The Epistemology of Geometry I: The Problem of Exactness

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Abstract

We show how an epistemology informed by cognitive science promises to shed light on an ancient problem in the philosophy of mathematics: the problem of exactness. The problem of exactness arises because geometrical knowledge is thought to concern perfect geometrical forms, whereas the embodiment of such forms in the natural world may be imperfect. There thus arises an apparent mismatch between mathematical concepts and physical reality. We propose that the problem can be solved by emphasizing the ways in which the brain can transform and organize its perceptual intake. It is not necessary for a geometrical form to be perfectly instantiated in order for perception of such a form to be the basis of a geometrical concept.

Keywords: geometrical knowledge; philosophy of mathematics; perception and mathematics; visualization

Mathematical Knowledge

The dominant problem in the epistemology of mathematics for many decades has been to give a naturalistic account of mathematical knowledge. Such naturalistic account will inevitably draw on cognitive science and what it shows about how the brain does mathematics. The major obstacle to giving such an account has been the assumption that mathematical objects are abstract. Philosophers do not completely agree on the notion of what it is for an object to be abstract. The truth may well be that 'abstract object' is a cluster concept that is largely defined by opposition to features associated with 'concrete object'. In the western philosophical tradition starting with ancient Greek philosophy, the notion of an abstract object does have a paradigm: the Platonic form. Platonic forms are ideals that exist in an intelligible realm, outside of the concrete, material spatiotemporal order.

These considerations suggest that the contrast between *abstract* and *concrete* objects is captured by the following:

Table 1: The contrast between abstract and concrete

Features	Abstract Objects	Concrete Objects
Spatial-temporal location?	No	Yes
Particular?	Some	Yes (usually)
Causally active?	No (?)	Yes
Material	No	Yes

As is evident from Table 1, abstract objects are typically defined negatively by contrast with concrete objects. There is no consensus as to whether abstract objects must be singular objects (particulars) or universals. Frege, for example, treats numbers as both particular objects *and* as abstract objects. By contrast, mathematical structuralists treat mathematical patterns as universals rather than particulars.

In contemporary metaphysics and epistemology, abstract objects are usually thought to be causally inactive on the grounds that efficient causal action requires location in the spatiotemporal order. If mathematical objects are abstract-- in the traditional sense captured in Table 1-- then it follows that they are causally inert. Benacerraf (1973) in his classic paper, 'What is Mathematical Truth?' points out that it is extremely mysterious how knowledge of abstract objects is possible. Our best naturalistic theory of knowledge appears to be—at least at a base level—to involve a causal condition on knowing. That is, generally we think that if a subject S knows that p (for some proposition p), then there must be a causal chain that connects S suitably with the fact that makes p true. This fact—the truthmaker for p—must be realized in the natural, spatiotemporal world somewhere. If mathematical objects are abstract, then the truthmakers for mathematical truths will not lie in the natural realm. This would be a clear violation of naturalism, which D.M. Armstrong helpfully characterizes as the view that 'space-time is all there is' (Armstrong 1997:5).

Benacerraf's problem is posed as a problem for realists about abstract objects. Such realists believe abstract objects entities exist independently of the human mind and that statements about them have a determinate truth-value even if that truth-value is yet to be discovered. To be sure, it is only a problem for realists about abstracta who feel the pull of naturalism. Thus, although Benacerraf's objection is posed as a problem for 'Platonism' in the generic sense of 'realism about abstract objects', it need not apply to Plato's realism.

Plato would not have granted the assumption that abstract objects are causally inert. Plato repeatedly speaks of concrete objects as 'partaking' or 'participating' in the Forms, which suggests at least a kind of one-way interaction between objects and Forms. Furthermore, Plato accepts the

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principle that power to affect and be affected is the mark of reality, and holds that the Forms—despite being immaterial-- are real.[1] This provides Plato with an argument for the reality and power of Platonic forms. We may not accept the argument, but it is highly plausible that Plato did view his Forms as having some kind of causal power (Fine 2003).[2] The problem, then, is not that Plato's forms lack causal power, but that the notion of efficient causality which is our contemporary scientific notion does not apply to the forms. Plato's realism is incompatible with scientific naturalism.

If we are to be naturalists, how then do we solve the Benacerraf problem for mathematics? Hartry Field (1989:25) points out that the Benacerraf problem survives even an objection to the causal theory of knowledge. The Benacerraf problem can be re-instated using the reigning epistemological theory, such as reliabilism. According to one version of reliabilism, if S knows that p, then there must be a reliable connection between the subject S and the fact that makes p true. Once again, the problem arises that there is *no explanation* for a subject's reliable connection to facts about abstract objects.

Nominalism is the view that there are no abstract objects. However, nominalism is not an attractive solution to the Benacerraf problem. Putnam (1971) argued persuasively that nominalism lacks the resources—the notion of a *linguistic type*, which is after all a universal that transcends its concrete instantiations--- to articulate its very doctrine. Aristotelian realism recommends itself as a variety of realism suitable for naturalism. In contemporary metaphysics, the position of Aristotelian realism is represented by D.M. Armstrong's theory of universals, which holds that they are immanent in the world. Applied to mathematics, Aristotelian realists hold that many basic mathematical entities (patterns, properties, facts, and objects) are instantiated in the natural world. As a consequence of this metaphysics, Aristotelian realism promises to give a naturalistic account of mathematical knowledge. On this account mathematical knowledge is grounded in perception of patterns (universals) in the world.

A major obstacle to locating mathematical patterns in the world is that, at least in some cases, the patterns do not appear to be there exactly. There seems to be a 'mismatch' between the perfect mathematical form and what is found (perceived, constructed) in the physical world. This problem is known as 'the problem of exactness' and it constitutes a major objection to Aristotelian realism.[3] Of course one possible solution to the problem is to give up the philosophy that generates it. However, it may be possible to retain the spirit of Aristotelian realism and naturalism while solving the problem. That is what we will suggest. We distinguish between three ways to solve the problem:

(a) inexactness theory: that geometry is about the real shapes of things: a cartwheel doesn't have an exact circle shape, but it does have an exact near-circular shape, and one can explain why studying circles is relevant to studying near-circles;

(b) the sub-perception theory, that there are perceivable shapes that are perceptually indistinguishable from perfect circles;

(c) "rectification" theory: where the mind actively extracts the perfect from the visibly imperfect shapes of things.

These three possible solutions to the problem can be jointly maintained and are mutually compatible. In what follows we will especially emphasize solutions (b) and (c), although we should be understood to accept (a).

The problem of exactness in ancient philosophy: Plato and Aristotle

Plato himself was reluctant to locate mathematical forms in the physical, sensible world because the perfection and precision of mathematical forms seems unparalleled aby many of the real-world exemplars of mathematical forms. In the *Phaedo*, Plato notes that sensible, concrete objects in the material world often fail to instantiate the perfect mathematical forms found in the intelligible world of the forms.[4] For example, at *Phaedo* 74a-c, Plato says that two sticks will not instantiate equality as perfectly as the form of the Equal itself. Presumably we cannot be sure that two sticks that look equal are actually equal, because our sense-perception may not be able to discriminate between small differences in size. Plato regards judgements about mathematical form made on the basis of perception as inherently less precise and prone to inaccuracy.

It is no accident that Plato's examples are geometrical, not arithmetical. The problem of exactness seems to have more of a bite in geometry than arithmetic. It is plausible to think that arithmetical forms are precisely instantiated. For example, a certain flower has an exact number of petals, say five petals. A certain book has an exact number of pages, such as two-hundred and twenty-nine pages. Once a sortal concept is supplied, we can count out a precise number of units of an item that falls under the appropriate sortal

[1] Plato entertains this line of thought in the *Sophist* at 247E. See F. Cornford (trans.), (1957). *Plato's Theory of Knowledge: The Theateatus and Sophist of Plato*, New York: Macmillan, 234. The principle has been called 'the Eleatic Principle' in contemporary literature on the grounds that Plato has the Eleatic stranger enunciate it. However, there is no reason to think that Plato did not accept it.

In contemporary metaphysics one is more likely to find the Eleatic principle—that to be real is to have causal power—used as an argument against the existence of abstract objects (as in Armstrong 1997: 41).

[2] Rosen (2009) suggests that the distinction between abstract and concrete does not go back to Plato's philosophy, but probably originates with Locke's transposition of the grammatical distinction between abstract terms (like 'whiteness') and concrete terms ('white') onto the realm of ideas.

[3] The related problem of 'the perfect circle' is discussed (not using the label 'the problem of exactness') in F. Copleston, (2003). *History of Western Philosophy* I, Ancient Greece and Rome, New York: Continuum Press, pp.297ff.

[4] Plato, (c.380BC), Phaedo. In J. Cooper (ed.), 1997, *Plato's Complete Works*, Indianapolis: Hackett.

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concept. In the case of geometrical forms, however, we deal with continuous variation of curves in (mathematical) space rather than with discrete differences in quantity. There seems to be plenty of scope for some patterns to fall short of ideal mathematical patterns. A table-top may not be perfectly square, as its edges may be bumpy.[5] An artist's drawing of a house, or a geometry teacher's drawing of a triangle on the blackboard, may be imperfect but sufficient to convey the appropriate ideas to their audience. Perhaps some geometric forms will be perfectly instantiated, but for those that are not the problem remains to give an account of how the brain recognizes in the imperfect illustration the perfect geometrical form.

Aristotle is well aware of the objection from exactness. He notes himself that a hoop in reality will not touch a straight edge normal to it just at a point as it is supposed to do in geometry:

> For neither are perceptible lines such lines as the geometer speaks of --for no perceptible thing is straight or curved in this way; for a hoop touches a straight edge not at a point, but as Protagoras said it did, in his refutation of the geometers…' (Metaphysics B2, 997b34-998a6).[6]

The philosophical problem, then, is that mathematical truth seems to be about exact mathematical objects with exact properties. Some idealization and approximation is involved in moving from the hoop of our everyday experience to the perfect circle of geometry.

The problem with idealization is that it is not always truth-preserving. If Don Quixote is in reality an old man, then his idealized conception of himself as a young knight is actually false. Similarly, if the earth is actually a lumpy oblate spheroid and not a perfect sphere, then it is actually false that the earth has the properties properly attributed to a sphere (such as every point on its surface being equidistant from its centre). If we engage in mathematical deduction concerning perfect objects (perfect spheres and the like), then there is no guarantee that the result of the deduction will perfectly apply to anything in the real world! Yet the beauty of the Aristotelian view is that it supposed to offer a straightforward explanation of how mathematics applies to the real world, and how mathematical knowledge is obtained by learning about features of the world.

Can Aristotelian realism survive? Some philosophers would say 'No'. Stewart Shapiro regards the problem of exactness as a very serious problem for Aristotelianism and Platonism (Shapiro 2000: 70) However, we still think the neo-Aristotelian view has a lot to offer (Franklin 2009). In what follows, we will focus on how cognitive science supports solutions (b) and (c) to the problem of exactness.

Cognitive science to the rescue?

Perhaps cognitive science can help us grapple with the problem of exactness. Giaquinto (2007) develops an account of the epistemology of mathematics that goes some way to solving the problem of exactness; moreover, he does so in a way that draws on the psychology of perception. In what follows here we endorse his solution to the problem of exactness and point out its limitations.

Consider the perfect square. How do we get the geometric concept of a perfect square? Not by mere perception. It may well be that we only come into contact with imperfect squares. For example, perhaps the squares in Susan's homemade brownies (biscuits) are not really square. The edges are not perfectly straight, or the symmetry isn't quite right. Nonetheless, an encounter with merely imperfect squares may suffice for us to acquire the concept of a perfect (geometrical) square. This geometrical concept may in turn structure our perceptual experience so that we take ourselves to be experiencing a perfect square. As Giaquinto explains,

> It can also be part of experience that a square is perfect. Since there is a finite limit to the acuity of experience, there are lower limits on perceptible asymmetry and perceptual deviation from (complete) straightness.' (Giaquinto 2007: 28).

Asymmetry or other imperfections that fall beneath our threshold of perceptual discrimination will not be perceived. Call this view 'the sub-perception theory'. The idea is that we can perceptually experience a perfect mathematical form even if objectively the form is not perfectly instantiated in nature. Curiously enough our perceptual limitations enable us to experience, as it were, perfect geometrical forms. There is thus no need to be committed absolutely to the existence of perfect forms in nature: it is enough if the forms in nature approximate mathematical forms.

To be sure, there are cases of imperfection that do not fall beneath the threshold of perfection: they are noticeably imperfect. We can speculate that such cases—such an oval (rather than a perfect circle) or a shape that fails to be an enclosed triangle—are something that we can either learn to recognize as approximating a perfect shape but failing in some respect, or else we can learn to recognize them as perfect exemplars of some new kind (ovals rather than circles, for example). Reflection on such cases might make some philosophers conclude that the debate between Platonists and Aristotelians focuses too much on the notion of pre-existing 'geometrical forms' that are ready to be imposed on the world. The aim of proper Aristotelianism, though, is to *discover* those forms already instantiated in the world.

Some allowance has to be made for imperfect instantiations of geometrical forms. We have suggested that an appeal to perceptual limitations can help the Aristotelian

[5] The neo-Platonists blamed matter for failing to receive form perfectly in some cases. See Glenn Morrow (ed. and trans.) (1970). *Proclus: A commentary on Euclid's First Book of Elements*, Princeton University Press, 1970, reprinted in 1992 with an introduction by I. Mueller.

6 Similar passages: Meta VII.10, 1035a25-b. ed. J. Barnes, (1984) "Metaphysics" (based on a translation by WD Ross), *The Complete Works of Aristotle: the Revised Oxford Translation*, vol. II, Princeton University Press. For discussion of the problem of exactness in Aristotle, see R. Pettigrew, (2009), 'Aristotle and the subject matter of geometry', *Phronesis* vol. 54.

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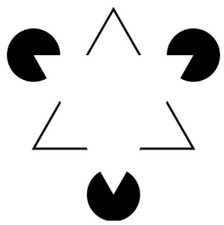
to explain how acquisition and knowledge of geometrical concepts is possible.

Another psychological phenomenon that helps in solving the problem of exactness is the ability to the visual system in the brain to organize perceptual scenes using *gestalt* principles. Consider the famous Kanizsa triangle:



[Figure 1]

When we look at these patterns, we seem to see an inverted and very bright white triangle covering a less bright upright triangle. The inverted triangle has illusory contours—its edges are not really there. Yet somehow the brain is tempted to see the bright inverted triangle as having edges of its own. Perhaps the black lines of the upright triangle 'spread' or 'smear' a bit making the upright triangle appear to be a grayish white and therefore darker than the bright white inverted triangle on top. The 'top-down' imposition of form of the bright white triangle depends also on lower-level perceptual phenomena, such as the visual system registering the dramatic shifts in colour across boundaries.[7] The Kanizsa triangle is a good example for the philosopher of geometry to consider because it is a case where the brain interprets the visual display in a way that adds something geometrically to what is strictly speaking present in the given visual form.

The implications of the preceding observations are that geometrical knowledge cannot be entirely a matter of direct perception of perfectly instantiated geometrical forms. That kind of naïve realism is *too* naïve. Geometrical knowledge does involve an element of abstraction which allows the perceiver to move away from some of the imperfections or limitations inherent in a perceptual scene. Perceptual experience does not have to be completely veridical in order to trigger or give rise to geometrical concepts and, in due course, geometrical knowledge.

If this account is right, then perceptual experience is not completely veridical, but it is not a hallucination either. Our perceptual concepts are formed in response to real objects with real properties which we perceive but whose flaws (deviations from ideal forms) we may or may not perceive. Our geometrical concepts are in turn closely linked to our perceptual concepts, abstracting away from the imperfections of some of the exemplars.

A Dangerous Concession? In his seminal book, *Mathematical Knowledge* (1984), Philip Kitcher discusses the problem of exactness in relation to Kant's constructivist philosophy of geometry. Kant's view of geometry is that it is a rule-governed construction that makes it possible for the geometer to 'discern the universal in the particular' (A714/B742).[8] That is to say, the geometer draws figures that represent universal geometrical patterns, and considers the features of those figures that would be common to *any* instantiation of the pattern. For example, if a teacher draws a triangle with white chalk, the whiteness of the chalk is irrelevant to the figure of the triangle. Only the purely geometrical properties are relevant: these are the angles and their total sum, the lengths of the three sides, and the arrangement of the sides in an enclosed figure. If the figure is used properly, these geometrical properties will be present in *any* figure of the triangle. Kant's philosophy of geometry, as with any such philosophy that gives perception (intuition) a central role has to explain how the geometrical figure that is the object of perception possesses *exactly* the properties of the ideal geometrical concept. As Kitcher puts it, the problem is that "we cannot assume that mental perception will give us exact knowledge even of the particular figures we construct" (Kitcher 1984: 51).[9]

How might a Kantian respond to the challenge? Kitcher suggests that the Kantian can appeal to the fallible and limited nature of our powers of perception. In particular, "We should concede that we might not be able to distinguish a straight line from a curved one" (Kitcher 1984: 51). This response is essentially the same as the 'sub-perception' view we found in Giaquinto (2007) and which we have endorsed as one solution to the problem of exactness. Kitcher goes on to suggest that this response is "a dangerous concession", because it would destroy the a priori warrant (justification) for the geometrical belief. That is, if we acknowledge the limits and fallibility of our perception, we may no longer be inclined to trust perception as a reliable source for forming geometrical beliefs.

We are not concerned here to defend the *a priori* nature of the justification of geometrical beliefs. However, we are concerned to defend the idea that geometrical *knowledge* can be obtained by having perceptual experiences. Kitcher attacks both claims in a serious assault on the notion of mathematical knowledge.

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7 For more on perceptual completion, see L. Pessoa, E. Thompson, and A. Noe, *Behavioral and Brain Sciences* (1998) 21, 723–802.

8 We are indebted to Quassim Cassam for drawing attention to this quotation from Kant. The quotation is from I. Kant, 'The Discipline of Pure Reason', in N. Kemp Smith, ed. (1911), *Immanuel Kant's Critique of Pure Reason,* London: Macmillan. A crisp and clear discussion of Kant's view of geometry is found in Q. Cassam (2007), *The Possibility of Knowledge*, Oxford: Clarendon Press.

9 We wish to acknowledge Giaquinto (2007) and Norman (2006) for putting us onto the problem of exactness, which is discussed by Norman (2006) with special reference to Kitcher (1984) in particular at p.121 as well. Our innovation is to illustrate the problem in ancient philosophy and to draw on psychology for its solution.

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Kitcher's objection is misguided. Perception can still be a reliable source of geometrical beliefs even if there is not an absolutely perfect fit between the approximate instantiation of a form in reality and a perfect ideal geometrical form. What matters is that the process that takes us from perception of geometrical figures in reality to geometrical beliefs should be a reliable one. Reliability does not require complete identical replication of an image. Rather what reliability requires is this: for a given perceptual input X there is a reliable transform f(X) that takes us to the output Y, which is the geometrical concept, every time. If background conditions are fixed, then perception can provide us with a reliable means of transforming perceptual intake into geometrical concepts and representations.

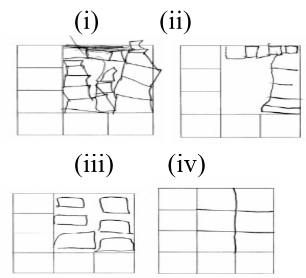
Platonism by the back door? Someone might object to our view that it looks like Platonism by the back door. After all, we have admitted that in many cases geometrical forms are not perfectly instantiated in the world. We have also hung onto the idea that geometrical forms are perfect. There seems to be nowhere to locate these perfect forms save in a Platonic realm. This problem is just an aspect of the general problem of where to locate perceptual experience, with its Janus-faced nature, both pertaining to the conscious subject and pertaining to the objective world. Our account is quite opposed to Platonism in seeking to ground basic geometrical knowledge in perception rather than in communion with Platonic forms.

It is worth noting that our account is neutral, though, on whether geometrical concepts are innate or acquired. There is some evidence that geometrical competence is shared across cultures, which may be taken as an argument for its innateness by some scientists (see Dehaene et. al. 2006). However, all that is required by our account is that perceptual experience—of some modality, not necessarily visual, but usually visual--- is a necessary trigger to the development of geometrical concepts. Some indirect evidence for the necessity of perceptual experience in acquiring geometrical knowledge comes from studies of congenitally blind children. The general consensus is that while congenitally blind children can acquire spatial and geometrical concepts through touch, they are delayed in proficiency of skills involving those concepts relative to their non-visually impaired peers (Millar 1994: 133). Presumably the explanation for this delay is that vision is one powerful but though not absolutely necessary means of acquiring geometrical knowledge.[10]

Relevant Empirical Work in Mathematics Education

Lastly we wish to draw the readers' attention to some relevant empirical work in the field of mathematics education. Mulligan, Prescott, and Mitchelmore (2004) found a significant correlation between ability to recognise and draw geometrical patterns and subsequent early school mathematical achievement.[11]

One of the interesting aspects of Mulligan's work is the way in which her longitudinal studies have exhibited the growth and development of a grasp of pattern ('mathematical form') in young children. For example, Mulligan's data set includes the following set of four illustrations of a child's progress in completing a simple geometrical drawing task:



[Figure 3] Taken from Mulligan's 'Understanding Young Children's Difficulties in Mathematics Learning', Macquarie University presentation, slide 24.

Mulligan et. al. rightly interpret the illustrations as showing the emergence of an ability to grasp structure and attend to the geometrically relevant aspects.

The relevance of this work to the philosophical problem of exactness is clear. This data demonstrates that children learn how to focus on the geometrically relevant and salient properties of a figure and to disregard—'abstract away from'—other aspects of the figure that may be interesting but not relevant for geometry. For example, only in (iv) is there the right number of squares with the right number of edges and approximately the right symmetry.

Drawing figures requires motor skills but also requires visual and spatial cognition. The development we see in the young children's drawings reflects a refinement of geometrical concepts: from a sloppy experimental working concept to the final pure geometrical concept. The geometrical concept is an ideal and somewhat abstract one that appears to distance itself from the messy world of perception. The idea of 'abstraction' figures heavily in Aristotelian philosophy of mathematics and has equally been heavily criticized by Frege. Frege's criticisms were immediately directed against John Stuart Mill's crude empiricism in which a number was identified with a collection of 'pure' units (Shapiro 2000: 67). However, Mill's empiricism is a descendant of Aristotle's empiricism,

[10] For a summary of the research on blind children's shape concepts, see S. Millar, *Understanding and Representing Space*, Oxford: Clarendon Press 1994/2002.

[11] Mulligan, J., Prescott, A., & Mitchelmore, M. C. (2004). Children's development of structure in early mathematics. In M. J. Høines & A. B. Fuglestad (Eds.), *Proceedings of the 28th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 393-400). Bergen, Norway: Program Committee.

and Frege's attack is an attack on the Aristotelian tradition as well.

Here in the empirical work, in children's drawings, we seem to have a demonstration of 'abstraction' at work. This suggests that perhaps we should not be too quick to discard this useful concept under pressure from the towering authority of Frege. Mathematics educators continue to find the concept of 'abstraction' useful in describing the development of mathematical concepts. What seems to be occurring in such cases is not the replacement of an object with a pure unit (the target of Frege's attack), but the omission of certain details and the emphasize on the structural (formal, universal) features of an object. It is not at all clear that such mental abstraction is necessarily shown to be untenable by Frege.

To be sure, we would not want to base our account of geometrical concept acquisition entirely on the notion of 'abstraction'. However, abstraction may be one technique along with the others we have examined for generating geometrical concepts and beliefs.

Conclusion

A scientific, naturalistic account of geometrical knowledge has to begin with our ability to discriminate and recognize geometrical forms in the physical world. Such an account will give a central role to perceptual experience in triggering the formation of geometrical concepts.

The problem of exactness is a two thousand year old problem. The problem concerns a mismatch between the perfect geometrical forms we contemplate in geometry and their sometimes rough instantiations (as disclosed by perception) in the natural, physical world.

We have suggested several different but compatible solutions to the problem of exactness. First, we have emphasized that geometrical forms do not have to be perfectly instantiated without exception in order for acquisition of geometrical concepts to occur. Perception of such approximately instantiated forms can yield geometric concepts provided the imperfection is beneath the threshold of perception, or else is "abstracted away". Second, as the gestalt psychologists emphasized, the brain plays an active role in organizing perceptual data according to gestalt principles. Thus, the top-down imposition of form may help to explain how the brain arrives at the notion of a perfect geometrical form out of the meagre materials of experience. Third, we have allowed that geometers can study irregular shapes in the real world while learning much from working with the tidy and regular patterns that figure in geometrical theory. Mathematics is an exact science, but the physical world and the process of perception is not exact. What matters is that there is a reliable means of moving from perception to geometrical concept.

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