gamma \Rightarrow \Delta.$$

Of course, validating this suspicion will require more investigation and thorough proof as we are able to provide in this thesis. This is due to space constraints and the expository nature of this section. Nevertheless, we hope that our deliberations are able to shed some light on possible obstacles a completeness proof for the * embedding would need to surpass.
Notwithstanding the above, both our model-theoretic and our proof-theoretic study clearly show that Fitting's * embedding can be regarded sound. It allows us to bridge the classical realm and the modal realm of S4.
In the following, final section of this thesis we want so ask: 'What does this mean?'.

7 Evaluation: the * embedding as notational variance

In this concluding section, we want to take stock and evaluate our results. We will do this by rationally reconstructing the quote from our introduction, *[T]ruth may be replaced by provable consistency* (Fitting, 1970, p. 529), based on what has been shown and discussed in this thesis in regards to the * embedding.
As we have seen in the beginning, 'truth' refers to (one possible philosophical reading of) classical logic, whereas 'provable consistency' relates to S4 (or a philosophical interpretation thereof). Hence, the part of the quote still requiring explication is what is meant by 'may be replaced by'. We want to offer a possible reading of replaceability in terms of notational variance and show that, consequently, we can spell out Fitting's quote as *Classical logic is a (one-way) notational variant of S4*.

However, what takes it to be a 'notational variant'? We want to base our following deliberations about this on French (2019), where one can find a comprehensive survey of the literature on the topic. We will introduce two[6] modified conceptions offered in French's paper, a syntactic and a semantic one, and evaluate them in the light of the * embedding.

7.1 Syntactic equivalence

We will start by giving the syntactic notion from French (cf. 2019, p. 325). We have changed the definition to fit our terminology (for instance, our concept of a faithful embedding) and our notation. We will also adapt it to our specific case right away without introducing a generalised version beforehand.

Definition 32 (Syntactic equivalence)
Be * a (forth-)translation, which translates the formulas of **G3c** into formulas of **G3s**, and be ^{*c} its converse, i.e. the (back-)translation, which translates the

[6] It would also be interesting to take a look at French's notion of definitional equivalence in this context. Unfortunately, the account has not yet been fully developed for first order languages but only for the propositional case (cf. French, 2019, p. 329).

formulas of **G3s** into formulas of **G3c**.
G3c and **G3s** are systactically equivalent :⇔

1. $\exists^*, {}^{*_c}$: the *(^{*_c}) embedding faithfully embeds **G3c**(s) in **G3s**(c),
2. the following 'inverse' conditions hold:
   (a) $A \vdash_{\mathbf{G3c}} (A^*)^{*_c}$,
   (b) $(A^*)^{*_c} \vdash_{\mathbf{G3c}} A$,
   (c) $A \vdash_{\mathbf{G3s}} (A^{*_c})^*$, and
   (d) $(A^{*_c})^* \vdash_{\mathbf{G3s}} A$.

We cannot say that this definition holds, i.e. that the * embedding establishes syntactic equivalence between classical logic and S4, based on our prior investigations. Although we have been able to show the soundness of the embedding syntactically, we still need to prove its completeness, the possibility of which we have doubted. We might also need some additional proofs to satisfy the 'inverse' conditions.
This and the fact that Fitting's proof sketch was for the soundness direction of a semantic embedding leads us to dismiss the thesis that replaceability in Fitting's quote should be analysed as syntactic equivalence.

7.2 Expressive equivalence

A second, more semantic notion of notational variance offered in French (cf. 2019, p. 326) is that of expressive equivalence. We will introduce the definition now, again modifying it to fit our notation.

Definition 33 (Expressive equivalence)
Be Mod_C, Mod_{S4} classes of Kripke models.
$\vDash_C$ and $\vDash_{S4}$ are expressively equivalent :⇔

1. $\forall A \in \mathcal{L}_C : \exists B \in \mathcal{L}_{S4} : \forall \mathcal{M}_C \in Mod_C : \mathcal{M}_C \vDash_C A \Leftrightarrow g(\mathcal{M}_C) \vDash_{S4} B$,
2. $\forall B \in \mathcal{L}_{S4} : \exists A \in \mathcal{L}_C : \forall \mathcal{M}_C \in Mod_C : \mathcal{M}_C \vDash_C A \Leftrightarrow g(\mathcal{M}_C) \vDash_{S4} B$,

where g is a bijection between Mod_C and Mod_{S4}: $\exists V' : g(\langle W, R, V\rangle) = \langle W, R, V'\rangle$.

This proposal for a concept of notational variance is a notably better candidate for the analysis of Fitting's replaceability relation, standalone for its semantic formulation. However, due to its use of equivalence and its branching into two cases, it is still stronger than our and, thus, Fitting's result.

7.3 Expressive implication

We want to use a simple fix for this: let us reduce expressive equivalence to expressive implication. Expressed differently, we want to constrain notational variance to a one-way case.
We will also use Def. 21 as a replacement for the g-bijection and the existence claim for B as they express the same (the former, the * translation, positioning the lever at a language level, whereas the latter does so at our models). We will also apply Def. 16. Hence, we obtain:

Definition 34 (Expressive implication)
$\vDash_C$ expressively implies $\vDash_{S4}$ $:\Leftrightarrow \forall A \in \mathcal{L}_C$:

$$\vDash_C A \Rightarrow \vDash_{S4} A^*.$$

This version of expressive equivalence, reduced to an implication, finally meets all of the criteria for the analysis of Fitting's replaceability claim.
In fact, this definition of expressive implication is identical to our main result in section 4: Theorem 2, the semantic embedding theorem. As it happens, this is also the result which Fitting's proof sketch arrives at.
Hence, we can hold that what Fitting means, when he writes *[T]ruth may be replaced by provable consistency*, is indeed (modulo translation):

$$\vDash_C \textit{ expressively implies } \vDash_{S4}.$$

If we consider expressive equivalence the right analysis of (full) notational variance, we can also say (modulo translation):

Classical logic is a one-way notational variant of S4.

or

Classical logic is a partial notational variant of S4.

If we express this more philosophically, also using the discussion from French (2019, cf. p. 321), we could re-phrase this as follows: the classical truths only trivially vary from the S4 truths, which are won by extending the former to the latter via the * translation.

7.3.1 Replaceability as extensibility

We have seen that classical logic can be considered a one-way or partial notational variant of S4, if we accept 'expressive equivalence' in the sense of Fitting as the right notion of notational variance. We have also seen that this can give us a conclusive explication of our initial quote.

We want to add one final thought: is this anything new? Indeed, this result was already foreshadowed in an earlier section of this thesis. When we defined our proof systems, we obtained **G3s** by adding rules for the modal operators to **G3c**. In other words, we obtained S4 by extending classical logic. This sounds suspiciously similar to our final result in the discussion of the embedding in the light of notational variance.

Therefore, we might want to conclude this thesis with an old insight: classical logic can be extended to S4 (cf. Troelstra and Schwichtenberg, 2000, p. 283). Nevertheless, the * translation could still offer richer insights: if our suspicion is false and the * embedding is indeed faithful, this might change our view on this matter. This, however, needs to be investigated in another paper.

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Appendix

List of definitions and remarks

List of theorems, lemmata, observations, and suspicions

List of examples

List of Tables