

Systems for non-reflexive consequence

Carlo Nicolai & Lorenzo Rossi
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ABSTRACT. Substructural logics and their application to logical and semantic paradoxes have been extensively studied, but non-reflexive systems have been somewhat neglected. Here, we aim to (at least partly) fill this lacuna, by presenting a non-reflexive logic and theory of naïve consequence (and truth). We also investigate the semantics and the proof-theory of the system. Finally, we develop a compositional theory of truth (and consequence) in our non-reflexive framework.

1. INTRODUCTION

Recent years have seen a considerable growth in the interest in substructural logics, and their application to logical and semantic paradoxes. Many recent works focus on non-transitive approaches to paradox [Rip12, CÉRvR12, CÉRVR13, Rip13a, Rip13b, BRT15, Rip15, BRT16, BPS20], and non-contractive approaches have also received considerable attention [Gri82, Pet00, Can03, Zar11, MP14, DRR18, Fje19, Ros19]. By contrast, non-reflexive theories have been investigated less.¹ Nevertheless, non-reflexive theories are especially promising to model the interplay between naïve truth and consequence [NR18]—in this respect, they are even more promising than their non-transitive rivals. However, a systematic presentation of the logic, the semantics, and the proof-theory of non-reflexive theories of naïve truth and consequence are currently lacking, and so does a thorough philosophical analysis (and defence).

The purpose of this paper is to (at least partly) fill this lacuna. First, we provide a self-contained introduction to non-reflexive logic(s) and semantics, and their basic extensions with naïve consequence (and truth) (§§2-3). Second, we investigate the proof-theory of non-reflexive systems, including cut-elimination proofs (§4). Finally (§5), we study the interaction between truth and consequence in non-reflexive systems: this is achieved by providing a compositional theory of truth and consequence and by establishing the adequacy of such a theory with respect to the semantics provided in §3.

The present work is mainly a technical study, which aims to consolidate non-reflexive logics as a viable basis to address semantic paradoxes, and to develop theories of naïve semantic notions.

2. LOGICS OF TRANSPARENT TRUTH AND CONSEQUENCE

2.1. Logical Rules. Let \mathcal{L} be a first-order language with logical constants \neg, \wedge, \forall , and $\mathcal{L}_C := \mathcal{L} \cup \{C\}$ its expansion with a binary predicate $C(x, y)$ intended to express *object-linguistic consequence*. Variables are denoted with x, y, z, \dots , and terms with r, s, t, \dots . We assume that \mathcal{L} contains constants $\ulcorner \varphi \urcorner$ for any formula φ of the language \mathcal{L}_C , and constants \top, \perp .² The nature of the names $\ulcorner \varphi \urcorner$ is not

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¹Some works touching upon local aspects of non-reflexive logics and their relationship to paradox include [Gre01, Rea03, Mea14, Fre16, Fje17, NR18].

²We define \forall, \exists in the usual way.

fully fixed by the theory: as customary practice when dealing with *logics* of semantic concepts, one can assume that the denotation of $\ulcorner \varphi \urcorner$ in all models of the theory is φ itself [Kre88, Rip12]. Such extra-theoretic assumptions will become redundant once a proper theory of syntax will be assumed in the final sections of the paper.

DEFINITION 1 (LPC). *The system LPC in \mathcal{L}_C contains the following initial sequents and rules, where $\Gamma, \Delta, \Theta, \Lambda \dots$ are finite multisets of formulas of \mathcal{L}_C .*

$$\begin{array}{ll}
(\text{REF}^-) \quad \frac{\Gamma, \varphi \Rightarrow \varphi, \Delta}{\text{with } \varphi \in \text{AtFml}_{\mathcal{L}}} & (\text{CUT}) \quad \frac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \\
(\top) \quad \Gamma \Rightarrow \top, \Delta & (\perp) \quad \Gamma, \perp \Rightarrow \Delta \\
(\text{CL}) \quad \frac{\Gamma \Rightarrow \Delta, \varphi \quad \psi, \Gamma \Rightarrow \Delta}{\Gamma, C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner) \Rightarrow \Delta} & (\text{CR}) \quad \frac{\Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner), \Delta} \\
(\neg\text{L}) \quad \frac{\Gamma \Rightarrow \varphi, \Delta}{\Gamma, \neg\varphi \Rightarrow \Delta} & (\neg\text{R}) \quad \frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \neg\varphi, \Delta} \\
(\wedge\text{L}) \quad \frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta} & (\wedge\text{R}) \quad \frac{\Gamma \Rightarrow \varphi, \Delta \quad \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi} \\
(\forall\text{L}) \quad \frac{\Gamma, \forall x\varphi, \varphi(s) \Rightarrow \Delta}{\Gamma, \forall x\varphi \Rightarrow \Delta} & (\forall\text{R}) \quad \frac{\Gamma \Rightarrow \varphi(y), \Delta}{\Gamma \Rightarrow \Delta, \forall x\varphi} \quad y \notin \text{FV}(\Gamma, \Delta, \forall x\varphi)
\end{array}$$

REMARK 2.

- (i) $\text{AtFml}_{\mathcal{L}}$ denotes the set of atomic formulae of \mathcal{L} , i.e. the language without the consequence predicate, and $\text{FV}(\Gamma)$ denotes the set of free variables of Γ .
- (ii) We can define a theory of full disquotational truth as a sub-theory of a definitional extension of LPC obtained by defining $\text{Tr}(x)$ as $C(\ulcorner \top \urcorner, x)$.³
- (iii) Secondly, the combination of the rules (CL) and (CR), unrestricted initial sequents, even in the absence of ($\neg\text{R}$), results in inconsistency. This is essentially a version of Curry's paradox that has recently received some attention [BM13, NR18].⁴ We will elaborate more on this point in section 3.
- (iv) As it happens in standard G3 systems on which it is based, the formulation of LPC with context-sharing rules is justified by the admissibility of weakening and contraction in the system, established below.

3. FIXED-POINT SEMANTICS

The semantics for the logical rules of LPC is provided by a substructural (non-reflexive) version of First-Degree Entailment (FDE), which includes both semantic value gaps and semantic value

³Similarly, we can define a theory of predication or 'true of' by generalizing the semantic rules to open formulas.

⁴Of course a contradiction arises only in the presence of contraction: however contraction is admissible in LPT. See Lemma 12 below.

gluts. This semantics can then be incorporated into a simple fixed-point construction (introduced in [NR18], and to be recalled in a moment), in order to interpret also the consequence predicate of LPC. Let us start from the former.

DEFINITION 3. *Let v be an FDE evaluation function, i.e. a function from the sentences of a first-order language \mathcal{L} to a set of four values $\{1, 0, \mathbf{n}, \mathbf{b}\}$ which obeys the FDE-evaluation clauses. The argument from Γ to Δ is PTS-valid (for Partial Tolerant-Strict), in symbols $\Gamma \models^{\text{PTS}} \Delta$ if: for any FDE evaluation function v , if for every $\varphi \in \Gamma$, $v(\varphi) = 1$ or \mathbf{b} or \mathbf{n} , then there is at least one $\psi \in \Delta$ s.t. $v(\psi) = 1$ or \mathbf{b} .*

Clearly, the inference from φ to φ is not unrestrictedly PTS-valid: to see this, just consider an FDE-evaluation which assigns value \mathbf{n} to φ . Informally, PTS can be thought of as applying the definition of consequence relation proper of the so-called *tolerant-strict* logic (TS) [CÉRvR12] to the four-valued evaluation function provided by FDE. More precisely, an argument from Γ to Δ is TS-valid if every Strong Kleene evaluation which assigns value 1 or \mathbf{n} to all the sentences in Γ assigns value $\mathbf{1}$ to at least one sentence in Δ . It is easy to see that structural reflexivity fails in TS as well. With the definition of TS-consequence in mind, it is clear that PTS merely generalizes it to the four-valued case provided by FDE evaluations (and, hence, explains the name PTS).

A few basic features of PTS are easily stated. Just like strong Kleene logic K3, PTS does not have any classical laws. In other words, no sequent of the form $\Rightarrow \varphi$, for φ classically valid, is PTS-valid. In addition, and unlike K3, PTS does not have any classical inferences, i.e. no classically valid sequent of the form $\Gamma \Rightarrow \Delta$ is PTS-valid. However, and again unlike K3, PTS is closed under all the classically valid meta-inferences: every classically valid sequent rule is also PTS-valid. This means that, as a pure logic, PTS contains no sequents, but this is not so once one combines it with initial sequents, as we will do in subsection 3.1.

As announced above, the consequence relation defined by PTS can be easily combined with a Kripke-style, fixed-point interpretation of the consequence predicate $C(x, y)$ [Kri75]. That this is generally possible is guaranteed by the fact that the PTS evaluation scheme is monotone in the evaluation ordering.⁵

3.1. Fixed-point Semantics. For simplicity and definiteness, we develop the model-theoretic construction in an arithmetical setting, thus identifying \top and \perp with some arithmetical truth and falsity, respectively.⁶ Let then $\mathcal{L}_{\mathbb{N}}$ be the language of arithmetic and $\mathcal{L}_{\mathbb{N}}^{\text{C}} := \mathcal{L}_{\mathbb{N}} \cup \{C\}$. We assume the language of arithmetic includes the signature $\{0, S, +, \times\}$ plus finitely many symbols for primitive recursive functions that facilitates the development of formal syntax. For instance, it will contain symbols for the syntactic operations:⁷

$$(1) \quad s, t \mapsto C(\ulcorner s \urcorner, \ulcorner t \urcorner)$$

$$(2) \quad n, r, s, t \mapsto C(\ulcorner r \urcorner, \underbrace{\ulcorner C(\ulcorner r \urcorner, \dots \ulcorner C(\ulcorner r \urcorner, \ulcorner C(\ulcorner s \urcorner, \ulcorner t \urcorner) \urcorner) \urcorner \dots \urcorner \urcorner}_{n-1\text{-times}})$$

The use of $\mathcal{L}_{\mathbb{N}}$ renders the formalization of the syntax of \mathcal{L}_{C} more uniform and coherent. The meaning of the Gödel quotes is now fixed by a canonical Gödel numbering and a standard formalisation of syntactic notions and operations. In what follows, we keep assuming a canonical coding of finite

⁵See the Fixed-Model Theorem, [AF80, Fef84].

⁶It is possible, but somewhat tedious, to generalize the construction to standard model of syntax theory, thus avoiding the usual arithmetical interpretation of the coding scheme. We stick to the arithmetical framework for simplicity and legibility.

⁷For readability, we do not distinguish between syntactic objects and their codes.

sets. A sequent is thus simply a pair of finite sets. We write $(\Gamma; \Delta)$ the sequent $\Gamma \Rightarrow \Delta$. For simplicity, we identify syntactic objects and their codes.

The semantic clauses of the jump given in the next definition simply correspond to the classically valid sequent rules, i.e. the classical rules to introduce complex formulae to the left and to the right of the sequent arrow, plus rules for the consequence predicate, that internalize them. As remarked above, the semantics for LPC is closed under all the classical sequent rules, so it's no surprise that a semantics for LPC follows the same patterns to interpret the logical vocabulary.

DEFINITION 4 (C-jump [NR18]). *Let $(\Gamma; \Delta)$ denote the Gödel code of the sequent $\Gamma \Rightarrow \Delta$. For $S \subseteq \omega$, the operator $\Psi : \mathcal{P}(\omega) \rightarrow \mathcal{P}(\omega)$ is defined as follows:*

$$\begin{aligned}
n \in \Psi(S) : & \leftrightarrow n \in S, \text{ or} \\
& n = (\Gamma; r = s, \Delta) \text{ and } \mathbb{N} \models r = s, \text{ or} \\
& n = (\Gamma, r = s; \Delta) \text{ and } \mathbb{N} \not\models r = s, \text{ or} \\
& n = (\Gamma; C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner), \Delta) \text{ and } (\Gamma, \varphi; \psi, \Delta) \in S, \text{ or} \\
& n = (\Gamma, C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner); \Delta) \text{ and } (\Gamma; \varphi, \Delta) \in S, (\Gamma, \psi; \Delta) \in S, \text{ or} \\
& n = (\Gamma, \neg \varphi; \Delta) \text{ and } (\Gamma; \varphi, \Delta) \in S, \text{ or} \\
& n = (\Gamma; \neg \varphi, \Delta) \text{ and } (\Gamma, \varphi; \Delta) \in S, \text{ or} \\
& n = (\Gamma, \varphi \wedge \psi; \Delta) \text{ and } (\Gamma, \varphi, \psi; \Delta) \in S, \text{ or} \\
& n = (\Gamma; \varphi \wedge \psi, \Delta) \text{ and } (\Gamma; \psi, \Delta) \in S, (\Gamma; \varphi, \Delta) \in S, \text{ or} \\
& n = (\Gamma, \forall x \varphi; \Delta) \text{ and } (\Gamma, \forall x \varphi, \varphi(t); \Delta) \in S \text{ for some closed } \mathcal{L}_C\text{-term } t, \text{ or} \\
& n = (\Gamma; \forall x \varphi, \Delta) \text{ and } (\Gamma; \varphi(t), \Delta) \in S \text{ for all closed } \mathcal{L}_C\text{-terms } t.
\end{aligned}$$

Iterations of Ψ can be defined as usual, by putting:⁸

$$\Psi^\alpha(S) = \Psi\left(\bigcup_{\beta < \alpha} \Psi^\beta(S)\right).$$

The operator Ψ is both increasing – i.e. $S \subseteq \Psi(S)$ for any S –, and monotonic: $S_0 \subseteq S_1$ entails $\Psi(S_0) \subseteq \Psi(S_1)$. The latter property entails the existence of fixed points of Ψ , i.e. sets T s.t. $\Psi(T) = T$. We are mainly interested in the minimal of these fixed points $\mathcal{J}_\Psi := \bigcup_{\alpha \in \text{Ord}} \Psi^\alpha(\emptyset)$.

The following lemma, proved in [NR18], shows that \mathcal{J}_Ψ is a model of a naïve, self-applicable consequence predicate.

LEMMA 5 ([NR18, Lemma 9]).

- (i) $(\Gamma; \Delta, \varphi) \in \mathcal{J}_\Psi$ and $(\psi, \Gamma; \Delta) \in \mathcal{J}_\Psi$ if and only if $(C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner), \Gamma; \Delta) \in \mathcal{J}_\Psi$;
- (ii) $(\Gamma, \varphi; \psi, \Delta) \in \mathcal{J}_\Psi$ if and only if $(\Gamma; C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner), \Delta) \in \mathcal{J}_\Psi$.

Via the definition $\text{Tr}(x) := C(\ulcorner 0 \urcorner, x)$ the minimal fixed point for self-referential truth from [Kri75] essentially ‘lives’ inside \mathcal{J}_Ψ . In particular, one can restrict the construction above to empty contexts, and the clauses for $C(x, y)$ then obviously can be restricted to:

$$\begin{aligned}
& (; C(\ulcorner 0 \urcorner, \ulcorner \varphi \urcorner)) \in \Psi(S), \text{ if } (; \varphi) \in S; \\
& (C(\ulcorner 0 \urcorner, \ulcorner \varphi \urcorner);) \in \Psi(S), \text{ if } (\varphi;) \in S.
\end{aligned}$$

⁸For more details, see [Mos74], Chapter 1.

This restriction of the monotone operator Ψ above reaches then fixed points X in which

$$(3) \quad (; \varphi) \in X \text{ iff } (; \text{Tr}^\Gamma \varphi^\neg) \in X$$

$$(4) \quad (; \neg\varphi) \in X \text{ iff } (; \neg\text{Tr}^\Gamma \varphi^\neg) \in X.$$

Properties (3) and (4) correspond to the so-called ‘intersubstitutivity’ or ‘symmetry’ of truth. For definiteness, let’s call \mathcal{J} the set of truths of the minimal fixed point X_0 so obtained, that is:

$$\mathcal{J} := \{\varphi \mid (; \varphi) \in X_0\}.$$

It’s clear that we can express $\Psi(\cdot)$ as a formula of the language \mathcal{L}_2 of second-order arithmetic in such a way that

$$\Psi(S) = \{n \mid \mathbb{N} \models F(x, X) [n, S]\}$$

for $F(x, X)$ arithmetical and X occurring only positively – i.e. not in the scope of an even number of negation symbols – in it. Therefore

$$n \in \mathcal{J}_\Psi \Leftrightarrow (\forall X)((\forall x)(F(x, X) \rightarrow x \in X) \rightarrow n \in X).$$

So $\mathcal{J}_\Psi \in \Pi_1^1$. Moreover, by the relationships between \mathcal{J} and \mathcal{J}_Ψ outlined above, and by Π_1^1 -hardness of \mathcal{J} ,⁹ we have:

COROLLARY 6. *\mathcal{J}_Ψ is Π_1^1 -complete.*

It is well-known that Π_1^1 -sets have a natural presentation in terms of cut-free infinitary derivability [Acz77, Poh09]. The case we are consider is not exception, and a suitable infinitary calculus LPC^∞ can be developed along the lines of the infinitary system for non-reflexive truth developed in [Nic20]. LPC^∞ is obtained from LPC by (essentially): replacing the axioms for \perp and \top with corresponding rules for arithmetical truth and falsity, and replacing $(\forall R)$ with an ω -rule.¹⁰ By adapting the analysis in [Nic20], it can be shown that LPC^∞ has nice proof-theoretical properties: weakening and contraction are admissible preserving the (possibly infinite) length of the derivation, its rules are invertible, and (crucially) cut is eliminable in it.

In addition, it is possible to show that (possibly infinitary) proofs in LPC^∞ closely ‘match’ the construction of \mathcal{J}_Ψ . More precisely: the ordinal stage of the inductive definition in which a sequent $\Gamma \Rightarrow \Delta$ enters in \mathcal{J}_Ψ — i.e. its ordinal norm — can be associated to the lengths of cut-free proofs of $\Gamma \Rightarrow \Delta$ in LPC^∞ . By a well-known result, this ordinal norm cannot exceed the first non-recursive (countable) ordinal ω_1^{CK} .¹¹

- (i) If there is a cut-free LPC^∞ -proof of length $\leq \alpha < \omega_1^{\text{CK}}$ of the sequent $\Gamma \Rightarrow \Delta$, then $(\Gamma; \Delta) \in \mathcal{J}_\Psi^{\alpha+1}$.
- (ii) If $(\Gamma; \Delta) \in \mathcal{J}_\Psi^\alpha$, $\alpha < \omega_1^{\text{CK}}$, then there there is a cut-free LPC^∞ -proof of length $\leq \alpha < \omega_1^{\text{CK}}$ of $\Gamma \Rightarrow \Delta$.

If one restricts her attention to pairs of sentences, the above result entails the existence of a tight correspondence between the extension of the consequence predicate in \mathcal{J}_Ψ , and the consequence ascriptions derivable in LPC^∞ . More specifically, for all $\varphi, \psi \in \mathcal{L}_C$, the following are equivalent

- (i) $\text{LPC}^\infty \vdash \varphi \Rightarrow \psi$;

⁹See [Kri75, McG91].

¹⁰Two more technical amendments are omitting free variables, and generalizing CL and CR to arbitrary terms which code formulae. See [Nic20] for details.

¹¹See for instance, [Poh09, Thm. 6.6.4].

- (ii) $\text{LPC}^\infty \vdash \Rightarrow \text{C}(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner)$;
- (iii) $(\varphi; \psi) \in \mathcal{I}_\Psi$.

4. PROOF THEORY OF LPC

In this section we focus on the proof-theoretic properties of LPC. Our analysis culminates in the full eliminability of cut in it. The key technical insight that makes cut fully eliminable—and that is extensively investigated in [Nic20]—is a strong form of invertibility of the C-rules (Lemma 10).

The notions of *length* of a derivation is standardly defined [Sch77, TS03].¹² Given a calculus with rules that are at most α -branching, the *length* of a derivation \mathcal{D} is the supremum of the lengths of its direct sub-derivations \mathcal{D}_γ – with $\gamma < \beta \leq \alpha$ increased by one:

$$d = \sup\{d_\gamma + 1 \mid \gamma < \beta\}$$

Clearly, for LPC, $\alpha = 2$ and the length of derivations is finite.

We now introduce the measure of complexity for applications of the C-rules to formulas in derivations that will play a crucial role in the cut-elimination proofs below. We will call it C-complexity. In the definition, formula occurrences are decorated with superscripts. Moreover, we follow the conventions introduced in [SH16, p. 346] and when writing, say,

$$\frac{\gamma_1^{j_1}, \dots, \gamma_n^{j_n} \Rightarrow \delta_1^{k_1}, \dots, \delta_m^{k_m}, \varphi}{\gamma_1^{j_1+1}, \dots, \gamma_n^{j_n+1} \Rightarrow \delta_1^{k_1+1}, \dots, \delta_m^{k_m+1}, \psi}$$

we assume that occurrences of γ_i^j , with $1 \leq i \leq n$ correspond precisely to occurrences of γ_i^{j+1} – i.e. they are distinct occurrences of the same formula – and similarly for the δ 's. As an abbreviation, this will be generalized to multisets of sentences: we occasionally write Γ^j instead of $\gamma_1^{j_1}, \dots, \gamma_n^{j_n}$. It should be clear that superscripts are not part of the language.

In a nutshell, the C-complexity of a formula keeps track of the applications of the C-rules: initial sequents and \mathcal{L} -formulas have complexity 0, and the only way to increase the C-complexity of a formula is by introducing the consequence predicate.

DEFINITION 7 (C-complexity). *The ordinal C-complexity $\kappa(\cdot)$ of a formula φ of \mathcal{L}_C in a derivation \mathcal{D} is defined inductively as follows:*

(i) *formulas of \mathcal{L} have C-complexity 0 in any \mathcal{D} ;*

(ii) *If \mathcal{D} is just*

$$\Gamma, \varphi \Rightarrow \varphi, \Delta$$

with $\varphi \in \mathcal{L}$, then $\kappa(\psi) = \kappa(\varphi) = 0$ for all $\psi \in \Gamma, \Delta$. Similarly for (T), (\perp).

(iii) *If \mathcal{D} ends with*

$$\frac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta}$$

then $\kappa(\varphi) = \kappa(\neg\varphi)$ and the C-complexity of the formulas in Γ, Δ is unchanged. Similarly for (\neg R) and (\forall R).

¹²Our notion of length amounts to what is called *depth* in [TS03].

(iv) If \mathcal{D} ends with

$$\frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}$$

then $\kappa(\varphi \wedge \psi) = \max(\kappa(\varphi), \kappa(\psi))$ and the C-complexity of the formulas in Γ, Δ is unchanged.

(v) If \mathcal{D} ends with

$$\frac{\gamma_1^{j_1}, \dots, \gamma_n^{j_n} \Rightarrow \delta_1^{k_1}, \dots, \delta_m^{k_m}, \varphi \quad \gamma_1^{j_1+1}, \dots, \gamma_n^{j_n+1} \Rightarrow \delta_1^{k_1+1}, \dots, \delta_m^{k_m+1}, \psi}{\gamma_1^{j_1+2}, \dots, \gamma_n^{j_n+2} \Rightarrow \delta_1^{k_1+2}, \dots, \delta_m^{k_m+2}, \varphi \wedge \psi}$$

then

$$\begin{aligned} \kappa(\varphi \wedge \psi) &= \max(\kappa(\varphi), \kappa(\psi)) \\ \kappa(\gamma_i^{j_i+2}) &= \max(\kappa(\gamma_i^{j_i}), \kappa(\gamma_i^{j_i+1})) \quad 1 \leq i \leq n \\ \kappa(\delta_j^{k_j+2}) &= \max(\kappa(\delta_j^{k_j}), \kappa(\delta_j^{k_j+1})) \quad 1 \leq l \leq m \end{aligned}$$

(vi) If \mathcal{D} ends with

$$\frac{\Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner), \Delta}$$

then $\kappa(C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner)) = \max(\kappa(\varphi), \kappa(\psi)) + 1$ and the C-complexity of the formulas in Γ, Δ is unchanged.

(vii) If \mathcal{D} ends with

$$\frac{\gamma_1^{j_1}, \dots, \gamma_n^{j_n} \Rightarrow \delta_1^{k_1}, \dots, \delta_m^{k_m}, \varphi \quad \psi, \gamma_1^{j_1+1}, \dots, \gamma_n^{j_n+1} \Rightarrow \delta_1^{k_1+1}, \dots, \delta_m^{k_m+1}}{\gamma_1^{j_1+2}, \dots, \gamma_n^{j_n+2}, C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner) \Rightarrow \delta_1^{k_1+2}, \dots, \delta_m^{k_m+2}}$$

then

$$\begin{aligned} \kappa(C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner)) &= \max(\kappa(\varphi), \kappa(\psi)) + 1 \\ \kappa(\gamma_i^{j_i+2}) &= \max(\kappa(\gamma_i^{j_i}), \kappa(\gamma_i^{j_i+1})) \quad 1 \leq i \leq n \\ \kappa(\delta_j^{k_j+2}) &= \max(\kappa(\delta_j^{k_j}), \kappa(\delta_j^{k_j+1})) \quad 1 \leq l \leq m \end{aligned}$$

(viii) If \mathcal{D} ends with

$$\frac{\Gamma, \forall x \varphi^k, \varphi(t) \Rightarrow \Delta}{\forall x \varphi^l, \Gamma \Rightarrow \Delta}$$

then $\kappa(\forall x \varphi^l) = \max(\kappa(\forall x \varphi^k), \kappa(\varphi(t)))$ and the C-complexity of the formulas in Γ, Δ is unchanged.

(ix) If \mathcal{D} ends with

$$\frac{\gamma_1^{j_1}, \dots, \gamma_n^{j_n} \Rightarrow \delta_1^{k_1}, \dots, \delta_m^{k_m}, \varphi \quad \varphi, \gamma_1^{j_1+1}, \dots, \gamma_n^{j_n+1} \Rightarrow \delta_1^{k_1+1}, \dots, \delta_m^{k_m+1}}{\gamma_1^{j_1+2}, \dots, \gamma_n^{j_n+2} \Rightarrow \delta_1^{k_1+2}, \dots, \delta_m^{k_m+2}}$$

then

$$\begin{aligned}\kappa(\gamma_i^{j_i+2}) &= \max(\kappa(\gamma_i^{j_i}), \kappa(\gamma_i^{j_i+1})) \quad 1 \leq i \leq n \\ \kappa(\delta_j^{k_l+2}) &= \max(\kappa(\delta_j^{k_l}), \kappa(\delta_j^{k_l+1})) \quad 1 \leq l \leq m\end{aligned}$$

In this case the complexity of the cut formula $\kappa(\varphi)$ is $\max(\kappa(\varphi^1), \kappa(\varphi^2))$.

From now on we occasionally write $\kappa(\Gamma^m) \leq \kappa(\Gamma^n)$ to express that the C-complexity of all formulas in Γ^m is smaller or equal than the C-complexity of the corresponding occurrence in Γ^n . Similarly for identity of C-complexities.

The cut-elimination theorem for LPC is based on the invertibility of the LPC-rules and the κ -admissibility of the structural rules of weakening and contraction. We will highlight in due course where the restriction of reflexivity differentiates the present setting from the one featuring unrestricted identity sequents. The cut-elimination strategy for LPC generalizes to a consequence predicate the argument provided for various systems of truth with a restriction of initial sequents in [Nic20].

We start by observing that, in proofs of sequents containing \top on the left, and \perp on the right, the occurrences of such constants can be omitted. Both claims follow by a straightforward induction on the length of the proof that preserves the κ -complexity of the formulas in the contexts.

LEMMA 8.

- (i) If $\vdash_{\text{LPC}} \top, \Gamma \Rightarrow \Delta$, then $\vdash_{\text{LPC}}^n \Gamma \Rightarrow \Delta$ and the C-complexity of the formulas in the contexts is unchanged.
- (ii) If $\vdash_{\text{LPC}} \Gamma \Rightarrow \Delta, \perp$, then $\vdash_{\text{LPC}}^n \Gamma \Rightarrow \Delta$ and the C-complexity of the formulas in the contexts is unchanged.

In both claims, the length of the derivation is preserved.

Next, we turn to substitution and weakening lemmata. Again, a proof by induction on the length of derivations is required. In the proof of weakening, the formulation of (REF), (\top), (\perp) with arbitrary contexts is of course essential.

LEMMA 9 (Substitution, Weakening).

- (i) If $\Gamma \Rightarrow \Delta$ is derivable in LPC, then $\Gamma^* \Rightarrow \Delta^*$ is LPC-derivable, where Γ^* , Δ^* are obtained by uniformly replacing in Γ , Δ , a variable x by a term t which is free for x and does not contain variables employed in applications of ($\forall\text{R}$) in the proof of $\Gamma \Rightarrow \Delta$. Moreover, the C-complexity of the formulas involved in the substitution and in the contexts does not change.
- (ii) Weakening is κ -admissible in LPC. That is, if we prove $\Gamma \Rightarrow \Delta$, we can prove $\Gamma \Rightarrow \varphi, \Delta$ (or $\Gamma, \varphi \Rightarrow \Delta$), so that $\kappa(\varphi) = 0$.

In both claims, the length of the derivation is preserved.

The next lemma marks out the key property of LPC which makes it possible to generalize the standard G3-strategy for the admissibility of cut to the present setting. All rules of LPC, including the rules for the consequence predicate, are invertible in a strong sense that preserves, and in the appropriate cases *reduces*, the κ -complexity of formulas.

LEMMA 10 (κ -invertibility of LPC-rules).

- (i) If $\mathcal{D} \vdash_{\text{LPC}} \Gamma, C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner) \Rightarrow \Delta$, then there are $\mathcal{D}' \vdash_{\text{LPC}} \Gamma \Rightarrow \Delta, \varphi$ and $\mathcal{D}'' \vdash_{\text{LPC}} \psi, \Gamma \Rightarrow \Delta$ with
 $\kappa(\varphi), \kappa(\psi) \leq \kappa(C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner))$, if $\kappa(C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner)) = 0$, or
 $\kappa(\varphi), \kappa(\psi) < \kappa(C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner))$, if $\kappa(C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner)) > 0$,
and in which the complexity of the side formulas does not increase.
- (ii) If $\mathcal{D} \vdash_{\text{LPC}} \Gamma \Rightarrow C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner), \Delta$, then there is a $\mathcal{D}' \vdash_{\text{LPC}} \Gamma, \varphi \Rightarrow \psi, \Delta$ with
 $\kappa(\varphi), \kappa(\psi) \leq \kappa(C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner))$, if $\kappa(C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner)) = 0$, or
 $\kappa(\varphi), \kappa(\psi) < \kappa(C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner))$, if $\kappa(C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner)) > 0$,
and in which the complexity of the side formulas is no greater than their κ -maximal occurrence in the premisses.
- (iii) If $\mathcal{D} \vdash_{\text{LPC}} \Gamma, \neg\varphi \Rightarrow \Delta$, then there is a $\mathcal{D}' \vdash_{\text{LPC}} \Gamma \Rightarrow \varphi, \Delta$ with $\kappa(\varphi) \leq \kappa(\neg\varphi)$ and in which the complexity of the side formulas does not increase.
- (iv) If $\mathcal{D} \vdash_{\text{LPC}} \Gamma \Rightarrow, \neg\varphi, \Delta$, then there is a $\mathcal{D}' \vdash_{\text{LPC}} \Gamma, \varphi \Rightarrow \Delta$ with $\kappa(\Gamma) \leq \kappa(\Gamma)$, with $\kappa(\varphi) \leq \kappa(\neg\varphi)$ and in which the complexity of the side formulas does not increase.
- (v) If $\mathcal{D} \vdash_{\text{LPC}} \Gamma, \varphi \wedge \psi \Rightarrow \Delta$, then there is a $\mathcal{D}' \vdash_{\text{LPC}} \Gamma, \varphi, \psi \Rightarrow \Delta$ with $\kappa(\varphi), \kappa(\psi) \leq \kappa(\varphi \wedge \psi)$ and in which the complexity of the side formulas does not increase.
- (vi) If $\mathcal{D} \vdash_{\text{LPC}} \Gamma \Rightarrow \varphi \wedge \psi, \Delta$, then there are $\mathcal{D}' \vdash_{\text{LPC}} \Gamma \Rightarrow \Delta, \varphi$ and $\mathcal{D}'' \vdash_{\text{LPC}} \Gamma \Rightarrow \Delta, \psi$ with $\kappa(\varphi), \kappa(\psi) \leq \kappa(\varphi \wedge \psi)$ and in which the complexity of the side formulas is no greater than their κ -maximal occurrence in the premisses.
- (vii) If $\mathcal{D} \vdash_{\text{LPC}} \Gamma \Rightarrow \Delta, \forall x\varphi$, then there is $\mathcal{D}' \vdash_{\text{LPC}} \Gamma \Rightarrow \Delta, \varphi(y)$, for any y not free in $\Gamma, \Delta, \forall x\varphi$, with $\kappa(\varphi(y)) \leq \kappa(\forall x\varphi)$ and in which the complexity of the side formulas does not increase.

Crucially, the invertibility of the rules preserves the length of the proof.

Proof. We proceed by induction on the length of the proof \mathcal{D} but only show (i). The other cases are similar or easier. For (vii), one essentially employs the substitution lemma (Lemma 9(i)).

If $\Gamma, C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner) \Rightarrow \Delta$ is an initial sequent, then it is for the form $\Gamma_0, \chi, C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner) \Rightarrow \chi, \Delta_0$, or $\Gamma := \Gamma_0, \perp$, or $\Delta := \Delta, \perp$. In all such cases the claim is trivially obtained since $\Gamma \Rightarrow \Delta, \varphi$ and $\psi, \Gamma \Rightarrow \Delta$ are also initial sequents.

If $\Gamma, C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner) \Rightarrow \Delta$ is *not* an initial sequent, there are two cases to consider. The first in which $C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner)$ is principal in the last inference of \mathcal{D} , the second in which it is not. In the former case, $\kappa(C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner)) > 0$, and the claim follows immediately by definition of C-complexity in the case of an application of (CL). In the latter case, suppose that \mathcal{D} ends with

$$\frac{\mathcal{D}_0 \quad \mathcal{D}_1}{\frac{C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner)^{k_0}, \Gamma \Rightarrow \Delta \quad C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner)^{k_1}, \Gamma \Rightarrow \Delta}{C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner)^{k_2}, \Gamma \Rightarrow \Delta}} \text{(R)}$$

The induction hypothesis applied to \mathcal{D}_0 and \mathcal{D}_1 yields derivations $\mathcal{D}'_i \vdash_{\text{LPC}} \Gamma_i \Rightarrow \Delta_i, \varphi$ and $\mathcal{D}''_i \vdash_{\text{LPC}} \psi, \Gamma_i \Rightarrow \Delta_i$, with $i \in \{0, 1\}$, such that $\kappa(\varphi), \kappa(\psi) \leq \kappa(C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner)^{k_2})$, if $\kappa(C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner)^{k_2}) = 0$, and $\kappa(\varphi), \kappa(\psi) < \kappa(C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner)^{k_2})$, otherwise. Therefore the required derivations are obtained by applications of (R). *qed.*

REMARK 11. In the presence of (REF), the inversion strategy considered above will not go through. For instance, the derivability of a sequent of the form $\Gamma, C(\Gamma\varphi^\neg, \Gamma\psi^\neg) \Rightarrow C(\Gamma\varphi^\neg, \Gamma\psi^\neg), \Delta$ does not guarantee, for instance, the derivability of a sequent $\Gamma, \psi \Rightarrow \varphi, \Delta$ with $\kappa(\varphi) \leq \kappa(C(\Gamma\varphi^\neg, \Gamma\psi^\neg))$. This fact is crucial for the next lemma, in which contraction is shown to be κ -admissible.

LEMMA 12 (κ -admissibility of contraction).

- (i) If $\mathcal{D} \vdash_{\text{LPC}} \Gamma, \varphi^{k_0}, \varphi^{k_1} \Rightarrow \Delta$, then there is a $\mathcal{D}' \vdash_{\text{LPC}} \Gamma, \varphi \Rightarrow \Delta$ with $\kappa(\varphi) \leq \max(\kappa(\varphi^{k_0}), \kappa(\varphi^{k_1}))$ and in which the complexity of the side formulas does not increase.
- (ii) If $\mathcal{D} \vdash_{\text{LPC}} \Gamma \Rightarrow \varphi^{k_0}, \varphi^{k_1}, \Delta$, then there is a $\mathcal{D}' \vdash_{\text{LPC}} \Gamma \Rightarrow \varphi, \Delta$ with $\kappa(\varphi) \leq \max(\kappa(\varphi^{k_0}), \kappa(\varphi^{k_1}))$ and in which the complexity of the side formulas does not increase.

Crucially, in both claims the length of the original derivation is preserved.

Proof. The proof is by induction on the length of \mathcal{D} . We prove (i) and (ii) simultaneously.

Case 1. $\Gamma, \varphi^{k_0}, \varphi^{k_1} \Rightarrow \Delta$ is an initial sequent. There are different subcases. If it is of the form $\Gamma, \varphi^{k_0}, \varphi^{k_1} \Rightarrow \Delta_0, \varphi$, then φ is an atomic formula of \mathcal{L} and therefore $\Gamma, \varphi \Rightarrow \Delta_0, \varphi$ is derivable with $\kappa(\varphi) = 0$. If $\Gamma, \varphi^{k_0}, \varphi^{k_1} \Rightarrow \Delta$ is of the form $\Gamma_0, \psi, \varphi^{k_0}, \varphi^{k_1} \Rightarrow \Delta, \psi$, then $\Gamma_0, \psi, \varphi \Rightarrow \psi, \Delta_0$ is an initial sequent. If $\varphi := \perp$ or $\Gamma := \Gamma_0, \perp$, then $\Gamma, \varphi \Rightarrow \Delta$ is also an initial sequent. In all cases it is clear that $\kappa(\varphi) \leq \max(\kappa(\varphi^{k_0}), \kappa(\varphi^{k_1}))$ by the definition of C-complexity. The proof of (ii) is symmetric.

Case 2.1. $\Gamma, \varphi^{k_0}, \varphi^{k_1} \Rightarrow \Delta$ is not an initial sequent, but neither φ^{k_0} nor φ^{k_1} are principal in the last inference. Then the derivability of $\Gamma, \varphi \Rightarrow \Delta$ – with the expected C-complexities – is obtained immediately by induction hypothesis. The proof of (ii) is analogous.

Case 2.2. $\Gamma, \varphi^{k_0}, \varphi^{k_1} \Rightarrow \Delta$ or $\Gamma \Rightarrow \varphi^{k_0}, \varphi^{k_1}, \Delta$ are not initial sequents, and one of φ^{k_0} or φ^{k_1} is principal in the last inference. We treat the crucial cases in which φ is $C(\Gamma\varphi^\neg, \Gamma\psi^\neg)$.

For (i), if $C(\Gamma\varphi^\neg, \Gamma\psi^\neg)^{k_0}$ is principal in the last inference, then \mathcal{D} is of the form:

$$(CL) \frac{\begin{array}{c} \mathcal{D}_0 \\ \Gamma, C(\Gamma\varphi^\neg, \Gamma\psi^\neg)^{k_{10}} \Rightarrow \Delta, \varphi^{k_{00}} \end{array} \quad \begin{array}{c} \mathcal{D}_1 \\ \psi^{k_{01}}, \Gamma, C(\Gamma\varphi^\neg, \Gamma\psi^\neg)^{k_{11}} \Rightarrow \Delta \end{array}}{\Gamma, C(\Gamma\varphi^\neg, \Gamma\psi^\neg)^{k_0}, C(\Gamma\varphi^\neg, \Gamma\psi^\neg)^{k_1} \Rightarrow \Delta}$$

such that $\kappa(\varphi^{k_{00}}), \kappa(\psi^{k_{01}}) < \kappa(C(\Gamma\varphi^\neg, \Gamma\psi^\neg)^{k_0})$.

We can then apply the inversion Lemma to \mathcal{D}_0 to obtain a

$$\mathcal{D}'_0 \vdash_{\text{LPC}} \Gamma \Rightarrow \Delta, \varphi^{k'_{00}}, \varphi^l$$

with

$$\begin{aligned} \kappa(\varphi^l) &\leq \kappa(C(\Gamma\varphi^\neg, \Gamma\psi^\neg)^{k_{10}}) \text{ if } \kappa(C(\Gamma\varphi^\neg, \Gamma\psi^\neg)^{k_{10}}) = 0 \\ \kappa(\varphi^l) &< \kappa(C(\Gamma\varphi^\neg, \Gamma\psi^\neg)^{k_{10}}) \text{ if } \kappa(C(\Gamma\varphi^\neg, \Gamma\psi^\neg)^{k_{10}}) \neq 0 \end{aligned}$$

Similarly, inversion applied to \mathcal{D}_1 yields

$$\mathcal{D}'_1 \vdash_{\text{LPC}} \psi^m, \psi^{k'_{01}}, \Gamma \Rightarrow \Delta$$

with

$$\begin{aligned} \kappa(\psi^m) &\leq \kappa(C(\Gamma\varphi^\neg, \Gamma\psi^\neg)^{k_{10}}) \text{ if } \kappa(C(\Gamma\varphi^\neg, \Gamma\psi^\neg)^{k_{10}}) = 0 \\ \kappa(\psi^m) &< \kappa(C(\Gamma\varphi^\neg, \Gamma\psi^\neg)^{k_{10}}) \text{ if } \kappa(C(\Gamma\varphi^\neg, \Gamma\psi^\neg)^{k_{10}}) \neq 0 \end{aligned}$$

By induction hypothesis, we obtain:

$$\mathcal{D}''_0 \vdash_{\text{LPC}} \Gamma \Rightarrow \varphi, \Delta$$

$$\mathcal{D}'' \vdash_{\text{LPC}} \psi, \Gamma \Rightarrow \Delta$$

An application of (CL) yields the desired

$$\mathcal{D}' \vdash_{\text{LPC}} \Gamma, C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner) \Rightarrow \Delta$$

with the required C-complexity

$$\begin{aligned} \kappa(C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner)) &= \max(\kappa(\varphi), \kappa(\psi)) + 1 \\ &\leq \max(\kappa(C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner)^{k_0}), \kappa(C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner)^{k_1})) \end{aligned}$$

For (ii), if $C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner)^{k_0}$ is principal in the last inference, then \mathcal{D} is of the form:

$$\frac{\mathcal{D}_0 \quad \Gamma, \varphi^{k_{00}} \Rightarrow \psi^{k_{01}}, C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner)^{k_{10}}, \Delta}{\Gamma \Rightarrow C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner)^{k_0}, C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner)^{k_1}, \Delta}$$

Inversion applied to \mathcal{D}_0 yields a proof \mathcal{D}'_0 ending with

$$\Gamma, \varphi^{l_0}, \varphi^{k'_{00}} \Rightarrow \psi^{k'_{01}}, \psi^{l_1}, \Delta$$

By two applications of the induction hypothesis, we obtain a proof \mathcal{D}'''_0 of

$$\Gamma, \varphi \Rightarrow \psi, \Delta$$

with

$$\begin{aligned} \kappa(\varphi) &\leq \max(\kappa(\varphi^{k'_{00}}), \kappa(\varphi^{l_0})) (=:\alpha) \\ \kappa(\psi) &\leq \max(\kappa(\psi^{k'_{01}}), \kappa(\psi^{l_1})) (=:\beta) \end{aligned}$$

Therefore, by (CL), one obtains a derivation of $\Gamma \Rightarrow C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner), \Delta$ with

$$\kappa(C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner)) = \max(\alpha, \beta) + 1 \leq \max(\kappa(C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner)^{k_0}), \kappa(C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner)^{k_1})).$$

It is worth noticing that the formulation of (\forall L) and its associated C-complexity renders the case of (i) in which one of the φ 's is principal in the last inference and of the form $\forall x\varphi$ straightforward. *qed.*

We can finally state and prove the cut-elimination lemma for LPC. We start with the reduction lemma.

LEMMA 13 (Reduction). *If \mathcal{D}_0 is a cut-free proof of $\Gamma \Rightarrow \Delta, \varphi^k$ in LPC, and \mathcal{D}_1 is a cut-free LPC-proof of $\varphi^l, \Gamma \Rightarrow \Delta$, then there is a cut-free proof \mathcal{D} of $\Gamma \Rightarrow \Delta$ in which the complexity of the side formulas is no greater than their κ -maximal occurrence in the premisses.*

Proof. The proof is by a main induction on $\kappa(\varphi) = \max(\kappa(\varphi^l), \kappa(\varphi^k))$, with side inductions on the logical complexity of φ and on the sum $d_0 + d_1$ of the lengths of \mathcal{D}_0 and \mathcal{D}_1 . We consider the main cases.

Case 1. One of $\mathcal{D}_0, \mathcal{D}_1$ is an initial sequent, say \mathcal{D}_0 . If φ is not principal, then $\Gamma \Rightarrow \Delta$ is already an initial sequent. If φ is principal in it, then we can distinguish two cases. If $\mathcal{D}_0 \vdash \Gamma_0, \varphi \Rightarrow \varphi^k, \Delta$, then we can apply Lemma 12 to \mathcal{D}_1 to obtain a derivation of $\Gamma \Rightarrow \Delta$ whose formulas have the required C-complexity. If $\mathcal{D}_0 \vdash \Gamma \Rightarrow \top^k, \Delta$, then $\mathcal{D}_1 \vdash \top^l, \Gamma \Rightarrow \Delta$. By lemma 8(i), $\Gamma \Rightarrow \Delta$ is derivable with the expected C-complexity.

Case 2. The cut formula is not principal in one of the premisses, say \mathcal{D}_1 . For instance the last inference of \mathcal{D}_1 is an application of (CL). Then, with $\Gamma := C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner), \Gamma_0$, the derivation \mathcal{D} ends with:

$$\frac{\mathcal{D}_0 \quad \frac{\mathcal{D}_{10} \quad \mathcal{D}_{11}}{\chi, \Gamma_0 \Rightarrow \Delta, \varphi \quad \chi, \psi, \Gamma_0 \Rightarrow \Delta}}{\chi, C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner), \Gamma_0 \Rightarrow \Delta}}{C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner), \Gamma_0 \Rightarrow \Delta}$$

By the weakening lemma, \mathcal{D} can be transformed into a derivation \mathcal{D}' whose last inference is an application of (CL), whose premisses are

$$\frac{\mathcal{D}'_0 \quad \mathcal{D}_{10}}{C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner), \Gamma_0 \Rightarrow \Delta, \varphi, \chi \quad \chi, C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner), \Gamma_0 \Rightarrow \Delta, \varphi}}{C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner), \Gamma_0 \Rightarrow \Delta, \varphi}$$

and

$$\frac{\mathcal{D}'_0 \quad \mathcal{D}_{11}}{\psi, C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner), \Gamma_0 \Rightarrow \Delta, \chi \quad \chi, \psi, C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner), \Gamma_0 \Rightarrow \Delta}}{\psi, C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner), \Gamma_0 \Rightarrow \Delta}$$

Therefore $\mathcal{D}' \vdash C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner), C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner), \Gamma_0 \Rightarrow \Delta$. The upper cuts in \mathcal{D}' be eliminated by side induction hypothesis, since $d'_0 + d'_{11}, d'_0 + d'_{10} < d_0 + d_1$. Moreover, since the weakened formulas can have lowest possible C-complexity, an application the contraction lemma to the transformed derivation yields the claim. The other cases in which the cut formula is not principal are easier.

Case 3. The cut formula is principal in the last inference of \mathcal{D}_0 and \mathcal{D}_1 . The case in which the cut formula if $C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner)$ is particularly easy, by main induction hypothesis, because the cut can be pushed upwards and applied to the ancestors of the cut formula, which have strictly smaller C-complexity. The case in which the cut formula is principal and of the form $\forall x\varphi$ is treated standardly as well but one has first to get rid of the universal quantifier in the premise of (\forall L). This involves an essential application of the substitution lemma [TS03, §4.1]. *qed.*

REMARK 14. Although our proof of lemma 13 above relies heavily on lemma 12, the role of κ -admissibility of contraction can be circumscribed to the role it plays in Case 1 – that is, the case in which one of the premisses is an axiom and the cut formula is principal.

In Case 2, and in the specific sub-case treated above, one can apply the inversion lemma to \mathcal{D}_0 to obtain LPC-proofs $\mathcal{D}'_{00} \vdash \Gamma_0 \Rightarrow \Delta, \varphi, \chi$ and $\mathcal{D}'_{01} \vdash \psi, \Gamma_0 \Rightarrow \Delta, \chi$. These can then be combined with \mathcal{D}_{10} and \mathcal{D}_{11} respectively, and then (CL) applied to the results of the shorter cuts. Such template, with inversion playing the fundamental role, can be applied to all other sub-cases of Case 2 except of course (\forall L). In such case, \mathcal{D} has the form:

$$\frac{\mathcal{D}_{00} \quad \frac{\Gamma_0, \varphi(s), \forall x\varphi \Rightarrow \Delta, \chi \quad \mathcal{D}_1}{\Gamma_0, \forall x\varphi \Rightarrow \Delta, \chi \quad \chi, \Gamma_0, \forall x\varphi \Rightarrow \Delta}}{\Gamma_0, \forall x\varphi \Rightarrow \Delta}$$

In such case, one can therefore weaken \mathcal{D}_1 , apply CUT to such weakened derivation and \mathcal{D}_{00} , and then apply (\forall L).

By repeated applications of the Reduction Lemma, we can then obtain:

COROLLARY 15. *The rule (CUT) is eliminable in LPC.*

Since cut-elimination proof above displays standard bounds for the reduction, Corollary 15 can be formalized in $\text{I}\Delta_0 + \text{superexp}$, where $\text{I}\Delta_0$ is the subsystem of PA featuring only bounded induction, and

$$\text{superexp} := \forall x \exists y (2_x^x = y),$$

with $2_0^x = x$, $2_{y+1}^x = 2^{2^y}$.

The strategy leading to the cut-elimination theorem above clearly generalizes to the case of the theory obtained by replacing the C-rules with the rules (Tr-L) and (Tr-R). One simply has to replace the C-complexity with a truth complexity measure (cf. section 5 below). Similarly, one can apply the strategy to a theory of naïve abstraction (or property predication) based on rules of the form

$$\begin{array}{c} \text{ER} \frac{\Gamma \Rightarrow \Delta, \varphi(t)}{\Gamma \Rightarrow \Delta, t \in \{x \mid \varphi\}} \\ \text{EL} \frac{\varphi(t), \Gamma \Rightarrow \Delta}{t \in \{x \mid \varphi\}, \Gamma \Rightarrow \Delta} \end{array}$$

where instead of a naming device one assumes a term-forming for abstraction operator $\{\cdot \mid \cdot\}$ — e.g. along the lines of the one employed for a contraction-free set theory in [Can03]. \in -complexity is then defined in the obvious way: given a derivation \mathcal{D} ending with EL , the \in -complexity of $t \in \{x \mid \varphi\}$ is defined as the \in -complexity of $\varphi(t)$ plus one. One can then follow the template of Definition 7. All results above then carry over with only minimal modifications.

5. A COMPOSITIONAL THEORY OF NON-REFLEXIVE TRUTH AND CONSEQUENCE

In their [HH06], Halbach and Horsten develop a formal system, called PKF (for *Partial Kripke-Feferman*), which axiomatizes Kripke’s fixed point models in a partial logic over Peano Arithmetic (PA). The logic of PKF amounts to what is sometimes called *Symmetric Strong Kleene* logic, in that it is a variant of Strong Kleene Logic which does not distinguish between gaps and gluts. In other words, PKF is also sound with respect to models in which the third value is ‘both true and false’. Later versions of PKF are based on a four-valued logic roughly corresponding to First-Degree Entailment (FDE) [Hal14, Hor12].¹³ When referring to PKF in what follows, we will implicitly refer to the four-valued version.¹⁴

PKF constitutes the basis of any theory of truth that extends Kripke’s theory with extra-resources — e.g. a new conditional [Fie08, Lei18]. In this section, we develop a twin-theory of (the FDE formulation of) PKF, which we call RKF, whose logic is based — somewhat unsurprisingly — on a restriction of (REF). PKF and RKF are twins in the sense that for X a fixed point model for the language \mathcal{L}_{Tr} obtained in the manner suggested in section 3,

$$(\mathbb{N}, X) \models^{\text{fde}} \text{PKF} \text{ iff } (\mathbb{N}, X) \models^{\text{pts}} \text{RKF}$$

This obviously entails that RKF is also a theory of naïve truth, and in fact an axiomatisation of Kripke’s theory of truth in partial (substructural) logic. Actually, RKF is still richer: it is *also* a theory of naïve consequence, whereas PKF cannot be. In fact, just as a naïve truth predicate can be defined from the naïve consequence predicate of LPC, so can a predicate for naïve consequence (obeying the rules (CL) and (CR)) be defined from the naïve truth predicate of RKF (the naïveté of the latter, in turn, follows from the compositional rules of RKF). Definition 16, Lemma 21, and Corollary 22 will establish this claim more precisely. By contrast, since PKF is a fully structural theory, the presence

¹³This logic is called BDM in [Nic18, FNH17].

¹⁴In keeping with the focus of the paper, we are mostly interested in the proof-theoretical aspects of the systems we consider here, and therefore we refer to the literature for more details on the semantic consequence relations we will mention.

of naïve consequence rules would immediately entail triviality by an internalized version of Curry's paradox—the V-Curry paradox by [BM13].

Despite the vicinity of RKF to well-known theories with restricted operational rules, it is not our purpose to motivate the theory on the basis of these relationships; however they suggest that, unlike other substructural approaches, the substructural theory we are about to introduce point to widely accepted conceptions of truth. Our main objective in this section is to extend the proof-theoretic study initiated above: just like LPC and LPC^∞ display better proof-theoretic properties than their fully structural rivals, RKF is amenable to a more natural and direct proof-theoretic analysis than its fully structural relative PKF.

As anticipated above, in order to formulate RKF, it is more convenient to take the truth predicate as primitive. Let \mathcal{L}_{Tr} be the language given by adding a fresh unary predicate Tr to the language of arithmetic, i.e. $\mathcal{L}_{\mathbb{N}} \cup \{\text{Tr}\}$. In this language, we can define the consequence predicate via a combination of truth and conditional, putting $C(x, y) : \leftrightarrow (\text{Tr}(x) \rightarrow \text{Tr}(y))$, i.e. $\neg(\text{Tr}(x) \wedge \neg\text{Tr}(y))$. Due to the fact that the logic PTS has all the classical meta-inferences, and thus the conditional can be introduced and eliminated just as the consequence predicate, one can easily define truth in terms of consequence and the other way around.

DEFINITION 16 (RKF). *The theory RKF in \mathcal{L}_{Tr} has the following components:*

- (i) *The logical component of LPC, that is the initial sequents and rules of LPC except (CL) and (CR)*
- (ii) *The initial sequents $\Gamma \Rightarrow \Delta$, φ for φ a basic axiom of PA, including identity axioms:*

$$\Gamma \Rightarrow \Delta, t = t; \quad \Gamma, P(r), r = s \Rightarrow \Delta, P(s) \text{ for } P \text{ an atom of } \mathcal{L}_{\mathbb{N}}.$$

- (iii) *All instances of the induction schema for all formulas $\varphi(v)$ of \mathcal{L}_{Tr} :*

$$\frac{\Gamma \Rightarrow \varphi(0), \Delta \quad \Gamma, \varphi(x) \Rightarrow \varphi(x+1), \Delta}{\Gamma \Rightarrow \varphi(t), \Delta}$$

with x not free in Γ, Δ , $\forall x\varphi$ and t arbitrary.

- (iv) *The following truth rules:*

$$\begin{array}{ll} \text{(=R)} \frac{\Gamma \Rightarrow s^\circ = t^\circ, \Delta}{\Gamma \Rightarrow \text{Tr}(s = t), \Delta} & \text{(=L)} \frac{\Gamma, s^\circ = t^\circ \Rightarrow \Delta}{\Gamma, \text{Tr}(s = t) \Rightarrow \Delta} \\ \text{(Tr1)} \frac{\Gamma \Rightarrow \text{Tr}(x), \Delta}{\Gamma \Rightarrow \text{Tr}^\ulcorner \text{Tr}(\dot{x}) \urcorner, \Delta} & \text{(Tr2)} \frac{\Gamma, \text{Tr}(x) \Rightarrow \Delta}{\Gamma, \text{Tr}^\ulcorner \text{Tr}(\dot{x}) \urcorner \Rightarrow \Delta} \\ \text{(Tr}\neg\text{1)} \frac{\Gamma \Rightarrow \neg\text{Tr}(x), \Delta}{\Gamma \Rightarrow \text{Tr}(\neg x), \Delta} & \text{(Tr}\neg\text{2)} \frac{\Gamma, \neg\text{Tr}(x) \Rightarrow \Delta}{\Gamma, \text{Tr}(\neg x) \Rightarrow \Delta} \\ \text{(Tr}\wedge\text{1)} \frac{\Gamma \Rightarrow \text{Tr}(x), \Delta \quad \Gamma \Rightarrow \text{Tr}(y), \Delta}{\Gamma \Rightarrow \text{Tr}(x \wedge y), \Delta} & \text{(Tr}\wedge\text{2)} \frac{\Gamma, \text{Tr}(x), \text{Tr}(y) \Rightarrow \Delta}{\Gamma, \text{Tr}(x \wedge y) \Rightarrow \Delta} \\ \text{(Tr}\forall\text{1)} \frac{\Gamma \Rightarrow \text{Tr}(x(s/v)), \Delta}{\Gamma \Rightarrow \text{Tr}(\forall vx), \Delta} & \text{(Tr}\forall\text{2)} \frac{\Gamma, \text{Tr}(x(t/v)) \Rightarrow \Delta}{\Gamma, \text{Tr}(\forall vx) \Rightarrow \Delta} \end{array}$$

s 'not free' (see remark) in Γ, Δ , $\forall u \text{Tr } y(u/v)$

A few comments on the definition:

REMARK 17.

- (i) Syntactic functions operating on codes of \mathcal{L}_{Tr} -expressions will be presented in simplified form for the sake of readability. For instance the operation $e_1, e_2 \mapsto e_1 = e_2$ represented in RKF by means of a function $\text{id}(x, y)$, is abbreviated as $(x = y)$. Similarly for the other syntactic operations employed in the definition.
- (ii) In $(=L)$ and $(=R)$, \cdot° stands for the PA-definable arithmetical evaluation function, such that $\ulcorner t^\circ = t$ for any closed term t of \mathcal{L}_{Tr} .
- (iii) In $(\text{Tr}1)$ and $(\text{Tr}2)$, terms of the form $\ulcorner \varphi(\dot{x}) \urcorner$ stand for the result of substituting, in the code of the formula $\varphi(v)$, the (code of the) variable v with the formal numeral for x (see e.g. [Smo77]). Moreover, $x(y/v)$ stands for the result of formally substituting y for the (code of) the variable v in x (we follow the conventions in [Hal14]).
- (iv) The Tr-rules for connectives and quantifiers are presented in simplified forms, with variables intended to range over sentences and terms, according to the form of the rules. For instance, $(=L)$ is short for:

$$\frac{\Gamma, s^\circ = t^\circ \Rightarrow \Delta}{\Gamma, \text{Sent}_{\mathcal{L}_{\text{Tr}}}(s = t), \text{Tr}(\ulcorner s = t \urcorner) \Rightarrow \Delta}$$

where s, t are arbitrary \mathcal{L}_{Tr} -terms, and $\text{Sent}_{\mathcal{L}_{\text{Tr}}}(\cdot)$ represents the set of \mathcal{L}_{Tr} -sentences in PA. Similarly, $(\text{Tr}\wedge 1)$ is short for:

$$\frac{\Gamma, \text{Sent}_{\mathcal{L}_{\text{Tr}}}(x) \Rightarrow \text{Tr}(x), \Delta \quad \Gamma, \text{Sent}_{\mathcal{L}_{\text{Tr}}}(y) \Rightarrow \text{Tr}(y), \Delta}{\Gamma, \text{Sent}_{\mathcal{L}_{\text{Tr}}}(\text{and}(x, y)) \Rightarrow \text{Tr}(\text{and}(x, y)), \Delta}$$

where ‘ $\text{and}(\cdot, \cdot)$ ’ represents the operation $e_1, e_2 \mapsto e_1 \wedge e_2$.

Finally, the non-abbreviated form of $(\text{Tr}\forall 1)$ reads:

$$\frac{\Gamma, \text{For}_{\mathcal{L}_{\text{Tr}}}(x, v), \text{Var}_{\mathcal{L}_{\text{Tr}}}(v), \text{CT}_{\mathcal{L}_{\text{Tr}}}(y) \Rightarrow \text{Tr}(x(y/v)), \Delta}{\Gamma, \text{For}_{\mathcal{L}_{\text{Tr}}}(x, v), \text{Var}_{\mathcal{L}_{\text{Tr}}}(v) \Rightarrow \text{Tr}(\forall v x), \Delta}$$

where $\text{For}_{\mathcal{L}_{\text{Tr}}}(x, v)$ represents in PA the set of \mathcal{L}_{Tr} -formulae x with the free \mathcal{L}_{Tr} -variable v only, $\text{Var}_{\mathcal{L}_{\text{Tr}}}$ represents the set of \mathcal{L}_{Tr} -variables, and $\text{CT}_{\mathcal{L}_{\text{Tr}}}$ the set of closed \mathcal{L}_{Tr} -terms.

The consistency of RKF will be a corollary of Proposition 23, whose proofs requires a few preliminary results that have also independent interest.

The following lemma, which follows from a simple *external* induction on the length of φ , indicates a form of recapture: RKF (and extensions thereof) features full initial sequents for the language $\mathcal{L}_{\mathbb{N}}$.

LEMMA 18. RKF proves $\varphi \Rightarrow \varphi$ for $\varphi \in \mathcal{L}_{\mathbb{N}}$.

Next, observe that weakening is length-preserving admissible in RKF, and inferences in RKF are ‘grounded’, in the sense specified in the following lemma.

LEMMA 19 (Length-preserving admissibility of Weakening). *If there is an RKF-derivation \mathcal{D} of length n of $\Gamma \Rightarrow \Delta$, then for any multisets $\Gamma' \supseteq \Gamma$ and $\Delta' \supseteq \Delta$, there is an RKF-derivation \mathcal{D}' of length n of $\Gamma' \Rightarrow \Delta'$.*

LEMMA 20 (Groundedness). *If there is an RKF-derivation \mathcal{D} of length n of $\Gamma \Rightarrow \Delta$, then either there is a formula $\varphi \in \Gamma$ such that there is an RKF-derivation \mathcal{D}_0 of length $\leq n$ of $\varphi \Rightarrow$, or there is a formula $\psi \in \Delta$ such that there is an RKF-derivation \mathcal{D}_1 of length $\leq n$ of $\Rightarrow \psi$.*

Proof sketch. The proof is by induction on the length of the derivations, and the only non-trivial case is the one involving the induction schema (as for the other cases, the proof is completely analogous

to the proof of semantic groundedness in [NR18], Lemma 6). Suppose that there is a derivation of length $n + 1$ ending with an instance of the induction schema:

$$\frac{\mathcal{D}_0 \quad \mathcal{D}_1}{\Gamma \Rightarrow \varphi(0), \Delta \quad \Gamma, \varphi(x) \Rightarrow \varphi(x + 1), \Delta} \Gamma \Rightarrow \varphi(t), \Delta$$

where $x \notin \text{FV}(\Gamma, \Delta, \forall x\varphi)$. Applying the IH to the last sequent of \mathcal{D}_0 and \mathcal{D}_1 gives us two cases:

- (i) There is a derivation \mathcal{D}'_0 or \mathcal{D}'_1 of length $\leq n$ of $\gamma \Rightarrow$ or $\Rightarrow \delta$ (for $\gamma \in \Gamma$ and $\delta \in \Delta$). In this case, the claim of the Lemma is established.
- (ii) There is a derivation \mathcal{D}'_0 of length $\leq n$ of $\Rightarrow \varphi(0)$ and a derivation \mathcal{D}'_1 of length $\leq n$ of $\varphi(x) \Rightarrow$. By Lemma 19, there is also a derivation \mathcal{D}''_1 of length $\leq n$ of $\varphi(x) \Rightarrow \varphi(x + 1)$. Then, an application of the induction schema proves the claim:

$$\frac{\mathcal{D}'_0 \quad \mathcal{D}''_1}{\Rightarrow \varphi(0) \quad \varphi(x) \Rightarrow \varphi(x + 1)} \Rightarrow \varphi(t)$$

The remaining case, in which the IH gives us a derivation \mathcal{D}'_1 of length $\leq n$ of $\Rightarrow \varphi(x + 1)$ is dealt with similarly.

qed.

We can now use the two results above to show that the naïve rules to introduce the truth predicate to the left and to the right of the sequent arrow are derivable in RKF.

LEMMA 21. *Every instance of the naïve truth rules is derivable in RKF:*

$$(\text{TrR}) \frac{\Gamma \Rightarrow \varphi, \Delta}{\Gamma \Rightarrow \text{Tr}(\ulcorner \varphi \urcorner), \Delta} \quad (\text{TrL}) \frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \text{Tr}(\ulcorner \varphi \urcorner) \Rightarrow \Delta}$$

Proof. We reason by cases and by induction on the complexity of φ . We will only do some cases.

Case 1. Suppose φ is an atomic sentence. Then it is either an atomic arithmetic sentence of the form $s^\circ = t^\circ$, or a truth predication of the form $\text{Tr}(t)$. In the former case, the rules =R and =L provide the corresponding instances of TrR and TrL, and in the latter case Tr1 and Tr2 provide the corresponding instances of TrR and TrL.

Case 2.1. Suppose φ is a negation $\neg\psi$, and suppose there is a derivation

$$\frac{\mathcal{D}}{\Gamma, \neg\psi \Rightarrow \Delta}$$

By Lemma 20, there are two possible cases:

Case 2.1.1. There is a derivation \mathcal{D}_0 of

$$\frac{\mathcal{D}_0}{\neg\psi \Rightarrow}$$

Since $\neg\psi \Rightarrow$ is not an axiom, it must have been obtained by one of the RKF rules, and the only possible rule is $\neg\text{L}$. Therefore, there is a derivation

$$(\neg\text{L}) \frac{\mathcal{D}'_0 \Rightarrow \psi}{\neg\psi \Rightarrow}$$

Applying the induction hypothesis (IH) to the ψ appearing at the end of \mathcal{D}'_0 , we get:

$$\begin{array}{c} \mathcal{D}'_0 \\ \Rightarrow \psi \\ \text{(IH)} \frac{}{\Rightarrow \text{Tr}(\Gamma\psi^\neg)} \\ \text{(\neg L)} \frac{}{\neg \text{Tr}(\Gamma\psi^\neg) \Rightarrow} \\ \text{(Tr}\neg\text{2)} \frac{}{\text{Tr}(\Gamma\neg\psi^\neg) \Rightarrow} \end{array}$$

Call the above derivation \mathcal{D}_1 . By Lemma 19, there is a derivation \mathcal{D}'_1 establishing:

$$\begin{array}{c} \mathcal{D}'_1 \\ \Gamma, \text{Tr}(\Gamma\neg\psi^\neg) \Rightarrow \Delta \end{array}$$

Case 2.1.2. Either there is a $\vartheta \in \Gamma$ and derivation \mathcal{D}_0 or a $\sigma \in \Delta$ and a derivation \mathcal{D}_1 s.t.

$$\begin{array}{cc} \mathcal{D}_0 & \mathcal{D}_1 \\ \vartheta \Rightarrow & \Rightarrow \sigma \end{array}$$

By Lemma 19, therefore, either there is a derivation \mathcal{D}'_0 or there is a derivation \mathcal{D}'_1 establishing:

$$\begin{array}{cc} \mathcal{D}'_0 & \mathcal{D}'_1 \\ \Gamma, \text{Tr}(\Gamma\neg\psi^\neg) \Rightarrow \Delta & \Gamma, \text{Tr}(\Gamma\neg\psi^\neg) \Rightarrow \Delta \end{array}$$

Case 2.2. Suppose φ is a conjunction $\psi \wedge \chi$, and suppose there is a derivation

$$\begin{array}{c} \mathcal{D} \\ \Gamma \Rightarrow \psi \wedge \chi, \Delta \end{array}$$

By Lemma 20, there are two possible cases:

Case 2.2.1. There is a derivation \mathcal{D}_0 of

$$\begin{array}{c} \mathcal{D}_0 \\ \Rightarrow \psi \wedge \chi \end{array}$$

Since $\Rightarrow \psi \wedge \chi$ is not an axiom, it must have been obtained by one of the RKF rules, and the only possible rule is $\wedge R$. Therefore, there is a derivation

$$\text{(\wedge R)} \frac{\begin{array}{c} \mathcal{D}'_0 \quad \mathcal{D}'_1 \\ \Rightarrow \psi \quad \Rightarrow \chi \end{array}}{\Rightarrow \psi \wedge \chi}$$

Applying the IH to the ψ and χ appearing at the end of \mathcal{D}'_0 and \mathcal{D}'_1 , we get:

$$\text{(Tr}\wedge\text{1)} \frac{\begin{array}{cc} \mathcal{D}'_0 & \mathcal{D}'_1 \\ \Rightarrow \psi & \Rightarrow \chi \\ \text{(IH)} \frac{}{\Rightarrow \text{Tr}(\Gamma\psi^\neg)} & \text{(IH)} \frac{}{\Rightarrow \text{Tr}(\Gamma\chi^\neg)} \end{array}}{\Rightarrow \text{Tr}(\Gamma\psi \wedge \chi^\neg)}$$

Call the above derivation \mathcal{D}_2 . By Lemma 19, there is a derivation \mathcal{D}'_2 establishing:

$$\begin{array}{c} \mathcal{D}'_2 \\ \Gamma \Rightarrow \text{Tr}(\Gamma\psi \wedge \chi^\neg), \Delta \end{array}$$

Case 2.2.2. This case is similar to case 2.1.2.

Case 2.3. Suppose φ is a universally quantified sentence $\forall x\psi(x)$ and suppose there is a derivation

$$\begin{array}{c} \mathcal{D} \\ \Gamma \Rightarrow \forall x\psi(x), \Delta \end{array}$$

By Lemma 20, there are two possible cases:

Case 2.3.1. There is a derivation \mathcal{D}_0 of

$$\begin{array}{c} \mathcal{D}_0 \\ \Rightarrow \forall x\psi(x) \end{array}$$

Since $\Rightarrow \forall x\psi(x)$ is not an axiom, it must have been obtained by one of the RKF rules, and the only possible rule is $\forall R$. Therefore, there is a derivation

$$\begin{array}{c} \mathcal{D}'_0 \\ (\forall R) \frac{\Rightarrow \psi(x)}{\Rightarrow \forall x\psi(x)} \end{array}$$

with $x \notin \text{FV}(\Gamma, \Delta, \forall x\psi)$ Applying the IH to the $\psi(x)$ appearing at the end of \mathcal{D}'_0 (noticing that $x \notin \text{FV}(\Gamma, \Delta, \forall x\text{Tr}(\ulcorner \psi(\dot{x}) \urcorner))$ as well), we get:

$$\begin{array}{c} \mathcal{D}'_0 \\ (\text{IH}) \frac{\Rightarrow \psi(x)}{\Rightarrow \text{Tr}(\ulcorner \psi(\dot{x}) \urcorner)} \\ (\forall R) \frac{\Rightarrow \text{Tr}(\ulcorner \psi(\dot{x}) \urcorner)}{\Rightarrow \forall x\text{Tr}(\ulcorner \psi(\dot{x}) \urcorner)} \\ (\text{Tr}\forall 1) \frac{\Rightarrow \forall x\text{Tr}(\ulcorner \psi(\dot{x}) \urcorner)}{\Rightarrow \text{Tr}(\ulcorner \forall x\psi(\dot{x}) \urcorner)} \end{array}$$

Call the above derivation \mathcal{D}_1 . By Lemma 19, there is a derivation \mathcal{D}'_1 establishing:

$$\begin{array}{c} \mathcal{D}'_1 \\ \Gamma \Rightarrow \text{Tr}(\ulcorner \forall x\psi(\dot{x}) \urcorner), \Delta \end{array}$$

Case 2.3.2. This case is similar to case 2.1.2.

qed.

The above Lemma also shows that, via the definition of $C(x, y)$ as $\text{Tr}(x) \rightarrow \text{Tr}(y)$, RKF unrestrictedly validates the naïve rules for consequence.

COROLLARY 22. *Every instance of the naïve consequence rules is derivable in RKF:*

$$(\text{CL}) \frac{\Gamma \Rightarrow \Delta, \varphi \quad \psi, \Gamma \Rightarrow \Delta}{\Gamma, C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner) \Rightarrow \Delta} \quad \frac{\Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner), \Delta} (\text{CR})$$

Proof. For CL:

$$\begin{array}{c} (\text{TrR}) \frac{\Gamma \Rightarrow \varphi, \Delta}{\Gamma \Rightarrow \text{Tr}(\ulcorner \varphi \urcorner), \Delta} \quad (\text{TrL}) \frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma, \text{Tr}(\ulcorner \psi \urcorner) \Rightarrow \Delta} \\ (\neg R) \frac{\Gamma \Rightarrow \neg \text{Tr}(\ulcorner \psi \urcorner), \Delta}{\Gamma \Rightarrow \neg \text{Tr}(\ulcorner \psi \urcorner), \Delta} \\ (\wedge R) \frac{\Gamma \Rightarrow \text{Tr}(\ulcorner \varphi \urcorner), \Delta \quad \Gamma \Rightarrow \neg \text{Tr}(\ulcorner \psi \urcorner), \Delta}{\Gamma \Rightarrow \text{Tr}(\ulcorner \varphi \urcorner) \wedge \neg \text{Tr}(\ulcorner \psi \urcorner), \Delta} \\ (\neg L) \frac{\Gamma \Rightarrow \text{Tr}(\ulcorner \varphi \urcorner) \wedge \neg \text{Tr}(\ulcorner \psi \urcorner), \Delta}{\Gamma, \neg(\text{Tr}(\ulcorner \varphi \urcorner) \wedge \neg \text{Tr}(\ulcorner \psi \urcorner)) \Rightarrow \Delta} \end{array}$$

For CR:

$$\begin{array}{c}
\frac{\Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma, \text{Tr}(\ulcorner \varphi \urcorner) \Rightarrow \psi, \Delta} \text{ (TrL)} \\
\frac{\Gamma, \text{Tr}(\ulcorner \varphi \urcorner) \Rightarrow \psi, \Delta}{\Gamma, \text{Tr}(\ulcorner \varphi \urcorner) \Rightarrow \text{Tr}(\ulcorner \psi \urcorner), \Delta} \text{ (TrR)} \\
\frac{\Gamma, \text{Tr}(\ulcorner \varphi \urcorner) \Rightarrow \text{Tr}(\ulcorner \psi \urcorner), \Delta}{\Gamma, \text{Tr}(\ulcorner \varphi \urcorner), \neg \text{Tr}(\ulcorner \psi \urcorner) \Rightarrow \Delta} \text{ (\neg L)} \\
\frac{\Gamma, \text{Tr}(\ulcorner \varphi \urcorner), \neg \text{Tr}(\ulcorner \psi \urcorner) \Rightarrow \Delta}{\Gamma, \text{Tr}(\ulcorner \varphi \urcorner) \wedge \neg \text{Tr}(\ulcorner \psi \urcorner) \Rightarrow \Delta} \text{ (\wedge L)} \\
\frac{\Gamma, \text{Tr}(\ulcorner \varphi \urcorner) \wedge \neg \text{Tr}(\ulcorner \psi \urcorner) \Rightarrow \Delta}{\Gamma \Rightarrow \neg(\text{Tr}(\ulcorner \varphi \urcorner) \wedge \neg \text{Tr}(\ulcorner \psi \urcorner)), \Delta} \text{ (\neg R)}
\end{array}$$

Since $C(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner)$ is defined as $\neg(\text{Tr}(\ulcorner \varphi \urcorner) \wedge \neg \text{Tr}(\ulcorner \psi \urcorner))$, this establishes the claim. *qed.*

Finally, thanks to the above Corollary, RKF can be shown to be *adequate* with respect to the semantics articulated in section 3.

PROPOSITION 23 (Adequacy). *For every multisets Γ, Δ of formulae of \mathcal{L}_C and every $S \subseteq \omega \times \omega$:*

$$\langle \mathbb{N}, S \rangle \models^{\text{PTS}} \text{RKF} \text{ if and only if } \Psi(S) = S$$

Proof sketch. The right-to-left direction is immediate: a quick inspection shows that if S is a fixed point of Ψ , then $\langle \mathbb{N}, S \rangle$ PTS-satisfies all the axiom and rules of RKF. For the left-to-right direction, notice that if $\langle \mathbb{N}, S \rangle \models^{\text{PTS}} \text{RKF}$, then the set of (codes of) sentences in S is closed under all the logical clauses of the operator Ψ (for otherwise $\langle \mathbb{N}, S \rangle$ would not PTS-satisfy the logical rules of RKF) and, by Corollary 22, also under the naïve consequence-theoretic clauses of Ψ . Therefore, $\Psi(S) = S$. *qed.*

6. FURTHER WORK

Much work remains to be done on non-reflexive systems and their applications. Just to mention a few: fully compositional, non-reflexive theories of consequence should be formulated and studied (by analogy with the compositional, non-reflexive theory of truth presented in section 5). Moreover, the relations between non-reflexive and other non-classical systems (paracomplete, paraconsistent, non-contractive, and non-transitive) should be fully investigated. Finally, other philosophical applications (beyond naïve truth and consequence) of non-reflexive logics should be considered, such as the notion of grounding.

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