Dynamics, Quantum mechanics and the Indeterminism of nature

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Abstract

We show that determinism is false assuming a realistic interpretation of quantum mechanics and considering the sensitive dynamics of macroscopical physical systems.

1 Introduction

One of the greatest philosophical question bothering mankind down the ages is whether every event that happens in the world is causally determined by prior occurrences. The question is especially virulent if we consider the behavior of our species on earth. Is it determined by our nature as a physical being or is there free will and thus responsibility beyond?[1]

In this paper we argue that nature is not determined at all. We claim that the evolution of a macroscopical physical systems that is sensitive, in a well defined mathematical sense, is indeterminate due to microscopical quantum effects. By natural condition only a probability distribution of possible states that may arise in the future can be derived. Our argument presupposes a certain realistic interpretation of the wave function in quantum mechanics, but no assumption on our role as conscious observers of physical reality and thus differs from other arguments in Quantum philosophy [5, 16]. Moreover we claim that major physical systems like the solar system, the climate system on earth and the central nervous system of humans are in fact sensitive and thus do not evolve deterministically. The results of natural science are in general stochastic, the fact that one of many possible events occurs in the temporal evolution of a system remains open to metaphysical interpretation.

The rest of the paper is organized as follows: In the next section we give a short introduction to the mathematical theory of dynamical systems, especially we introduce the notion of sensitivity. In section three we consider Quantum mechanics and present our minimal

[1]Consider the philosophical works of Baruch Spinoza (1632-1677) [34], Gottfried Wilhelm Leibnitz (1646-1716) [22], David Hume (1711-1776) [10], Imanuel Kant (1724-1804) [12], Pierre Laplace (1749-1824) [14], Arthur Schppenhauer(1788-1860) [19] and Daniel Dennett (1942-) [6].

realistic interpretation. In section four the reader will find our argument against determinism assuming sensitivity and in the following section we indicate that natural systems are in general sensitive. The last section contains some metaphysical speculations how consciousness affects undetermined spatial-temporal systems.

2 Mathematical theory of dynamical systems

Dynamical systems which have a complicated long term behavior were intensively studied in mathematics since the pioneering work of Poincare on the three body problem coming from Newtonian mechanics [17]. In the 20th century many concepts were clarified and mathematical tools to analyze dynamical systems were developed. The field was popularized under the label "chaos theory" and had wide influence on other scientific developments.

We choose here a general approach to introduce the mathematical concepts we need, without going very deep into mathematical details.[2]

In the mathematical theory of dynamical systems we consider a space X of states of a system. We may assume that X is equipped with a metric d which measures the distance between two states. If X models states of nature this assumption is necessary to quantifies the difference os states. Moreover we may assume that X is complete which means that all convergent sequences have a limit inside X. The evolution of systems is described by a family of maps E_t where t is the time variable. The family of maps acts on the space of states:

$$E_t : X \longmapsto X \qquad \text{with} \qquad E_{t+s} = E_t \circ E_s$$

This condition describes time consistence: the state after $t + s$ time units of evolution equals the state we get by first s and then t time units of evolution. Here s and t are real numbers, if we assume that time is a continuum and s and t are integers if we assume time to be discrete. Depending on what kind of process we describe E_t will have additional mathematical properties, as (piecewise) differentiability, continuity or least measurability.

Now we introduce the concept of sensitivity which will be crucial for our argumentation. We say that a state $x \in X$ is sensitive for E if any neighborhood of x, as small as it may be, contains at least one different state y such that

$$\limsup_{t \longmapsto \infty} d(E_t(x), E_t(y)) > 1.$$

This condition means that there is a sequences of moments t_n for $n \in \mathbb{N}$ such that the evolution of the initial states x and y become sperate with distance ta least one for these

[2] A popular introduction to "chaos theory" is given in [8], as introduction to the mathematical theory of dynamical systems [1] may serve and for advances studies we recommend [13].

moments. Our definition goes basically back to the influential work of Li and York [15]. The bound one we use is somehow arbitrary, since the scaling of the metric is. In a application of the mathematical theory to the physical world we may assume the metric is scaled in a way such the $d(x, y) > 1$ means that there is a macroscopic difference between the states x and y big enough to be recognized without technical help.

We have defined the notion of single sensitive state, a whole system (X, E) is sensitive if there are many sensitive states in X, but what does "many" mean here? From a topological point of view we may say that (X, E) is sensitive if the set $S \subseteq X$ of sensitive states is residual in X, the complement of S is than negligible as the union of nonwhere dense sets. If X is a finite dimensional manifold we may also choose a measure theoretical approach and say that the whole system is sensitive if S has full measure, $m(S) = m(X)$. If the space X has a fractal geometry, for instance as the attractor or hyperbolic set of larger system, than an appropriate notion is that S has full Hausdorff dimension, $\dim_H S = \dim_H X$ [28].

It is a difficult mathematical problem to prove rigourously that a given dynamical system is sensitive. This is especially true for systems describing "real world" phenomena. One approach that goes back to the work of Steven Smale is to find a symbolic coding of the dynamics of the system and then infer from symbolic to metric dynamics [35, 7]. Another strategy is to prove the existences of positive Lyapunov exponents $\lambda \in \mathbb{R}$ and related unstable manifolds M. The evolution then expands distances along these manifolds exponentially with factor e^λ and thus guarantees the existence of sensitive states. The Lyapunov time used by physicist is the inverse of the largest Lyapunov exponent λ^{-1}, it is the time in which distances are expanded by factor e. Another approach to prove sensitive uses entropy, as a measure of the information produced by a dynamical system. Entropy is intimately related Lyapunov exponents via dimension and helps to demonstrate sensitivity [23, 24, 2]. In contemporary mathematics we have a long list of dynamical systems that were proved to be sensitive, the list contains unimodal maps, holomorphic maps on their Julia set, Horseshoe maps and maps possing heteroclinic orbits, Solenoidial maps and certain coupled oscillators etc. On the other hand at this time the classification of all dynamical system with respect to sensitivity is far out of reach.

Note at the end of this section that the dynamics described so far is fully deterministic for points in x in the state space X, the future $E_t(x)$ is uniquely determined for all moments t and if E_t is invertible even the past is.

3 A realistic Interpretation of Quantum mechanics

Quantum mechanics is the best physical theory to describe microscopic reality we have. It is spectacularly successful in terms of power and precision. Our approach here is again

very general just introducing basic concepts.[3]
A physical state in Quantum theory is represented by a wave functions ϕ which is an element of a complex Hilbert space $\mathfrak{H}$. Beside being compete metric a Hilbert space admits an inner product $< .,. >$ and is thus the best infinite-dimensional generalization of the Euclidian space. An observable, like position or momentum of a particle, is in quantum mechanics formally described as an Hermitian operator O acting on $\mathfrak{H}$. Possible observations performing a measurement on a system are given by the eigenvalues $\mu \in \mathbb{R}$ of O defined by the invariance relation

$$O\psi = \mu\psi$$

for some $\psi \in \mathfrak{H}$. Note that value μ that could be observed is not unique given a wave function ϕ, there is a whole spectrum $S(\phi)$ of values that could be observed. A state ϕ only determines the probability of observing λ by Born's rule

$$P(\mu) = | < \phi, P_\mu \phi > |^2$$

where $P_\mu \phi$ is the projection of ϕ onto the subspace determined by the eigenstates ψ. If we have two Observable O_1 and O_2 the famous Heisenberg uncertainty principles states

$$\Delta_\phi O_1 \Delta_\phi O_2 \geq \frac{1}{2} | \, [O_1 O_2 + O_2 O_1 \,]_\phi |$$

Here Δ_ϕ denotes the standard deviation of an observable in a state ϕ and $[.]_\phi$ is the mean of an observable in state ϕ. Especially we get for place x and momentum p of a particle

$$\Delta x \Delta p \geq \frac{h}{4\pi}$$

where $h = 6.626 \times 10^{-34} Js$ is Plank's constant. The uncertainty principle means that the precision of measurement of one observable bounds the precision of measurement of other observables. We can not measure both quantities with arbitrary precision at the same time. So far there is an agreement of physicists. On the other hand there is still an substantial dispute among physicist and philosophers on the interpretation of quantum mechanic. There are two main questions. The first is what is the meaning of the wave function, is it just a mathematical tool in the sense of an instrumentalist interpretation or does it have a realistic meaning? The second is what is the role of the observer in a measurement. It is claimed that a measurement is a physical process that "collapses" or "reduces" the wave function ϕ to the eigenstate ψ corresponding to the measured value μ. There exits many interpretations of the "collapse" of the wave function which are in part quite mysterious or even esoteric.[4] We hold here a minimal realistic interpretation of

[3] An introduction to Quantum physics is given in [3], an advanced text book is [33]

[4] Have for instance a look at the entries in Stanford Encyclopedia of Philosophy concerning this.

quantum theory: There exists an observer independent physical reality which is described at microscopical level by quantum theory, the wave function in quantum theory refers to a state of this observer independent microscopical reality. This interpretation is minimal in the sense that we do not comment on the collapse of the wave function or the role of the observer. The main point of the paper is to show that this is not necessary to refute determinism. The classical Copenhagen interpretation of quantum mechanics, held by many physicists, leaves the philosophical question of the ontology status of the wave function largely open, so our thesis is consistent with this interpretation.[5]
Now if the wave function refers to something real what it is? Consider a classical state space X containing variables for place, momentum of particles or other macroscopical observables. Assume that this space models the phenomenology reality given to us by perception. A wave function from ϕ from a Hilbert space $\mathfrak{H}$ in quantum mechanics projects to a subset $U_\phi \subseteq X$. The subset contains all possible observation of the system. Due to our realistic interpretation of quantum mechanics a macroscopical physical state is not a point $x \in X$ but a subset $U \subseteq X$. The size of this set is determined by Heisenberg's uncertainty principle. This means that physical states at microscopical scale are not correctly described as a tuple of numbers but by tuple of of sets of numbers. States of physical reality are in this sense uncertain with respect to their quantification. Bounded quantificationablity seems to us an essential feature of spatial-temporal nature.

4 Indeterminism

The main argument of this paper reads, to put it in a nutshell, as follows:

- The states of the physical world are uncertain at microscopical scale.
- The macroscopic dynamics of the physical world is sensitive.

- Hence the evolution of the physical world is not deterministic.

Let us develop the details of our argument. Due to our realist interpretation of quantum mechanics a physical state of a natural system is given by a subset U of the space X of variables quantifying the system. Let $d(U)$ be the diameter of the state with respect to the metric d on X. Since a physical state is not given by a point in x we have $d(U) > 0$. The diameter of U describes the microscopical uncertainty determined by a wave function in quantum mechanics, respective by the uncertainty principle. Now let E_t be a macroscopical evolution with time variable t acting on the space X . Let us assume

[5] The Copenhagen interpretation is not a homogenous view concerning this issue, Heisenbergs position seems to be more realistic than Bohr's original view, compare [9]

that the system (X, E_t) is sensitive. Than for some sequence of moments t_n we have $d(E_{t_n}(U)) > 1$, the diameter of the evolved set $E_{t_n}(U)$. By this lower bound E_{t_n} is not one macroscopical physical state; distance one means by our assumption on d in section one a macroscopical differences. $E_{t_n}U$ consists of a whole spectrum

$$S_{t_n} = \{U_i \subseteq X | i \in I\}$$

of physical states U_i that possibly arise in the temporal evolution of the system. We see that a sensitive macroscopical system is not deterministic due to quantum physical uncertainty. Given an initial physical state we can determine possible states but not one state that arises in the future. Moreover using Born's rule we may introduce a probability measure μ on U and using stochastic dynamics we can derive probabilities $\mu(U_i)$ of the macroscopical states U_i after a time of evolution. Which state really arises is not determined by physical conditions.
So far we assumed sensitivity of physical systems in their mathematical description. To conclude the argument we have to show that systems in physical reality are in fact sensitive, in the sense that good physical models for their evolution have this mathematical property. In the next section we will argue for this claim in the the case of three major examples and conclude that spatial-temporal nature does not evolve deterministic in general.

5 Sensitivity of nature

5.1 The evolution of the solar system

Consider the solar system, consisting of the sun and the planets bounded by gravity to it. The motion of the system is described by the laws of celestial mechanics with some corrections due to general relativity and existence of satellites. Scientist believed for a long time that the solar system is deterministic, and even tody many people think that the position of the planets in the future is fixed and may be calculated. The evolution of solar system is thus a paradigmatic examples in classical determinism.
We assert here that the solar system is sensitive. It is a difficult and still unsolved mathematical problem to prove that the solar system is sensitive, in the mathematical sense defined in section one. We can not solve the equations governing systems analytically and a proof of the existences of a positive Lyapunov or positive entropy is not available today. On the other hand we have strong numerical evidence that the solar system admits a positive Lyapunov exponent and is thus sensitive. Let us present some details concerning this issue.
The first evidence for sensitivity of the solar system was found by Sussman and Wisdom

[36] at the end of the eighties using numerical integrations. Their results suggests that the orbit of Pluton gets unpredictable after a Lypunov time of 20Ma. Further numerical investigations of the full Solar System performed by Laskar [20, 21] included the Newtonian interaction of 8 major planets and gave an estimate of Lyapunov time of 5 million years for the whole system. The Lypunov exponent was later on confirmed in an investigation of a more realistic model by Sussmann and Wisdom. An error of 1m in the Earth's initial position gives rise to an error of about 10 m after 10 Ma; but this same error grows to 10 million km after 100 Ma. Thus it becomes essentially impossible to predict the motion of the planets with precision beyond 100 million years. [37]. We safely conclude that our assertion is true. On the very(!) long run even small Quantum uncertainty could thus lead to indeterministic behavior of solar system. Of course much earlier fluctuation of the mass distribution of the sun and the planets and other macroscopic effects lead to unpredictability. But these fluctuation themselves are likely sensitive processes and hence indeterministic. So we have good reasons to belief that not even the evolution of solar system is deterministic.

5.2 Climate change on the earth

At the beginning of the 21th century we observe a real hype on climate prognosis. What will be the mean temperature and the distribution of temperature on earth in hundred ore two hundred years? A lot of effort and money is spent for answers of these questions. Of course there good reason for this. If we look at the increase of the global surface temperature in the last fifty years there is likely to be a correlation to the increasing concentrations of greenhouse gases, resulting from human activity such as fossil fuel burning and deforestation [11].

We conjecture here that climate dynamics is a sensitive process and hence the mean temperature on earth in hundred years is not at all determined. The first argument for this conjecture is simple. Human behavior on earth is sensitive and not determined (compare with the next section) so the climate system is, if we assume effects of human activity. Models for climate prognosis taking into account this constrain include different emission scenarios. But even if we prescribe future production of greenhouse gases by mankind in our model, we have good reason to belief that the system remains sensitive.

One of the earliest examples in applied sciences of a sensitive system is the Lorenz attractor from meteorology [27, 38]. It represent the convective motion of fluid cell which is warmed from below and cooled from above and models the unpredictable behavior of the weather. The differential equations given the Lorenz attractor are obtained from the Navier-Stokes equations in fluid dynamics by simplification and reduction. Now fluid dynamics and the Navier-Stockes equation play in central role in climate models. Coupled atmosphere-ocean circulation models provide numerical solutions of these equations devised for simulating mesoscale to large-scale atmospheric and oceanic dynamics. Thereby the Navier-Stockes equation are at present mathematically intractable. The existences and the properties of solutions are not at all known and brute force numerical methods are mathematically not verified [4]. In view of the Lorenz attractor we suppose that the evolution given by Navier-Stockes equation is sensitive and thus climate system is. To prove this rigorously could be a task for further generations of mathematicians. If our conjecture is true we could conclude that mean temperature on earth in 100 years is not determined, since climate system is not deterministic. Perhaps natural science will give trustworthy estimates on the probabilities(!) of long term temperature changes in a few centuries. Today neither our mathematical skills nor our models are not sufficient do this honestly.

5.3 Human behavior

Consider a human being viewed as part of spatial-temporal nature. The state of a human in this sense is described by the state and interaction of around 100 Trillion cells and the causal effects of the environment on this system. From human biology we learn that the behavior of living human is governed by his cental nervous system. Now if we had a complete description of the central nervous system of a human and its stimulation, can we predict the behavior of the human? Is the temporal dynamics and thus output on the peripheral nervous system determinate given state and input of the central nervous system? We claim here that the brain and thus the complete nervous system of humans and other animals is sensitive and thus not deterministic. Even smallest uncertainty may lead to different dynamics and in consequences to different behavior. A future neurobiology may once perfectly predict the probability of possible behavior of a human. But there remains a whole spectrum of possible behavior of humans and animals under physically indistinguishable conditions.

Let us provide some evidence for our thesis. The brain may be modeled by a neuronal network which corresponds, in a mathematical description, to a networks of coupled oscillators [41]. In neuroscience usually layer neuronal networks are considered as appropriate models [19]. In general it is difficult to analyze these models rigorously. Nevertheless we know that network-level interactions can lead to dynamical instabilities which are not present in isolated neurons. The instability may cause a networks response to depend sensitively on its internal state and the stimulus that is presented [40, 39]. Especially in the single and two layer neuronal we have resent results on the Lyapunov exponent of such systems. Single layer networks do not have a positive Lyapunov exponent in general and are thus fairly reliable. Ont the other hand two-layer networks can be show to have a positive Lyapunov exponent if recurrent connections, even with a small amount of feedback, exists [25, 26]. Now if we think about the brain as a multi-layer network with many

recurrent connections it is reasonable to assume that this system has a positive Lyapunov exponent and is highly sensitive.
Microscopical indeterminacy leads via sensitivity to indeterminacy of the brain. We hence hold the controversial position that quantum effects are relevant for human behavior. Roger Penrose [31, 32] tried to locate the effects of quantum gravity in the microtubuli of the cytoskeleton at the subneuronal level of the brain. He speculates about the "orchestrated objective reduction" of the wave function as the origin of consciousness and the way its effects nature. Also this approach is attractive we do not need an interpretation of the reduction of wave function to refute determinism and argue metaphysically for effects of consciousness.

6 Effects of consciousness

To put the thing as simply as possible on the one hand there are physical states of spatial nature and on the other hand there are mental states of consciousness, which are not spatial. Assuming classical dualism there is no identity or reduction between these entities. Now nature evolves by physical laws which are, as we have shown here, essential stochastic; some probability distribution of a spectrum possible states that may arise in the future is determined. Is the human brain hence a huge gambling house dicing out our behavior and our consciousness remains epiphenomenal in nature? This position seems to us unacceptable, since it ignores the phenomenon of will power. Our decisions, as irreducible mental events in our consciousness, do have some impact on our behavior. To make our point explicit, define a unit of will power as the mental power that is needed to make a behavior that appears with probability 1/2 under physical conditions certainly determined. The will power that we need for a physically impossible action is infinite and the will power for a physically certain action is obviously zero. Therefore we suggest a mental-physical low of will as follows:

$$\mathfrak{W}(P) = \frac{1-P}{P}.$$

Here P the probability that an action appears under physical conditions and $\mathfrak{W}(P)$ is the will power that is needed to make this action certain. Probability is in this sense the acceptance resp. resistance of nature to will power of consciousness. The will power of individuals is bounded and subjected to temporal changes that depend on preceding mental episodes. Therefore physical possible behavior is sometimes difficult to obtain, we need some luck to do what we want to do. [6]

[6]Of course the argument here has strong metaphysical assumption. The reader who is interested in our position may consider [30, 29]

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