Systems for non-reflexive consequence

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ABSTRACT. Substructural logics and their application to logical and semantic paradoxes have been extensively studied. In the paper, we study theories of naïve consequence and truth based on a non-reflexive logic. We start by investigating the semantics and the proof-theory of a system based on schematic rules for object-linguistic consequence. We then develop a fully compositional theory of truth and consequence in our non-reflexive framework.

1. INTRODUCTION

Interest in substructural logics and in their application to logical and semantic paradoxes has grown considerably in recent years. Many recent works focus on non-transitive approaches to paradox [Rip12, CÉRvR12, CÉRVR13, Rip13a, Rip13b, BRT15, Rip15, BRT16, BPS20], and non-contractive approaches have also received considerable attention [Gri82, Pet00, Can03, Zar11, MP14, DRR18, Fje19, Ros19]. By contrast, non-reflexive theories have been investigated less.[1] Nevertheless, non-reflexive theories are especially promising to model the interplay between naïve truth and consequence [NR18]—in this respect, they are even more promising than their non-transitive rivals. However, a systematic study of the logic, the semantics, and the proof-theory of non-reflexive theories of naïve truth and consequence is currently lacking, and so does a thorough philosophical analysis (and defence).

The purpose of this paper is to fill this lacuna, at least in part. First, we introduce the basics of non-reflexive logic(s) and semantics, and their extensions with naïve consequence (and truth) rules (§2-3). The two main sections of the paper are §4 and §5. In the former, we first build on the work carried out in [Nic21] on logics of truth to investigate the proof-theory of non-reflexive logics of consequence, with a special focus on cut-elimination proofs. In §5 we study the interaction between truth and consequence in non-reflexive systems: this is achieved by providing a compositional theory of truth and consequence, by establishing the adequacy of such a theory with respect to the semantics provided in §3, and by investigating its proof-theoretic properties.

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[1] Some works touching upon non-reflexive logics and their relationship to paradox include [Gre01, Rea03, Mea14, Fre16, Fje17, NR18]. A brief comparison of our paper with those work might be helpful: [Gre01], [Rea03], and [Fre16] motivate the non-reflexive approach without providing a model-theory or a proof-theory; our work integrates those with a more comprehensive model- and proof-theoretic analysis. As explained in [NR18], a model-theory for a non-reflexive "validity" predicate is provided in [Mea14], but the construction differs form our in that it does not validate many desirable principles (e.g. contraction and cut). [Fje17] studies a pure logic of disquotational truth based on a three-sided logical system: our two-sided approach combines well with a semantics, and can be neatly extended to a fully fledged compositional theory of truth over a non-logical syntactic base.

The present work is mainly a technical study, which aims to consolidate non-reflexive logics as a viable basis to address semantic paradoxes, and to develop satisfactory theories of naïve semantic notions. Its main findings are that (i) naive consequence rules can be added to a non-reflexive logic while preserving an intuitive semantics and remarkable proof-theoretic properties (above all, the eliminability of cut); (ii) it is possible to provide a fully compositional theory of truth over a non-reflexive logic that admits naïve rules for truth and consequence and that axiomatizes a generalization of a standard fixed-point construction for truth.

2. Logics of transparent truth and consequence

Let $\mathcal{L}$ be a first-order language with logical constants $\neg, \wedge, \forall$, and $\mathcal{L}_C := \mathcal{L} \cup \{C\}$ its expansion with a binary predicate $C(x, y)$ intended to express *object-linguistic consequence*.[2] Variables are denoted with $x, y, z, ...$, and terms with $r, s, t, \dots$. We assume that $\mathcal{L}$ contains constants $\ulcorner\varphi\urcorner$ for any formula φ of the language $\mathcal{L}_C$, and constants $\top, \bot$. The nature of the names $\ulcorner\varphi\urcorner$ is not fully fixed by the theory: as customary practice when dealing with *logics* of semantic concepts, one can assume that the denotation of $\ulcorner\varphi\urcorner$ in all models of the theory is φ itself [Kre88, Rip12]. Such extra-theoretic assumptions will become redundant once a proper theory of syntax will be assumed in the final sections of the paper.

DEFINITION 1 (LPC). *The system* LPC *in* $\mathcal{L}_C$ *contains the following initial sequents and rules, where* $\Gamma, \Delta, \Theta, \Lambda \dots$ *are finite multisets of formulae of* $\mathcal{L}_C$.

$$(\text{REF}^-)\quad \Gamma, \varphi \Rightarrow \varphi, \Delta \quad \text{with } \varphi \in \text{AtFml}_{\mathcal{L}} \qquad\qquad (\text{CUT})\ \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

$$(\top)\quad \Gamma \Rightarrow \top, \Delta \qquad\qquad (\bot)\ \Gamma, \bot \Rightarrow \Delta$$

$$(\text{CL})\ \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \psi, \Gamma \Rightarrow \Delta}{\Gamma, C(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner) \Rightarrow \Delta} \qquad\qquad (\text{CR})\ \frac{\Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow C(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner), \Delta}$$

$$(\neg\text{L})\ \frac{\Gamma \Rightarrow \varphi, \Delta}{\Gamma, \neg\varphi \Rightarrow \Delta} \qquad\qquad (\neg\text{R})\ \frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \neg\varphi, \Delta}$$

$$(\wedge\text{L})\ \frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta} \qquad\qquad (\wedge\text{R})\ \frac{\Gamma \Rightarrow \varphi, \Delta \qquad \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}$$

$$(\forall\text{L})\ \frac{\Gamma, \forall x\varphi, \varphi(s) \Rightarrow \Delta}{\Gamma, \forall x\varphi \Rightarrow \Delta} \qquad\qquad (\forall\text{R})\ \frac{\Gamma \Rightarrow \varphi(y), \Delta}{\Gamma \Rightarrow \Delta, \forall x\varphi}\ y \notin \text{FV}(\Gamma, \Delta, \forall x\varphi)$$

REMARK 2.

(i) $\text{AtFml}_{\mathcal{L}}$ denotes the set of atomic formulae of $\mathcal{L}$, i.e. the language without the consequence predicate, and $\text{FV}(\Gamma)$ denotes the set of free variables of Γ.

(ii) We can define a theory of full disquotational truth as a sub-theory of a definitional extension of LPC obtained by defining $\text{Tr}(x)$ as $C(\ulcorner\top\urcorner, x)$.[3]

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[2]We define $\vee, \exists$ in the usual way.

[3]Similarly, we can define a theory of predication or 'true of' by generalizing the semantic rules to open formulae.

(iii) The combination of the rules (CL) and (CR) with unrestricted initial sequents, even in the absence of (¬R), results in inconsistency. This is essentially a version of Curry's paradox that has recently received some attention [BM13, NR18].[4] Our predicate validates the principles for "validity" introduced by Beall and Murzi. The intended reading of $\mathrm{C}(x, y)$ is "grounded consequence" in the sense of [NR18]: this provides a model-theoretic interpretation for the validity predicate, as well as a consistency (and non-triviality) proof for the rules of [BM13]. As explained in [NR18, §2.3], $\mathrm{C}(x, y)$ differs also from a formal provability predicate, in that it does not generate a hierarchical concept of consequence – by Löb's theorem, formal provability in arithmetic can in fact only validate naive principles for $\mathrm{C}(x, y)$ if one ascends to a stronger system. We will elaborate more on this point in section 3.

(iv) As it happens in the standard G3 systems on which it is based, the formulation of LPC with context-sharing rules is justified by the admissibility of weakening and contraction in the system, established below.

3. Fixed-point Semantics

The semantics for the logical rules of LPC is provided by a substructural (non-reflexive) logical consequence relation defined over strong Kleene semantics (K3), i.e. the *tolerant-strict* consequence relation (TS) defined in [CÉRvR12]. This semantics can then be incorporated into a simple fixed-point construction (introduced in [NR18], and to be recalled in a moment), in order to interpret also the consequence predicate of LPC. Let us start from the former. In the following, we take for granted the notion of a strong Kleene (K3) evaluation – see for instance [DRP22].

Definition 3. *Let v be a* K3 *evaluation function. The argument from Γ to Δ is* TS-*valid (for* Tolerant-Strict*), in symbols $\Gamma \models^{\mathrm{ts}} \Delta$ if: for any* K3 *evaluation function v, if for every $\varphi \in \Gamma$, $v(\varphi) = 1$ or $\mathbf{n}$, then there is at least one $\psi \in \Delta$ s.t. $v(\psi) = 1$.*

A few basic features of TS are easily stated. Just like strong Kleene logic K3, TS does not have any classical laws. In other words, no sequent of the form $\Rightarrow \varphi$, for φ classically valid, is TS-valid. In addition, and unlike K3, TS does not have any classical inferences, i.e. no classically valid sequent of the form $\Gamma \Rightarrow \Delta$ is TS-valid. This includes, of course, reflexivity: to see that the inference from φ to φ is not unrestrictedly TS-valid, just consider a K3-evaluation which assigns value $\mathbf{n}$ to φ. However, and again unlike K3, TS is closed under all the classically valid meta-inferences: every classically valid sequent rule is also TS-valid. This means that, as a pure logic, TS contains no sequents, but this is not so once one combines it with initial sequents, as we will do below.

As announced above, the consequence relation defined by TS can be easily combined with a Kripke-style, fixed-point interpretation of the consequence predicate $\mathrm{C}(x, y)$ [Kri75]. That this is generally possible is guaranteed by the fact that the TS evaluation scheme is monotone in the evaluation ordering.[5] For simplicity and definiteness, we develop the model-theoretic construction in an arithmetical setting, thus identifying $\top$ and $\bot$ with some arithmetical truth and falsity, respectively.[6] Let then $\mathcal{L}_{\mathbb{N}}$ be the language of arithmetic and $\mathcal{L}_{\mathbb{N}}^{\mathrm{C}} := \mathcal{L}_{\mathbb{N}} \cup \{\mathrm{C}\}$. We assume that the language of

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[4] Of course a contradiction arises only in the presence of contraction: however contraction is admissible in LPT. See Lemma 12 below. For an alternative approach, based on a restriction of cut and of the side sequents in the validity rules, see [BRT16].

[5] See the Fixed-Model Theorem, [AF80, Fef84].

[6] It is possible, but somewhat tedious, to generalize the construction to a standard model of syntax theory, thus avoiding the usual arithmetical interpretation of the coding scheme. We stick to the arithmetical framework for simplicity and legibility.

arithmetic includes the signature $\{0, \mathrm{S}, +, \times\}$ plus finitely many symbols for primitive recursive functions, which facilitates the development of formal syntax. For instance, it will contain symbols for the syntactic operations:

(1) $$s, t \mapsto \mathrm{C}(\ulcorner s \urcorner, \ulcorner t \urcorner)$$

(2) $$n, r, s, t \mapsto \mathrm{C}(\ulcorner r \urcorner, \underbrace{\ulcorner \mathrm{C}(\ulcorner r \urcorner, \dots \ulcorner \mathrm{C}(\ulcorner r \urcorner,}_{n-1\text{-times}} \ulcorner \mathrm{C}(\ulcorner s \urcorner, \ulcorner t \urcorner)\urcorner)^{\dots}\urcorner)$$

The use of $\mathcal{L}_{\mathbb{N}}$ presupposes a more comprehensive formalization of the syntax of $\mathcal{L}_{\mathrm{C}}$. The meaning of the Gödel quotes is now fixed by a canonical Gödel numbering and a standard formalisation of syntactic notions and operations. In what follows, we keep assuming a canonical coding of finite sets. A sequent is thus simply a pair of finite sets. We write $(\Gamma; \Delta)$ for (the code of) the sequent $\Gamma \Rightarrow \Delta$. For simplicity, we identify syntactic objects and their codes.

The semantic clauses of the jump given in the next definition correspond to standard, classically valid sequent rules – i.e. classical rules to introduce complex formulae to the left and to the right of the sequent arrow – plus rules for the consequence predicate that internalize them. As remarked above, the semantics for LPC is closed under a version of all classical sequent rules, so it's no surprise that a semantics for LPC follows the same patterns to interpret the logical vocabulary.

Definition 4 (C-jump [NR18]). *Let* $(\Gamma; \Delta)$ *denote the Gödel code of the sequent* $\Gamma \Rightarrow \Delta$. *For* $S \subseteq \omega$, *the operator* $\Psi : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ *is defined as follows:*

$$\begin{aligned} n \in \Psi(S) :\leftrightarrow\ & n \in S, \textit{ or} \\ & n = (\Gamma; \overline{j} = \overline{k}, \Delta) \textit{ and } \mathbb{N} \models \overline{j} = \overline{k}, \textit{ or} \\ & n = (\Gamma, \overline{j} = \overline{k}; \Delta) \textit{ and } \mathbb{N} \not\models \overline{j} = \overline{k}, \textit{ or} \\ & n = (\Gamma; \mathrm{C}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner), \Delta) \textit{ and } (\Gamma, \varphi; \psi, \Delta) \in S, \textit{ or} \\ & n = (\Gamma, \mathrm{C}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner); \Delta) \textit{ and } (\Gamma; \varphi, \Delta) \in S, (\Gamma, \psi; \Delta) \in S, \textit{ or} \\ & n = (\Gamma, \neg\varphi; \Delta) \textit{ and } (\Gamma; \varphi, \Delta) \in S, \textit{ or} \\ & n = (\Gamma; \neg\varphi, \Delta) \textit{ and } (\Gamma, \varphi; \Delta) \in S, \textit{ or} \\ & n = (\Gamma, \varphi \wedge \psi; \Delta) \textit{ and } (\Gamma, \varphi, \psi; \Delta) \in S, \textit{ or} \\ & n = (\Gamma; \varphi \wedge \psi, \Delta) \textit{ and } (\Gamma; \psi, \Delta) \in S, (\Gamma; \varphi, \Delta) \in S, \textit{ or} \\ & n = (\Gamma, \forall x\varphi; \Delta) \textit{ and } (\Gamma, \forall x\varphi, \varphi(\overline{m}); \Delta) \in S \textit{ for some } m, \textit{ or} \\ & n = (\Gamma; \forall x\varphi, \Delta) \textit{ and } (\Gamma; \varphi(\overline{m}), \Delta) \in S \textit{ for all } m. \end{aligned}$$

Iterations of Ψ can be defined as usual, by putting:[7]

$$\Psi^{\alpha}(S) = \Psi(\bigcup_{\beta<\alpha} \Psi^{\beta}(S)).$$

The operator Ψ is both increasing – i.e. $S \subseteq \Psi(S)$ for any S –, and monotonic: $S_0 \subseteq S_1$ entails $\Psi(S_0) \subseteq \Psi(S_1)$. The latter property entails the existence of fixed points of Ψ, i.e. sets T s.t. $\Psi(T) = T$. A fixed point T is said to be *inconsistent* if, for some sentence φ, both $(;\varphi)$ and $(;\neg\varphi)$ are in T, and *consistent* otherwise. We are mainly interested in the minimal of these fixed points $\mathcal{I}_{\Psi} := \bigcup_{\alpha \in \mathrm{Ord}} \Psi^{\alpha}(\varnothing)$. It can be shown that the minimal fixed point is indeed consistent [NR18].

[7]For more details, see [Mos74], Chapter 1.

The following lemma, proved in [NR18], shows that $\mathcal{I}_\Psi$ is a model of a naïve, self-applicable consequence predicate.

LEMMA 5 ([NR18, Lemma 9]).

(i) $(\Gamma;\Delta,\varphi) \in \mathcal{I}_\Psi$ *and* $(\psi,\Gamma;\Delta) \in \mathcal{I}_\Psi$ *if and only if* $(\mathrm{C}(\ulcorner\varphi\urcorner,\ulcorner\psi\urcorner),\Gamma;\Delta) \in \mathcal{I}_\Psi$*;*

(ii) $(\Gamma,\varphi;\psi,\Delta) \in \mathcal{I}_\Psi$ *if and only if* $(\Gamma;\mathrm{C}(\ulcorner\varphi\urcorner,\ulcorner\psi\urcorner),\Delta) \in \mathcal{I}_\Psi$.

Via the definition $\mathrm{Tr}(x) :\leftrightarrow \mathrm{C}(\ulcorner 0=0\urcorner, x)$ the minimal fixed point for self-referential truth from [Kri75] essentially "lives" inside $\mathcal{I}_\Psi$. In particular, one can restrict the construction above to empty contexts, and the clauses for $\mathrm{C}(x,y)$ then obviously can be restricted to:

$$(;\mathrm{C}(\ulcorner 0=0\urcorner,\ulcorner\varphi\urcorner)) \in \Psi(S), \text{ if } (;\varphi) \in S;$$
$$(\mathrm{C}(\ulcorner 0=0\urcorner,\ulcorner\varphi\urcorner);) \in \Psi(S), \text{ if } (\varphi;) \in S.$$

This restriction of the monotone operator Ψ above reaches then fixed points X in which

$$(;\varphi) \in X \text{ iff } (;\mathrm{Tr}\ulcorner\varphi\urcorner) \in X \tag{3}$$
$$(;\neg\varphi) \in X \text{ iff } (;\neg\mathrm{Tr}\ulcorner\varphi\urcorner) \in X. \tag{4}$$

Properties (3) and (4) correspond to the so-called "intersubstitutivity" of truth. For definiteness, let's call $\mathcal{J}$ the set of truths of the minimal fixed point X_0 so obtained, that is:

$$\mathcal{J} := \{\varphi \mid (;\varphi) \in X_0\}.$$

It's clear that we can express $\Psi(\cdot)$ as a formula of the language $\mathcal{L}_2$ of second-order arithmetic in such a way that

$$\Psi(S) = \{n \mid \mathbb{N} \vDash \mathrm{F}(x,X)\,[n,S]\}$$

for $\mathrm{F}(x,X)$ arithmetical and X occurring only positively – i.e. not in the scope of an even number of negation symbols – in it. Therefore

$$n \in \mathcal{I}_\Psi \Leftrightarrow (\forall X)\big((\forall x)(\mathrm{F}(x,X) \to x \in X) \to n \in X\big).$$

So $\mathcal{I}_\Psi \in \Pi^1_1$. Moreover, by the relationships between $\mathcal{J}$ and $\mathcal{I}_\Psi$ outlined above, and by Π^1_1-hardness of $\mathcal{J}$,[8] we have:

COROLLARY 6. *$\mathcal{I}_\Psi$ is Π^1_1-complete.*

It is well-known that Π^1_1-sets have a natural presentation in terms of cut-free infinitary derivability [Acz77, Poh09]. The case we are considering is not an exception, and a suitable infinitary calculus LPC^∞ can be developed along the lines of the infinitary system for non-reflexive truth developed in [Nic21]. LPC^∞ is obtained from LPC by (essentially): replacing the axioms for $\bot$ and $\top$ with corresponding rules for arithmetical truth and falsity, and replacing ($\forall$R) with an ω-rule.[9] By adapting the analysis in [Nic21], it can be shown that LPC^∞ has nice proof-theoretical properties: weakening and contraction are admissible preserving the (possibly infinite) length of the derivation, its rules are invertible, and (crucially) cut is eliminable in it.

In addition, it is possible to show that (possibly infinitary) proofs in LPC^∞ closely "match" the construction of $\mathcal{I}_\Psi$. More precisely: the ordinal stage of the inductive definition in which a sequent $\Gamma \Rightarrow \Delta$ enters in $\mathcal{I}_\Psi$ — i.e. its ordinal norm — can be associated to the lengths of cut-free proofs of

[8] See [Kri75, McG91].

[9] Two more technical amendments are omitting free variables, and generalizing CL and CR to arbitrary terms which code formulae. See [Nic21] for details.

$\Gamma \Rightarrow \Delta$ in LPC^∞. By a well-known result, this ordinal norm cannot exceed the first non-recursive (countable) ordinal ω_1^{CK}:[10]

(i) If there is a cut-free LPC^∞-proof of length $\leq \alpha < \omega_1^{\mathrm{CK}}$ of the sequent $\Gamma \Rightarrow \Delta$, then $(\Gamma; \Delta) \in \mathcal{J}_\Psi^{\alpha+1}$.

(ii) If $(\Gamma; \Delta) \in \mathcal{J}_\Psi^{\alpha}$, $\alpha < \omega_1^{\mathrm{CK}}$, then there there is a cut-free LPC^∞-proof of length $\leq \alpha + n < \omega_1^{\mathrm{CK}}$ of $\Gamma \Rightarrow \Delta$, for some $n \in \omega$.

If one restricts their attention to pairs of sentences, the above result entails the existence of a tight correspondence between the extension of the consequence predicate in $\mathcal{J}_\Psi$, and the consequence ascriptions derivable in LPC^∞. More specifically, for all $\varphi, \psi \in \mathcal{L}_\mathrm{C}$, the following are equivalent

(i) $\mathrm{LPC}^\infty \vdash \varphi \Rightarrow \psi$;

(ii) $\mathrm{LPC}^\infty \vdash \Rightarrow \mathrm{C}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner)$;

(iii) $(\varphi; \psi) \in \mathcal{J}_\Psi$.

4. Proof Theory of LPC

In this section we focus on the proof-theoretic properties of LPC. Our analysis culminates in the full eliminability of cut in it. The key technical insight that makes cut fully eliminable—and that is extensively investigated in [Nic21]—is a strong form of invertibility of the C-rules (Lemma 10).

The notions of *length* of a derivation is standardly defined [Sch77, TS03].[11] Given a calculus with rules that are at most α-branching, the *length* of a derivation $\mathcal{D}$ is the supremum of the lengths of its direct sub-derivations $\mathcal{D}_\gamma$ increased by one:

$$d = \sup\{d_\gamma + 1 \mid \gamma < \alpha\}$$

Clearly, for LPC, $\alpha = 2$ and the length of derivations is finite. We will write $\vdash_{\mathrm{LPC}} \Gamma \Rightarrow \Delta$ to indicate that there is a derivation of the sequent $\Gamma \Rightarrow \Delta$ in LPC, and $\mathcal{D} \vdash_{\mathrm{LPC}} \varphi$ to indicate that $\mathcal{D}$ is a proof of φ in LPC.

In developing the proof theory of LPC, it is convenient to work in a system that is *extensionally equivalent* to LPC, but that features an explicit labelling of formulae in sequents in a proof. Extensional equivalence means in this context that labelling does not allow one to obtain new proofs, but only to keep track of existing ones. This machinery is left implicit in work on the restriction of identity sequents [SH16, Fis18], but it's required for a formally precise cut-elimination argument, and in particular to define the main measure of complexity – called C-complexity – for applications of the C-rules to formulae in derivations. For each proof $\mathcal{D}$, we assume a labelling function $\mathfrak{l}_\mathcal{D} : \mathrm{Form}_{\mathcal{L}_\mathrm{C}} \to \omega \setminus \{0, 1\}$ applying to formulae in initial sequents. Labels then expand in a uniform way depending on the rule employed. For instance, for different rules configurations, we have:

$$\frac{\gamma^k, \varphi^l \Rightarrow \psi^m, \delta^n}{\gamma^{(1,k)}, \Rightarrow \mathrm{C}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner)^{(l,m)}, \delta^{(1,n)}}, \quad \frac{\gamma^k, \varphi^l \Rightarrow \delta^m}{\gamma^{(1,k)}, \Rightarrow \neg\varphi^{(1,l)}, \delta^{(1,m)}}, \quad \frac{\gamma^k \Rightarrow \varphi^l, \delta^m \qquad \gamma^n \Rightarrow \psi^p, \delta^q}{\gamma^{(k,n)} \Rightarrow \varphi \wedge \psi^{(l,p)}, \delta^{(m,q)}}.$$

All other rules conform to one of these patterns, and are labelled in an analogous way. Full details of the labelling machinery, including its extension to infinitary rules, can be found in [Nic21]. Once we know in principle that we can always employ labels to uniquely refer to formulae and their "history"

[10] See for instance, [Poh09, Thm. 6.6.4].

[11] Our notion of length amounts to what is called *depth* in [TS03].

throughout a proof, we can choose to omit labels for the sake of readability. We will often choose to do so.

In a nutshell, the C-complexity of a formula keeps track of the applications of the C-rules: initial sequents and $\mathcal{L}$-formulae have complexity 0, and the only way to increase the C-complexity of a formula is by introducing the consequence predicate.

DEFINITION 7 (C-complexity). *The ordinal* C*-complexity* $\kappa(\cdot)$ *of a formula* φ *of* $\mathcal{L}_{\mathrm{C}}$ *in a derivation* $\mathcal{D}$ *is defined inductively as follows:*

(i) formulae of $\mathcal{L}$ *have* C*-complexity* 0 *in any* $\mathcal{D}$*;*

(ii) If $\mathcal{D}$ *is just*

$$\Gamma, \varphi \Rightarrow \varphi, \Delta$$

with $\varphi \in \mathcal{L}$*, then* $\kappa(\psi) = \kappa(\varphi) = 0$ *for all* $\psi \in \Gamma, \Delta$*. Similarly for* $(\top)$, $(\bot)$.

(iii) If $\mathcal{D}$ *ends with*

$$\frac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta}$$

then $\kappa(\varphi) = \kappa(\neg\varphi)$ *and the* C*-complexity of the formulae in* Γ, Δ *is unchanged. Similarly for* $(\neg\mathrm{R})$ *and* $(\forall\mathrm{R})$.

(iv) If $\mathcal{D}$ *ends with*

$$\frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}$$

then $\kappa(\varphi \wedge \psi) = \max(\kappa(\varphi), \kappa(\psi))$ *and the* C*-complexity of the formulae in* Γ, Δ *is unchanged.*

(v) If $\mathcal{D}$ *ends with*

$$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}$$

then $\kappa(\varphi \wedge \psi) = \max(\kappa(\varphi), \kappa(\psi))$ *and the complexity of occurrences in side formulae is the maximum of the corresponding occurrences of side formulae in premisses.*

(vi) If $\mathcal{D}$ *ends with*

$$\frac{\Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \mathrm{C}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner), \Delta}$$

then $\kappa(\mathrm{C}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner)) = \max(\kappa(\varphi), \kappa(\psi)) + 1$ *and the* C*-complexity of the formulae in* Γ, Δ *is unchanged.*

(vii) If $\mathcal{D}$ *ends with*

$$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \psi, \Gamma \Rightarrow \Delta}{\Gamma, \mathrm{C}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner) \Rightarrow \Delta}$$

then $\kappa(\mathrm{C}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner)) = \max(\kappa(\varphi), \kappa(\psi)) + 1$ *and the complexity of occurrences in side formulae is the maximum of the corresponding occurrences of side formulae in premisses.*

(viii) If $\mathcal{D}$ ends with

$$\frac{\Gamma, \forall x\varphi^k, \varphi(t) \Rightarrow \Delta}{\forall x\varphi^l, \Gamma \Rightarrow \Delta}$$

then $\kappa(\forall x\varphi^l) = \max(\kappa(\forall x\varphi^k), \kappa(\varphi(t)))$ *and the* C*-complexity of the formulae in* Γ, Δ *is unchanged.*

(ix) If $\mathcal{D}$ ends with

$$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

then the complexity of occurrences in side formulae is the maximum of the corresponding occurrences of side formulae in premisses. In this case the complexity of the cut formula is the maximum of its two active occurrences.

We start by observing that, in proofs of sequents containing $\top$ on the left, and $\bot$ on the right, the occurrences of such constants can be omitted. Both claims follow by a straightforward induction on the length of the proof that preserves the C-complexity of the formulae in the contexts.

LEMMA 8.

(i) *If* $\vdash_{\text{LPC}} \top, \Gamma \Rightarrow \Delta$*, then* $\vdash_{\text{LPC}} \Gamma \Rightarrow \Delta$ *and the* C*-complexity of the formulae in the contexts is unchanged.*

(ii) *If* $\vdash_{\text{LPC}} \Gamma \Rightarrow \Delta, \bot$*, then* $\vdash_{\text{LPC}} \Gamma \Rightarrow \Delta$ *and the* C*-complexity of the formulae in the contexts is unchanged.*

In both claims, the length of the derivation is preserved.

Next, we turn to subsitution and weakening lemmata. Again, a proof by induction on the length of derivations is required. In the proof of weakening, the formulation of (REF), ($\top$), ($\bot$) with arbitrary contexts is of course essential.

LEMMA 9 (Substitution, Weakening).

(i) *If* $\Gamma \Rightarrow \Delta$ *is derivable in* LPC*, then* $\Gamma^* \Rightarrow \Delta^*$ *is* LPC*-derivable, where* Γ^**,* Δ^* *are obtained by uniformly replacing in* Γ*,* Δ*, a variable* x *by a term* t *which is free for* x *and does not contain variables employed in applications of* ($\forall$R) *in the proof of* $\Gamma \Rightarrow \Delta$*. Moreover, the* C*-complexity of the formulae involved in the substitution and in the contexts does not change.*

(ii) *Weakening is* κ*-admissible in* LPC*. That is, if we prove* $\Gamma \Rightarrow \Delta$*, we can prove* $\Gamma \Rightarrow \varphi, \Delta$ *(or* $\Gamma, \varphi \Rightarrow \Delta$*), so that* $\kappa(\varphi) = 0$*.*

In both claims, the length of the derivation is preserved.

The next lemma marks out the key property of LPC which makes it possible to generalize the standard G3-strategy for the admissiblity of cut to the present setting. All rules of LPC, including the rules for the consequence predicate, are invertible in a strong sense that preserves, and in the appropriate cases *reduces*, the C-complexity of formulae.

LEMMA 10 (κ-invertibility of LPC-rules).

(i) *If* $\mathcal{D} \vdash_{\text{LPC}} \Gamma, \mathrm{C}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner) \Rightarrow \Delta$*, then there are* $\mathcal{D}' \vdash_{\text{LPC}} \Gamma \Rightarrow \Delta, \varphi$ *and* $\mathcal{D}'' \vdash_{\text{LPC}} \psi, \Gamma \Rightarrow \Delta$ *with*

$$\kappa(\varphi), \kappa(\psi) = \kappa(\mathrm{C}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner)), \textit{ if } \kappa(\mathrm{C}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner)) = 0, \textit{ or}$$

$$\kappa(\varphi), \kappa(\psi) < \kappa(\mathrm{C}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner)), \text{ if } \kappa(\mathrm{C}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner)) > 0,$$

and in which the C*-complexity of the side formulae does not increase.*

(ii) *If* $\mathcal{D} \vdash_{\mathrm{LPC}} \Gamma \Rightarrow \mathrm{C}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner), \Delta$*, then there is a* $\mathcal{D}' \vdash_{\mathrm{LPC}} \Gamma, \varphi \Rightarrow \psi, \Delta$ *with*

$$\kappa(\varphi), \kappa(\psi) = \kappa(\mathrm{C}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner)), \text{ if } \kappa(\mathrm{C}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner)) = 0, \text{ or}$$
$$\kappa(\varphi), \kappa(\psi) < \kappa(\mathrm{C}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner)), \text{ if } \kappa(\mathrm{C}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner)) > 0,$$

and in which the C*-complexity of the side formulae is no greater than their* κ*-maximal occurrence in the premisses.*

(iii) *If* $\mathcal{D} \vdash_{\mathrm{LPC}} \Gamma, \neg\varphi \Rightarrow \Delta$*, there there is a* $\mathcal{D}' \vdash_{\mathrm{LPC}} \Gamma \Rightarrow \varphi, \Delta$ *with* $\kappa(\varphi) \leq \kappa(\neg\varphi)$ *and in which the* C*-complexity of the side formulae does not increase.*

(iv) *If* $\mathcal{D} \vdash_{\mathrm{LPC}} \Gamma \Rightarrow, \neg\varphi, \Delta$*, there there is a* $\mathcal{D}' \vdash_{\mathrm{LPC}} \Gamma, \varphi \Rightarrow \Delta$ *with* $\kappa(\varphi) \leq \kappa(\neg\varphi)$ *and in which the* C*-complexity of the side formulae does not increase.*

(v) *If* $\mathcal{D} \vdash_{\mathrm{LPC}} \Gamma, \varphi \wedge \psi \Rightarrow \Delta$*, then there is a* $\mathcal{D}' \vdash_{\mathrm{LPC}} \Gamma, \varphi, \psi \Rightarrow \Delta$ *with* $\kappa(\varphi), \kappa(\psi) \leq \kappa(\varphi \wedge \psi)$ *and in which the* C*-complexity of the side formulae does not increase.*

(vi) *If* $\mathcal{D} \vdash_{\mathrm{LPC}} \Gamma \Rightarrow \varphi \wedge \psi, \Delta$*, then there are* $\mathcal{D}' \vdash_{\mathrm{LPC}} \Gamma \Rightarrow \Delta, \varphi$ *and* $\mathcal{D}'' \vdash_{\mathrm{LPC}} \Gamma \Rightarrow \Delta, \psi$ *with* $\kappa(\varphi), \kappa(\psi) \leq \kappa(\varphi \wedge \psi)$ *and in which the* C*-complexity of the side formulae is no greater than their* κ*-maximal occurrence in the premisses.*

(vii) *If* $\mathcal{D} \vdash_{\mathrm{LPC}} \Gamma \Rightarrow \Delta, \forall x\varphi$*, then there is* $\mathcal{D}' \vdash_{\mathrm{LPC}} \Gamma \Rightarrow \Delta, \varphi(y)$*, for any* y *not free in* $\Gamma, \Delta, \forall x\varphi$*, with* $\kappa(\varphi(y)) \leq \kappa(\forall x\varphi)$ *and in which the* C*-complexity of the side formulae does not increase.*

Crucially, the invertibility of the rules preserves the length of the proof.

Proof. We proceed by induction on the length of the proof $\mathcal{D}$ but only show (i). The other cases are similar or easier. For (vii), one essentially employs the substitution lemma (Lemma 9(i)).

If $\Gamma, \mathrm{C}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner) \Rightarrow \Delta$ is an initial sequent, then it is for the form $\Gamma_0, \chi, \mathrm{C}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner) \Rightarrow \chi, \Delta_0$, or $\Gamma := \Gamma_0, \bot$, or $\Delta := \Delta, \bot$. In all such cases the claim is trivially obtained since $\Gamma \Rightarrow \Delta, \varphi$ and $\psi, \Gamma \Rightarrow \Delta$ are also initial sequents.

If $\Gamma, \mathrm{C}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner) \Rightarrow \Delta$ is *not* an axiom, there are two cases to consider. The first in which $\mathrm{C}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner)$ is principal in the last inference of $\mathcal{D}$, the second in which it is not. In the former case, $\kappa(\mathrm{C}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner)) > 0$, and the claim follows immediately by definition of C-complexity in the case of an application of (CL). In the latter case, suppose that $\mathcal{D}$ ends with

$$\dfrac{\begin{array}{c}\mathcal{D}_0\\ \mathrm{C}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner)^{k_0}, \Gamma_0 \Rightarrow \Delta_0\end{array} \qquad \begin{array}{c}\mathcal{D}_1\\ \mathrm{C}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner)^{k_1}, \Gamma_1 \Rightarrow \Delta_1\end{array}}{\mathrm{C}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner)^{k_2}, \Gamma \Rightarrow \Delta}\ (\mathrm{R})$$

(we treat the case of an arbitrary binary rule, the case of unary rules is simpler). The induction hypothesis applied to $\mathcal{D}_0$ and $\mathcal{D}_1$ yields derivations

$$\mathcal{D}_{00} \vdash_{\mathrm{LPC}} \Gamma_0 \Rightarrow \Delta_0, \varphi^{k_{00}} \qquad \mathcal{D}_{01} \vdash_{\mathrm{LPC}} \psi^{k_{01}}, \Gamma_0 \Rightarrow \Delta_0$$
$$\mathcal{D}_{10} \vdash_{\mathrm{LPC}} \Gamma_1 \Rightarrow \Delta_1, \varphi^{k_{10}} \qquad \mathcal{D}_{11} \vdash_{\mathrm{LPC}} \psi^{k_{11}}, \Gamma_1 \Rightarrow \Delta_1$$

such that, in both cases, $\kappa(\varphi), \kappa(\psi) = \kappa(\mathrm{C}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner))$, if $\kappa(\mathrm{C}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner)) = 0$, and $\kappa(\varphi), \kappa(\psi) < \kappa(\mathrm{C}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner))$, otherwise. Therefore the required derivations are obtained by applications of (R) to $\mathcal{D}_{00}$ and $\mathcal{D}_{11}$, and $\mathcal{D}_{10}$ and $\mathcal{D}_{01}$, respectively. *qed.*

REMARK 11. In the presence of (REF), the inversion strategy considered above will not go through. For instance, the derivability of a sequent of the form $\Gamma, \mathrm{C}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner) \Rightarrow \mathrm{C}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner), \Delta$ does not guarantee, for instance, the derivability of a sequent $\Gamma, \varphi \Rightarrow \psi, \Delta$ with $\kappa(\varphi) \leq \kappa(\mathrm{C}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner))$. This fact is crucial for the next lemma, in which contraction is shown to be κ-admissible.

LEMMA 12 (κ-admissibility of contraction).

(i) *If* $\mathcal{D} \vdash_{\mathrm{LPC}} \Gamma, \varphi^{k_0}, \varphi^{k_1} \Rightarrow \Delta$*, then there is a* $\mathcal{D}' \vdash_{\mathrm{LPC}} \Gamma, \varphi \Rightarrow \Delta$ *with* $\kappa(\varphi) \leq \max(\kappa(\varphi^{k_0}), \kappa(\varphi^{k_1}))$ *and in which the* C*-complexity of the side formulae does not increase.*

(ii) *If* $\mathcal{D} \vdash_{\mathrm{LPC}} \Gamma \Rightarrow \varphi^{k_0}, \varphi^{k_1}, \Delta$*, then there is a* $\mathcal{D}' \vdash_{\mathrm{LPC}} \Gamma \Rightarrow \varphi, \Delta$ *with* $\kappa(\varphi) \leq \max(\kappa(\varphi^{k_0}), \kappa(\varphi^{k_1}))$ *and in which the* C*-complexity of the side formulae does not increase.*

Crucially, in both claims the length of the original derivation is preserved.

Proof. The proof is by induction on the length of $\mathcal{D}$. One proves (i) and (ii) simultaneously.

The case of initial sequents follows immediately by the definition of C-complexity. The sub-case of the induction step in which neither φ^{k_0} nor φ^{k_1} is principal in the last inference is immediate by induction hypothesis.

What remains is the case in which $\Gamma, \varphi^{k_0}, \varphi^{k_1} \Rightarrow \Delta$ or $\Gamma \Rightarrow \varphi^{k_0}, \varphi^{k_1}, \Delta$ are not initial sequents, and one of φ^{k_0} or φ^{k_1} is principal in the last inference. We treat the crucial cases in which φ is $\mathrm{C}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner)$.

For (i), if $\mathrm{C}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner)^{k_0}$ is principal in the last inference, then $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c}\mathcal{D}_0\\ \Gamma, \mathrm{C}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner)^{k_{10}} \Rightarrow \Delta, \varphi^{k_{00}}\end{array} \qquad \begin{array}{c}\mathcal{D}_1\\ \psi^{k_{01}}, \Gamma, \mathrm{C}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner)^{k_{11}} \Rightarrow \Delta\end{array}}{\Gamma, \mathrm{C}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner)^{k_0}, \mathrm{C}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner)^{k_1} \Rightarrow \Delta}\ (\mathrm{CL})$$

such that $\kappa(\varphi^{k_{00}}), \kappa(\psi^{k_{01}}) < \kappa(\mathrm{C}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner)^{k_0})$.
We can then apply the inversion Lemma to $\mathcal{D}_0$ to obtain a

$$\mathcal{D}_0' \vdash_{\mathrm{LPC}} \Gamma \Rightarrow \Delta, \varphi^{k_{00}'}, \varphi^{l}$$

with

$$\kappa(\varphi^{l}) = \kappa(\mathrm{C}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner)^{k_{10}}) \text{ if } \kappa(\mathrm{C}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner)^{k_{10}}) = 0$$
$$\kappa(\varphi^{l}) < \kappa(\mathrm{C}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner)^{k_{10}}) \text{ if } \kappa(\mathrm{C}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner)^{k_{10}}) \neq 0$$

Similarly, inversion applied to $\mathcal{D}_1$ yields

$$\mathcal{D}_1' \vdash_{\mathrm{LPC}} \psi^{m}, \psi^{k_{01}'}, \Gamma \Rightarrow \Delta$$

with

$$\kappa(\psi^{m}) = \kappa(\mathrm{C}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner)^{k_{10}}) \text{ if } \kappa(\mathrm{C}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner)^{k_{10}}) = 0$$
$$\kappa(\psi^{m}) < \kappa(\mathrm{C}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner)^{k_{10}}) \text{ if } \kappa(\mathrm{C}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner)^{k_{10}}) \neq 0$$

By induction hypothesis, we obtain:

$$\mathcal{D}_0'' \vdash_{\mathrm{LPC}} \Gamma \Rightarrow \varphi, \Delta$$
$$\mathcal{D}_1'' \vdash_{\mathrm{LPC}} \psi, \Gamma \Rightarrow \Delta$$

An application of (CL) yields the desired

$$\mathcal{D}' \vdash_{\mathrm{LPC}} \Gamma, \mathrm{C}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner) \Rightarrow \Delta$$

with the required C-complexity

$$\begin{aligned}\kappa(\mathrm{C}(\ulcorner\varphi\urcorner,\ulcorner\psi\urcorner)) &= \max(\kappa(\varphi),\kappa(\psi))+1\\ &\leq \max(\kappa(\mathrm{C}(\ulcorner\varphi\urcorner,\ulcorner\psi\urcorner)^{k_0},\kappa(\mathrm{C}(\ulcorner\varphi\urcorner,\ulcorner\psi\urcorner)^{k_1}))\end{aligned}$$

For (ii), if $\mathrm{C}(\ulcorner\varphi\urcorner,\ulcorner\psi\urcorner)^{k_0}$ is principal in the last inference, then $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c}\mathcal{D}_0\\ \Gamma,\varphi^{k_{00}}\Rightarrow\psi^{k_{01}},\mathrm{C}(\ulcorner\varphi\urcorner,\ulcorner\psi\urcorner)^{k_{10}},\Delta\end{array}}{\Gamma\Rightarrow\mathrm{C}(\ulcorner\varphi\urcorner,\ulcorner\psi\urcorner)^{k_0},\mathrm{C}(\ulcorner\varphi\urcorner,\ulcorner\psi\urcorner)^{k_1},\Delta}$$

Inversion applied to $\mathcal{D}_0$ yields a proof $\mathcal{D}_0'$ ending with

$$\Gamma,\varphi^{l_0},\varphi^{k'_{00}}\Rightarrow\psi^{k'_{01}},\psi^{l_1},\Delta$$

By two applications of the induction hypothesis, we obtain a proof $\mathcal{D}_0''$ of

$$\Gamma,\varphi\Rightarrow\psi,\Delta$$

with

$$\kappa(\varphi)\leq\max(\kappa(\varphi^{k'_{00}}),\kappa(\varphi^{l_0}))\ (=:\ \alpha)$$

$$\kappa(\psi)\leq\max(\kappa(\psi^{k'_{01}}),\kappa(\psi^{l_1}))\ (=:\ \beta)$$

Therefore, by (CR), one obtains a derivation of $\Gamma\Rightarrow\mathrm{C}(\ulcorner\varphi\urcorner,\ulcorner\psi\urcorner),\Delta$ with

$$\kappa(\mathrm{C}(\ulcorner\varphi\urcorner,\ulcorner\psi\urcorner))=\max(\alpha,\beta)+1\leq\max(\kappa(\mathrm{C}(\ulcorner\varphi\urcorner,\ulcorner\psi\urcorner)^{k_0}),\kappa(\mathrm{C}(\ulcorner\varphi\urcorner,\ulcorner\psi\urcorner)^{k_1})).$$

It is worth noticing that the formulation of ($\forall$L) and its associated C-complexity renders the case of (i) in which one of the φ's is principal in the last inference and of the form $\forall x\varphi$ straightforward. *qed.*

We can finally state and prove the cut-elimination lemma for LPC. We start with the reduction lemma.

LEMMA 13 (Reduction). *If $\mathcal{D}_0$ is a cut-free proof of $\Gamma\Rightarrow\Delta,\varphi^k$ in* LPC*, and $\mathcal{D}_1$ is a cut-free* LPC*-proof of $\varphi^l,\Gamma\Rightarrow\Delta$, then there is a cut-free proof $\mathcal{D}$ of $\Gamma\Rightarrow\Delta$ in which the* C*-complexity of the side formulae is no greater than their κ-maximal occurrence in the premisses.*

Proof. The proof is by a main induction on $\kappa(\varphi)=\max(\kappa(\varphi^l),\kappa(\varphi^k))$, with side inductions on the logical complexity of φ and on the sum d_0+d_1 of the lengths of $\mathcal{D}_0$ and $\mathcal{D}_1$. We consider the main cases.

Case 1. One of $\mathcal{D}_0,\mathcal{D}_1$ is an initial sequent, say $\mathcal{D}_0$. If φ is not principal, then $\Gamma\Rightarrow\Delta$ is already an initial sequent. If φ is principal in it, then we can distinguish two cases. If $\mathcal{D}_0\vdash\Gamma_0,\varphi\Rightarrow\varphi^k,\Delta$, then we can apply Lemma 12 to $\mathcal{D}_1$ to obtain a derivation of $\Gamma\Rightarrow\Delta$ whose formulae have the required C-complexity. If $\mathcal{D}_0\vdash\Gamma\Rightarrow\top^k,\Delta$, then $\mathcal{D}_1\vdash\top^l,\Gamma\Rightarrow\Delta$. By lemma 8(i), $\Gamma\Rightarrow\Delta$ is derivable with the expected C-complexity.

Case 2. The cut formula is not principal in one of the premises, say $\mathcal{D}_1$. For instance the last inference of $\mathcal{D}_1$ is an application of (CL). Then, with $\Gamma\ :=\ \mathrm{C}(\ulcorner\varphi\urcorner,\ulcorner\psi\urcorner),\Gamma_0$, the derivation $\mathcal{D}$ ends with:

$$\dfrac{\begin{matrix}\mathcal{D}_0 \\ \mathrm{C}(\ulcorner\varphi\urcorner,\ulcorner\psi\urcorner),\Gamma_0 \Rightarrow \Delta,\chi\end{matrix} \qquad \dfrac{\begin{matrix}\mathcal{D}_{10} \\ \chi,\Gamma_0 \Rightarrow \Delta,\varphi\end{matrix} \qquad \begin{matrix}\mathcal{D}_{11} \\ \chi,\psi,\Gamma_0 \Rightarrow \Delta\end{matrix}}{\chi,\mathrm{C}(\ulcorner\varphi\urcorner,\ulcorner\psi\urcorner),\Gamma_0 \Rightarrow \Delta}}{\mathrm{C}(\ulcorner\varphi\urcorner,\ulcorner\psi\urcorner),\Gamma_0 \Rightarrow \Delta}$$

By the weakening lemma, $\mathcal{D}$ can be transformed into a derivation $\mathcal{D}'$ whose last inference is an application of (CL), whose premises are

$$\dfrac{\begin{matrix}\mathcal{D}'_0 \\ \mathrm{C}(\ulcorner\varphi\urcorner,\ulcorner\psi\urcorner),\Gamma_0 \Rightarrow \Delta,\varphi,\chi\end{matrix} \qquad \begin{matrix}\mathcal{D}'_{10} \\ \chi,\mathrm{C}(\ulcorner\varphi\urcorner,\ulcorner\psi\urcorner),\Gamma_0 \Rightarrow \Delta,\varphi\end{matrix}}{\mathrm{C}(\ulcorner\varphi\urcorner,\ulcorner\psi\urcorner),\Gamma_0 \Rightarrow \Delta,\varphi}$$

and

$$\dfrac{\begin{matrix}\mathcal{D}'_0 \\ \psi,\mathrm{C}(\ulcorner\varphi\urcorner,\ulcorner\psi\urcorner),\Gamma_0 \Rightarrow \Delta,\chi\end{matrix} \qquad \begin{matrix}\mathcal{D}'_{11} \\ \chi,\psi,\mathrm{C}(\ulcorner\varphi\urcorner,\ulcorner\psi\urcorner),\Gamma_0 \Rightarrow \Delta\end{matrix}}{\psi,\mathrm{C}(\ulcorner\varphi\urcorner,\ulcorner\psi\urcorner),\Gamma_0 \Rightarrow \Delta}$$

Therefore $\mathcal{D}' \vdash \mathrm{C}(\ulcorner\varphi\urcorner,\ulcorner\psi\urcorner),\mathrm{C}(\ulcorner\varphi\urcorner,\ulcorner\psi\urcorner),\Gamma_0 \Rightarrow \Delta$. The upper cuts in $\mathcal{D}'$ can be eliminated by side induction hypothesis, since $d'_0 + d'_{11}, d'_0 + d'_{10} < d_0 + d_1$. Moreover, since the weakened formulae can have lowest possible C-complexity, an application of the contraction lemma to the transformed derivation yields the claim. The other cases in which the cut formula is not principal are easier.

Case 3. The cut formula is principal in the last inference of $\mathcal{D}_0$ and $\mathcal{D}_1$. The case in which the cut formula is $\mathrm{C}(\ulcorner\varphi\urcorner,\ulcorner\psi\urcorner)$ is particularly easy, by main induction hypothesis, because the cut can be pushed upwards and applied to the ancestors of the cut formula, which have strictly smaller C-complexity. The case in which the cut formula is principal and of the form $\forall x\varphi$ is treated standardly as well but one has first to get rid of the universal quantifier in the premise of (∀L). This involves an essential application of the substitution lemma [TS03, §4.1]. *qed.*

REMARK 14. Although our proof of lemma 13 above relies heavily on lemma 12, the role of κ-admissibility of contraction can be circumscribed to the role it plays in Case 1 – that is, the case in which one of the premisses is an axiom and the cut formula is principal.

In Case 2, and in the specific sub-case treated above, one can apply the inversion lemma to $\mathcal{D}_0$ to obtain LPC-proofs $\mathcal{D}'_{00} \vdash \Gamma_0 \Rightarrow \Delta,\varphi,\chi$ and $\mathcal{D}'_{01} \vdash \psi,\Gamma_0 \Rightarrow \Delta,\chi$. These can then be combined with $\mathcal{D}_{10}$ and $\mathcal{D}_{11}$ respectively, and then (CL) applied to the results of the shorter cuts. Such template, with inversion playing the fundamental role, can be applied to all other sub-cases of Case 2 except of course (∀L). In such case, $\mathcal{D}$ has the form:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_{00} \\ \Gamma_0,\varphi(s),\forall x\varphi \Rightarrow \Delta,\chi\end{matrix}}{\Gamma_0,\forall x\varphi \Rightarrow \Delta,\chi} \qquad \begin{matrix}\mathcal{D}_1 \\ \chi,\Gamma_0,\forall x\varphi \Rightarrow \Delta\end{matrix}}{\Gamma_0,\forall x\varphi \Rightarrow \Delta}$$

In such case, one can therefore weaken $\mathcal{D}_1$, apply CUT to such weakened derivation and $\mathcal{D}_{00}$, and then apply (∀L).

By repeated applications of the Reduction Lemma, we can then obtain:

COROLLARY 15. *The rule* (CUT) *is eliminable in* LPC.

Since the cut-elimination proof above displays standard bounds for the reduction, Corollary 15 can be formalized in $\mathrm{I}\Delta_0 + \mathsf{superexp}$, where $\mathrm{I}\Delta_0$ is the subsystem of PA featuring only bounded induction, and

$$\mathsf{superexp} := \forall x \exists y (2^x_x = y),$$

with $2^x_0 = x$, $2^x_{y+1} = 2^{2^x_y}$.

The strategy leading to the cut-elimination theorem above clearly generalizes to the case of the theory obtained by replacing the C-rules with the rules (Tr-L) and (Tr-R). One simply has to replace the C-complexity with a truth complexity measure (cf. section 5 below). Similarly, one can apply the strategy to a theory of naïve abstraction (or property predication) based on rules of the form

$$\in\mathrm{R}\ \frac{\Gamma \Rightarrow \Delta, \varphi(t)}{\Gamma \Rightarrow \Delta, t \in \{x \mid \varphi\}} \qquad\qquad \in\mathrm{L}\ \frac{\varphi(t), \Gamma \Rightarrow \Delta}{t \in \{x \mid \varphi\}, \Gamma \Rightarrow \Delta}$$

where instead of a naming device one assumes a term-forming abstraction operator $\{\cdot \mid \cdot\}$ — e.g. along the lines of the one employed for a contraction-free set theory in [Can03]. $\in$-complexity is then defined in the obvious way: given a derivation $\mathcal{D}$ ending with $\in$L, the $\in$-complexity of $t \in \{x \mid \varphi\}$ is defined as the $\in$-complexity of $\varphi(t)$ plus one. On can then follow the template of Definition 7. All results above then carry over with only minimal modifications.

5. A compositional theory of non-reflexive truth and consequence

In their [HH06], Halbach and Horsten develop a formal system, called PKF (for *Partial Kripke-Feferman*), which axiomatizes Kripke's fixed point models over Peano Arihtmetic (PA) in strong Kleene logic. PKF constitutes the basis of any theory of truth that extends Kripke's theory with extra-resources – e.g. a new conditional [Fie08, Lei18]. In this section, we develop a twin-theory of PKF, which we call RKF, whose logic is based – somewhat unsurprisingly – on a restriction of (REF). PKF and RKF are twins in the sense that for X a fixed point model for the language $\mathcal{L}_{\mathrm{Tr}}$ obtained in the manner suggested in section 3,

$$(\mathbb{N}, X) \vDash^{\mathrm{k3}} \mathrm{PKF} \text{ iff } (\mathbb{N}, X) \vDash^{\mathrm{ts}} \mathrm{RKF}$$

This obviously entails that RKF is also a theory of naïve truth, and in fact an axiomatisation of Kripke's theory of truth in partial (substructural) logic. Actually, RKF is still richer: it is *also* a theory of naïve consequence, whereas PKF cannot be. In fact, just as a naïve truth predicate can be defined from the naïve consequence predicate of LPC, so can a predicate for naïve consequence (obeying the rules (CL) and (CR)) be defined from the naïve truth predicate of RKF (the naïveté of the latter, in turn, follows from the compositional rules of RKF). Definition 16, Lemma 20, and Corollary 21 will establish this claim more precisely. By contrast, since PKF is a fully structural theory, the presence of naïve consequence rules would immediately entail triviality by an internalized version of Curry's paradox—the V-Curry paradox by [BM13].

In addition to the vicinity of RKF to well-known theories with restricted operational rules, RKF displays some important theoretical virtues. First, it admits a nice semantics (via the simple, inductive construction reviewed in section 3) which is matched by the axiomatic theory. More specifically, RKF enjoys an adequacy result with respect to the fixed points of the inductive construction (Proposition 22). Adequacy results have been defended as a theoretical virtue for theories of truth, e.g.

by [FHKS15]. Moreover, non-reflexive approaches of the kind we discuss here admit a full interdefinability of naïve validity and naïve truth (via the conditional), and both notions enjoy fully symmetric rules (Lemma 20 and Corollary 21). Here, we leave open the question of which approach is ultimately preferable as an environment to formalize naïve semantic notions. This work is aimed at producing new results concerning non-reflexive theories, in order to better assess their prospects as formal approaches to naïve semantic notions.

As anticipated above, in order to formulate RKF, it is more convenient to take the truth predicate as primitive. Let $\mathcal{L}_{\mathrm{Tr}}$ be the language given by adding a fresh unary predicate Tr to the language of arithmetic, i.e. $\mathcal{L}_{\mathbb{N}} \cup \{\mathrm{Tr}\}$. In this language, we can define the consequence predicate via a combination of truth and conditional, putting $\mathrm{C}(x, y) :\leftrightarrow (\mathrm{Tr}(x) \to \mathrm{Tr}(y))$, i.e. $\neg(\mathrm{Tr}(x) \wedge \neg\mathrm{Tr}(y))$. Due to the fact that the logic TS has all the classical meta-inferences, and thus the conditional can be introduced and eliminated just as the consequence predicate, one can easily define truth in terms of consequence and the other way around.

One last piece of notation, following [Fef91]. Let $\mathrm{num}(x)$ be the function symbol representing the primitive recursive function that sends each number to its numeral. Given a formula $\varphi(v)$, we write $\ulcorner\varphi(\dot{x})\urcorner$ for the result of formally substituting the variable v for the numeral of x in φ (see e.g. [Smo77]). Moreover, $x(y/v)$ stands for the result of formally substituting y for the (code of) the variable v in x (we follow the conventions in [Hal14])

Definition 16 (RKF). *The theory* RKF *in* $\mathcal{L}_{\mathrm{Tr}}$ *has the following components:*

(i) *The logical component of* LPC*, that is the initial sequents and rules of* LPC *except* (Cl) *and* (Cr)

(ii) *The initial sequents* $\Gamma \Rightarrow \Delta, \varphi$ *for* φ *a basic axiom of* PA*, including identity axioms:*

$$\Gamma \Rightarrow \Delta, t = t; \qquad\qquad \Gamma, P(r), r = s \Rightarrow \Delta, P(s) \;\; \textit{for } P \textit{ an atom of } \mathcal{L}_{\mathrm{Tr}}.$$

(iii) *All instances of the induction schema for all formulae* $\varphi(v)$ *of* $\mathcal{L}_{\mathrm{Tr}}$*:*

$$\frac{\Gamma \Rightarrow \varphi(0), \Delta \qquad \Gamma, \varphi(x) \Rightarrow \varphi(x+1), \Delta}{\Gamma \Rightarrow \forall x\varphi, \Delta}$$

with x *not free in* $\Gamma, \Delta, \forall x\varphi$.

(iv) *The following truth rules:*

$$(\mathrm{TrAT1})\ \frac{\Gamma \Rightarrow P(x_1, \dots, x_n), \Delta}{\Gamma \Rightarrow \mathrm{Tr}(\ulcorner P(\dot{x}_1, \dots, \dot{x}_n)\urcorner), \Delta} \qquad (\mathrm{TrAT2})\ \frac{\Gamma, P(x_1, \dots, x_n) \Rightarrow \Delta}{\Gamma, \mathrm{Tr}(\ulcorner P(\dot{x}_1, \dots, \dot{x}_n)\urcorner) \Rightarrow \Delta}$$

$P(x_1, \dots, x_n)$ *an atom of* $\mathcal{L}_{\mathbb{N}}$

$$(\mathrm{Tr1})\ \frac{\Gamma \Rightarrow \mathrm{Tr}(x), \Delta}{\Gamma \Rightarrow \mathrm{Tr}\ulcorner\mathrm{Tr}(\dot{x})\urcorner, \Delta} \qquad (\mathrm{Tr2})\ \frac{\Gamma, \mathrm{Tr}(x) \Rightarrow \Delta}{\Gamma, \mathrm{Tr}\ulcorner\mathrm{Tr}(\dot{x})\urcorner \Rightarrow \Delta}$$

$$(\mathrm{Tr}\neg 1)\ \frac{\Gamma \Rightarrow \neg\mathrm{Tr}(x), \Delta}{\Gamma \Rightarrow \mathrm{Tr}(\underset{\cdot}{\neg} x), \Delta} \qquad (\mathrm{Tr}\neg 2)\ \frac{\Gamma, \neg\mathrm{Tr}(x) \Rightarrow \Delta}{\Gamma, \mathrm{Tr}(\underset{\cdot}{\neg} x) \Rightarrow \Delta}$$

$$(\mathrm{Tr}\wedge 1)\ \frac{\Gamma \Rightarrow \mathrm{Tr}(x), \Delta \qquad \Gamma \Rightarrow \mathrm{Tr}(y), \Delta}{\Gamma \Rightarrow \mathrm{Tr}(x \underset{\cdot}{\wedge} y), \Delta} \qquad (\mathrm{Tr}\wedge 2)\ \frac{\Gamma, \mathrm{Tr}(x), \mathrm{Tr}(y) \Rightarrow \Delta}{\Gamma, \mathrm{Tr}(x \underset{\cdot}{\wedge} y) \Rightarrow \Delta}$$

$$(\mathrm{Tr}\forall 1)\ \frac{\Gamma \Rightarrow \mathrm{Tr}(x(\dot{y}/v)), \Delta}{\Gamma \Rightarrow \mathrm{Tr}(\underset{\cdot}{\forall} v x), \Delta} \qquad (\mathrm{Tr}\forall 2)\ \frac{\Gamma, \mathrm{Tr}(x(t/v)) \Rightarrow \Delta}{\Gamma, \mathrm{Tr}(\underset{\cdot}{\forall} v x) \Rightarrow \Delta}$$

y *'not free' (see remark) in* $\Gamma, \Delta, \mathrm{Tr}(\underset{\cdot}{\forall} v x)$

A few comments on the definition:

REMARK 17.

(i) Syntactic functions operating on codes of $\mathcal{L}_{\mathrm{Tr}}$-expressions will be presented in simplified form for the sake of readability. For instance the operation $e_1, e_2 \mapsto e_1 = e_2$ represented in RKF by means of a function $\mathrm{id}(x, y)$, is abbreviated as $(x = y)$. Similarly for the other syntactic operations employed in the definition.

(ii) The Tr-rules for connectives and quantifiers are presented in simplified forms, with variables intended to range over sentences and terms, according to the form of the rules. For instance, (Tr∧1) is short for:

$$\frac{\Gamma, \mathrm{Sent}_{\mathcal{L}_{\mathrm{Tr}}}(x) \Rightarrow \mathrm{Tr}(x), \Delta \qquad \Gamma, \mathrm{Sent}_{\mathcal{L}_{\mathrm{Tr}}}(y) \Rightarrow \mathrm{Tr}(y), \Delta}{\Gamma, \mathrm{Sent}_{\mathcal{L}_{\mathrm{Tr}}}(\mathrm{and}(x, y)) \Rightarrow \mathrm{Tr}(\mathrm{and}(x, y)), \Delta}$$

where 'and(·, ·)' represents the operation $e_1, e_2 \mapsto e_1 \wedge e_2$.

Finally, the non-abbreviated form of (Tr∀1) reads:

$$\frac{\Gamma, \mathrm{Sent}_{\mathcal{L}_{\mathrm{Tr}}}(\underset{\cdot}{\forall} v x) \Rightarrow \mathrm{Tr}(x(\dot{y}/v)), \Delta}{\Gamma, \mathrm{Sent}_{\mathcal{L}_{\mathrm{Tr}}}(\underset{\cdot}{\forall} v x) \Rightarrow \mathrm{Tr}(\underset{\cdot}{\forall} v x), \Delta}$$

The consistency of RKF will be a corollary of Proposition 22, whose proof requires a few preliminary results that have also independent interest.

The following lemma, which follows from a simple *external* induction on the length of φ, indicates a form of recapture: RKF (and extensions thereof) features full initial sequents for the language $\mathcal{L}_{\mathbb{N}}$.

LEMMA 18. RKF *proves* $\varphi \Rightarrow \varphi$ *for* $\varphi \in \mathcal{L}_{\mathbb{N}}$.

Next, observe that weakening is length-preserving admissible in RKF, and inferences in RKF are "grounded", in the sense specified in the following lemma.

LEMMA 19 (Length-preserving admissibility of Weakening). *If there is an* RKF*-derivation* $\mathcal{D}$ *of length* n *of* $\Gamma \Rightarrow \Delta$, *then for any multisets* $\Gamma' \supseteq \Gamma$ *and* $\Delta' \supseteq \Delta$, *there is an* RKF*-derivation* $\mathcal{D}'$ *of length* n *of* $\Gamma' \Rightarrow \Delta'$.

We can now show that the naïve rules to introduce the truth predicate to the left and to the right of the sequent arrow are derivable in RKF.

LEMMA 20. *Every instance of the naïve truth rules is derivable in* RKF*:*

$$(\mathrm{TrR})\ \frac{\Gamma \Rightarrow \varphi(x_1, \dots, x_n), \Delta}{\Gamma \Rightarrow \mathrm{Tr}(\ulcorner \varphi(\dot{x}_1, \dots, \dot{x}_n)\urcorner), \Delta} \qquad (\mathrm{TrL})\ \frac{\Gamma, \varphi(x_1, \dots, x_n) \Rightarrow \Delta}{\Gamma, \mathrm{Tr}(\ulcorner \varphi(\dot{x}_1, \dots, \dot{x}_n)\urcorner) \Rightarrow \Delta}$$

Proof. We prove the admissibility of both TrR and TrL by simultaneous induction, with the main induction on the logical complexity of φ and secondary induction on the length of the derivations. We do only some cases, and only for the rule TrR, for the sake of brevity. Call $\mathcal{D}$ the derivation of $\Gamma \Rightarrow \varphi, \Delta$ in the premiss of TrR.

Case 1. Suppose φ has logical complexity 0. Therefore, it is atomic. There are two cases.

Case 1.1. φ is an atomic formula of $\mathcal{L}_{\mathbb{N}}$. In this case, the rule TrAt1 provides the desired conclusion:

$$(\mathrm{TrAt1})\ \frac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow P(x_1, \dots, x_n), \Delta\end{array}}{\Gamma \Rightarrow \mathrm{Tr}(\ulcorner P(\dot{x}_1, \dots, \dot{x}_n)\urcorner), \Delta}$$

Case 1.2. φ is of the form $\mathrm{Tr}(x)$. In this case, the rule Tr1 provides the desired conclusion:

$$\begin{array}{c}\vdots\\ (\mathrm{Tr1})\ \dfrac{\Gamma \Rightarrow \mathrm{Tr}(x), \Delta}{\Gamma \Rightarrow \mathrm{Tr}\ulcorner \mathrm{Tr}(\dot{x})\urcorner, \Delta}\end{array}$$

Case 2. φ has logical complexity $n+1$. There are 3 main cases.

Case 2.1. φ is a negation $\neg\psi$. We omit this case (it is similar to, and easier than, the others).

Case 2.2. φ is a conjunction $\psi \wedge \chi$. There are two sub-cases.

Case 2.2.1. $\psi \wedge \chi$ is principal in $\mathcal{D}$. Then, the last rule applied to $\psi \wedge \chi$ in $\mathcal{D}$ is $\wedge$R, and $\mathcal{D}$ has this form:

$$(\wedge\mathrm{R})\ \dfrac{\begin{array}{c}\mathcal{D}_0\\ \Gamma \Rightarrow \psi, \Delta\end{array} \qquad \begin{array}{c}\mathcal{D}_1\\ \Gamma \Rightarrow \chi, \Delta\end{array}}{\Gamma \Rightarrow \psi \wedge \chi, \Delta}$$

By the main induction hypothesis, there is a derivation $\mathcal{D}_0^*$ of $\Gamma \Rightarrow \mathrm{Tr}(\ulcorner\psi\urcorner), \Delta$ from $\Gamma \Rightarrow \psi, \Delta$, and a derivation $\mathcal{D}_1^*$ of $\Gamma \Rightarrow \mathrm{Tr}(\ulcorner\chi\urcorner), \Delta$ from $\Gamma \Rightarrow \chi, \Delta$. Then, we reason as follows:

$$(\mathrm{Tr}\wedge 1)\ \dfrac{\begin{array}{c}\mathcal{D}_0^*\\ \Gamma \Rightarrow \mathrm{Tr}(\ulcorner\psi\urcorner), \Delta\end{array} \qquad \begin{array}{c}\mathcal{D}_1^*\\ \Gamma \Rightarrow \mathrm{Tr}(\ulcorner\chi\urcorner), \Delta\end{array}}{\Gamma \Rightarrow \mathrm{Tr}(\ulcorner\psi \wedge \chi\urcorner), \Delta}$$

Case 2.2.2. $\psi \wedge \chi$ is not principal in $\mathcal{D}$. Therefore, some other rule R is the last one in $\mathcal{D}$. Suppose R has two premises (the case with one premiss is analogous). Therefore, $\mathcal{D}$ has this form:

$$(\mathrm{R})\ \dfrac{\begin{array}{c}\mathcal{D}_0\\ \Gamma_0 \Rightarrow \psi \wedge \chi, \Delta_0\end{array} \qquad \begin{array}{c}\mathcal{D}_1\\ \Gamma_1 \Rightarrow \psi \wedge \chi, \Delta_1\end{array}}{\Gamma \Rightarrow \psi \wedge \chi, \Delta}$$

By the secondary induction hypothesis, there is a derivation $\mathcal{D}_0^*$ of $\Gamma_0 \Rightarrow \mathrm{Tr}(\ulcorner\psi \wedge \chi\urcorner), \Delta_0$ from $\Gamma_0 \Rightarrow \psi \wedge \chi, \Delta_0$, and a derivation $\mathcal{D}_0^*$ of $\Gamma_1 \Rightarrow \mathrm{Tr}(\ulcorner\psi \wedge \chi\urcorner), \Delta_1$ from $\Gamma_1 \Rightarrow \psi \wedge \chi, \Delta_1$. Then, we reason as follows:

$$(\mathrm{R})\ \dfrac{\begin{array}{c}\mathcal{D}_0^*\\ \Gamma_0 \Rightarrow \mathrm{Tr}(\ulcorner\psi \wedge \chi\urcorner), \Delta_0\end{array} \qquad \begin{array}{c}\mathcal{D}_1\\ \Gamma_1 \Rightarrow \mathrm{Tr}(\ulcorner\psi \wedge \chi\urcorner), \Delta_1\end{array}}{\Gamma \Rightarrow \mathrm{Tr}(\ulcorner\psi \wedge \chi\urcorner), \Delta}$$

Case 2.3. φ is a universally quantified formula $\forall x\psi$. There are three sub-cases.

Case 2.3.1. $\forall x\psi$ is principal in $\mathcal{D}$. There are two sub-cases.

Case 2.3.1.1. The last rule applied to $\forall x\psi$ in $\mathcal{D}$ is $\forall$R. Therefore, $\mathcal{D}$ has the following form:

$$(\forall\mathrm{R})\ \dfrac{\begin{array}{c}\mathcal{D}_0\\ \Gamma \Rightarrow \psi(y), \Delta\end{array}}{\Gamma \Rightarrow \forall x\psi, \Delta}$$

where $y \notin \mathrm{FV}(\Gamma, \Delta, \forall x\psi)$. By the main induction hypothesis, there is a derivation $\mathcal{D}_0^*$ of $\Gamma \Rightarrow \mathrm{Tr}(\ulcorner\psi(\dot{y})\urcorner), \Delta$ from $\Gamma \Rightarrow \psi(y), \Delta$. We then reason as follows:

$$\begin{array}{c}\mathcal{D}_0^*\\ (\forall\mathrm{R})\ \dfrac{\Gamma \Rightarrow \mathrm{Tr}(\ulcorner\psi(\dot{y})\urcorner), \Delta}{\Gamma \Rightarrow \forall x\mathrm{Tr}(\ulcorner\psi(\dot{x})\urcorner), \Delta}\\ (\mathrm{Tr}\forall 1)\ \dfrac{\qquad\qquad}{\Gamma \Rightarrow \mathrm{Tr}(\ulcorner\forall x\psi\urcorner), \Delta}\end{array}$$

noticing that our assumption entails that $y \notin \mathrm{FV}(\Gamma, \Delta, \forall x \mathrm{Tr}(\ulcorner \psi(\dot{x}) \urcorner))$ as well.

Case 2.3.1.2. The last rule applied to $\forall x \psi$ in $\mathcal{D}$ is IND. Therefore, $\mathcal{D}$ has the following form:

$$\text{(IND)}\ \dfrac{\begin{array}{c}\mathcal{D}_0\\ \Gamma \Rightarrow \psi(0), \Delta\end{array} \qquad \begin{array}{c}\mathcal{D}_1\\ \Gamma, \psi(x) \Rightarrow \psi(x+1), \Delta\end{array}}{\Gamma \Rightarrow \forall x \psi, \Delta}$$

By the main induction hypothesis, there is a derivation $\mathcal{D}_0^*$ of $\Gamma \Rightarrow \mathrm{Tr}(\ulcorner \psi(0) \urcorner), \Delta$ from $\Gamma \Rightarrow \psi(0), \Delta$. Moreover, by the secondary induction hypothesis, there is a derivation $\mathcal{D}_1^*$ of $\Gamma, \mathrm{Tr}(\ulcorner \psi(\dot{x}) \urcorner) \Rightarrow \psi(x + 1), \Delta$ from $\Gamma, \psi(x) \Rightarrow \psi(x+1), \Delta$, and by the main induction hypothesis, there is also a derivation $\mathcal{D}_1^{**}$ of $\Gamma, \mathrm{Tr}(\ulcorner \psi(\dot{x}) \urcorner) \Rightarrow \mathrm{Tr}(\ulcorner \psi(\dot{x}+1) \urcorner), \Delta$ from the latter (i.e. from $\Gamma, \mathrm{Tr}(\ulcorner \psi(\dot{x}) \urcorner) \Rightarrow \psi(x+1), \Delta$). We then reason as follows:

$$\text{(Tr}\forall 1\text{)}\ \dfrac{\text{(IND)}\ \dfrac{\begin{array}{c}\mathcal{D}_0\\ \Gamma \Rightarrow \mathrm{Tr}(\ulcorner \psi(0) \urcorner), \Delta\end{array} \qquad \begin{array}{c}\mathcal{D}_1^{**}\\ \Gamma, \mathrm{Tr}(\ulcorner \psi(\dot{x}) \urcorner) \Rightarrow \mathrm{Tr}(\ulcorner \psi(\dot{x}+1) \urcorner), \Delta\end{array}}{\Gamma \Rightarrow \forall x \mathrm{Tr}(\ulcorner \psi(\dot{x}) \urcorner, \Delta}}{\Gamma \Rightarrow \mathrm{Tr}(\ulcorner \forall x \psi \urcorner), \Delta}$$

Case 2.3.2. $\forall x \psi$ is not principal in $\mathcal{D}$. Therefore, some other rule R is the last one in $\mathcal{D}$. Suppose R has one premiss (the case with two premises is, as always, analogous). $\mathcal{D}$ has the following form:

$$\text{(R)}\ \dfrac{\begin{array}{c}\mathcal{D}_0\\ \Gamma_0 \Rightarrow \forall x \psi, \Delta_0\end{array}}{\Gamma \Rightarrow \forall x \psi, \Delta}$$

By the secondary induction hypothesis, there is a derivation $\mathcal{D}_0^*$ of $\Gamma_0 \Rightarrow \mathrm{Tr}(\ulcorner \forall x \psi \urcorner), \Delta_0$ from $\Gamma_0 \Rightarrow \forall x \psi, \Delta_0$. We then we can simply apply the rule R.

qed.

The above Lemma also shows that, via the definition of $\mathrm{C}(x, y)$ as $\mathrm{Tr}(x) \rightarrow \mathrm{Tr}(y)$, RKF unrestrictedly validates the naïve rules for consequence.

COROLLARY 21. *Every instance of the naïve consequence rules is derivable in* RKF*:*

$$\text{(CL)}\ \dfrac{\Gamma \Rightarrow \Delta, \varphi \qquad \psi, \Gamma \Rightarrow \Delta}{\Gamma, \mathrm{C}(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner) \Rightarrow \Delta} \qquad \dfrac{\Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \mathrm{C}(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner), \Delta}\ \text{(CR)}$$

Proof. For CL:

$$\text{(}\neg\text{L)}\ \dfrac{\text{(}\wedge\text{R)}\ \dfrac{\text{(TrR)}\ \dfrac{\Gamma \Rightarrow \varphi, \Delta}{\Gamma \Rightarrow \mathrm{Tr}(\ulcorner \varphi \urcorner), \Delta} \qquad \text{(}\neg\text{R)}\ \dfrac{\text{(TrL)}\ \dfrac{\Gamma, \psi \Rightarrow \Delta}{\Gamma, \mathrm{Tr}(\ulcorner \psi \urcorner) \Rightarrow \Delta}}{\Gamma \Rightarrow \neg \mathrm{Tr}(\ulcorner \psi \urcorner), \Delta}}{\Gamma \Rightarrow \mathrm{Tr}(\ulcorner \varphi \urcorner) \wedge \neg \mathrm{Tr}(\ulcorner \psi \urcorner), \Delta}}{\Gamma, \neg(\mathrm{Tr}(\ulcorner \varphi \urcorner) \wedge \neg \mathrm{Tr}(\ulcorner \psi \urcorner)) \Rightarrow \Delta}$$

For CR:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma, \mathrm{Tr}(\ulcorner \varphi \urcorner) \Rightarrow \psi, \Delta}\ \text{(TrL)}}{\Gamma, \mathrm{Tr}(\ulcorner \varphi \urcorner) \Rightarrow \mathrm{Tr}(\ulcorner \psi \urcorner), \Delta}\ \text{(TrR)}}{\Gamma, \mathrm{Tr}(\ulcorner \varphi \urcorner), \neg \mathrm{Tr}(\ulcorner \psi \urcorner) \Rightarrow \Delta}\ \text{(}\neg\text{L)}}{\Gamma, \mathrm{Tr}(\ulcorner \varphi \urcorner) \wedge \neg \mathrm{Tr}(\ulcorner \psi \urcorner) \Rightarrow \Delta}\ \text{(}\wedge\text{L)}}{\Gamma \Rightarrow \neg(\mathrm{Tr}(\ulcorner \varphi \urcorner) \wedge \neg \mathrm{Tr}(\ulcorner \psi \urcorner)), \Delta}\ \text{(}\neg\text{R)}$$

Since $\mathrm{C}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner)$ is defined as $\neg(\mathrm{Tr}(\ulcorner\varphi\urcorner) \land \neg\mathrm{Tr}(\ulcorner\psi\urcorner))$, this establishes the claim. *qed.*

Finally, thanks to the above Corollary, RKF can be shown to be *adequate* with respect to the semantics articulated in section 3.

PROPOSITION 22 (Adequacy). *For every multisets* Γ, Δ *of formulae of* $\mathcal{L}_{\mathrm{C}}$ *and every* $S \subseteq \omega \times \omega$*:*

$$\langle \mathbb{N}, S\rangle \vDash^{\mathrm{ts}} \mathrm{RKF} \textit{ if and only if } \Psi(S) = S, \textit{ and } S \textit{ is consistent}$$

Proof sketch. The right-to-left direction is immediate: a quick inspection shows that if S is a consistent fixed point of Ψ, then $\langle \mathbb{N}, S\rangle$ TS-satisfies all the axiom and rules of RKF. For the left-to-right direction, notice that if $\langle \mathbb{N}, S\rangle \vDash^{\mathrm{ts}}$ RKF, then the set of (codes of) sentences in S is consistent and closed under all the logical clauses of the operator Ψ (for otherwise $\langle \mathbb{N}, S\rangle$ would not TS-satisfy the logical rules of RKF) and, by Corollary 21, also under the naïve consequence-theoretic clauses of Ψ. Therefore, $\Psi(S) = S$. *qed.*

We now turn to the proof-theoretic analysis of RKF. We will establish an upper-bound for RKF-provability. We show that RKF can be embedded in the theory PKF – first proposed by [HH06]. We assume a sequent calculus formulation of K3 with identity (see, e.g., [NS21, Appendix A]).

DEFINITION 23 (PKF). *The system* PKF *extends first-order* K3 *formulated in* $\mathcal{L}_{\mathrm{Tr}}$ *with the basic axioms of* PA *as initial sequents, the induction principle*

$$\frac{\Gamma, A(x) \Rightarrow A(x+1), \Delta}{\Gamma, A(\overline{0}) \Rightarrow A(t), \Delta}\ \mathtt{IND}(\mathcal{L}_{\mathrm{Tr}})$$

for $A(v) \in \mathcal{L}_{\mathrm{Tr}}$ *(and* x *not free in* $\Gamma, \Delta, A(\overline{0})$ *and the following initial sequents:*

(PKF1)	$\Gamma, \mathrm{Tr}(\ulcorner P(\dot{x}_1, \dots, \dot{x}_n)\urcorner) \Rightarrow P(x_1, \dots, x_n), \Delta$
(PKF2)	$\Gamma, P(x_1, \dots, x_n) \Rightarrow \mathrm{Tr}(\ulcorner P(\dot{x}_1, \dots, \dot{x}_n)\urcorner), \Delta$
(PKF3)	$\Gamma, \mathrm{Sent}_{\mathcal{L}_{\mathrm{Tr}}}(x), \neg\mathrm{Tr}x \Rightarrow \mathrm{Tr}\underset{\cdot}{\neg}x, \Delta$
(PKF4)	$\Gamma, \mathrm{Sent}_{\mathcal{L}_{\mathrm{Tr}}}(x), \mathrm{Tr}\underset{\cdot}{\neg}x \Rightarrow \neg\mathrm{Tr}x, \Delta$
(PKF5)	$\Gamma, \mathrm{Sent}_{\mathcal{L}_{\mathrm{Tr}}}(x\underset{\cdot}{\land}y), \mathrm{Tr}(x\underset{\cdot}{\land}y) \Rightarrow \mathrm{Tr}x \land \mathrm{Tr}y, \Delta$
(PKF6)	$\Gamma, \mathrm{Sent}_{\mathcal{L}_{\mathrm{Tr}}}(x\underset{\cdot}{\land}y), \mathrm{Tr}x \land \mathrm{Tr}y \Rightarrow \mathrm{Tr}(x\underset{\cdot}{\land}y), \Delta$
(PKF7)	$\Gamma, \mathrm{Sent}_{\mathcal{L}_{\mathrm{Tr}}}(\underset{\cdot}{\forall}vx), \mathrm{Tr}(\underset{\cdot}{\forall}vx) \Rightarrow \forall y \mathrm{Tr}\, x(\dot{y}/v), \Delta$
(PKF8)	$\Gamma, \mathrm{Sent}_{\mathcal{L}_{\mathrm{Tr}}}(\underset{\cdot}{\forall}vx), \forall y \mathrm{Tr}\, x(\dot{y}/v) \Rightarrow \mathrm{Tr}(\underset{\cdot}{\forall}vx), \Delta$
(PKF9)	$\Gamma, \mathrm{Tr}(x) \Rightarrow \mathrm{Tr}\ulcorner\mathrm{Tr}(\dot{x})\urcorner, \Delta$
(PKF10)	$\Gamma, \mathrm{Tr}\ulcorner\mathrm{Tr}(\dot{x})\urcorner \Rightarrow \mathrm{Tr}(x), \Delta$

Notational abbreviations have been applied as in the definition of RKF.

The idea of the reduction is as follows: since the sequent arrow of RKF is modelled after the material conditional of K3, we can translate the provability of a sequent as provability of the corresponding material conditional, plus the condition that the sentences in the conclusion are fully classical in PKF.

LEMMA 24. *If* $\mathrm{RKF} \vdash \Gamma \Rightarrow \Delta$*, then* $\mathrm{PKF} \vdash \Rightarrow \neg \bigwedge \Gamma, \bigvee \Delta$*, where* $\bigwedge \varnothing := \top$*, and* $\bigvee \varnothing := \bot$*.*

Proof. The proof is by induction on the length of the derivation in RKF.

For the base case: if $\Gamma \Rightarrow \Delta$ is an initial sequent in RKF, then $\varphi \in \Delta$ may be either $t = t$ or a basic axiom of PA. In both cases, PKF $\Rightarrow \varphi$; the claim then follows by logic. If there is a $\varphi \in \mathcal{L}_{\mathbb{N}} \cap \Gamma \cap \Delta$, then again $\Rightarrow \varphi, \neg\varphi$ is derivable in PKF, and the claim is obtained again by logic.

For the induction step, each rule must be considered. We report the key cases of induction, where an additional lemma is required, and of the logical rules of negation, where it shown how the shift of the principal formulae between antecedent and consequent of the sequent in the rules is dealt with within PKF.

If $\Gamma \Rightarrow \Delta, \forall x\varphi$ results from an application of induction, then one has by induction hypothesis (and inversion for $\vee$-rules) that the following sequents are provable in PKF:

$$\Rightarrow \neg \bigwedge \Gamma, \varphi(0), \bigvee \Delta \tag{5}$$

$$\Rightarrow \neg \bigwedge \Gamma, \neg\varphi(x), \varphi(x+1), \bigvee \Delta \tag{6}$$

By induction on the length of the proof in PKF one can show that the following rule is admissible in PKF:

$$\frac{\Rightarrow \psi, \neg\varphi, \chi}{\varphi \Rightarrow \psi, \chi} \tag{7}$$

The rule (7) applied to (6) yields

$$\varphi(x) \Rightarrow \neg \bigwedge \Gamma, \varphi(x+1), \bigvee \Delta \tag{8}$$

The sequent (8), together with (5), gives

$$\Rightarrow \neg \bigwedge \Gamma, \forall x\varphi, \bigvee \Delta \tag{9}$$

by the induction rule of PKF.

For the logical negation rules, let's consider ($\neg$L). The case of ($\neg$R) is analogous. One has (again by employing the inversion properties of $\vee$-rules):

$$\Rightarrow \neg \bigwedge \Gamma, \varphi, \bigvee \Delta. \tag{10}$$

We would like to obtain

$$\Rightarrow \neg(\bigwedge \Gamma \wedge \neg\varphi), \bigvee \Delta. \tag{11}$$

However, (10) and (11) are interderivable in PKF by pure logic. *qed.*

Since PKF proves $\Rightarrow \varphi \vee \neg\varphi$ for $\varphi \in \mathcal{L}_{\mathbb{N}}$, we have the desidered corollary:

COROLLARY 25. *For* $\varphi \in \mathcal{L}_{\mathbb{N}}$, *if* RKF $\vdash \Rightarrow \varphi$, *then* PKF $\vdash \Rightarrow \varphi$.

The study of the proof-theoretic lower bound for RKF appears to be more involved. Assuming a standard notation for ordinals $< \varepsilon_0$, one would hope to define in RKF the truth predicates of the theory of ramified truth up to the ordinal ω^ω ($\mathrm{RT}_{<\omega^\omega}$) – see [Hal14, §9.1] for a definition. If one succeeded, then it would follow that any arithmetical theorem of PKF is also a theorem of RKF: this is because $\mathrm{RT}_{<\omega^\omega}$ is an upper bound for the arithmetical theorems of PKF. This strategy would be realized if one could show that the rule

$$\mathrm{TI}_{\mathcal{L}_{\mathrm{Tr}}}(\alpha) := \frac{\Gamma, \forall \zeta \prec \eta\, A(\zeta) \Rightarrow A(\eta), \Delta}{\Gamma \Rightarrow \forall \xi \prec \alpha\, A(\xi), \Delta}$$

is admissible in RKF for each $\alpha < \omega^\omega$. Since each $\alpha < \omega^\omega$ has the form ω^k, one may try to mimic the classical proof and prove the claim by first establishing

$$\Gamma, \forall\alpha \prec \beta\, A(\alpha) \Rightarrow \forall\alpha \prec \beta + \omega^k\, A(\alpha), \Delta$$

via an external induction on k. Troubles already appear in the base step of this induction. In fact, the main assumption

$$\Gamma, \forall\alpha \prec \beta\, A(\alpha) \Rightarrow A(\beta), \Delta$$

on the progressiveness of A does not suffice to conclude

$$\Gamma, \forall\alpha \prec \beta\, A(\alpha) \Rightarrow \forall\alpha \prec \beta + 1\, A(\alpha), \Delta$$

because of the potential failure of

$$\forall\alpha \prec \beta\, A(\alpha) \Rightarrow \forall\beta \prec \alpha\, A(\alpha).$$

We therefore list the claim as an open, although we conjecture a positive answer to it:

OPEN PROBLEM 26. RKF *defines the truth predicates of* $\mathrm{RT}_{<\omega^\omega}$. *That is, there is a relative interpretation of* $\mathrm{RT}_{<\omega^\omega}$ *in* RKF *that leaves the arithmetical vocabulary unchanged. Therefore, all arithmetical theorems of* $\mathrm{RT}_{<\omega^\omega}$ *are theorems of* RKF.

6. Further Work

Much work remains to be done on non-reflexive systems and their applications. Just to mention a few: fully compositional, non-reflexive theories of consequence should be formulated and studied (by analogy with the compositional, non-reflexive theory of truth presented in section 5). Moreover, the relations between non-reflexive and other non-classical systems (paracomplete, paraconsistent, non-contractive, and non-transitive) should be fully investigated. For instance, the non-reflexive logic TS is known to be *dual* to the non-transitive logic ST, in a precise technical sense:[12] therefore, TS-based theories could be dual, in the same sense, to ST-based theories. Another nonclassical system in the vicinity of RKF may involve logical constants interpreted by means of other truth functions such as the ones of Weak-Kleene logic.

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