Supplement 1 to "'Adding Up' Reasons": Proof of Theorem 2

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1 Introduction

Recall the basic set up of our discussion quoted from the main text:

> We assume that propositions are elements of an algebra based on a partition $U = \{A_1, A_2, \ldots, A_n\}$ where the A_i's are the cells of the partition and $n \geq 3$. So a proposition is a (possibly empty) set of cells of the partition. We adopt some shorthand for designating particular propositions: $\top = U, \bot = \emptyset$. If P, Q are propositions, we will use the following notation when it is convenient: $\neg P = \top - P, P \vee Q = P \cup Q, P \wedge Q = P \cap Q$. We will frequently omit the braces around propositions that are singletons so we will write $\{A_i\}$ as A_i. Finally we say P entails Q exactly if $P \subseteq Q$.

Against this background, our aim is to prove:

Theorem 2. *For any* regular *probability function, Pr, there is a reasons weighing function, $\mathbf{r}_b$, such that (i) for any proposition P, $Pr(P) = f_{\mathbf{r}_b}(P)$ and (ii) for any propositions H, E, either*

$$log_b\left(\frac{Pr(E \mid H)}{Pr(E \mid \neg H)}\right) = \mathbf{r}_b(H, E)$$

or $log_b\left(\frac{Pr(E|H)}{Pr(E|\neg H)}\right)$ and $\mathbf{r}_b(H, E)$ are both undefined.

We will for the remainder of our discussion suppress our assumption that Pr is a regular probability function (in the sense that if $P \neq \bot, Pr(P) \neq 0$). Recall that:

$$l_b(H, E) = log_b\left(\frac{Pr(E \mid H)}{Pr(E \mid \neg H)}\right)$$

It is easiest to start by proving (ii) of Theorem 2. We will then consider (i) of Theorem 2.

2 $l_b = \mathbf{r}_b$

To prove this, it suffices to that $l_b(H,E)$ satisfies the axioms in Definition 1:

Definition 1. *A function from pairs of propositions from the algebra based on* U *to the interval* $(-\infty,\infty)$, $\mathbf{r}_b$, *is a* reasons weighing function *exactly if it satisfies the following axioms:*

Base Propriety: $b > 1$

Undefined Reasons: *if* (H,E) *is extreme,* $\mathbf{r}_b(H,E)$ *is undefined*

No Reason: *if* (H,E) *is vacuous,*

$$\mathbf{r}_b(H,E) = log(1) = 0$$

Complimentary Reasons: *if* (H,E) *is not extreme,*

$$\mathbf{r}_b(\neg H,E) = -\mathbf{r}_b(H,E)$$

Entailed Reason: *if* (H,E) *is not trivial and* H *entails* E,

$$\mathbf{r}_b(H,E) > log_b(1) = 0$$

Negatively Correlated Reasons: *if* (P,Q) *is a non-trivial determiner,*

$$\mathbf{r}_b(\neg P \wedge \neg Q, \neg P) = log_b\left(\frac{b^{\mathbf{r}_b(\neg P \wedge \neg Q, \neg Q)}}{b^{\mathbf{r}_b(\neg P \wedge \neg Q, \neg Q)} - 1}\right)$$

Positively Correlated Reasons: *if* (P,Q), (Q,R), *and* (P,R) *are non-trivial determiners,*

$$\mathbf{r}_b(\neg P \wedge \neg R, \neg R) = log_b\left(\left(b^{\mathbf{r}_b(\neg Q \wedge \neg R, \neg R)} - 1\right)\left(b^{\mathbf{r}_b(\neg P \wedge \neg Q, \neg Q)} - 1\right) + 1\right)$$

Aggregative Reasons: *if* (P,Q) *is a non-trivial determiner,*

$$\mathbf{r}_b\left(\neg P \wedge \neg Q, \neg Q\right) = log_b\left(\left(\left(\sum_{Q_i \in Q} b^{\mathbf{r}_b(\neg P \wedge \neg Q_i, \neg Q_i)} - 1\right) + 1\right)\right)$$

Factored Reasons: *if* (H,E) *is not trivial,* H *does not entail* E, *and* $\neg H$ *does not entail* E, *then for any* D, D' *such that* (H,D) *and* $(\neg H, D')$ *are non-trivial determiners,*

$$\mathbf{r}_b(H,E) = log_b\left(\frac{\left(b^{\mathbf{r}_b(\neg D \wedge \neg (H \wedge E), \neg (H \wedge E))} - 1\right)\left(b^{\mathbf{r}_b(\neg H \wedge \neg D, \neg D)} - 1\right)}{\left(b^{\mathbf{r}_b(\neg D' \wedge \neg (\neg H \wedge E), \neg (\neg H \wedge E))} - 1\right)\left(b^{\mathbf{r}_b(H \wedge \neg D', \neg D')} - 1\right)}\right)$$

Thought it is often left implicit in discussions of this matter, we stipulate that l_b is defined so that that $b > 1$. Thus, it is immediate that:

Proposition 2.1 (l_b satisfies Base Propriety)**.** $b > 1$

The remaining axioms make use of some terminology for classifying pairs of propositions. The terminology is this:

(H, E) is *extreme* exactly if E entails H or E entails $\neg H$.

(H, E) is *vacuous* exactly if (H, E) is not extreme and $E = \top$.

(H, E) is *trivial* exactly if (H, E) is extreme or vacuous.

(P, Q) is *a non-trivial determiner* exactly if $P \neq \bot, Q \neq \bot$, $P \vee Q \neq \top$, and $P \wedge Q = \bot$

Based on these definitions, in the main text there is a proof of the following fact:

Notational Variants: *If (H, E) is not trivial and H entails E, then there is exactly one (P, Q) such that (P, Q) is a non-trivial determiner and $H = \neg P \wedge \neg Q$ and $E = \neg Q$. And if (P, Q) is a non-trivial determiner, then $(\neg P \wedge \neg Q, \neg Q)$ is not trivial and $\neg P \wedge \neg Q$ entails $\neg Q$.*

With this in mind, we now show l_b satisfies each of the axioms that are of interest to us.

2.1 Well-Known Features of l_b

Undefined Reasons-Entailed Reason are well known features of l_b. But for completeness, I shall provide proofs of them here.

Proposition 2.2 (l_b satisfies Undefined Reasons)**.** *if (H, E) is extreme, $l_b(H, E)$ is undefined*

Proof of Proposition 2.2. Consider (H, E) that are extreme in the sense that E entails H or E entails $\neg H$. Suppose E entails H. . In this setting,

$$0 = Pr(E \wedge \neg H) = \frac{Pr(E \wedge \neg H)}{Pr(\neg H)} = Pr(E \mid \neg H)$$

so $l_b(H, E)$ is undefined because the term inside the log involves division by 0. Suppose instead E entails $\neg H$. In this setting,

$$0 = Pr(E \wedge H) = \frac{Pr(E \wedge H)}{Pr(H)} = Pr(E \mid H) = \frac{Pr(E \mid H)}{Pr(E \mid \neg H)}$$

so $l_b(H, E)$ is undefined because $log(0)$ is undefined. □

Proposition 2.3 (l_b satisfies No Reason)**.** *If (H, E) is vacuous,*

$$l_b(H, E) = log(1) = 0$$

Proof of Proposition 2.3. Consider (H, E) that are vacuous in the sense that $E = \top$ and $H \neq \top, \bot$.[1] So $Pr(H) \neq 0$, $Pr(\neg H) \neq 0$, $E \wedge H = H$, and $E \wedge \neg H = \neg H$. Thus:

$$Pr(E \mid H) = \frac{Pr(E \wedge H)}{Pr(H)} = \frac{Pr(H)}{Pr(H)} = 1$$

and

$$Pr(E \mid \neg H) = \frac{Pr(E \wedge \neg H)}{Pr(\neg H)} = \frac{Pr(\neg H)}{Pr(\neg H)} = 1$$

Therefore as desired:

$$l_b(H, E) = log_b \left(\frac{1}{1} \right) = log_b(1) = 0$$

□

Proposition 2.4 (l_b satisfies Complimentary Reasons)**.** *If (H, E) is not extreme,*

$$l_b(\neg H, E) = -l_b(H, E)$$

Proof of Proposition 2.4. Consider (H, E) that are not extreme. It follows $(\neg H, E)$ is also not extreme.[2] It is a fact about log's that $log(\frac{a}{b}) = -log(\frac{b}{a})$ when these terms are defined. And the terms are undefined only if the denominator of the faction inside the log is 0 or the numerator of the fraction inside the log is 0. So

$$l_b(\neg H, E) = -l_b(H, E)$$

if $Pr(E \mid H)$ is non-zero and $Pr(E \mid \neg H)$ is non-zero. Since (H, E) and $(\neg H, E)$ are not extreme $Pr(E \wedge H)$, $Pr(H)$, $Pr(E \wedge \neg H)$, and $Pr(\neg H)$ are all non-zero. So $Pr(E \mid H)$ and $Pr(E \mid \neg H)$ are non-zero. □

Proposition 2.5 (l_b satisfies Entailed Reason)**.** *if (H, E) is not trivial and H entails E,*

$$l_b(H, E) > log_b(1) = 0$$

Proof of Proposition 2.5. Consider (H, E) that are not trivial (so E does not entail H, E does not entail $\neg H$, and $E \neq \top$) and such that H entails E. In this case, $Pr(E \wedge H) = Pr(H)$ so $Pr(E \mid H) = 1$. On the other hand, $\neg H$ does not entail E[3] so $Pr(\neg H \wedge E) < Pr(\neg H)$. Thus, $Pr(E \mid \neg H) < 1$. So $\frac{Pr(E|H)}{Pr(E|\neg H)} > 1$.[4] Thus $l_b(H, E) > 0$ □

We can now turn to some less well-known properties of l_b.

[1] If $H = \top$, then E entails H so (H, E) is extreme and hence not vacuous. If $H = \bot$, E entails $\neg H$ so (H, E) is extreme and hence not vacuous.

[2] Since (H, E) is not extreme, E does not entail H and $\neg E$ does not entail H. Thus $\neg E$ does not entail H and $\neg\neg E$ does not entail H. So $(\neg H, E)$ is not extreme.

[3] The only super set of both H and $\neg H$ is $\top$ but $E \neq \top$.

[4] $\frac{Pr(E|H)}{Pr(E|\neg H)}$ must also be defined because $Pr(E \mid \neg H) \neq 0$. This is because $Pr(E \wedge \neg H)$ and $Pr(\neg H)$ are non-zero (because E does not entail H so $E \wedge \neg H \neq \bot$ and $Pr(\neg H) \neq \bot$)).

2.2 Less Well-Known Features of l_b

In proving that l_b satisfies these axioms. We will often rely on the following lemma whose proof can be found in the main text.

Lemma 1.4.1: *For any (P, Q) that is a non-trivial determiner,*

$$l_b(\neg P \wedge \neg Q, \neg Q) = log_b\left(\frac{Pr(\neg Q \mid \neg P \wedge \neg Q)}{Pr(\neg Q \mid \neg(\neg P \wedge \neg Q))}\right) = log_b\left(\frac{Pr(Q)}{Pr(P)} + 1\right)$$

We now consider each axiom in order.

Proposition 2.6 (l_b satisfies Negatively Correlated Reasons)**.** *if (P, Q) is a non-trivial determiner,*

$$l_b(\neg P \wedge \neg Q, \neg P) = log_b\left(\frac{b^{l_b(\neg P \wedge \neg Q, \neg Q)}}{b^{l_b(\neg P \wedge \neg Q, \neg Q)} - 1}\right)$$

Proof of Proposition 2.6. Consider (P, Q) that are non-trivial determiners. Since $b^{log_b(x)} = x$, Lemma 1.4.1 tell us:

$$b^{l_b(\neg P \wedge \neg Q, \neg Q)} = \frac{Pr(Q)}{Pr(P)} + 1$$

So

$$\frac{b^{l_b(\neg P \wedge \neg Q, \neg Q)}}{b^{l_b(\neg P \wedge \neg Q, \neg Q)} - 1} = \frac{\frac{Pr(Q)}{Pr(P)} + 1}{\frac{Pr(Q)}{Pr(P)}} = \frac{Pr(Q)Pr(P)}{Pr(P)Pr(Q)} + \frac{Pr(P)}{Pr(Q)} = 1 + \frac{Pr(P)}{Pr(Q)}$$

Thus:

$$l_b\left(\frac{b^{l_b(\neg P \wedge \neg Q, \neg Q)}}{b^{l_b(\neg P \wedge \neg Q, \neg Q)} - 1}\right) = l_b\left(1 + \frac{Pr(P)}{Pr(Q)}\right)$$

Finally we know from Lemma 1.4.1 that:

$$l_b(\neg Q \wedge \neg P, \neg P) = l_b(\neg P \wedge \neg Q, \neg P) = log_b\left(\frac{Pr(P)}{Pr(Q)} + 1\right)$$

Therefore as desired:

$$l_b(\neg P \wedge \neg Q, \neg P) = l_b\left(\frac{b^{l_b(\neg P \wedge \neg Q, \neg Q)}}{b^{l_b(\neg P \wedge \neg Q, \neg Q)} - 1}\right)$$

□

Proposition 2.7 (l_b satisfies Positively Correlated Reasons)**.** *if (P, Q), (Q, R), and (P, R) are non-trivial determiners,*

$$l_b(\neg P \wedge \neg R, \neg R) = log_b\left(\left(b^{l_b(\neg Q \wedge \neg R, \neg R)} - 1\right)\left(b^{l_b(\neg P \wedge \neg Q, \neg Q)} - 1\right) + 1\right)$$

Proof of Proposition 2.7. Consider (P, Q), (Q, R) and (P, R) that are non-trivial determiners. Lemma 1.4.1 tells us that:

$$l_b(\neg P \wedge \neg Q, \neg Q) = log_b \left(\frac{Pr(Q)}{Pr(P)} + 1 \right)$$

$$l_b(\neg Q \wedge \neg R, \neg R) = log_b \left(\frac{Pr(R)}{Pr(Q)} + 1 \right)$$

$$l_b(\neg P \wedge \neg R, \neg R) = log_b \left(\frac{Pr(R)}{Pr(P)} + 1 \right)$$

We know that

$$b^{l_b(\neg Q \wedge \neg R, \neg R)} - 1 = \frac{Pr(R)}{Pr(Q)}$$

$$b^{l_b(\neg P \wedge \neg Q, \neg Q)} - 1 = \frac{Pr(Q)}{Pr(P)}$$

Thus:

$$\left(b^{l_b(\neg Q \wedge \neg R, \neg R)} - 1\right)\left(b^{l_b(\neg P \wedge \neg Q, \neg Q)} - 1\right) + 1 = \left(\frac{Pr(R)}{Pr(Q)}\right)\left(\frac{Pr(Q)}{Pr(P)}\right) + 1 = \frac{Pr(R)}{Pr(P)} + 1$$

Therefore as desired:

$$l_b(\neg P \wedge \neg R, \neg R) = log_b \left(\frac{Pr(R)}{Pr(P)} + 1 \right) = log_b \left(\left(b^{l_b(\neg Q \wedge \neg R, \neg R)} - 1\right)\left(b^{l_b(\neg P \wedge \neg Q, \neg Q)} - 1\right) + 1 \right)$$

□

Proposition 2.8 (l_b satisfies Aggregative Reasons)**.** *if (P, Q) is a non-trivial determiner,*

$$l_b(\neg P \wedge \neg Q, \neg Q) = log_b \left(\left(\sum_{Q_i \in Q} b^{l_b(\neg P \wedge \neg Q_i, \neg Q_i)} - 1 \right) + 1 \right)$$

Proof of Proposition 2.8. Consider (P, Q) that is a non-trivial determiner. For any $Q_i \in Q$, (P, Q_i) is also a non-trivial determiner.[5]. Thus we know from Lemma 1.4.1:

$$l_b(\neg P \wedge \neg Q, \neg Q) = log_b \left(\frac{Pr(Q)}{Pr(P)} + 1 \right)$$

$$l_b(\neg P \wedge \neg Q_i, \neg Q_i) = log_b \left(\frac{Pr(Q_i)}{Pr(P)} + 1 \right)$$

[5] It is obvious that $Q_i \neq \bot$. It must be $P \vee Q_i \neq \top$ because $P \vee Q \neq \top$. And it must be that $P \wedge Q_i = \bot$ because $P \wedge Q = \bot$.

Since $Q = \{Q_1, Q_2, \cdots, Q_n\}$, $Pr(Q) = \sum_{Q_i \in Q} Pr(Q_i)$.[6] Thus:

$$\sum_{Q_i \in Q} b^{l_b(\neg P \wedge \neg Q_i, \neg Q_i)} - 1 = \frac{Pr(Q_1)}{Pr(P)} + \frac{Pr(Q_2)}{Pr(P)} + \cdots + \frac{Pr(Q_n)}{Pr(P)} = \frac{Pr(Q)}{Pr(P)}$$

Thus as desired:

$$log_b\left(\left(\sum_{Q_i \in Q} b^{l_b(\neg P \wedge \neg Q_i, \neg Q_i)} - 1\right) + 1\right) = log_b\left(\frac{Pr(Q)}{Pr(P)} + 1\right) = l_b(\neg P \wedge \neg Q, \neg Q)$$

□

Proposition 2.9 (l_b satisfies Factored Reasons)**.** *if (H, E) is not trivial, H does not entail E, and $\neg H$ does not entail E, then for any D, D' such that (H, D) and $(\neg H, D')$ are non-trivial determiners,*

$$l_b(H, E) = log_b\left(\frac{\left(b^{l_b(\neg D \wedge \neg(H \wedge E), \neg(H \wedge E))} - 1\right)\left(b^{l_b(\neg H \wedge \neg D, \neg D)} - 1\right)}{\left(b^{l_b(\neg D' \wedge \neg(\neg H \wedge E), \neg(\neg H \wedge E))} - 1\right)\left(b^{l_b(H \wedge \neg D', \neg D')} - 1\right)}\right)$$

Proof. Consider (H, E) that is not trivial and such that H does not entail E and $\neg H$ does not entail E. And consider some D, D' such that (H, D) and $(\neg H, D')$ are non-trivial determiners. Since (H, D) is a non-trivial determiner, it follows that $(D, H \wedge E)$ is a non-trivial determiner too.[7] Similarly, since $(\neg H, D')$ is a non-trivial determiner, it follows that $(D', \neg H \wedge E)$ is a non-trivial determiner too. So Lemma 1.4.1 tell us:

$$l_b(\neg H \wedge \neg D, \neg D) = log_b\left(\frac{Pr(D)}{Pr(H)} + 1\right)$$
$$l_b(\neg D \wedge \neg(H \wedge E), \neg(H \wedge E)) = log_b\left(\frac{Pr(H \wedge E)}{Pr(D)} + 1\right)$$
$$l_b(\neg D' \wedge \neg(\neg H \wedge E), \neg(\neg H \wedge E)) = log_b\left(\frac{Pr(\neg H \wedge E)}{Pr(D')} + 1\right)$$
$$l_b(H \wedge \neg D', \neg D') = log_b\left(\frac{Pr(D')}{Pr(\neg H)} + 1\right)$$

So:

$$\left(b^{l_b(\neg D \wedge \neg(H \wedge E), \neg(H \wedge E))} - 1\right)\left(b^{l_b(\neg H \wedge \neg D, \neg D)} - 1\right) = \frac{Pr(D)Pr(H \wedge E)}{Pr(H)Pr(D)} = \frac{Pr(H \wedge E)}{Pr(H)} = Pr(E \mid H)$$

And by analogous reasoning:

$$\left(b^{l_b\left(\neg D' \wedge \neg(\neg H \wedge E), \neg(\neg H \wedge E)\right)} - 1\right)\left(b^{l_b\left(H \wedge \neg D', \neg D'\right)} - 1\right) = Pr(E \mid \neg H)$$

[6] The summation claim in the text follows from the finite additivity property of Pr.

[7] It follows from $H \vee D \neq \top$ that $D \vee (H \wedge E) \neq \top$. It follows from $H \wedge D = \perp$ that $D \wedge (H \wedge E) = \perp$. And it follows (H, E) being not trivial that $H \wedge E \neq \perp$. So $(D, H \wedge E)$ is a non-trivial determiner.

Thus as desired:

$$log_b\left(\frac{\left(b^{l_b(\neg D\wedge\neg(H\wedge E),\neg(H\wedge E))}-1\right)\left(b^{l_b(\neg H\wedge\neg D,\neg D)}-1\right)}{\left(b^{l_b(\neg D'\wedge\neg(\neg H\wedge E),\neg(\neg H\wedge E))}-1\right)\left(b^{l_b(H\wedge\neg D',\neg D')}-1\right)}\right)=log_b\left(\frac{Pr(E\mid H)}{Pr(E\mid\neg H)}\right)=l_b(H,E)$$

□

3 $Pr = f_{l_b}$

Since we have seen l_b and $\mathbf{r}_b$ are equivalent, in order to show (i) of Theorem 2 it suffices to show that the function the probability function Pr that defines l_b is equivalent to f_{l_b} as defined by Definition 2:

Definition 2. *A function from propositions from the algebra based U to the interval $(-\infty,\infty)$, $f_{\mathbf{r}_b}$,* is the prior based on $\mathbf{r}_b$ function *exactly if it satisfies the following axioms:*

Ratios of Cells: *If $U=\{A_1,A_2,\cdots A_n\}$ then,*

$$\begin{aligned}1&=f_{\mathbf{r}_b}(A_1)+f_{\mathbf{r}_b}(A_2)+\cdots+f_{\mathbf{r}_b}(A_n)\\f_{\mathbf{r}_b}(A_2)&=(b^{\mathbf{r}_b(\neg A_1\wedge\neg A_2,\neg A_2)}-1)f_{\mathbf{r}_b}(A_1)\\f_{\mathbf{r}_b}(A_3)&=(b^{\mathbf{r}_b(\neg A_1\wedge\neg A_3,\neg A_3)}-1)f_{\mathbf{r}_b}(A_1)\\&\vdots\\f_{\mathbf{r}_b}(A_n)&=(b^{\mathbf{r}_b(\neg A_1\wedge\neg A_n,\neg A_n)}-1)f_{\mathbf{r}_b}(A_1)\end{aligned}$$

Sum of Cells: *For any proposition P,*

- *if $P=\bot$, $f_{\mathbf{r}_b}(P)=0$*
- *if $P\neq\bot$, $f_{\mathbf{r}_b}(P)=\sum_{A_i\in P}f_{\mathbf{r}_b}(A_i)$*

Proposition 2.10. *If $l_b(H,E)=log_b\left(\frac{Pr(E|H)}{Pr(E|\neg H)}\right)$, then for any proposition P, $Pr(P)=f_{l_b}(P)$*

Proof. For each A_i such that $i>1$, Ratios of Cells tell us:

$$f_{l_b}(A_i)=(b^{l_b(\neg A_1\wedge\neg A_i,\neg A_i)}-1)f_{l_b}(A_1)$$

Therefore, substituting this in the first equation, we have:

$$1=f_{l_b}(A_1)+(b^{l_b(\neg A_1\wedge\neg A_2,\neg A_2)}-1)f_{l_b}(A_1)+\cdots+(b^{l_b(\neg A_1\wedge\neg A_n,\neg A_n)}-1)f_{l_b}(A_1)$$

So:

$$\begin{aligned}1-f_{l_b}(A_1)&=(b^{l_b(\neg A_1\wedge\neg A_2,\neg A_2)}-1)f_{l_b}(A_1)+\cdots+(b^{l_b(\neg A_1\wedge\neg A_n,\neg A_n)}-1)f_{l_b}(A_1)\\&=f_{l_b}(A_1)\left((b^{l_b(\neg A_1\wedge\neg A_2,\neg A_2)}-1)+\cdots+(b^{l_b(\neg A_1\wedge\neg A_n,\neg A_n)}-1)\right)\\\frac{1-f_{l_b}(A_1)}{f_{l_b}(A_1)}&=(b^{l_b(\neg A_1\wedge\neg A_2,\neg A_2)}-1)+\cdots+(b^{l_b(\neg A_1\wedge\neg A_n,\neg A_n)}-1)\end{aligned}$$

Since (A_1, A_i) is a non-trivial determiner, Lemma 1.4.1 together with some reasoning tells us:

$$b^{l_b(\neg A_1 \wedge \neg A_i, \neg A_i)} - 1 = \frac{Pr(A_i)}{Pr(A_1)}$$

So:

$$\frac{1 - f_{l_b}(A_1)}{f_{l_b}(A_1)} = \frac{Pr(A_2)}{Pr(A_1)} + \cdots + \frac{Pr(A_n)}{Pr(A_1)}$$

We can now engage in some ordinary reasoning about probabilities:

$$\begin{aligned} \frac{Pr(A_2)}{Pr(A_1)} + \cdots + \frac{Pr(A_n)}{Pr(A_1)} &= \frac{Pr(A_2) + \cdots + Pr(A_n)}{Pr(A_1)} \\ &= \frac{Pr(\neg A_1)}{Pr(A_1)} \\ &= \frac{1 - Pr(A_1)}{Pr(A_1)} \end{aligned}$$

Thus:

$$\frac{1 - f_{l_b}(A_1)}{f_{l_b}(A_1)} = \frac{1 - Pr(A_1)}{Pr(A_1)}$$

So $f_{l_b}(A_1) = Pr(A_1)$.[8]

For each A_i such that $i > 1$, we already know:

$$f_{l_b}(A_i) = (b^{l_b(\neg A_1 \wedge \neg A_i, \neg A_i)} - 1) f_{l_b}(A_1)$$

Two substitutions gets us:

$$f_{l_b}(A_i) = \left(\frac{Pr(A_i)}{Pr(A_1)} \right) Pr(A_1) = Pr(A_i)$$

Thus, we have shown for each $A_i \in U f_{l_b}(A_i) = Pr(A_i)$.

Sum of Cells tell us $f_{l_b}(\bot) = 0$. Therefore $f_{l_b}(\bot) = 0 = Pr(\bot)$.

Sum of Cells tells us that if $P \neq \bot$, $f_{l_b}(P) = \sum_{A_i \in P} f_{l_b}(A_i)$. Given our previous results and the finite additivity of Pr, we know $f_{l_b}(P) = \sum_{A_i \in P} f_{l_b}(A_i) = \sum_{A_i \in P} Pr(A_i) = Pr(P)$. □

[8] We can fill in the details of this reasoning as follows:

$$\begin{aligned} \frac{1 - f_{l_b}(A_1)}{f_{l_b}(A_1)} &= \frac{1 - Pr(A_1)}{Pr(A_1)} \\ (1 - f_{l_b}(A_1))Pr(A_1) &= (1 - Pr(A_1))f_{l_b}(A_1) \\ Pr(A_1) - f_{l_b}(A_1)Pr(A_1) &= f_{l_b}(A_1) - Pr(A_1)f_{l_b}(A_1) \\ Pr(A_1) &= f_{l_b}(A_1) \end{aligned}$$