On Ramsey's reason to amend *Principia Mathematica*'s logicism and Wittgenstein's reaction

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Final Version: 06/10/2020 (to be published in Synthese)

Abstract In the *Foundations of Mathematics* (1925), Ramsey attempted to amend *Principia Mathematica*'s logicism to meet serious objections raised against it. While Ramsey's paper is well known, some questions concerning Ramsey's motivations to write it and its reception still remain. This paper considers these questions afresh. First, an account is provided for why Ramsey decided to work on his paper instead of simply accepting Wittgenstein's account of mathematics as presented in the *Tractatus*. Secondly, evidence is given supporting that Wittgenstein was not moved by Ramsey's objection against the Tractarian account of arithmetic, and a suggestion is made to explain why Wittgenstein reconsidered Ramsey's account in the early thirties on several occasions. Finally, a reading is formulated to understand the basis on which Wittgenstein argues against Ramsey's definition of identity in his 1927 letter to Ramsey.

Keywords: logicism, identity, Wittgenstein, Ramsey.

> The introduction of any new device into the symbolism of logic is necessarily a momentous event.
> – Wittgenstein, *Tractatus Logico-Philosophicus*, 5.452

1 Introduction

In his very influential paper *The Foundations of Mathematics* (1925, henceforth **FoM**), Ramsey defended a logicist account of mathematics, holding it to be part of logic. Instead of proposing an entirely new version of logicism, he took Russell's version, as espoused in *Principia Mathematica* (henceforth

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PM), as “a basis for discussion and amendment” [Ramsey, 1931, 1]. Amendments in **PM** were required because of “serious objections which caused its rejection by the majority of German authorities.” Ramsey believed, however, that he discovered how **PM** could be rendered free from these objections “by using the work of Mr Ludwig Wittgenstein” (obviously the *Tractatus Logico-Philosophicus*).

While Ramsey’s paper is now well known among philosophers in the analytic tradition, some questions concerning Ramsey’s motivations to write it and its reception still remain. First, to my knowledge, there is no adequate account of why Ramsey decided to work on his paper instead of simply accepting Wittgenstein’s account of mathematics as presented in the *Tractatus*. According to Potter [2000, 216], the “by now familiar reason” Ramsey had to reject a Tractarian account of mathematics is that “it does not explain the application of arithmetic.” In spite of all its familiarity however, I shall demonstrate that this reason, in the case of Potter’s reconstruction, rests on an erroneous explanation. I then show how to correct it to give a better account of the reason why Ramsey had to distance himself from the *Tractatus*.

Secondly, as a consequence of the first point, there is no proper discussion of how Wittgenstein reacted to “Ramsey’s reason” and whether he found it persuasive or not. After briefly presenting Ramsey’s proposal to amend **PM**, I give evidence that Wittgenstein was not moved by Ramsey’s objection. I then argue that, despite this fact, he was led to carefully consider Ramsey’s proposal in the early thirties since his main argument at that time against logicism depended, for its cogency, on a previous critique of this proposal. On this basis, I suggest that he thought deeply about Ramsey’s paper to see more closely the premises of his critique of logicism and the reach of his critique of Ramsey.

Finally, I consider the argument Wittgenstein advanced against Ramsey’s definition of identity (which was at the core of Ramsey’s proposal in **FoM** to amend **PM**) in a letter he wrote to Ramsey in 1927. Although this argument has already received some attention in the literature[1], no consensus about Wittgenstein’s point was reached. I give a reading that, if correct, explains on which premise Wittgenstein’s argument was based. Moreover, I connect this reading with a Tractarian point about the difference between a tautology and an equation.

2 The tractarian heritage

I begin by describing what Ramsey took from the *Tractatus* and which conclusions he drew regarding Russell’s version of logicism. Ramsey agreed, first and foremost, with the general account of logic presented in the *Tractatus*. In particular, he agreed with *i)* the conception of propositions of logic as tautologies and *ii)* the idea that every genuine proposition is truth-functionally built

[1] See Fogelin [1983], Sullivan [1995], Marion [1998, 55-72], and Trueman [2011].

from elementary propositions. As Marion [1998, 50] points out, to be consonant with these ideas, Wittgenstein had to eliminate the identity sign from his conceptual notation. Wittgenstein therefore proposed either to translate identity away from some propositions or to consider some apparent propositions as nonsensical. So, for instance, the proposition "there is only one green thing," usually expressed in predicate calculus by $\exists x(Fx \wedge \forall y(Fy \supset x = y))$, is expressed instead by the identity-free formula $(\exists x Fx \wedge \neg \exists x \exists y (Fx \wedge Fy))$. In this formula, it is supposed that different variables take different values, and the same holds for names (this convention was later called by logicians "the exclusive interpretation" of variables and of names). By contrast, sentences such as "there are at least two different things," usually expressed in predicate calculus by $\exists x \exists y\ x \neq y$, are catalogued as nonsensical. The existence of two different things would then, according to Wittgenstein, be *shown* by the meaningful use of two different names in the conceptual notation, but the notation would be unable to *say* it, because the very rules for the meaningful use of the notation would presuppose the existence of two different things. Wittgenstein also rejected Russell's attempt to define identity by means of a version of Leibniz's law of identity of indiscernibles, namely $x = y \overset{Def.}{=} \forall F(Fx \supset Fy)$, on the grounds that this definition rules out the genuine possibility of saying that two objects have all their (material) properties in common (cf. 5.5302). Ramsey agreed with this criticism as well and called the treatment of identity in **PM** its third serious defect [Ramsey, 1931, 29].[2]

The first thing Ramsey investigated in this connection was the destructive effects of the elimination of identity to **PM**'s conception of arithmetic, based on an intensional theory of classes. The most important effect was that, without identity available, there could be classes of objects given extensionally with no corresponding propositional function as their intension. Given, say, the class $\{a, b, c\}$, it is then just by accident that there is a propositional function Fx which is satisfied only by the objects a, b, and c. With identity available, by contrast, such propositional function might be constructed as $Fx \overset{Def.}{=} (x = a \vee x = b \vee x = c)$. This consequence was fatal for **PM**'s theory, which admitted classes only as correlates of propositional functions. In particular, the theory could not guarantee that the numbers 1 and 2, defined respectively as the class of all singletons and the class of all pairs, were not both equal and identical to the empty class.

One can further see in Ramsey's manuscript [002-26-01] that he also investigated whether it was possible to save **PM**'s theory by considering a hybrid theory that makes use of some ideas from the *Tractatus*. This attempt was conducted as follows: first, Ramsey supposed the definition of the n-th term of the formal series that corresponds to the sentence "the number of φ's is greater than or equal to n." Using the exclusive interpretation of variables, this formal series can be defined as follows:

[2] The other two defects were *i)* the neglect of the possibility of indefinable classes and relations in extension and *ii)* the failure to overcome the difficulties raised by the contradictions.

$$
\begin{aligned}
Nc`\hat{x}(\varphi x) \geq 1 &\overset{Def.}{=} \exists x \varphi x \\
Nc`\hat{x}(\varphi x) \geq 2 &\overset{Def.}{=} \exists x \exists y (\varphi x \wedge \varphi y) \\
&\dots
\end{aligned} \tag{1}
$$

Similarly, a formal series could be defined that corresponds to "the number of φ's is less than or equals to n":

$$
\begin{aligned}
Nc`\hat{x}(\varphi x) \leq 1 &\overset{Def.}{=} \neg \exists x \exists y (\varphi x \wedge \varphi y) \\
Nc`\hat{x}(\varphi x) \leq 2 &\overset{Def.}{=} \neg \exists x \exists y \exists z (\varphi x \wedge \varphi y \wedge \varphi z) \\
&\dots
\end{aligned} \tag{2}
$$

Ascription of numbers would then be trivially defined as: $Nc`\hat{x}(\varphi x) = n \overset{Def.}{=} (Nc`\hat{x}(\varphi x) \geq n \wedge Nc`\hat{x}(\varphi x) \leq n)$. Strict inequalities could be also defined straightforwardly.[3]

The next step Ramsey took was to ask for the meaning, according to this theory, of equations such as $n \geq m$, $n = m$ and $n \leq m$ (n and m conceived as schematic letters). For the first case, he suggested the following formula: $\forall \varphi (Nc`\hat{x}(\varphi x) \geq n \supset Nc`\hat{x}(\varphi x) \geq m)$. This formula is a tautology whenever the symbol replacing "n" is the symbol of a number greater than or equal to the number which is symbolized by the symbol replacing "m." It is clear from this case that Ramsey was trying to find an expression for arithmetic equations within the symbolism of logic, an expression that would "capture" arithmetical truths by tautologies. The other two cases above were not explicitly given, but it is not hard to imagine how he could have had defined them, so that we would have had, for the equations considered, the following definitions (I add a definition for $n \neq m$ as well, which will be useful below):

$$
\begin{aligned}
n \geq m &\overset{Def.}{=} \forall \varphi (Nc`\hat{x}(\varphi x) \geq n \supset Nc`\hat{x}(\varphi x) \geq m) \\
n \leq m &\overset{Def.}{=} \forall \varphi (Nc`\hat{x}(\varphi x) \leq n \supset Nc`\hat{x}(\varphi x) \leq m) \\
n = m &\overset{Def.}{=} \forall \varphi (Nc`\hat{x}(\varphi x) = n \supset Nc`\hat{x}(\varphi x) = m) \\
n \neq m &\overset{Def.}{=} \forall \varphi (Nc`\hat{x}(\varphi x) = n \supset \neg Nc`\hat{x}(\varphi x) = m)
\end{aligned} \tag{3}
$$

Now, how would this proposal save **PM**'s theory? Let us consider the case of $1 \neq 2$. As we previouly saw, without the identity sign, this inequality could be false in **PM**. If we consider Ramsey's theory, on the other hand, this inequality would still be a tautology, for its meaning would be $\forall \varphi (Nc`\hat{x}(\varphi x) = 1 \supset \neg Nc`\hat{x}(\varphi x) = 2)$, which is a tautology. And this holds in general for all values of m and n that result in true arithmetic equations. Considered in this

[3] It is perhaps relevant to note that Ramsey assumes here that each member of the formal series above is a meaningful proposition. This assumption draws on the hypothesis that there are infinitely many propositions of the form φx. Although the topic of infinity is a central theme of the intellectual exchange between Wittgenstein and Ramsey, it will not play any significant role in this paper.

light, arithmetical truths become tautologies, which apparently establishes the logicist thesis.

The problem that bothered Ramsey at this point was that, while this translation would make true arithmetic equations and inequalities tautologies, false arithmetic equations and inequalities would not be contradictions, but meaningful propositions. For let us consider the case of the false arithmetic inequality $3 \leq 2$. Its translation would be given by $\forall\varphi(Nc`\hat{x}(\varphi x) \leq 3 \supset Nc`\hat{x}(\varphi x) \leq 2)$. But this formula is not a contradiction, since it is true in case there are no propositional functions $\varphi\hat{x}$ that are satisfied by less than four things. The consequence Ramsey draws at this point is that $n \geq m$ and $n < m$ are not incompatible: they can be both true. And his conclusion is that "this attempt to make maths tautologies breaks down." He also finally suggested that this reasoning leads to Wittgenstein's idea that, though logic consists of tautologies, mathematics does not.

In the next Section, I shall first give a brief presentation of some features of the account of arithmetic found in the *Tractatus*. Thereafter, I shall explain why Ramsey was not satisfied with this account and why he believed that it was essential to consider false arithmetic equations as contradictions.

3 Abandoning the Tractatus

Arithmetic is dealt with in the *Tractatus* by two groups of remarks. The first group is composed of remarks 6.01-6.031, in which the main concern is the clarification of the concept of number as the exponent of an operation. The second group, composed of remarks 6.2-6.241, deals with the method of arithmetic, characterized as a logical method (6.2). The remarks between these two groups, namely, the remarks belonging to section 6.1, are devoted to the clarification of the status of the propositions of logic. As I've pointed out before, these propositions are identified as tautologies. It is hard to see why Wittgenstein chose to keep both remarks about arithmetic apart. However, it is clear from his presentation that he wished to make a clear contrast between the propositions of logic and of mathematics, and this contrast is of fundamental importance to understand Ramsey's reasons to abandon the *Tractatus* at this point.

According to the *Tractatus*, the essence of mathematical method is that it employs equations (6.2341). Equations express the substitutability of two expressions (6.24). However, that two expressions can be substituted for one another is something that must show itself in the two expressions themselves (6.23) and cannot be *asserted* by a proposition. Therefore, an equation tries to say something that cannot be said: it is, thus, a pseudo-proposition (6.2) and *nonsensical* as such (as nonsensical as the attempt to assert that there are at least two objects). Tautologies, on the other hand, although they lack sense (being for this reason *sinnlos*), are not nonsensical (*unsinnig*), because they do not try to assert anything. As Frascolla emphasizes, the logical symbols that occur in a proposition must be understood not as trying to assert that a certain meta-logical relation obtains [Frascolla, 2007, 190], but only as operations used for the construction of propositions. Therefore, in a formula such as "$\neg(\neg p \lor \neg p) \equiv ((q \supset p) \land (\neg q \supset p))$," the biconditional "$\equiv$" is not to be read as *asserting* the logical equivalence between two expressions, but only as another logical connective that is employed to construct the relevant truth-function of elementary propositions, which turns out to be a tautology in this case.

I would like to emphasize this point due to both its relevance and to the fact that the early reception of Wittgenstein's view on mathematics tended to disregard the distinction he drew between equations and tautologies. In the 1929 *manifesto*, for instance, members of the Vienna Circle characterized their view as defending the tautological character of mathematics, a conception that was "based on the investigations of Russell and Wittgenstein" [Hahn et al., 1973, 311]. It is also known that, in 1941, the mathematician G. H. Hardy attributed the thesis that mathematics consists of tautologies to Wittgenstein.[4] However, this (widespread?) view is simply false: Wittgenstein always argued against the identification of equations with tautologies. In *Philosophical Remarks*, he expressed the difference of these two notions in this way:

> It seems to me that you may compare mathematical equations only with significant (*sinnvollen*) propositions, not with tautologies. For an equation contains precisely this assertoric (*aussagende*) element – the equals sign – which is not designed for showing something. Since whatever shows itself, shows itself without the equals sign. The equals sign doesn't correspond to the '$. \supset .$' in '$p \land (p \supset q). \supset .q$' since the '$. \supset .$' is only one element among others which go to make up the tautology. It doesn't drop out of its context, but belongs to the proposition, in the same way that the '$\land$' or '$\supset$' do. But the '$=$' is a copula, which alone makes the equation into something propositional. A tautology shows something, an equation shows nothing: rather, it indicates that its sides show something. [Wittgenstein, 1975, XI:120]

Therefore, by characterizing the equals sign that occurs in mathematical equations as a "copula," Wittgenstein drew a severe distinction between the role of logical connectives in logical propositions (*i.e.*, tautologies) and the role of '$=$' in mathematical propositions (*i.e.*, equations). Equations try to *say* that the left-hand side of the equation can be replaced by its right-hand side, but this can only be shown by a demonstration. The Tractarian conclusion is that equations are pseudo-propositions. In Section 5, I argue that Wittgenstein interpreted Ramsey's attempt in **FoM** to define identity as an attempt at

[4] This occurred during a meeting of the Moral Sciences Club held in Broad's rooms. On this day, Hardy gave a talk on "Mathematical Reality," in which he discussed §20-22 of his book, *A Mathematician's Apology* Hardy [1992]. Wittgenstein was in the audience at this occasion. Wolfe Mays recalls, "Hardy mentioned that he did not accept Wittgenstein's view that mathematics consisted of tautologies. Wittgenstein denied that he had ever said this, and pointed to himself saying in an incredulous tone of voice, 'Who, I?' " [Mays, 1967, 82]

saying something by means of a tautology. Seen in this light, his argument against Ramsey's definition goes in the direction of keeping apart equations, which, as he says, do have an "assertoric element," and tautologies, which do not.

I now turn to consider Ramsey's views about Wittgenstein's conception of mathematics in the *Tractatus* (which he called the "identity theory of mathematics"). In the **FoM**, Ramsey wrote,

> As it stands this is obviously a ridiculously narrow view of mathematics, and confines it to simple arithmetic; but it is interesting to see whether a theory of mathematics could not be constructed with identities for its foundation. I have spent a lot of time developing such a theory, and found that it was faced with what seemed to me insuperable difficulties. It would be out of place here to give a detailed survey of this blind alley, but I shall try to indicate in a general way the obstructions which block its end. [Ramsey, 1931, 17]

One can indeed see in Ramsey's manuscripts (in particular in MSS 002-26-01 and 004-03-01) that he tried to develop a mathematical theory based on Wittgenstein's ideas, but in the end he could not see how to overcome certain difficulties. I do not have enough space here to go through this material in detail but, before returning to **FoM**, it will be worthwhile to examine two difficulties mentioned in these manuscripts.

The first difficulty Ramsey mentioned is that we cannot apply truth-functions to identities if the latter are conceived as nonsensical propositions. However, the application of truth-functions to equations is needed, he argues, to express some mathematical propositions. This is the case, first, of inequalities and, secondly, of propositions such as $\forall x(x^2 - 3x + 2 = 0 \supset (x = 2 \vee x = 1))$, in which the applications of "$\supset$" and "$\vee$" are apparently unavoidable.[5] Ramsey then remarks that one possible way out of this problem is to conceive of equations not as asserting the logical equivalence between two expressions but as propositions about "marks or other descriptions of the propositions" [002-26-01]. In the **FoM**, Ramsey calls these propositions "verbal propositions" and cashes them out in terms of *meaning.* The above-mentioned proposition is, for instance, interpreted as follows: for every numeric expression x, if "$x^2 - 3x + 2$" means 0, then "x" means 2 or 1. Expressions such as "$x = 2$" or "$x = 1$" would then behave as truth-arguments of truth-functions of verbal propositions.[6]

[5] In his *Critical Notice* of the *Tractatus*, published in 1923, Ramsey had already expressed his dissatisfaction with the Tractarian account of arithmetic in this way: "[this account] is evidently incomplete since there are also inequalities" [Ramsey, 1931, 282].

[6] To the reader that is not convinced that Ramsey's trick would be allowed by the *Tractatus*, I would like just to remark that Wittgenstein himself considered this possibility at the beginning of his 1929 manyscript: "Alle Gleichungen – nicht nur die Definitionen – sind Zeichenregeln. (...) Kann man Zeichenregeln durch *Sätze* – die von den Zeichen handeln – ersetzen? Wenn ja, so ist es klar daß ich die ganze Logik auf Zeichenregeln, also in einem

The second and somewhat related difficulty Ramsey alludes to concerns the application of arithmetic equations. As said above, equations express, according to the *Tractatus*, the substitutability of two expressions. But, for Ramsey, equations are not only ordinarily used as replacement rules, but also as a means to build up propositions. We do not use numbers and equations just to infer from the fact that I have $2 + 2$ coins in my pockets that I have 4 coins in my pockets, but we also use numbers and equations to say, for instance, that the number of coins in my left pocket is equal to twice the number of coins in my right pocket. A common way of analyzing this proposition is: $\exists m \exists n((Nc\text{`}\hat{x}(\varphi x) = m \wedge Nc\text{`}\hat{x}(\psi x) = n) \wedge m = 2n)$. But in this expression the equation $m = 2n$, a pseudo-proposition according to the *Tractatus*, appears as a truth-argument, which is not allowed.

Potter [2000, 216] explains Ramsey's dissatisfaction with the *Tractatus* at this point as follows. First, he mentions that Ramsey considered the possibility of working with two hierarchies of propositions: those formed by applying truth-functions and quantification to genuine propositions and those formed by applying truth-functions and quantification to verbal propositions (Potter calls these propositions *formal* propositions). Then, he says,

> Ramsey spent some time considering this possibility, but he had eventually to reject it for the by now familiar reason that it does not explain the application of arithmetic. What we have is two distinct hierarchies, connected at the base by the fact that the atomic formal propositions are equations whose meaning consists in certain schematic propositions of the genuine hierarchy. But we have no way of coping with cases where propositional functions of the genuine hierarchy occur within the scope of quantifiers from the formal hierarchy. Ramsey considered the example of the proposition 'The square of the number of φ's is greater by two than the cube of the number of ψ's'. (...) The rules which enable us to translate from Russell's notation into Wittgenstein's break down in the face of examples such as this because they involve terms for numbers: the problem is that Wittgenstein's convention cannot be extended to number terms occurring inside the scope of quantifiers. [Potter, 2000, 216]

"Wittgenstein's convention" refers to the exclusive quantification of variables and of names. Now, the proposition $\exists m \exists n((Nc\text{`}\hat{x}(\varphi x) = m \wedge Nc\text{`}\hat{x}(\psi x) = n) \wedge m = 2n)$ involves terms for numbers occurring inside the scope of quantifiers. However, it is simply false that we cannot translate this proposition into Wittgenstein's notation. The Tractarian way of expressing this proposition is as follows. We first construct the formal series:

übertragenen Sinn auf Gleichungen anwenden kann" [Wittgenstein, 1999, 7]. The fact that he at least considered this possibility indicates, I think, that this was not obviously wrong or absurd.

$$
\begin{array}{c}
(Nc\text{‘}\hat{x}(\varphi x) = 2 \wedge Nc\text{‘}\hat{x}(\psi x) = 1) \\
(Nc\text{‘}\hat{x}(\varphi x) = 4 \wedge Nc\text{‘}\hat{x}(\psi x) = 2) \\
(Nc\text{‘}\hat{x}(\varphi x) = 6 \wedge Nc\text{‘}\hat{x}(\psi x) = 3) \\
\dots
\end{array}
\tag{4}
$$

Next, we take the disjunction of the propositions of this series, obtaining in this way the required proposition. Thus, Potter's explanation does not capture the specificity of Ramsey's example. It is not too difficult, however, to correct the explanation to get a better account of what Ramsey's reason was to distance himself from the *Tractatus*. One needs only to note that, in the case of the proposition above, we have succeeded in eliminating the formula $m = 2n$ of our expression because we know beforehand that the pairs $(2, 1)$, $(4, 2)$, $(6, 3)$, etc. satisfy the equation for m and n. That is, we were only able to express the proposition *in virtue of our mathematical knowledge*. But for Ramsey it is clear that we can express propositions with the help of elementary mathematics without having to know first which values satisfy the relevant equations. I can say, for instance, that "the square of the number of Englishmen is greater by two than the cube of the number of Frenchmen" without having to know first which numbers satisfy the equation $m^2 = n^3 + 2$. As a consequence, this latter formula is, Ramsey argues, not eliminable from our expression.[7]

To sum up, Ramsey faced two crucial difficulties when he considered the Tractarian account of mathematical equations. The first is that, as nonsensical propositions, equations cannot be arguments of truth-functions. Ramsey coped with this difficulty by allowing for verbal propositions. The second is that there are cases in which we cannot eliminate the equation of the proposition because we do not know which values satisfy it. In the **FoM**, Ramsey combines both difficulties to show that the Tractarian account of arithmetic is unable to generally explain the use of equations in ordinary language. He says first that the proposition "the square of the number of Englishmen is greater by two than the cube of the number of Frenchmen" can only be analyzed as $\exists m \exists n((Nc\text{‘}\hat{x}(\varphi x) = m \wedge Nc\text{‘}\hat{x}(\psi x) = n) \wedge m^2 = n^3 + 2)$ if we take the equation $m^2 = n^3 + 2$ "to be about symbols, thereby making the whole proposition to be partly about symbols" [Ramsey, 1931, 18]. However, the proposition is clearly not about symbols, but about Englishmen and Frenchmen. Such a use of $m^2 = n^3 + 2$, he concludes, "the identity theory of mathematics is quite inadequate to explain" [Ramsey, 1931, 19].

On the other hand, the "tautology theory," he continues, "would do everything which is required" [Ramsey, 1931, 19]. If true equations are tautologies and false equations are contradictions, the expression $\exists m \exists n((Nc\text{‘}\hat{x}(\varphi x) = m \wedge Nc\text{‘}\hat{x}(\psi x) = n) \wedge m^2 = n^3 + 2)$ would be equivalent to the logical sum of the propositions $(Nc\text{‘}\hat{x}(\varphi x) = m \wedge Nc\text{‘}\hat{x}(\psi x) = n)$ for all values of m and n which satisfy $m^2 = n^3 + 2$. True equations would have the effect of selecting

[7] Landini [2007, 186] argues, contrary to Ramsey, that the proposition in question could be analyzed as "$({}^{m}\exists x)\varphi x \& ({}^{n}\exists x)\varphi x . \& . ({}^{m^2}\exists x)\psi x \equiv_{\psi} ({}^{n^3+2}\exists x)\varphi x$". I cannot see, however, how this expression could give the right analysis of the proposition.

the right disjuncts while false equations would cancel out the wrong disjuncts. Ramsey's conclusion is that "this difficulty, which seems fatal for the identity theory, is escaped altogether by the tautology theory, which we are therefore encouraged to pursue" [Ramsey, 1931, 19].

4 Ramsey's proposal and Wittgenstein's reaction

Before continuing, let me briefly summarize what we have been able to gather from our discussion so far:

1. Ramsey's intention was to defend a logicist foundation for arithmetic. Before reaching his final position in the **FoM**, he considered three theories as candidates:
2. (a) The theory advanced in **PM**, which he regarded as suffering from serious defects. In particular, Ramsey agreed with the *Tractatus* that the **PM**'s definition of identity was untenable.
   (b) A hybrid theory, in which arithmetic equations are translated into the symbolism of logic. The problem of this theory was that, while it considers true equations as tautologies, false equations are not considered as contradictions.
   (c) The "identity theory" of the *Tractatus*. This theory, however, was unable to explain all the uses we make of numbers and equations in ordinary language.
3. Ramsey was led then to his "tautology theory," which considers true equations to be tautologies and false equations to be contradictions.

To defend his theory, however, Ramsey needed the identity sign (or an analogue of it). For without a tool to collect an arbitrary set of objects by a propositional function, the prospects for an intensional theory of classes were pretty dim. And it was to recover the identity sign (after having jettisoned **PM**'s account of identity) that Ramsey opted for the introduction of the notion of a propositional function in extension. It is true that, after having introduced this notion, arbitrary classes could be defined without identity (see [Ramsey, 1931, 54]). But, as Potter [2000, 217] remarks, the notions of class, identity, and propositional function in extension form a "job lot, to be accepted or rejected together." These notions are, in Ramsey's framework, glued together, and it is expected that any trouble with one of these notions will similarly affect the treatment given to the others.

A propositional function in extension is defined in **FoM** as any functional map between individuals and propositions. Unlike Russellian propositional functions, a propositional function in extension φa need not say something about the individual a. Thus, to use Ramsey's memorable examples, φ(Socrates) may be the proposition "Queen Anne is dead" and φ(Plato) may be the proposition "Einstein is a great man." Equipped with this notion, Ramsey was able to define identity as $x = y \stackrel{Def.}{=} \forall\varphi_e(\varphi_e x \equiv \varphi_e y)$, where the quantifier $\forall\varphi_e$ ranges over all propositional functions in extension. This definition is adequate, according to Ramsey, because when x and y denote the same individual, the *definiens* becomes a tautology, and otherwise it becomes a contradiction. In a footnote, Ramsey notes that the proposition $\forall\varphi(\varphi x \equiv \varphi y)$, by contrast, is not a contradiction when x and y denote two different individuals, in spite of being a tautology when x and y denote the same individual. That is, it is meaningful to say that two different individuals have the same material properties, but it is a contradiction to say that these individuals are always mapped to equivalent propositions when we quantify over all propositional functions in extension.

Now, what was Wittgenstein's reaction, first, to "Ramsey's reason" and, second, to Ramsey's "tautology theory"? I begin by considering the first question. In Section 3, I have described why Ramsey abandoned the account of mathematical equations given by Wittgenstein in the *Tractatus*. His reason was that this account was unable to explain our use of equations to construct propositions. To repeat the example, I can very well say that "the square of the number of Englishmen is greater by two than the cube of the number of Frenchmen" without having to know first which numbers satisfy the equation $m^2 = n^3 + 2$. The account of mathematical equations given in the *Tractatus* explain the use of equations as rules of symbol substitution, but not as rules of proposition construction. While we could still translate an equation away using Wittgenstein's notation when we know which values satisfy the equation, this is not possible when this knowledge is not available.

As a matter of fact, Ramsey's equation has no solution in the positive integers. This implies that the proposition above is not a meaningful (true/false) proposition. We can see that Wittgenstein was sensitive to this fact when he discussed a similar example in *Philosophical Remarks*. This example occurs in Chapter XV, which is devoted to a critique of the "theory of aggregates" (another name he gives to the theory of classes). It is significant that this example appears in a chapter devoted to a discussion of the theory of classes since, as we have seen, Ramsey wanted to make the discourse about classes legitimate in logic with his notion of a propositional function in extension. The example is as follows:

> Is there a sense in saying: 'I have as many shoes as the value of a root of the equation $x^3 + 2x - 3 = 0$'? Even if solving it were to yield a positive integer? For, on my view, we would have here a notation in which we cannot immediately tell sense from nonsense.[Wittgenstein, 1975, XV:175]

The mentioned equation does have a solution in the integers, namely, 1. But Wittgenstein argues that, even in this case, there is a problem with this notation: it does not allow an immediate distinction between legitimate propositions and pseudo-propositions. In the continuation of the text, Wittgenstein rejects the idea that I can incorporate the equation in a proposition and *use* the proposition without knowing beforehand whether it makes sense:

> I cannot use a proposition before knowing whether it makes sense, whether it is a proposition. And this I don't know in the above case of an unsolved equation, since I don't know whether cardinal numbers correspond to the roots in the prescribed manner. (...) I mustn't chance my luck and incorporate the equation in the proposition, I may only incorporate it if I know that it determines a cardinal number, for in that case it is simply a different notation for the cardinal number. Otherwise, it's just like throwing the signs down like so many dice and leaving it to chance whether they yield a sense or not. [Wittgenstein, 1975, XV:175]

Therefore, we have evidence that Wittgenstein was not moved by Ramsey's objection to his account of arithmetic in the *Tractatus*. I move now to the second question mentioned above. It is a well known fact that Wittgenstein critically considered Ramsey's proposal again and again in the early thirties.[8] His dissatisfaction with the account presented by Ramsey in **FoM** is already present, in fact, in a letter he wrote in June 1927, in which he formulates an objection against Ramsey's definition of identity. Now, why did Wittgenstein bother so much, in the early thirties, with a theory *i)* that was born because of an objection by which he was not moved and *ii)* against which he had already argued earlier?

Two suggestions may arise here. The first is that it is possible that Wittgenstein was not satisfied with his objection to Ramsey's theory, and he then returned to it to see if he could formulate a stronger objection. In Section 5, I consider the argument Wittgenstein presented in his 1927 letter to Ramsey and conclude that the argument was mainly based on the Tractarian saying/showing distinction. Now, in the early thirties, Wittgenstein arguably continued to reason on the basis of this distinction.[9] Therefore, it is unlikely that he was dissatisfied with his earlier reason against Ramsey. Moreover, even if he was, this would be no sufficient reason, I think, for him to go back to Ramsey's theory if the theory was not in some way relevant to his own views. This last point leads to a second suggestion, namely, that Ramsey's theory of identity acquires a new significance in the face of changes in Wittgenstein's thought that occurred independently. I develop this second suggestion as follows. First, I describe some changes that occurred in Wittgenstein's account of arithmetic at the beginning of his middle period. These changes led him to reconsider the logicist theories of his predecessors. For different reasons than those given in

[8] See, *e.g.*, [Wittgenstein, 1975, XI:120−121], [Wittgenstein, 2005, §113] and [Wittgenstein, 1969, II-§16].

[9] On this point, see Marion [2012].

the *Tractatus*, the middle Wittgenstein rejected these theories and formulated new arguments against them. I then show that the main argument he lauched against logicism at that time depended on a previous critique of the use of identity as a legitimate propositional function. This dependency may have led him, I suggest, to reconsider Ramsey's theory of identity to see more closely what the premises of his critique of logicism were and the reach of his critique of Ramsey.

On page 19 of MS 105, written soon after Wittgenstein returned to Cambridge in 1929, Wittgenstein says: "I am apparently thrown back against my will on arithmetic." From this page on he started working on a theory of arithmetic that was very distinct from the one described in the *Tractatus*. The literature is usually silent on the ultimate reasons for this reconsideration of arithmetic, essentially because the reconstruction of Wittgenstein's thought at the beginning of 1929, particularly before he started to write the remarks that served as the basis for the discussion of *Some Remarks on Logical Form*, is a task yet to be done. Here I can only give some hints about what was happening in this period of Wittgenstein's thought with regard to arithmetic. In MS 105, one page after having said that he needed to return to thinking about arithmetic, he wrote:

> How does this theory relate to the theory of Frege and Russell? The first difference is that in Frege's theory a one-one relation is constructed; this is disallowed and presupposes a false conception of identity. Secondly, a class is constructed with a certain number of members, and that is disallowed for the same reason. This fundamental class (*Grundklasse*) would be in my theory the class of substantives in a certain correlation (and, to be sure, *in extenso*).[Wittgenstein, 1999, 8]

It is clear that the theory to which Wittgenstein refers above as "my theory" is *not* the theory he set out in the *Tractatus*, since this latter theory does not mention anything about "substantives" and "fundamental classes." It is a theory he was drafting and which could supposedly overcome some essential difficulty which he realized was present in his earlier views. It is also clear that, by comparing his theory to the theory of Frege and Russell, he manifests a need for a theory similar to those of his mentors. At the same time he could not accept Frege and Russell's theories in their original form due to the false conceptions of identity that they each presupposed. Some pages later[10], he continued to work on his theory and started to flirt with the idea that number is a "scheme" or a "picture" of an extension. And these ideas will later develop into a theory of arithmetic, in *Philosophical Remarks*, which is very distinct from the theory of the *Tractatus*. Two key differences are that:

[10] See, in particular, p. 100-6 (even pages) and p. 109-129 (odd pages).

1. Numbers are not presented anymore as “exponents of operations,” but as “pictures of the extensions of concepts” [Wittgenstein, 1975, X:100].
2. The applicability of arithmetic no longer depends on the general form of an operation, but arithmetic is described as “autonomous,” thereby guaranteeing its applicability [Wittgenstein, 1975, X:111].

Wittgenstein was thus led to admit that numbers do not have only an *operational* function in language, but additionally have a *symbolic* (pictorial) function. At the end, then, Ramsey was right: the “identity theory” presented in the *Tractatus* was unable to explain all the uses we make of numbers and equations in language.

In spite of this change, however, Wittgenstein kept his fundamental insight in the *Tractatus* that equations are not tautologies. This insight appears in *Philosophical Remarks* as a rejection of the possibility of reducing a calculus of extensions to a calculus of tautologies. The main argument he invokes against such a reduction thematizes the heart of the logicist project, namely, Hume’s Principle (henceforth **HP**). This principle states a bi-implication between, on one hand, an assertion of identity between two number ascriptions (say, to concepts φ and ψ) and, on the other hand, the existence of a one-one correlation (here $\approx$) between the objects that are φ and the objects that are ψ:

$$(\varphi \approx \psi) \equiv (Nc\text{‘}\hat{x}(\varphi x) = Nc\text{‘}\hat{x}(\psi x)) \tag{5}$$

As it is well known, **HP** is used by Frege and Russell to define numbers as equivalence classes of the $\approx$ relation.[11] Wittgenstein objected to this definition by arguing that it confuses the *possibility* of a one-one correlation with the *existence* of a one-one correlation. The argument can be briefly stated as follows (here I closely follow Marion and Okada [2014], who label the argument as “the modality argument”):

1. Although there always *can be* a one-one correlation between the objects belonging to any two equinumerous classes, it is not the case that there always *is* such a one-one correlation.
2. The *existence* of a one-one correlation can establish that two classes of things are equinumerous. But the mere *possibility* of a one-one correlation cannot establish this, for this possibility is not a *condition* for two classes of things being equinumerous, but a *consequence* of this equinumerosity.
3. Therefore, sameness of number can be defined
   (a) neither by the existence of one-one correlations (because in some cases there are no such correlations),
   (b) nor by the possibility of one-one correlations (due to the conceptual priority of sameness of number).

[11] I abstract here from the differences of Russell and Frege with regard to how classes are conceived by each author.

One can readily see that the modality argument depends for its cogency on a previous critique of the use of identity as a propositional function. For, if identity is allowed, from the possibility of a one-one correlation between two classes one could prove, at least for finite classes, the existence of such a correlation. For instance, from the possibility of correlating one to one the two classes $\{a, b\}$ and $\{c, d\}$, one could exhibit a one-one correlation, *e.g.*, $(x = a \wedge y = c) \vee (x = b \wedge y = d)$. But then the argument would fail, for Wittgenstein acknowledges that from the existence of a one-one correlation one can establish sameness of number. I conjecture that this dependency of the argument on a previous critique of identity was the reason Wittgenstein had to reconsider again and again Ramsey's theory of identity in the early thirties.

5 The 1927 objection to Ramsey's treatment of identity

In this Section, I give a reading that could help us understand the way Wittgenstein argues in his 1927 letter to Ramsey. For the reader's convenience, I quote the body of the letter in full. I follow here Sullivan [1995]'s subdivisions, but I add at the end a part unremarked by Sullivan (part **I** below). This last part contains an important indication of the fact that Wittgenstein conceived of the saying/showing distinction as playing a prominent role in his argument. The absense of this last part in Sullivan's reconstruction of the argument reveals, I think, a blind spot in his very detailed reading of the argument.

> [**A. Preliminaries**] You define $x = y$ by
>
> $$(\varphi_e) : \varphi_e x \equiv \varphi_e y \text{ - - - - - - - - - - - - } Q(x, y)$$
>
> and you justify this definition by saying that $Q(x, y)$ is a tautology whenever 'x' and 'y' have the same meaning, and a contradiction, when they have different meanings.
> I will try to show that this definition won't do nor any other that tries to make $x = y$ a tautology or a contradiction.
>
> [**B. Case 1: Setting out**] It is clear that $Q(x, y)$ is a logical product. Let 'a' and 'b' be two names having different meanings. Then amongst the members of our product there will be one such that $f(a)$ means p and $f(b)$ means $\neg p$. Let me call such a function a critical function f_k.
>
> [**C. Case 1: Argument**] Now although we know that 'a' and 'b' have different meanings, still to say that $a = b$ cannot be nonsensical, if $a \neq b$ is to have any sense. For if $a = b$ were nonsensical the negative proposition, i.e., that they [do not][12] have the same meaning, would be

[12] As Sullivan [1995, 132] points out, this addition is plainly required by Wittgenstein's argument.

nonsensical too, *for the negation of nonsense is nonsense.* Now let us suppose, wrongly, that $a = b$, then, by substituting a for b (which must be legitimate if we have given $a = b$ the right meaning[13]) in our logical product the critical function $f_k(a)$ becomes nonsensical (ambiguous) and, consequently, the whole product too.

[**D. Case 2: Setting out**] On the other hand, let 'c' and 'd' be two names having the same meaning, then it is true that $Q(c, d)$ becomes a tautology.

[**E. Case 2: Argument**] But suppose now (wrongly) $c \neq d$. $Q(c, d)$ remains a tautology still, for there is no critical function in our product. And even if it could be supposed (which it cannot) that $c \neq d$[14]; surely a critical function f_k (such that $f_k(c)$ means p, $f_k(d)$ means $\neg p$) cannot be supposed to exist, for the sign becomes meaningless.

[**F. Case 1: Résumé**] Therefore, if $x = y$ were a tautology or a contradiction and correctly defined by $Q(x, y)$, $Q(a, b)$ would not be contradictory but nonsensical (as this supposition, if it were the supposition that 'a' and 'b' had the same meaning, would make the critical function nonsensical). And therefore $\neg Q(a, b)$ would be nonsensical too, for the negation of nonsense is nonsense.

[**G. Case 2: Résumé**] In the case of c and d, $Q(c, d)$ remains tautologous, even if c and d could be supposed to be different (for in this case a critical function cannot even be supposed to exist).
I conclude: $Q(x, y)$ is a very interesting function, but cannot be substituted for $x = y$.

[**H. Consequence**] The mistake becomes still clearer in its consequences, when you try to say 'there is an individual'. You are aware of the fact that the supposition of there being no individual makes 'absolute nonsense'. But if $(\exists x).x = x$ (E) is to say 'there is an individual', $\neg E$ says: 'there is no individual'. Therefore from $\neg E$ follows that E is nonsense. Therefore $\neg E$ must be nonsense itself, and therefore again so must be E.

[**I. Possible answer and rejoinder**] The case lies as before. E, according to your definition of the sign '=' may be a tautology right enough, but does not say 'there is an individual'. Perhaps you will answer: of course it does not say 'there is an individual' but it *shows* what we really mean when we say 'there is an individual'. But this is not shown by E, but simply by the legitimate use of the symbol $(\exists x)$..., and therefore

[13] This sentence inside brackets occurs as a footnote in the letter.

[14] The letter received by Ramsey has "$c = d$," but it is plain that "$c \neq d$" is intended.

just as well (and as badly) by the expression $\neg(\exists x).x = x$. The same, of course, applies to your expressions 'there are at least two individuals' and so on.

Wittgenstein's argument is clearly an argument by cases. He first analyses the situation in which two names 'a' and 'b' have different meanings (parts **B** and **C**), and then the situation in which two names 'c' and 'd' have the same meaning (parts **D** and **E**). The first thing that should be noticed about the argument is that there is a fundamental asymmetry regarding the two cases. As it can be seen from the summary of the argument (parts **F** and **G**), in the first case the conclusion is that $Q(a, b)$ is nonsensical and in the second case the conclusion is that $Q(c, d)$ is a tautology no matter what is supposed about the identity between the meanings of 'c' and 'd'. This is what allows him to say that $Q(x, y)$ is "a very interesting function," instead of being merely nonsensical. The function is interesting, I suppose, because it is only significant (and tautologous) when both arguments have the same meaning (being nonsensical otherwise).

The second important point that should be noticed is that, in the second case, Wittgenstein is apparently also complaining that $Q(c, d)$ behaves differently from how it should behave *if* $Q(x, y)$ could be substituted for $x = y$. He says, "$Q(c, d)$ remains tautologous, even if 'c' and 'd' could be supposed to be different." It seems, then, that he expected that $Q(c, d)$ would not be tautologous in this case. Thus, Wittgenstein is presupposing that a difference in the supposition regarding the truth value of an identity would make some difference in the *definiens* of Ramsey's definition. As far as I can see, he is therefore interpreting Ramsey's definition of $x = y$ as an attempt at *saying* (not *showing*) something, namely, that the names on the left and on the right-hand side of the equation have the same meaning and that, therefore, there is no critical function in $Q(c, d)$. The supposition that $c = d$ is false would then be the supposition that 'c' and 'd' do not have the same meaning and, consequently, that there is a critical function in $Q(x, y)$. But this critical function, Wittgenstein argues, cannot be supposed (at the risk of nonsensicality) and, as a consequence, $Q(c, d)$ remains tautologous in this case.

My reason for advancing the above hypothesis is that the notion of *showing* is not sensitive to different suppositions: what shows itself in the notation does so independently of any supposition regarding the truth value of a proposition. Therefore, we shall assume that Wittgenstein is not interpreting Ramsey's definition as an attempt at showing something. Against this conclusion, it may be argued that Ramsey's definition was intended to be either a tautology or a contradiction and, therefore, could not say anything at all. But I think that here we should read Wittgenstein as considering, just *for the sake of the argument*, the possibility of a content being said by tautology or a contradiction. It would not say, of course, something about reality, but it would still say something about language or the meaning of our expressions. I think that Wittgenstein is drawing mainly on his idea, mentioned in Section 3 above, that the equals sign is an assertoric element, designed for saying something. Part **H** of the letter contains evidence for the correctness of my reading. There, Wittgenstein says, "E, according to your definition of the sign '$=$' may be a tautology right enough, but does not say 'there is an individual'." Now, if the possibility of saying something with a tautology was out of the question from the start, then this remark would be of course correct, but at the same time pointless.

Therefore, if my analysis above is right, Wittgenstein presupposes that Ramsey's definition is an attempt at saying the identity of meaning. Against Ramsey, he argues that this presupposition leads, in the first case, to the fact that $Q(a, b)$ is nonsensical and, for this reason, cannot say anything at all and, in the second case, to the fact that $Q(c, d)$ remains tautologous independently of our suppositions regarding the meaning of 'c' and 'd' and, for this reason, cannot express the content of these different suppositions. It is worth noticing that the argument in no way depends on the "knowledge" of the identity relation between 'a' and 'b', or between 'c' and 'd', but rests solely on the analysis of the consequences of the supposition that $Q(x, y)$ could work as a substitute for identity. Instead of saying "although we know that 'a' and 'b' have different meanings" (see part **C**), he might as well have started the argument as follows: "let 'a' and 'b' be two names. They either have different meanings, or they do not. In the first case, ... In the second case ..."

In part **H**, again, Wittgenstein considers a possible answer to his line of reasoning, namely, that, although E does not say something, it *shows* what we really mean when we say "there is an individual." To this answer he offers a rejoinder in the context of part **H**, but it is fruitful to consider this possible answer (and how a rejoinder could be made) in the second case as well. Consider then, the following possible answer in defense of Ramsey with respect to **E**: "of course '$c = d$' does not say that 'c' and 'd' have the same meaning, but it *shows* what we really mean when we say that." This would be indeed an obvious answer to Wittgenstein's criticism since, as we have seen, $c = d$ turns out to be a tautology in this case. So the question is then why is it that the fact $c = d$ is a tautology cannot *show* that 'c' and 'd' have the same meaning?

Here I think that Wittgenstein would answer that it is not the fact that $c = d$ is a tautology in light of Ramsey's definition that shows that 'c' and 'd' have the same meaning, but this is shown instead by the very sensicality of $Q(c, d)$ (as opposed to the nonsensicality of $Q(a, b)$). Therefore, to paraphrase part **I** of the letter, this would be shown as well (and as badly) by the expression $c \neq d$. The conclusion is, thus, that $c = d$ is neither capable of *saying* nor of *showing* the identity of meaning.

6 Final remarks

In this paper, I have attempted to fill some gaps in the interpretation of certain questions concerning the historical context of Ramsey's **FoM**. First, an explanation for why Ramsey tried to amend **PM** instead of accepting Wittgenstein's views in the *Tractatus* was offered. According to my explanation, Ramsey thought that the Tractarian account of the application of equations was incomplete, for it does not explain how we use general equations (like $m^2 = n^3 + 2$) to build up propositions when we do not know which values satisfy the equation. Secondly, evidence was presented for the fact that Wittgenstein was not moved by Ramsey's objection. However, the fact that he returned again and again to Ramsey's proposal in the early thirties indicates that he thought of it as relevant to the understanding of his own views. I showed that this is the case, because his main argument against logicism at that time depended, for its cogency, on a critique of Ramsey's definition of identity. I then suggested that he was led to reconsider Ramsey's theory of identity to see more exactly the interconnection between these two topics.

Finally, I have considered the argument Wittgenstein made against Ramsey in his 1927 letter. According to the reading I have offered, Wittgenstein was advocating for the impossibility both of showing and of saying the identity of meaning by means of Ramsey's definition. His main argument in the letter, as I described in Section 5, runs against the possibility of *saying* the identity of meaning. I have indicated, however, why Ramsey's definition is unable also to *show* the identity of meaning. Wittgenstein thus reaffirms in his letter the Tractarian intention to keep equations and tautologies apart: the former containing an assertoric element which is not designed for showing something, and the latter being able to show something but without asserting anything and, therefore, without being able to capture the assertoric character of an identity statement.

Acknowledgments

I acknowledge FAPESP for providing financial support during this work. For discussion and criticism, I am grateful to Timm Lampert and Markus Säbel. I would also like to thank Husna Farooqui, Tianyu Wu, and Landon Elkind for their precious editorial suggestions.

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