Making Mathematics Visible: Mathematical Understanding and How It Differs from Mathematical Knowledge

By Anne Newstead [1]

Project

Significance of the research problem

The inscription on the gate of Plato's Academy is commonly believed to have read: "Let no one enter here who is ignorant of geometry". Mathematical knowledge, specifically knowledge of geometry, was once considered an essential prerequisite for further academic studies. Yet in Australia today, many students are entering university who lack basic knowledge and understanding of mathematics (Slattery & Perpitch 2010; Brown 2009). In order to begin to remedy this significant problem, though, we need to know what mathematical understanding is and only then can we promote mathematical understanding.

Recently some philosophers have argued that understanding rather than knowledge is the real goal of inquiry (Kvanvig 2003). Yet, how can understanding be different from knowledge, and if it is, why does it matter? Some philosophers contend that understanding a phenomenon is simply a matter of knowing its cause. Others claim that understanding is distinct from knowledge, and involves discerning patterns (Zagzebski 2001), or seeing connections between propositions that form a coherent theory (Kvanvig 2003). If knowledge is distinct from understanding, then perhaps epistemologists have been spending too much time trying to analyse the concept of knowledge rather than seeking to understand the concept of understanding.

Understanding as an epistemic value is typically neglected in the sceptical modern tradition in epistemology that prizes certainty and the guarantee of knowledge above all else. Understanding is the main goal of agents in confronting well-established bodies of knowledge; it therefore flourishes in non-sceptical eras (Zagzebski 2001). Understanding is therefore a particularly appropriate goal in looking at mathematics, a well-established body of knowledge if ever there was one.

Given mathematics' role as the paradigm of knowledge for rationalist philosophers, it is surprising to find that contemporary epistemological discussions omit to consider in depth the case of mathematics. That is particularly surprising since mathematics has always been the main source of examples of pure a priori understanding, where on a certain appealing naïve picture, a mathematician may come to appreciate why the Pythagorean Theorem holds true in a certain instance even though nothing about the world has changed except her understanding.

The aim of this research project is to develop a rich theory of mathematical understanding that is grounded in mathematical practice as well as philosophical epistemology. The project has three central research questions:

1. What is mathematical understanding in a given problem domain, and how does such understanding differ, if at all, from knowledge?

2. In what way might mathematical knowledge be a kind of practical knowledge, and how does such practical knowledge relate to understanding?

3. What is the role of visualization and technology in promoting mathematical and scientific understanding and what does this tell us about the nature of understanding?

Significance and Innovation

The project will develop a theory of mathematical understanding grounded in both contemporary epistemology and the practice-based philosophy of science and mathematics. It will be the first to distinguish sharply between mathematical knowledge and mathematical understanding, demonstrating that they are distinct epistemic kinds. It will apply this distinction to studies of visualization's role in mathematical thinking, asking how and whether visualization promotes mathematical understanding.

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[1] When this proposal was written, I had just finished work as a Research Associate on an ARC Discovery Project, "Mathematics, the Science of Structure", based at UNSW in Sydney, Australia. I would like to acknowledge the assistance I received from Professor James Franklin in revising the prose of this research description, the encouragement I received to create it in the first place, and for his helpful guidance throughout my pursuit of this topic.

Most epistemology of mathematics until recently has tried to assimilate mathematical knowledge to other kinds of scientific knowledge, and so naturally it applied the reigning model of scientific understanding and explanation to mathematics (e.g. Colyvan 2001). Yet this approach risks forgetting the norms and methods distinctive to mathematics such as proof and axiom selection (Maddy 2001).

This project starts from the default assumption that mathematical knowledge is unique. Moreover, this project does not assume that mathematical understanding is a kind of causal knowledge, or that knowledge and understanding are interchangeable kinds. This project argues for and accepts the premise that knowledge and understanding are distinct epistemic kinds, and applies it to the mathematical domain.

Moreover, this project operates within the new genre of 'practice-based' philosophy of mathematics, which seeks to ground philosophical theories of mathematics in mathematical practice. Practitioners of 'practice-based' philosophy of mathematics use case studies and actual examples of mathematical practice in order to generalize about the nature of the objects of mathematics and mathematical knowledge. Although not abandoning foundational studies, practice-based philosophers of mathematics think that philosophy of mathematics has proceeded with too much isolation from mathematics itself (Mancosu 2008a). The remedy for this problem is for philosophers to collaborate with mathematicians.

Advancement of Knowledge

With regard to [1], there is very little sustained discussion in the philosophy of mathematics that bears directly on the distinction between mathematical knowledge and understanding. It is intuitive, however, that there is such a distinction because one can know that Fermat's Last Theorem is true without really understanding why this is so. Furthermore, different proofs of the same theorem may deepen our understanding without adding to our knowledge that the theorem is true. Finally, some explanations and proofs in mathematics do not really promote understanding. For example, proofs by contradiction can be unsatisfactory when they don't really show why the theorem in question is true.

The closest existing literature to our topic concerns mathematical explanation. Mancosu (2008b) surveys theories of mathematical explanation. He notes that philosophers of mathematics have tended to reach for philosophy of science in order to give an account of understanding, but this approach doesn't work well for mathematics. A prominent account of scientific understanding is causal (Salmon 1984). Roughly, on causal approaches, to understand something is to know its cause. Yet it is not obvious that mathematical understanding really is a matter of having causal knowledge or indeed that mathematical reality is causally active (cf. Newstead & Franklin 2011; Colyvan 2001).

Another theory of scientific understanding is that such understanding results when disparate theories are unified (Kitcher 1989). The phenomenon of theory unification does have an echo in mathematical practice. The explanatoriness of Andrew Wiles' proof of Fermat's Last Theorem is due in part to the unification effected between the theories of elliptical curves and modular forms. Yet unification cannot account for what it is to understand a solitary theorem, and it cannot be the whole story about mathematical explanation and understanding.

With regard to [2], there is an existing theory of understanding that holds that understanding is in part a matter of practical knowledge. Zagzebski (2001) develops such a theory of understanding, on which understanding is tied to mastery of a practice (a techne). Understanding, unlike knowledge, is less vulnerable to sceptical challenges, because the criteria for successful understanding on this view are wholly internal to it. In determining whether one understands, say, the art of gourmet cooking, the understanding is demonstrated by engaging in the practice skilfully.

This project will pursue in depth the proposal that understanding is a kind of practical knowledge as applied to the example of mathematical understanding. Historically there has been an emphasis on geometrical knowledge as a kind of practical knowledge based on the activity of performing constructions based on diagrams as in Euclid's Elements (Hintikka 1974). Avigad (2008) puts forward a Wittgensteinian view of understanding as a matter of knowing how to give proofs and engage in other kinds of activities tied to mathematical practice. Yet this 'practicalist' conception of understanding needs to be related to other epistemic kinds, including propositional knowledge (cf. Hetherington 2011). By providing a kind of 'road map' to contemporary epistemology on this matter, the present project will deepen our understanding of the relations between practical knowledge, understanding, and theoretical knowledge.

With regard to [3], in philosophy of mathematics proper there is great excitement about the role of visualization in understanding mathematics, much of it motivated by the fact that computer graphic and visualization packages are now actually used in mathematical research (Palais 1999). Still, not much is known, and even less is understood, about how visualization contributes to understanding. One of the best philosophical works on visual thinking is Giaquinto (2007), which argues that visual thinking is invaluable to most kinds of mathematical discovery and knowledge. However, Giaquinto does not explicitly consider knowledge and understanding as distinct epistemic kinds. His discussion concerns knowledge, and for the most part, he espouses a reliabilist epistemology focused on concept formation and obtaining true beliefs. Unlike Giaquinto's work, this project would employ the internalist notion of understanding as its main concept, rather than the notion of reliably produced true belief.

The topic of whether visualization and visual aids such as diagrams, figures, graphs, and pictures are relevant to proof is still not settled in philosophy of mathematics. Brown (2008) defends the extreme view that pictures and diagrams can be proofs in themselves and thus constitute a kind of mathematical knowledge, but never discusses the distinction between knowledge and understanding. Once the distinction is drawn between understanding and knowledge in mathematics, it becomes possible to answer questions about the admissibility of visual aids in generating mathematical 'knowledge'. The recurrent debate about the illegitimacy of reliance on visual intuition in mathematical justifications, for example, can be recast as a debate between those who find visual representations useful for understanding aspects of a mathematical problem and those who find that visual representations have no place in formal proofs used to justify mathematical knowledge (Newstead & Franklin 2009; Franklin 2014). Thus the present project will differentiate between the various roles that visualization might play in a range of formal and informal mathematical activities rather than restricting the discussion to the role of visualization in proof per se.

The topic of understanding and visualization's role is discussed increasingly in educational circles (e.g. Presmeg 2006) but typically—for reasons of a different emphasis-- without much epistemological reflection. Bobis, Mulligan & Lowrie (2009) discuss how to foster children's developing understandings of mathematics in the classroom, but do not say much about what such understanding is or how it differs from knowledge. Educational researchers such as Sierpinksa (1994) and Pirie & Kieren (1994) are concerned with describing stages of understanding (a genetic description), not with normative epistemology. Nelsen (2000) and Alsina & Nelsen (2006) provide classroom resources for teaching mathematics visually. Such educational research does have the virtue—quite opposite to that found in philosophy—of taking into account real-life cases of mathematical understanding in students.

Hanna (2000) sounds a cautionary note amidst the new enthusiasm for 'visual maths', noting that visualization does not necessarily deliver the kind of understanding obtained by deductive proof. Hanna, Jahnke, and Pulte (2009) are one of the very few volumes to connect approaches in philosophy of mathematics and mathematics education, with most of the discussion revolving around proof. Linn (2011) has also warned that when students claim to understand something based on visualization, they may be taken in by the 'deceptive clarity' of visual modes of presentation. That is to say, they may think they understand more than they do about how something works after viewing a seductive visualization. In the age of iPADs, these epistemological warnings are important.

Approach and Methodology

The project applies the 'practice-based' philosophy of mathematics (Mancosu 2008). As Mancosu (2008b) notes there is a need for more case studies of mathematical knowledge and mathematical explanation in order to construct a rich, empirically grounded theory of mathematical explanation. The practice-based philosophy of mathematics starts from the bottom-up, using actual case studies to generalize and draw inferences about the nature of mathematical explanation and understanding.

Successful pursuit of practice-based philosophy of mathematics requires some knowledge of mathematics, but basic mathematical knowledge suffices. Examples from geometry are particularly good for case studies, since they touch on the question of whether mathematical understanding need be propositional in form. Case studies from Euclid's geometry will be discussed, probably the propositions that cannot be demonstrated without essential reliance on diagrams (Norman 2006; Mumma 2010). Examples from geometry are discussed in Giaquinto (2007) but as noted his discussion of mathematical knowledge employs a basically reliabilist epistemology, which seems inappropriate for mathematical understanding. The project will use an internalist approach to understanding. Furthermore, full account will be taken of the new possibilities afforded by digital visual technology (e.g. Jackiew's geometer's sketchpad) that can be used to teach geometry.

REFERENCES

Alsina,C.& R.B. Nelsen (2006). Math Made Visual. Washington, DC: Mathematics Association of America.

Avigad, J. (2008). 'Understanding Mathematical Proof'. The Philosophy of Mathematical Practice. P. Mancosu. Oxford, Oxford University Press.

Bobis, J. Mulligan, J. and Lowrie, T. 2009. Mathematics for children: Challenging children to think mathematically. Pearson Education, Australia.

Brogaard, Berit. (2011). 'I Know, therefore I understand', unpublished typescript.

Brown, J. (1999/2008). Philosophy of Mathematics: An introduction to the world of proofs and pictures. New York, Routledge.

Brown, G. (2009). 'Report to the Group of Eight: Review of Education in Mathematics, Data Sciences, and Quantitative Disciplines', Turner, ACT: Group of Eight.

Colyvan, M. 2001. The Indispensability of Mathematics. Oxford: Oxford University Press.

Elgin, C. (2006). 'From Knowledge to Understanding'. Epistemology Futures. S. Hetherington. Oxford, Oxford University Press: 199-216.

Feferman, S. (1998). 'Mathematical Intuition vs Mathematical Monsters'. Twentieth World Congress of Philosophy. Boston, Massachusetts.

Franklin, J. (2008). 'Aristotelian realism'. Handbook of Philosophy of Mathematics. J. Wood and A. Irvine. North Holland Elsevier Press.

Giaquinto, M. (2007). Visual Thinking in Mathematics. Oxford: Oxford University Press.

Grimm, S. (2006). "Is Understanding a Species of Knowledge?" British Journal for the Philosophy of Science 57:3 (2006).

Grimm, S. (2010). 'Understanding', Routledge Encyclopedia of Philosophy, forthcoming.

Hanna, G. 2000.'Proof, Exploration, and Explanation'. Educational Studies in Mathematics v. 44 n.1-3, 5-23.

Jackiw, N.: 1991, *The geometer's sketchpad* (computer software), Key Curriculum Press,
Berkeley, CA.

Hanna, G. (2000). Proof, explanation and exploration: An overview. Educational Studies in Mathematics, Special issue on "Proof in Dynamic Geometry Environments", 44 (1-2), 5-23.

Hanna, G. , Hans Niels Jahnke, Helmut Pulte. (2010). Explanation and Proof in Mathematics: Philosophical and Educational Perspectives, New York: Springer.

Hetherington, S. (2011). How to Know: A Practicalist Conception of Knowledge. Oxford: Wiley-Blackwell.

Hintikka, J. (1974). 'Practical and Theoretical Reason—An Ambiguous Legacy', in S. Körner, ed. Practical Reason , New Haven: Yale University Press, 83-102.

Kitcher, P., (1989). "Explanatory Unification and the Causal Structure of the World", in P. Kitcher & W. Salmon, eds., Scientific Explanation, vol. XIII of Minnesota Studies in the Philosophy of Science, 1989, University of Minnesota Press, Minneapolis, 410-505.

Kvanvig, J. (2003), The Value of Knowledge and the Pursuit of Understanding, Cambridge: Cambridge University Press.

Lakatos, I. (1976/1994). Proofs and Refutations: The Logic of Mathematical Discovery. Cambridge University Press.

Linn, M. and Eyelon, B. (2011). Science Learning and Instruction. New York: Routledge.

Maddy, P. (2001). Naturalism in Mathematics. Oxford: Oxford University Press.

Mancosu, P. (2005). Mathematical reasoning and visualization. Visualization, explanation and reasoning styles in mathematics. P. Mancosu, Jørgensen and Pedersen. Dordrecht, Springer.

Mancosu, P. (ed.) (2008a). The Philosophy of Mathematical Practice. Oxford: Oxford University Press.

Mancosu, P. (2008b). "Explanation in Mathematics", The Stanford Encyclopedia of Philosophy (Fall 2008 Edition), Edward N. Zalta (ed.), URL <http://plato.stanford.edu/archives/fall2008/entries/mathematics-explanation/>.

Mumma, J. (2010). "Proofs, pictures, and Euclid", Synthese 175: 255-287.

Nelsen, R.B. (2000). Proofs without Words I and II, Washington, DC: Mathematics Association of America.

Netz, Reviel (1999). The Shaping of Deduction in Greek Mathematics. Cambridge: Cambridge University Press.

Newstead, A. (2001). Aristotle and Modern Mathematical Theories of the Continuum, Aristotle and Contemporary Science II. Demetra Sfendoni-Mentzou, J. R. Brown and J. Hattiangadi. Frankfurt am Main, Peter Lang 113-129.

Newstead, A. (2006). 'Knowledge by Intention? On the Possibility of Agents' Knowldge', in S. Hetherington (ed.), Aspects of Knowing, Elsevier,183-203.

Newstead, A., Franklin, J. (2009). 'The Role of Visual Intuition in Mathematical Understanding', special stream/symposium, Proofs and Pictures: The Role of Diagrams in Logic and Mathematics (ed. C. Montelle, C. Legg), Australasian Association of Philosophy-New Zealand, Massey University, December 7-10, 2009.

Newstead, A., Franklin, J. (2010). The Epistemology of Geometry I: The Problem of Exactness. In W. Christensen, E. Schier, and J. Sutton (Eds.), ASCS09: Proceedings of the 9th Conference of the Australasian Society for Cognitive Science). Sydney: Macquarie Centre for Cognitive Science, pp. 254-260.

Newstead, A and Franklin, J. (2011). 'Indispensability of Mathematics without Platonism', in A. Bird, H. Sankey, and B. Ellis (eds.), Powers , Structures, and Properties: Issues in the Metaphysics of Realism, New York: Routledge (forthcoming).

Norman, J. 2006. After Euclid. Stanford: CSLI Publications.

Palais, R.S.: 1999, 'The visualization of mathematics: Toward a mathematical exploratorium', Notices of the AMS 46(6), 647–658.

Presmeg, N. (2006). Research on visualization in learning and teaching mathematics, in A. Gutiérrez and P. Boero, eds, Handbook of Research on the Psychology of Mathematics Education: Past, Present and Future, 205-235. Amsterdam, Sense Publishers.

Pirie, S. & Kieren, T. 1994. Growth in mathematical understanding: how can we characterise it and how can we represent it? Educational studies in mathematics 26, 165-190.

Pritchard, D, A. Millar, and A. Haddock (eds). 2010. The Nature and Value of Knowledge. Oxford: Oxford University Press.

Salmon, W., 1984, Scientific Explanation and the Causal Structure of the World, Princeton: Princeton University Press.

Sierpinksa, Anna. 1994. Understanding in Mathematics.London: Falmer Press.

Slattery, L. and Perpitch, N.2010. 'Mathematics students in serious decline'. The Australian. Canberra, March 10, 2010.(http://www.theaustralian.com.au/news/nation/mathematics-students-in-serious-decline/story-e6frg6nf-1225838901032; accessed 22 April 2011)

Steiner, Mark. 1978. 'Mathematical Explanation', Philosophical Studies 34, 135-51.

Zagzebski, L. (2001). Recovering understanding, in M. Steup, ed, Knowledge, Truth and Duty: Essays on Epistemic Justification, Responsibility and Virtue, 235-258. New York, Oxford University Press.
Zimmerman, W. (1991). Visual Thinking in Calculus. Visualization in Teaching and Learning Mathematics: A Project. W. Zimmerman. Washington DC, Mathematical Association of America: 127-137.