Accuracy and Probabilism in Infinite Domains

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Abstract

The best accuracy arguments for probabilism apply only to credence functions with finite domains, that is, credence functions that assign credence to at most finitely many propositions. This is a significant limitation. It reveals that the support for the accuracy-first program in epistemology is a lot weaker than it seems at first glance, and it means that accuracy arguments cannot yet accomplish everything that their competitors, the pragmatic (Dutch book) arguments, can. In this paper, I investigate the extent to which this limitation can be overcome. Building on the best arguments in finite domains, I present two accuracy arguments for probabilism that are perfectly general—they apply to credence functions with arbitrary domains. I then discuss how the arguments' premises can be challenged. We will see that it is particularly difficult to characterize admissible accuracy measures in infinite domains.

Introduction

Probabilism is the principle that it is irrational to have credences that violate the axioms of probability theory. What reason is there to accept this principle? There is an influential argument, which has its roots in results due to de Finetti (1974), based on the idea that probabilism promotes accuracy. There is a precise sense in which non-probabilistic credence functions are less accurate than probabilistic ones. So, the argument goes, insofar as it is irrational to have inaccurate credences, there is good reason to accept probabilism.

Several variations of this accuracy argument for probabilism have been developed over the years, but there is a major gap in all of the arguments that have been most influential in the philosophical literature: they all assume that credence functions have *finite domains*.[1] The domain of a credence function is a collection of propositions. To say that a credence function has a finite domain is to say that it assigns credences to at most finitely many propositions. To say that a credence function has an infinite domain is to say that it assigns credences to infinitely many propositions.

It is widely accepted that rationality does not preclude credence functions with infinite domains. Consider that probability functions with infinite domains are ubiquitous throughout the sciences. If there are some circumstances in which it is rationally permissible to align one's credences with such scientifically approved probabilities, then credence functions with

[1] I have in mind Joyce (1998, 2009); Predd et al. (2009); Pettigrew (2016a, 2022).

infinite domains are permissible. And it seems very plausible that there are circumstances in which it is permissible to align one's credences with the probabilities endorsed by science. This is a very weak claim. To deny it would be to claim that rationality always forbids aligning one's credences with the probabilities used by scientists. As far as I know, no one defends this.

But if credence functions can have infinite domains, then accuracy arguments give us little reason to accept probabilism *in general*. The most that the arguments show is that probabilism promotes accuracy *in certain special cases*, namely for credence functions with finite domains. If an agent finds herself in a circumstance in which she has credences in infinitely many propositions, then, for all the accuracy arguments show, there is no reason for her to obey the axioms of probability.

This gap in the accuracy arguments has been pointed out elsewhere. In his book on accuracy and credence, Richard Pettigrew identifies infinite domains as the first of several open problems for future research (2016a, p. 222). But I think Pettigrew undersells the problem that infinite domains pose.

Pettigrew uses accuracy arguments not only to justify particular principles like probabilism but also to defend a sweeping program in epistemology called *accuracy-first*.[2] According to the accuracy-first program, accuracy is the fundamental epistemic good. Many will agree that it's good to have accurate credences and that it's also good to have credences that are probabilistic. But, according to accuracy-firsters, the goodness of accuracy is basic, whereas probabilism is good *because* it promotes accuracy. And what goes for probabilism goes in general: the principles of rationality promote accuracy. This is the claim with which accuracy-first program stands or falls.[3]

The main support for the accuracy-first program is of an inductive variety.[4] The accuracy-firster selects a particular principle of rationality and argues that it promotes accuracy. As successful instances of such arguments multiply, the accuracy-firster's view that *every* principle of rationality promotes accuracy becomes more plausible. Surveying the literature, one gets the impression that the inductive base of this argument is strong. Since Joyce's (1998) accuracy argument for probabilism, research in the accuracy-first program has flourished, producing arguments for conditionalization[5], the principal principle[6], the principle of indifference[7], and more.[8]

But how much support do these arguments really lend to the accuracy-first program? In view of the gap between finite and infinite domains, arguably not very much. For accuracy

[2] I discuss accuracy-first only as it applies to credences and not as it applies to full belief. For a defense of the latter, see Goldman (2002, ch. 3).

[3] What I and Pettigrew (2022) call "accuracy-first" is closer to what Pettigrew (2016a) calls "veritism." According to what Pettigrew (2016a, p. 29) calls "accuracy-first," there may be rational principles—for example, the principle that one ought to respect one's evidence—that do not promote accuracy in general; but when there are conflicts between principles that promote accuracy and those that do not, the accuracy promoting principles have *priority*. The distinction between this version of accuracy-first and veritism is mostly irrelevant for my purposes because the two views agree that "non-evidential" principles like probabilism promote accuracy.

[4] Pettigrew (2016a, pp. 6–8)

[5] Greaves and Wallace (2006); Leitgeb and Pettigrew (2010); Briggs and Pettigrew (2020).

[6] Pettigrew (2013).

[7] Pettigrew (2016b).

[8] Many of these arguments are given in Pettigrew's (2016a) book.

arguments do *not* show that probabilism promotes accuracy, as they are supposed to do. What they show is that probabilism promotes the accuracy *of credence functions with finite domains*. At best, then, the accuracy arguments for probabilism support only a highly qualified version of accuracy-first—that accuracy is the fundamental epistemic good for agents with credences in at most finitely many propositions. This is a far cry from what accuracy-firsters claim.[9]

Because of this, I would argue that the problem of infinite domains does not represent merely one of "the ways in which the [accuracy-first] project may be extended" (Pettigrew, 2016a, p. 221). Rather, the restriction to finite domains severely limits the support for the accuracy-first program that accuracy arguments for probabilism are supposed to provide.

There is another reason the gap between finite and infinite domains is important. Part of the accuracy argument's initial appeal comes by way of contrast with the pragmatic ("Dutch book") arguments championed by Ramsey (1931) and de Finetti (1974). Joyce writes, for example, "The pragmatic character of the Dutch book argument makes it unsuitable as an 'epistemic' justification of the fundamental probabilist dogma that rational partial beliefs must conform to the axioms of probability" (1998, p. 575). Whatever one thinks of the justificatory powers of pragmatic arguments, it is clear that, in infinite domains, they fare much better than accuracy arguments. De Finetti's Dutch book argument for probabilism is perfectly general; it applies to any credence functions whatsoever, not only to those with finite domains.[10] And a number of other pragmatic arguments extend to infinite domains as well.[11] Insofar as accuracy arguments for probabilism are meant to replace or compete with pragmatic arguments, then, it is important to see them extended to infinite domains.

That is what I will do in this paper. I will develop two accuracy arguments for probabilism that apply to credence functions with arbitrary domains. I begin, in the next section, by outlining the general structure of accuracy arguments.

1 The Structure of Accuracy Arguments

All accuracy arguments for probabilism have a similar form, which consists of three components (Pettigrew, 2022). First, there is a premise that specifies which measures of accuracy are admissible. Second, there is a normative premise that connects rationality and the promotion of accuracy. Third, there is a mathematical theorem showing that non-probabilistic credence functions fail to promote accuracy. It then follows that non-probabilistic credence functions are irrational.

There is fairly widespread agreement among purveyors of accuracy arguments about how to flesh out normative premises. Start by saying that a credence function *fails* to promote accuracy if there is a second credence function that is more accurate than it *in every possible*

[9] What about accuracy arguments for principles other than probabilism? Do they too apply only to credences with finite domains? One can run accuracy-first arguments for conditionalization in infinite domains using the results in Easwaran (2013) and Huttegger (2013). But these results *assume* that credences are probabilistic from the outset (this is explicit on p. 121 of Easwaran's paper and on p. 416 of Huttegger's). Since an argument is only as strong as its weakest premise, and since there is no accuracy argument for probabilism in infinite domains, it's not clear that these arguments for conditionalization lend much additional support to the accuracy-first program.

[10] For a detailed discussion, see Nielsen (2021c).

[11] See, for instance, Rescorla (2018), Huttegger and Nielsen (2020), and Nielsen (2021a).

world. We say that the second credence function *accuracy-dominates* the first. The first normative premise that we will consider says that a credence function is irrational if, according to any admissible measure of accuracy, it is accuracy-dominated. In section 3.1, we will consider an objection to this premise as well as an alternative normative premise.

There is less agreement about how to handle the accuracy arguments' first component. The disagreement is over how to answer the question, What properties characterize the measures of accuracy that are admissible when it comes to assessing the rationality of credence functions? I will delay giving any answers to this question until the next section of the paper, where we develop a formal framework in which the question can be answered precisely. Admissible inaccuracy measures will also be discussed at length in 3.2.

This leaves the mathematical theorems, which we will refer to as *accuracy-dominance theorems*. In order to conclude that probabilism is a *general* principle of rationality, accuracy arguments require a theorem with the following form.

General accuracy-dominance theorem According to any admissible measure of accuracy, every non-probabilistic credence function is accuracy-dominated by some probabilistic credence function.[12]

However, no accuracy-dominance theorem like this has ever been established. The best theorems on offer have the following form.

Finitary accuracy-dominance theorem According to any admissible measure of accuracy, *for credence functions with finite domains*, every non-probabilistic credence function is accuracy-dominated by some probabilistic credence function.

Because the best accuracy-dominance theorems apply only to credence functions with finite domains, they do not in fact support probabilism. At best, an accuracy argument that uses a finitary accuracy-dominance theorem supports the principle that it is irrational for a credence function *with a finite domain* to violate the axioms of probability theory. Obviously, this principle is much weaker than probabilism.

In order to bridge the gap between what accuracy arguments seek to establish (probabilism) and what they actually establish (probabilism for credence functions with finite domains), a general accuracy-dominance theorem is needed. I will provide such a theorem in the next section.

2 An Accuracy Argument for Probabilism

In this section, I will present the paper's first accuracy argument for probabilism. The form of the argument was sketched in the previous section. Here, I will fill in the argument's details, starting with formal definitions of the concepts introduced above (2.1). A general accuracy-dominance theorem, which is the argument's main innovation, is then stated in 2.2.

[12]A theorem with this form actually proves more than needed, given our current normative premise. To show that non-probabilistic credence functions are irrational, we need only show that they are accuracy-dominated by *some* credence function, whether probabilistic or not. We focus our attention on the stronger result for three reasons: (1) the literature is mainly concerned with results of this form, (2) the stronger result is more interesting from a purely mathematical point of view, and (3) the added strength will be useful when we consider an objection to our normative premise in 3.1.

2.1 Formal Framework

In this subsection, we define (probabilistic) credence functions (2.1.1), admissible inaccuracy measures (2.1.2), and accuracy-dominance (2.1.3). The section concludes by recalling the normative premise of the paper's first argument for probabilism.

2.1.1 Credence Functions

Let W be a set of worlds that are epistemically possible for an agent. Subsets of W are called *propositions*. The set of all propositions (the powerset of W) is denoted by 2^W. A *credence function* c is function from propositions to numbers in $[0, 1]$, that is $c : 2^W \to [0, 1]$.[13] The number $c(A)$ represents an agent's confidence that the proposition A is true.

A credence function c is called *probabilistic* iff it has the following properties.

Normalization $c(W) = 1$.

Additivity If A and B are disjoint propositions, then $c(A \cup B) = c(A) + c(B)$.[14]

An example of a probabilistic credence function is the *valuation function at world* w, defined by

$$v_w(A) = \begin{cases} 1, & \text{if } w \in A \\ 0, & \text{if } w \notin A. \end{cases} \tag{1}$$

Let $\mathcal{C}$ be the set of all credence functions, and let $\mathcal{P}$ be the set of all probabilistic credence functions.

If $W = \{w_1, ..., w_n\}$ is finite, then so is 2^W. In fact, $|2^W| = 2^n$. This allows us to identify credence functions with vectors in $\mathbb{R}^{2^n}$. In particular, $\mathcal{C} = [0, 1]^{2^n}$. When W is finite, then, the analysis of credence functions is just the analysis of familiar Euclidean space. If W is infinite, we still have $\mathcal{C} = [0, 1]^{2^W}$, but now this space has infinite dimension, which complicates the analysis.

2.1.2 Admissible Inaccuracy Measures

Up to this point, I have been speaking of measures of *accuracy*. It is more common, however, to measure *inaccuracy* in the formal framework. Formally, an *inaccuracy measure* $\mathcal{J}$ is a function from $\mathcal{C} \times W$ into the extended half-line $[0, \infty]$,[15] that is $\mathcal{J} : \mathcal{C} \times W \to [0, \infty]$.[16] An inaccuracy measure represents how inaccurate a credence function is in a possible world. Given an inaccuracy measure $\mathcal{J}$, one can think of $-\mathcal{J}$ as an *accuracy* measure, and vice versa. So there is no harm in shifting the focus from accuracy to inaccuracy.

One inaccuracy measure that has received a lot of attention is the Brier score, which was used by de Finetti to establish the first accuracy-dominance theorems.

[13]The arguments in this paper can be adapted to credence functions defined on an algebra of subsets of W by introducing various measurability requirements. I have avoided this generalization in order to ease the exposition.

[14] This is *finite* additivity. It would certainly be interesting to investigate accuracy-based justifications for *countable* additivity, but that project is beyond the scope of this paper. See Kelley (2021) for some partial results.

[15]Since we are using extended-real numbers, it is worth reminding the reader that $\infty - \infty$ is undefined.

[16]In order to adapt the argument to an arbitrary algebra of subsets of W, one would need to require $\mathcal{J}$ to be measurable in a suitable algebra over $\mathcal{C} \times W$.

Brier score $\mathcal{I}_{Br}(c,w) = \sum_{A\subseteq W}(v_w(A) - c(A))^2$.

Notice that if W is finite, then the Brier score is *bounded* for each credence function c. That is, for all c, there is some real number r such that $\mathcal{I}_{Br}(c,w) \leq r$ for every possible world w. This is because the Brier score is *real-valued*; it never attains the value ∞. So, for a fixed c, we obtain the bound r by letting $r = \max_w \mathcal{I}_{Br}(c,w)$.

Other scoring rules are unbounded even when W is finite. For example:

Log score $\mathcal{I}_{log}(c,w) = -\log c(\{w\})$.

If $c(\{w\}) = 0$, then the log score takes the value ∞.

We now return to the question of how to characterize admissible inaccuracy measures. I will begin by following Pettigrew (2022), who makes two claims about admissible inaccuracy measures in finite domains. First, he claims that the currently *best* accuracy arguments for probabilism require only that measures of inaccuracy be *strictly proper* and *continuous*. These properties will be defined momentarily. Second, he claims that weaker characterizations of admissibility give rise to better accuracy arguments.[17] The second claim is meant to be just a logical point: the weaker an argument's premises, the stronger the argument. We will question these claims in section 3.2.

Let us look at the formal definitions of strict propriety and continuity. In order to define strict propriety, we need some notation for mathematical expectation with respect to a probability function.[18] If p is a probability function, and c is a credence function, let $\mathbb{E}_p(\mathcal{I}(c))$ be the expectation of the function $\mathcal{I}(c,\cdot)$ according to p. Strict propriety is then defined as follows.

Strict propriety For all $p \in \mathcal{P}$ and all $c \in \mathcal{C} \setminus \{p\}$, $\mathbb{E}_p(\mathcal{I}(p)) < \mathbb{E}_p(\mathcal{I}(c))$.

Intuitively, strict propriety requires that probabilistic credence functions are the unique minimizers of their own expected inaccuracy. In finite domains, the Brier score is strictly proper, but the log score is not.[19]

Continuity is defined as follows.

Continuity For all $p \in \mathcal{P}$, the function $\mathbb{E}_p(\mathcal{I}(\cdot))$ is continuous on $\mathcal{C}$.[20]

Both the Brier score and the log score are continuous in finite domains.

[17] Pettigrew (2022) begins his paper by arguing that the best accuracy-first argument for probabilism is based on the accuracy-dominance theorem of Predd et al. (2009). In addition to strict propriety and continuity, Predd et al. require accuracy measures to be *additive* (not to be confused with additivity for probability functions). But Pettigrew claims to improve the Predd et al. theorem, and hence the argument for probabilism that it supports, by dropping the additivity requirement.

[18] Section A of the appendix summarizes the theory of finitely additive expectation (integration) for extended-real functions that I use in this paper.

[19] The log score is strictly proper when it is restricted to probabilistic credence functions on finite domains but not when it is defined for all credence functions, as it is in this paper. For example, if c assigns credence 1 to every proposition, then it is non-probabilistic but has minimal expected inaccuracy.

[20] We equip $\mathcal{C}$ with the product topology, where $[0,1]$ has its standard topology. It follows that both $\mathcal{C}$ and $\mathcal{P}$ are compact. I provide some details about convergence in the product topology in Section B of the appendix. In finite domains, the definition of continuity is equivalent to the following: if $w \in W$, $c \in \mathcal{C}$, and (c_n) is a sequence of credence functions such that $\lim_{n\to\infty} c_n(A) = c(A)$ for every proposition A, then $\lim_{n\to\infty} \mathcal{I}(c_n,w) = \mathcal{I}(c,w)$.

The general accuracy-dominance theorem that I will prove weakens both strict propriety and continuity, and in that way is at least as good as the accuracy arguments that Pettigrew considers best.[21] On the other hand, my theorem relies on a third property that some may wish to relax. This issue will be discussed more in section 3.2.1.

Here are the weaker properties that replace strict propriety and continuity, respectively.

Quasi-Strict Propriety An inaccuracy measure $\mathcal{J}$ is *quasi-strictly proper* iff $\mathbb{E}_p(\mathcal{J}(p)) \leq \mathbb{E}_p(\mathcal{J}(c))$ for all $p \in \mathcal{P}$ and all $c \in \mathcal{C}$, with strict inequality if $c \in \mathcal{C} \setminus \mathcal{P}$.

Continuity on $\mathcal{P}$ For all $p \in \mathcal{P}$, the function $\mathbb{E}_p(\mathcal{J}(\cdot))$ is continuous on $\mathcal{P}$.

Unlike strict propriety, quasi-strict propriety does not require probability functions to be *unique* minimizers of their own expected inaccuracy. It allows for $\mathbb{E}_p(\mathcal{J}(p)) = \mathbb{E}_p(\mathcal{J}(p'))$ provided p' is probabilistic. A strict inequality is required only for non-probabilistic credence functions. Continuity on $\mathcal{P}$ differs from continuity by allowing the function $\mathbb{E}_p(\mathcal{J}(\cdot))$ to be discontinuous on non-probabilistic credence functions.

The third property that we will need is the following.

Local boundedness on $\mathcal{P}$ For all $p \in \mathcal{P}$, the function $\mathcal{J}(p, \cdot)$ is bounded on W.[22]

There are a few things to note about this new property. First, in finite domains, it is satisfied by several common inaccuracy measures. As pointed out above, the Brier score is one example. Second, local boundedness on $\mathcal{P}$ is much weaker than requiring $\mathcal{J}$ to be a bounded function. We can see this in two ways. First, local boundedness on $\mathcal{P}$ allows the inaccuracy measure $\mathcal{J}$ to be unbounded on $\mathcal{P} \times W$.[23] Second, it allows *non*-probabilistic credences to have infinite inaccuracy. For all it says, there could be a non-probabilistic credence c and world w such that $\mathcal{J}(c, w) = \infty$. So, as boundedness properties go, this one is not exceedingly strong.

We will say that an inaccuracy measure is *admissible* if it satisfies quasi-strict propriety, continuity on $\mathcal{P}$, and local boundedness on $\mathcal{P}$.[24]

[21] Here I follow, Nielsen (2021b), which also corrects some errors in Pettigrew's formal results.

[22] That is, $\sup_{w \in W} \mathcal{J}(p, w) < \infty$.

[23] That is, it allows $\sup_{p \in \mathcal{P}} \sup_{w \in W} \mathcal{J}(p, w) = \infty$. While for each probability function there is some bound on how inaccurate it can be, different probability functions can have different bounds, and there is no uniform bound for all of them.

[24] Do admissible inaccuracy measures exist in infinite domains? Yes. Here is a trivial example. For all $w \in W$, let $\mathcal{J}(p, w) = 0$ if $p \in \mathcal{P}$, and let $\mathcal{J}(c, w) = 1$ if $c \in \mathcal{C} \setminus \mathcal{P}$. Of course, one wonders what less trivial examples might look like. Constructing such examples turns out to be a difficult mathematical problem, which I have not been able to resolve completely. I thank Johannes Jahn and Daniel Daners for discussing this problem with me. I can show, however, that the notion of admissibility used in this paper avoids the most obvious triviality result. Let us say that an admissible inaccuracy measure $\mathcal{J}$ is *trivial* if either $\mathcal{J}$ is constant on $\mathcal{P} \times W$ or, for all $c \in \mathcal{C} \setminus \mathcal{P}$, every $p \in \mathcal{P}$ accuracy-dominates c (definition below). On this definition, non-trivial, admissible inaccuracy measures exist for arbitrary W. Here is an example that Alex Pruss shared with me. Fix $w_1, w_2 \in W$. For all $p \in \mathcal{P}$, define $\mathcal{J}$ by

$$\mathcal{J}(p, w) = \frac{1}{4} \sum_{A \subseteq \{w_1, w_2\}} (v_w(A) - p(A))^2.$$

It's clear that $\mathcal{J}$ is locally bounded on $\mathcal{P}$ by 1, and, because $\mathcal{J}$ is defined on $\mathcal{P}$ by a finite sum, it follows that $\mathcal{J}$ is continuous on $\mathcal{P}$. The reader can use the standard proof that the Brier score is proper to verify that $\mathcal{J}$

2.1.3 Accuracy-Dominance

There are several notions of accuracy-dominance in the literature. We will focus on what Predd et al. (2009) call *strong* accuracy-dominance.

Accuracy-dominance Let c and c' be credence functions. We say that c *accuracy-dominates* c' (according to $\mathcal{J}$) iff $\mathcal{J}(c, w) < \mathcal{J}(c', w)$ for all worlds w.

In other words, c accuracy-dominates c' if c is strictly less inaccurate than c' in every possible world.

As discussed in Section 1, accuracy arguments rely on a normative premise that connects accuracy-dominance with rationality. We will start with the following normative premise, taken from Pettigrew (2022).

Normative Premise for Probabilism If, according to every admissible measure of inaccuracy, a credence function is accuracy-dominated, then it is irrational.

We will consider objections to this premise in section 3.1.

2.2 A General Accuracy-Dominance Theorem

We have two of the three components that we need for an accuracy argument for probabilism. We have specified which inaccuracy measures are admissible, and we have a normative premise connecting accuracy-dominance and rationality. The final component of our argument is the following theorem.

Theorem 1. *Let $\mathcal{J}$ be an admissible inaccuracy measure, and let c be a non-probabilistic credence function. Then, there is a probabilistic credence function that accuracy-dominates c according $\mathcal{J}$.*

Proof. See section C in the appendix. □

If c is any non-probabilistic credence function, then, by Theorem 1 it is accuracy-dominated according to any admissible measure of inaccuracy. By the Normative Premise for Probabilism, c is irrational. So, probabilism follows: it is irrational for credences to violate the axioms of probability theory.

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is proper when restricted to $\mathcal{P}$. Moreover, since $\mathcal{J}$ behaves exactly like the (normalized) Brier score when it is restricted to probability functions supported by $\{w_1, w_2\}$, one can see that $\mathcal{J}$ is strictly proper when restricted to probability functions supported by $\{w_1, w_2\}$. It follows that $\mathcal{J}$ is not constant on $\mathcal{P} \times W$. Now, for every $c \in \mathcal{C} \setminus \mathcal{P}$, let p_c be a probability function supported by $\{w_1, w_2\}$ such that $p_c(\{w_1\}) \in (0, 1)$, and define

$$\mathcal{J}(c, w) = \begin{cases} \mathcal{J}(p_c, w_1), & \text{if } w = w_1 \\ 2, & \text{otherwise.} \end{cases}$$

By this definition, it is not the case that every $c \in \mathcal{C} \setminus \mathcal{P}$ is accuracy-dominated by every probability function. So $\mathcal{J}$ is non-trivial. It remains to observe that $\mathcal{J}$ is quasi-strictly proper. It suffices to show that $\mathbb{E}_p(\mathcal{J}(p)) < \mathbb{E}_p(\mathcal{J}(c))$ for all $c \in \mathcal{C} \setminus \mathcal{P}$ and $p \in \mathcal{P}$. The reader can verify this by considering three cases: (i) $p \neq p_c$ and $p(\{w_1\}) < 1$; (ii) $p \neq p_c$ and $p(\{w_1\}) = 1$; (iii) $p = p_c$. Hopefully, future work on this topic will focus more attention on the existence and construction of non-trivial admissible inaccuracy measures.

3 Discussion

This completes the presentation of the paper's first accuracy argument for probabilism. The first component of the argument is a premise characterizing the admissible inaccuracy measures as those satisfying quasi-strict propriety, continuity on $\mathcal{P}$, and local boundedness on $\mathcal{P}$. The second component is the Normative Premise for Probabilism, and the third component is Theorem 1.

Like de Finetti's Dutch book argument for probabilism, this accuracy argument is totally general. It applies to any credence functions whatsoever, not only to those with finite domains. In developing the first two premises, however, we have relied quite heavily on Pettigrew (2022), and it is certainly worth questioning some of the decisions we have made. We will begin by raising an objection to our normative premise. This will lead us to the paper's second accuracy argument for probabilism.

3.1 Normative Premises

One might worry that the Normative Premise for Probabilism is too strong. Recall that it says the following: If, according to every admissible measure of inaccuracy, a credence function is accuracy-dominated, then it is irrational. The idea behind this premise is that when a credence function is accuracy-dominated, it fails to promote accuracy, and failures to promote accuracy indicate failures of rationality. The worry that the Normative Premise for Probabilism is too strong arises by observing that while the argument from the previous section concludes that non-probabilistic credence functions are irrational, it leaves open the question whether probabilistic credence functions are irrational as well. That is, nothing we said in section 2 rules out the possibility that probabilistic credence functions, in addition to non-probabilistic ones, are accuracy-dominated and therefore irrational. If probabilistic credence functions turned out to be accuracy-dominated, this would provide a means for arguing that the Normative Premise for Probabilism is too strong. Arguably, a normative premise according to which all credence functions are irrational is heavy-handed.

Pettigrew (2016a, 2.1) suggests a way of pressing this objection directly with a concrete example. Let $\{c_x : x \in (0, \infty)\}$ be an enumeration of the credence functions defined on $W = \{w_1, w_2\}$, and suppose that the only admissible inaccuracy measure is defined by $\mathcal{J}(c_x, w) = x$ for every $x \in \mathbb{R}$ and $w \in W$.[25] Then, every credence function, including the probabilistic ones, is accuracy-dominated: if $x < y$, then c_x accuracy-dominates c_y. But, arguably, it would be incorrect to conclude that every credence function in this example is irrational. Since every credence function is accuracy-dominated anyway, the best any agent can do is adopt some c_x, with x "very small"—and perhaps this is enough to avoid charges of irrationality. But this line of thought tells against the Normative Premise for Probabilism, which forces us to say that every credence function in the example is irrational.

In response, one might complain about the artificiality of the example. No one thinks, for instance, that the $\mathcal{J}$ defined therein might be the only admissible inaccuracy measure. In response to this, however, one might insist that normative premises be evaluated independently of an accuracy argument's admissibility component: what's needed is a normative premise that's plausible regardless of which inaccuracy measures are counted as admissible.

[25]This is a variation on the Name Your Fortune example given by Pettigrew (2016a, p. 20).

Although I can't settle once and for all what the correct normative premise is, I think the concerns raised above are troubling enough to consider weakening the Normative Premise for Probabilism. A standard way of weakening the Normative Premise for Probabilism invokes

Weak accuracy-dominance Let c and c' be credence functions. We say that c *weakly accuracy-dominates* c' (according to $\mathcal{J}$) iff $\mathcal{J}(c,w) \leq \mathcal{J}(c',w)$ for all worlds w and $\mathcal{J}(c,w) < \mathcal{J}(c',w)$ for some world w.

We now consider the following normative premise.

Weak Normative Premise If, according to every admissible measure of inaccuracy, c is accuracy-dominated by some c', and c' is not weakly accuracy-dominated by any credence function, then c is irrational.[26]

Unlike the previous Normative Premise for Probabilism, the Weak Normative Premise does not entail that every credence function in the example above is irrational because, although every credence function c_x in the example is dominated by some c_y with $y < x$, the dominating credence function c_y is itself dominated by another c_z with $z < y$. The Weak Normative Premise also has some intuitive appeal. A credence function might not be irrational when it's accuracy-dominated since every credence function might be accuracy-dominated (as the example suggested). But if there are credence functions that aren't even weakly accuracy-dominated, and one of them accuracy-dominates c, then this provides better grounds for declaring c irrational.

If we simply replace the Normative Premise for Probabilism with the Weak Normative Premise, then we no longer have a valid argument for probabilism. Theorem 1 doesn't show enough for us to conclude that non-probabilistic credence functions are irrational. All it shows is that every non-probabilistic credence function is accuracy-dominated by some probabilistic credence function. In order to apply the Weak Normative Premise, we need to know that probabilistic credence functions aren't weakly accuracy-dominated. The next result establishes this.

Theorem 2. *Let $\mathcal{J}$ be an admissible inaccuracy measure, and let p be a probabilistic credence function. Then, there is no credence function that weakly accuracy-dominates p according $\mathcal{J}$.*

Proof. See section D in the appendix. □

With Theorem 2 in hand, we now have a valid argument for probabilism that uses the Weak Normative Premise. According to any admissible measure of inaccuracy, if c is a non-probabilistic credence function, then, by Theorem 1 it is accuracy-dominated by some probabilistic credence function. Probabilistic credence functions, in turn, are not even weakly accuracy-dominated (Theorem 2). By the Weak Normative Premise, c is irrational. That is, probabilism holds. This is the paper's second accuracy argument for probabilism.[27]

[26] This is similar to what Pettigrew (2016a) calls Undominated Dominance (p. 22).

[27] It is worth remarking that the admissibility hypothesis in Theorem 2 can be weakened. An inspection of the theorem's proof reveals that local boundedness on $\mathcal{P}$ is not used. So Theorem 2 continues to hold under the weaker assumption that $\mathcal{J}$ is quasi-strictly proper and continuous on $\mathcal{P}$.

3.2 Admissibility

Let us now revisit the first component of our accuracy arguments, which characterizes the admissible inaccuracy measures. We began by following Pettigrew (2022) in asserting that the best accuracy arguments require only strict propriety (or something weaker) and continuity (or something weaker). We have also made the additional assumption that admissible inaccuracy measures are locally bounded on $\mathcal{P}$. The notion of admissibility characterized by quasi-strict propriety, continuity on $\mathcal{P}$, and local boundedness on $\mathcal{P}$ can be criticized in a number of ways. My discussion of these criticisms will set continuity aside, as I have nothing to add to the detailed discussion in Pettigrew (2016a, 4.2). To my mind, quasi-strict propriety and local boundedness on $\mathcal{P}$ are the more interesting and controversial properties.

3.2.1 Local Boundedness on $\mathcal{P}$

The first objection I will consider is that requiring local boundedness on $\mathcal{P}$ makes our notion of admissibility too exclusive. In finite domains, local boundedness on $\mathcal{P}$ rules out inaccuracy measures that allow probabilistic credence functions to be infinitely inaccurate. As discussed in section 2.1.2, the log score is one such inaccuracy measure. Ruling it out might strike some as too restrictive. (But it should be noted that this particular version of the objection is not available to proponents of strict propriety because the log score is not strictly proper.)

I do not have a principled defense of local boundedness on $\mathcal{P}$. Frankly, it's a technical assumption that's needed in the proof of Theorem 1. As remarked at the end of the previous section, however, local boundedness on $\mathcal{P}$ is *not* needed in the proof of Theorem 2. This raises the question whether local boundedness on $\mathcal{P}$ is necessary for the conclusion of Theorem 1 to hold: Is there an inaccuracy measure $\mathcal{J}$ that is not locally bounded on $\mathcal{P}$ according to which every non-probabilistic credence function is accuracy-dominated by a probabilistic credence function? The answer to this question is affirmative, as a simple example will illustrate. Assume that $|W| \geq 2$, and let $w' \in W$. Let p' be some probabilistic credence function that assigns $\{w'\}$ probability 0, and define $\mathcal{J}$ by

$$\mathcal{J}(c,w) = \begin{cases} 1, \text{ if } c \in \mathcal{C} \setminus \mathcal{P}; \\ 0, \text{ if } c \in \mathcal{P} \setminus \{p'\}; \\ 0, \text{ if } c = p' \text{ and } w \neq w'; \\ \infty, \text{ if } c = p' \text{ and } w = w'. \end{cases}$$

It's clear that $\mathcal{J}$ is not locally bounded on $\mathcal{P}$ because $\mathcal{J}(p', w') = \infty$. On the other hand, every non-probabilistic c is accuracy-dominated by a probabilistic p (just choose any $p \neq p'$). It should also be noted that $\mathcal{J}$ is quasi-strictly proper. This observation notwithstanding, the example is of limited interest because $\mathcal{J}$ is not continuous on $\mathcal{P}$. In particular, we can see that the conclusion of Theorem 2 fails to hold: p' is weakly-accuracy dominated by every probabilistic $p \neq p'$.

Where does this leave us? Although local boundedness on $\mathcal{P}$ is not *implied* by the conclusion of Theorem 1, it is essential to its current proof. I leave it to future research to answer whether Theorem 1 can be generalized by relaxing local boundedness on $\mathcal{P}$.

3.2.2 (Quasi-)Strict Propriety

We now turn away from boundedness and toward propriety. Once again our discussion begins with an objection: allowing inaccuracy measures to be merely quasi-strictly proper makes our notion of admissibility too *inclusive*. By a *merely* quasi-strictly proper inaccuracy measure, I mean an inaccuracy measure that is quasi-strictly proper but *not* strictly proper. The objection, then, claims that quasi-strictly proper inaccuracy measures are not admissible unless they are also strictly proper.

The purported problem with merely quasi-strictly proper inaccuracy measures is that they exhibit an unjustified bias in favor of probabilistic credence functions that renders them unsuitable for use in an argument seeking to justify probabilism. Merely quasi-strictly proper inaccuracy measures have too much probabilism "baked in." If $\mathscr{J}$ is merely quasi-strictly proper, then every probabilistic p expects itself to be less inaccurate than any non-probabilistic c, i.e. $\mathbb{E}_p(\mathscr{J}(p)) < \mathbb{E}_p(\mathscr{J}(c))$; but p does *not* expect itself to be less inaccurate than every other probabilistic p', i.e. there is some $p' \in \mathcal{P}$ such that $\mathbb{E}_p(\mathscr{J}(p)) = \mathbb{E}_p(\mathscr{J}(p'))$. Put more simply, the inaccuracy measure $\mathscr{J}$ treats probabilistic and non-probabilistic credence functions differently: in terms of expected inaccuracy, probabilistic credence functions can be equals, but non-probabilistic credence functions are always inferiors. It is this differential treatment that the objection targets, alleging that it represents an unacceptable bias in favor probabilism.

I should be forthright and admit that I do not have a decisive argument for quasi-strict propriety. But I do have several points to make in response to the objection just raised. The first is dialectical: the aim of this paper is not to defend quasi-strict propriety. Rather, my main aim, which I think has been accomplished with the results above, has been to generalize the best extant accuracy arguments for probabilism to infinite domains. Those arguments assume strict propriety, and anyone who objects to merely quasi-strictly proper inaccuracy measures will have to admit that, all else equal, an accuracy argument that weakens the assumption of strict propriety is stronger, in a strictly logical sense, than an accuracy argument that relies on the full strength of strict propriety. Concerns about unacceptably biased quasi-strictly inaccuracy measures, then, do little to undermine what I have set out to achieve. Still, one can grant all of this this while at the same time having good reason to favor a more exclusive notion of admissibility, one that requires strict propriety.

It turns out, however, that this position is untenable due to the remarkable fact that strictly proper inaccuracy measures *do not exist* in infinite domains. This can be shown with a fairly straightforward counting argument. If W is infinite, then the cardinality of the set of probabilistic credence functions is greater than the cardinality of the set of functions from W into $[0, \infty]$.[28] But this means that for every inaccuracy measure $\mathscr{J}$ there are two probabilistic credence functions, p and p', that have identical inaccuracy scores in the sense that $\mathscr{J}(p, w) = \mathscr{J}(p', w)$ for all $w \in W$. And this means that $\mathscr{J}$ cannot be strictly proper. It will fail to satisfy $\mathbb{E}_p(\mathscr{J}(p)) < \mathbb{E}_p(\mathscr{J}(p'))$, for instance. So, in infinite domains, insisting on strict propriety is a non-starter.[29]

[28]See http://alexanderpruss.blogspot.com/2021/03/scoring-rules-for-finitely-additive.html.

[29]Strictly speaking, accuracy arguments remain valid when the set of admissible inaccuracy measures is empty. In particular, Theorems 1 and 2 remain true—but trivially so. The problem here is not with the arguments' validity but with the fact that proponents of accuracy arguments are committed to the claim that inaccuracy can be measured *somehow*.

But why insist on strict propriety in the first place? There is a well known way of arguing for strict propriety that appeals to a kind of rational immodesty.[30] A rational agent, the argument goes, expects his own credence function to minimize inaccuracy. That's because if he expected another credence function to be less inaccurate than his, then he would be compelled to adopt that other credence function instead, which would indicate that his actual credence function wasn't rationally held in the first place.

In response to this, it has been argued that even if we accept this line of thought, it doesn't support *strict* propriety.[31] According to the immodesty argument, rational credences must minimize their own expected inaccuracy, but they needn't do so *uniquely*. Rational agents aren't compelled to adopt credence functions that they expect to be *as* inaccurate as their own. So far, then, the immodesty argument supports only

Weak propriety For all $p \in \mathcal{P}$ and all $c \in \mathcal{C}$, $\mathbb{E}_p(\mathcal{J}(p)) \leq \mathbb{E}_p(\mathcal{J}(c))$.

But weak propriety, even in the presence of continuity and boundedness, is not enough to secure an accuracy-dominance theorem. For example, *constant* inaccuracy measures are weakly proper, continuous, and uniformly bounded.[32] But if $\mathcal{J}$ is constant, then no credence functions are (even weakly) accuracy-dominated.

In a recent paper, Campbell-Moore and Levinstein (2021) try to salvage the immodesty argument by showing that weak propriety, together with some additional assumptions, *implies* strict propriety. We need not concern ourselves with the details of their argument, but I emphasize the familiar theme: Campbell-Moore and Levinstein's argument assumes a finite domain. It's far from clear how their argument might extend to infinite domains, so it's far from clear that rational immodesty supports strict propriety in general.[33]

Finally, there is a more direct, albeit speculative, line of argument worth considering that attempts to defend the bias in favor of probabilistic credence functions that quasi-strictly inaccuracy measures exhibit. The idea is that this bias may well be acceptable to those who are already inclined to endorse probabilism. The accuracy arguments presented above, the idea goes, showcase a kind of virtuous circularity by which friends of probabilism can assure themselves that the principle promotes accuracy.[34] On this view, accuracy arguments will do little on their own to convert agnostics about probabilism, but, for those with some faith in *other* arguments for probabilism, the accuracy arguments may provide additional support.

It's not clear to me that this defense of quasi-strict propriety is viable. What seems clearer, however, is that it's not compatible with the accuracy-first program. If defending quasi-strict propriety requires arguments for probabilism *other* than the accuracy arguments (for example, pragmatic arguments), then it seems likely that the total case for probabilism—that is, the virtuously circular version of the accuracy arguments—will end up relying on epistemic goods besides accuracy. In particular, it will rely on those epistemic goods invoked by whatever

[30]See Joyce (2009) and Campbell-Moore and Levinstein (2021).

[31]See Pettigrew (2009) and Mayo-Wilson and Wheeler (2016).

[32]An inaccuracy measure $\mathcal{J}$ is constant if there is some $x \in [0, \infty]$ such that $\mathcal{J}(c, w) = x$ for all $c \in \mathcal{C}$ and $w \in W$.

[33]Moreover, since strictly proper inaccuracy measures do not exist in infinite domains, a straightforward generalization of Campbell-Moore and Levinstein's argument for strict propriety would only show that the assumptions of the argument cannot be jointly satisfied in general. I thank an anonymous referee for pointing this out.

[34]I thank an anonymous referee for suggesting this idea to me.

auxiliary argument for probabilism is used to defend quasi-strict propriety. So, if the virtuous circle offers a promising defense of quasi-strict propriety, it might turn out, surprisingly, that the accuracy arguments are most useful to those who reject the accuracy-first program.

So far, we have been considering the objection that our notion of admissibility is too inclusive, that it admits inaccuracy measures with an unacceptable bias in favor of probabilism. Before concluding, it's natural to consider what happens if we relax (quasi-)strict propriety. A manuscript by Mikayla Kelley (2021) contains some intriguing mathematical results, but I don't think they can support a general accuracy argument for probabilism. Kelley studies inaccuracy measures that are determined by "generalized quasi-additive Bregman divergences" and "Bregman distances." These inaccuracy measures exhibit dramatic failures of quasi-strict propriety. For instance, Kelley gives examples (4.1 and 4.2) of probabilistic credence functions p such that $\mathscr{J}(p, w) = \infty$ for every world w, where $\mathscr{J}$ is a generalization of the Brier score.[35] Using these radically improper inaccuracy measures, Kelley is able to obtain some fairly general accuracy-dominance theorems.

But her most general results (in section 5 of her paper)—those most comparable to the results in this paper—don't support probabilism in the traditional sense. I will explain this at a high level as the details are quite technical. Kelley's inaccuracy measures are "quasi-additive" in the sense that, in finite domains, they are *sums* of local scoring rules. The standard example here is the Brier score. In infinite domains, these sums must be replaced by *integrals*; and in order to integrate, one must have a reference measure with respect to which the integration is defined. Let us call this reference measure μ, and let us say that a proposition holds μ *almost surely* if the set of worlds at which the proposition is false has measure 0 according to μ. Now, Kelley is able to show that every non-probabilistic credence function c is accuracy-dominated by a credence function c' such that c' is probabilistic μ almost surely.[36] This result does not guarantee that c' is probabilistic, rather it guarantees only that c' is non-probabilistic with μ measure 0. But, in general, μ will assign measure 0 to non-empty sets of worlds, so, in general, c' will not be probabilistic.[37]

This might not be a problem for someone who endorses the Normative Premise for Probabilism. For such a person to infer that non-probabilistic credence functions are irrational, all that's needed is to show that they are accuracy-dominated by *something*, and Kelley's result shows this. But for someone who favors the Weak Normative Premise, Kelley's results don't lead to probabilism. And regardless of which normative premises one supports, it's just not clear that probabilism μ almost surely is an interesting and well-motivated principle.[38] For

[35] It follows that $\mathbb{E}_p(\mathscr{J}(p)) = \infty$, which is incompatible with quasi-strict propriety.

[36] What I am glossing as a "credence function" here is actually what Kelley calls a "measurable credence function," which is a function that takes not just propositions as inputs but also worlds. In addition to the problems pointed out in the main text, then, one might also object that Kelley's framework does not study credence functions in the traditional sense.

[37] Kelley admits as much: "In some measure spaces, like the weighted counting measure spaces underlying generalized legitimate inaccuracy measures, we lose nothing since every credence function is measurable and only the empty set is measure zero. However, in other cases, these assumptions are substantive" (p. 16). Moreover, the result mentioned above (Theorem 5.3), assumes that μ is finite, which means it cannot be the counting measure over an infinite domain.

[38] As Kelley concludes: "Further, while the measure theoretic framework introduced in Section 5 to score inaccuracy of credence functions over opinion sets of arbitrary cardinality seems like a natural extension of the finite and countably infinite frameworks, is it well motivated that inaccuracy does not track the behavior of a credence function on measure zero sets?" (p. 17).

those who would relax (quasi-)strict propriety, it seems like there is still a lot of work to be done in extending the accuracy arguments for probabilism to infinite domains.

Conclusion

I have presented two accuracy arguments for probabilism. They are perfectly general in that they apply to credence functions with arbitrary domains. The arguments help to close the significant gap between accuracy arguments and their pragmatic counterparts. And they might be taken as providing additional inductive support to the accuracy-first program. Whether they do depends, of course, on whether the arguments' premises are defensible. We have seen that characterizing admissibility in infinite domains is especially complicated. On the one hand, strictly proper inaccuracy measures simply don't exist. On the other, allowing inaccuracy measures to be merely quasi-strictly proper might introduce an objectionable bias in favor of probabilism. Relaxing propriety constraints altogether leads to unknown territory. Perhaps a framework like Kelley's can vindicate probabilism in the end, but this remains to be seen. It also remains to be seen whether the arguments can be generalized away from all boundedness assumptions. So, there is plenty of work to do—as always—but the picture is becoming clearer.

Appendix

A Finitely Additive Expectation of Extended-Real Functions

I begin this appendix by summarizing the theory of finitely additive expectation for extended-real functions that is used below. Let us start by defining some notation. Let $\mathscr{B}$ be the set of all bounded real functions on W. Let $\mathscr{F}^+$ be the set of non-negative extended-real functions on W. If f and g are two extended real functions on W, let $f \wedge g$ be the function defined by $(f \wedge g)(w) = \min\{f(w), g(w)\}$, and let $f \vee g$ be the function defined by $(f \vee g)(w) = \max\{f(w), g(w)\}$. If $A \subseteq W$, let 1_A be the indicator function for A.

Let $p \in \mathcal{P}$ be given. The theory of expectation used in this paper starts with the standard theory of expectation over $\mathscr{B}$ (Aliprantis and Border, 2006, 11.2). For $f \in \mathscr{F}^+$, define

$$\mathbb{E}_p(f) = \sup\big\{\mathbb{E}_p(h) : h \in \mathscr{B},\ h \leq f\big\}. \tag{2}$$

Now, for a general extended-real function f on W, let $f^+ = f \vee 0$ and $f^- = (-f) \vee 0$. Then, $f^+, f^- \in \mathscr{F}^+$ and $f = f^+ - f^-$. If at least one of $\mathbb{E}_p(f^+)$ or $\mathbb{E}_p(f^-)$ is finite, then we say that $\mathbb{E}_p(f)$ *exists* and let

$$\mathbb{E}_p(f) = \mathbb{E}_p(f^+) - \mathbb{E}_p(f^-). \tag{3}$$

It is now straightforward to verify the following proposition, following Lemma 9 (p. 257) in Schervish et al. (2020), for example.

Proposition 1. *Let f, g be extended-real functions on W and $p, q \in \mathcal{P}$. Let $c \in \mathbb{R}$, and let $\lambda \in [0, 1]$. If $f + g$ is well defined and $\mathbb{E}_p(f)$, $\mathbb{E}_p(g)$, $\mathbb{E}_p(f + g)$, and $\mathbb{E}_q(f)$ exist, then*

1. *$f \leq g$ implies $\mathbb{E}_p(f) \leq \mathbb{E}_p(g)$*
2. *$\mathbb{E}_p(f + g) = \mathbb{E}_p(f) + \mathbb{E}_p(g)$,*
3. *$\mathbb{E}_p[cf] = c\mathbb{E}_p(f)$,*
4. *$\mathbb{E}_{\lambda p+(1-\lambda)q}(f) = \lambda\mathbb{E}_p(f) + (1 - \lambda)\mathbb{E}_q(f)$.*

I will appeal to Proposition 1 freely throughout the rest of the appendix.

B Convergent Nets

Some of the results below rely on the notion of a convergent net in $\mathcal{P}$. I'll review the main concepts here.

A *directed set* is a set D equipped with a binary relation $\succeq$ that is reflexive, transitive, and such that for every $\alpha, \beta \in D$ there exists $\gamma \in D$ for which $\gamma \succeq \alpha$ and $\gamma \succeq \beta$. A net (x_α) in a set X is a function from a directed set into X. Every sequence is a net; in this case, $D = \mathbb{N}$ with the usual ordering. If X is a topological space, and (x_α) is a net in X, then we say that (x_α) *converges* to a point $x \in X$ and write $x_\alpha \to x$ iff for every neighborhood V of x there is some β such that $x_\alpha \in V$ for all $\alpha \succeq \beta$. A basic result about convergent nets is that a subset C of a topological space is closed iff $x \in C$ whenever (x_α) is a net in C such that $x_\alpha \to x$.[39]

[39] For more details, see Aliprantis and Border (2006, 2.4). Since $\mathcal{C}$ is equipped with the product topology (footnote 20), a net (p_α) in $\mathcal{P}$ converges to $p \in \mathcal{P}$ iff $p_\alpha(A) \to p(A)$ for all $A \subseteq W$.

The following lemma illustrates these ideas and will be needed later.

Lemma 1. *Let $h \in \mathscr{B}$, and let (p_α) be a net in $\mathcal{P}$ that converges to $p \in \mathcal{P}$. Then,*

$$\lim_\alpha \mathbb{E}_{p_\alpha}(h) = \mathbb{E}_p(h).$$

Proof. The result is immediate if h is a step function because in that case the expectation of h is given by a finite sum. For general $h \in \mathscr{B}$, let $\epsilon > 0$, and use the fact that step functions are dense in $\mathscr{B}$ under the supremum norm topology to find a step function g such that $\sup_w |g(w) - h(w)| \le \epsilon/4$. Then, for all α,

$$\left|\mathbb{E}_{p_\alpha}(h) - \mathbb{E}_p(h)\right| \le 2 \sup_w |g(w) - h(w)| + \left|\mathbb{E}_{p_\alpha}(g) - \mathbb{E}_p(g)\right|. \tag{4}$$

But, because g is a step function, there is some β such that for all $\alpha \succeq \beta$, $|\mathbb{E}_{p_\alpha}(g) - \mathbb{E}_p(g)| \le \epsilon/2$. Thus, by (4), $|\mathbb{E}_{p_\alpha}(h) - \mathbb{E}_p(h)| < \epsilon$ for all $\alpha \succeq \beta$, which proves the result. □

C Proof of Theorem 1

For all $c \in \mathcal{C}$, let the function $\mathcal{J}(c) : W \to [0, \infty]$ be defined by $\mathcal{J}(c)(w) = \mathcal{J}(c, w)$ for all $w \in W$. We note that $\mathbb{E}_p(\mathcal{J}(c))$ exists, according to the definition in section A, because $\mathcal{J}(c) \in \mathscr{F}^+$.

If $\mathcal{J}$ is a quasi-strictly proper inaccuracy measure, let the *divergence* $\mathcal{D}_\mathcal{J} : \mathcal{C} \times \mathcal{P} \to [0, \infty]$ associated with $\mathcal{J}$ be defined by $\mathcal{D}_\mathcal{J}(c, p) = \mathbb{E}_p(\mathcal{J}(c)) - \mathbb{E}_p(\mathcal{J}(p))$ for all $c \in \mathcal{C}$ and $p \in \mathcal{P}$. Quasi-strict propriety implies that $\mathbb{E}_p(\mathcal{J}(p)) < \infty$ for all $p \in \mathcal{P}$, so $\mathcal{D}_\mathcal{J}$ is well defined.

Definition 1. Let X be a topological space. A function $f : X \to [-\infty, \infty]$ is *lower semicontinuous (lsc)* on X iff $\{x \in X : f(x) \le r\}$ is closed for every $r \in \mathbb{R}$.

Lemma 2. *Let $\mathcal{J}$ be any inaccuracy measure, and let $c \in \mathcal{C}$. The function $\mathbb{E}_{(\cdot)}(\mathcal{J}(c))$ is lsc on $\mathcal{P}$.*

Proof. Let $r \in \mathbb{R}$ be given and let $F_r = \{q \in \mathcal{P} : \mathbb{E}_q(\mathcal{J}(c)) \le r\}$. Let (p_α) be a net in F_r that converges to $p \in \mathcal{P}$. Let $h \in \mathscr{B}$ and $h \le \mathcal{J}(c)$. Then, for all α,

$$r \ge \mathbb{E}_{p_\alpha}(\mathcal{J}(c)) \ge \mathbb{E}_{p_\alpha}(h),$$

which, by Lemma 1, implies

$$r \ge \lim_\alpha \mathbb{E}_{p_\alpha}(h) = \mathbb{E}_p(h). \tag{5}$$

Taking a supremum over h on the right-hand side of (5) and using (2) we get

$$r \ge \mathbb{E}_p(\mathcal{J}(c)).$$

Thus, $p \in F_r$, and the lemma is proved. □

Lemma 3. *Let $\mathcal{J}$ be quasi-strictly proper and locally bounded on $\mathcal{P}$. Let $c \in \mathcal{C}$. Then the function $\mathcal{D}_\mathcal{J}(c, \cdot)$ is lsc on $\mathcal{P}$.*

Proof. Let $r \in \mathbb{R}$ be given and let $F_r = \{q \in \mathcal{P} : \mathcal{D}_{\mathcal{J}}(c, q) \leq r\}$. Let (p_α) be a net in F_r that converges to $p \in \mathcal{P}$. Then, for all α, by quasi-strict propriety,

$$r \geq \mathcal{D}_{\mathcal{J}}(c, p_\alpha) = \mathbb{E}_{p_\alpha}(\mathcal{J}(c)) - \mathbb{E}_{p_\alpha}(\mathcal{J}(p_\alpha)) \geq \mathbb{E}_{p_\alpha}(\mathcal{J}(c)) - \mathbb{E}_{p_\alpha}(\mathcal{J}(p)). \tag{6}$$

Now let $\epsilon > 0$. By Lemma 1, there is some β such that $\mathbb{E}_{p_\alpha}(\mathcal{J}(p)) - \mathbb{E}_p(\mathcal{J}(p)) \leq \epsilon$ for all $\alpha \succeq \beta$. So if $\alpha \succeq \beta$, then (6) implies

$$r + \mathbb{E}_p(\mathcal{J}(p)) + \epsilon \geq \mathbb{E}_{p_\alpha}(\mathcal{J}(c)). \tag{7}$$

But the net $(p_\alpha)_{\alpha \succeq \beta}$ also converges to p, so (7) and Lemma 2 imply

$$r + \mathbb{E}_p(\mathcal{J}(p)) + \epsilon \geq \mathbb{E}_p(\mathcal{J}(c)).$$

As ϵ is arbitrary, this implies $r \geq \mathbb{E}_p(\mathcal{J}(c)) - \mathbb{E}_p(\mathcal{J}(p)) = \mathcal{D}_{\mathcal{J}}(c, p)$. That is, $p \in F_r$, and the lemma is proved. □

Lemma 4. *Let X be a compact topological space, and let $f : X \to [-\infty, \infty]$ be a lsc function. Then, f attains a minimum on X.*

Proof. Let $A = f(X)$. If $-\infty \in A$, then the result is immediate, so assume this is not the case. For all $r \in A$, let $F_r = \{x \in X : f(x) \leq r\}$. If $r \in \mathbb{R}$, then F_r is closed by lower semicontinuity, and if $r = \infty$, then $F_r = X$ is closed as well. If $\{r_1, ..., r_n\} \subseteq A$, then $\bigcap_{i=1}^n F_{r_i} = F_{\min_i r_i}$ is non-empty. Since X is compact, this implies that $\bigcap_{r \in A} F_r$ is non-empty. But f attains a minimum at any point in $\bigcap_{r \in A} F_r$.[40] □

Theorem 1. *Let $\mathcal{J}$ be an admissible inaccuracy measure, and let c be a non-probabilistic credence function. Then, there is a probabilistic credence function that accuracy-dominates c.*

Proof. If $\mathcal{J}(c, w) = \infty$ for all $w \in W$, then the result is immediate: by local boundedness on $\mathcal{P}$, any $p \in \mathcal{P}$ accuracy-dominates c. So assume that $\mathcal{J}(c, w) < \infty$ for some $w \in W$. Using Lemmas 3 and 4, let $p^* \in \mathcal{P}$ minimize $\mathcal{D}_{\mathcal{J}}(c, \cdot)$ on $\mathcal{P}$. Since there is some $w \in W$ such that $\mathcal{J}(c, w) < \infty$, we have

$$\mathcal{D}_{\mathcal{J}}(c, p^*) \leq \mathcal{D}_{\mathcal{J}}(c, v_w) = \mathcal{J}(c, w) - \mathcal{J}(v_w, w) < \infty.$$

This, in turn, implies that $\mathbb{E}_{p^*}(\mathcal{J}(c)) < \infty$.

Now let $w \in W$ be arbitrary. If $n \in \mathbb{N}$, let $p_n = n^{-1} v_w + (1 - n^{-1}) p^*$. Note that because $\mathbb{E}_{p_n}(\mathcal{J}(p_n)) < \infty$ and

$$\mathbb{E}_{p_n}(\mathcal{J}(p_n)) = n^{-1} \mathcal{J}(p_n, w) + (1 - n^{-1}) \mathbb{E}_{p^*}(\mathcal{J}(p_n)),$$

we must have $\mathbb{E}_{p^*}(\mathcal{J}(p_n)) < \infty$ for all n. Using the fact that p^* is a minimizer of $\mathcal{D}_{\mathcal{J}}(c, \cdot)$, we have, for all n,

$$\begin{aligned}
0 &\leq n\big(\mathcal{D}_{\mathcal{J}}(c, p_n) - \mathcal{D}_{\mathcal{J}}(c, p^*)\big) \\
&= n\big(\mathbb{E}_{p_n}(\mathcal{J}(c)) - \mathbb{E}_{p_n}(\mathcal{J}(p_n)) - \mathbb{E}_{p^*}(\mathcal{J}(c)) + \mathbb{E}_{p^*}(\mathcal{J}(p^*))\big) \\
&= n\big(n^{-1}\mathcal{J}(c, w) + (1 - n^{-1})\mathbb{E}_{p^*}(\mathcal{J}(c)) - n^{-1}\mathcal{J}(p_n, w) - (1 - n^{-1})\mathbb{E}_{p^*}(\mathcal{J}(p_n)) - \mathbb{E}_{p^*}(\mathcal{J}(c)) + \mathbb{E}_{p^*}(\mathcal{J}(p^*))\big) \\
&= \big(\mathcal{J}(c, w) - \mathbb{E}_{p^*}(\mathcal{J}(c)) - \mathcal{J}(p_n, w) + \mathbb{E}_{p^*}(\mathcal{J}(p_n))\big) + n\big(\mathbb{E}_{p^*}(\mathcal{J}(p^*)) - \mathbb{E}_{p^*}(\mathcal{J}(p_n))\big) \\
&\leq \mathcal{J}(c, w) - \mathbb{E}_{p^*}(\mathcal{J}(c)) - \mathcal{J}(p_n, w) + \mathbb{E}_{p^*}(\mathcal{J}(p_n)).
\end{aligned} \tag{8}$$

[40] For completeness, I have reproduced the proof of this result from Nielsen (2021b).

where the final inequality uses quasi-strict propriety. Next, (8) rearranges to

$$\mathcal{J}(p_n, w) \leq \mathcal{J}(c, w) - \mathbb{E}_{p^*}(\mathcal{J}(c)) + \mathbb{E}_{p^*}(\mathcal{J}(p_n)). \tag{9}$$

Since $p_n \to p^*$, continuity on $\mathcal{P}$ and (9) imply

$$\mathcal{J}(p^*, w) \leq \mathcal{J}(c, w) - \mathcal{D}_{\mathcal{J}}(c, p^*). \tag{10}$$

Quasi-strict propriety implies $\mathcal{D}_{\mathcal{J}}(c, p^*) > 0$ because $c \in \mathcal{C} \setminus \mathcal{P}$. So from (10) we have $\mathcal{J}(p^*, w) < \mathcal{J}(c, w)$ if $\mathcal{J}(c, w) < \infty$. And if $\mathcal{J}(c, w) = \infty$, then local boundedness on $\mathcal{P}$ implies $\mathcal{J}(p^*, w) < \mathcal{J}(c, w)$. Thus, p^* accuracy-dominates c. □

D Proof of Theorem 2

Theorem 2. *Let $\mathcal{J}$ be an admissible inaccuracy measure, and let p be a probabilistic credence function. Then, there is no credence function that weakly accuracy-dominates p according $\mathcal{J}$.*

Proof. Suppose for contradiction that c is a credence function that weakly accuracy-dominates p. It follows from weak accuracy-dominance that $\mathbb{E}_p(\mathcal{J}(c)) \leq \mathbb{E}_p(\mathcal{J}(p))$, and the reverse inequality holds by quasi-strict propriety. Thus,

$$\mathbb{E}_p(\mathcal{J}(c)) = \mathbb{E}_p(\mathcal{J}(p)). \tag{11}$$

Another consequence of weak accuracy-dominance is that there is some $w \in W$ such that

$$\mathcal{J}(c, w) < \mathcal{J}(p, w). \tag{12}$$

Now, if $n \in \mathbb{N}$, let $p_n = n^{-1}v_w + (1 - n^{-1})p$. Using (11) and quasi-strict propriety, we have, for all n,

$$\begin{aligned} n^{-1}\mathcal{J}(c, w) + (1 - n^{-1})\mathbb{E}_p(\mathcal{J}(p)) &= n^{-1}\mathcal{J}(c, w) + (1 - n^{-1})\mathbb{E}_p(\mathcal{J}(c)) \\ &= \mathbb{E}_{p_n}(\mathcal{J}(c)) \\ &\geq \mathbb{E}_{p_n}(\mathcal{J}(p_n)) \\ &= n^{-1}\mathcal{J}(p_n, w) + (1 - n^{-1})\mathbb{E}_p(\mathcal{J}(p_n)) \\ &\geq n^{-1}\mathcal{J}(p_n, w) + (1 - n^{-1})\mathbb{E}_p(\mathcal{J}(p)). \end{aligned} \tag{13}$$

Since quasi-strict propriety implies that $\mathbb{E}_p(\mathcal{J}(p)) < \infty$, from (13) we deduce that

$$\mathcal{J}(c, w) \geq \mathcal{J}(p_n, w) \tag{14}$$

holds for all n. Since $p_n \to p$ and $\mathcal{J}$ is continuous on $\mathcal{P}$, from (14) we deduce

$$\mathcal{J}(c, w) \geq \mathcal{J}(p, w). \tag{15}$$

But (15) contradicts (12).[41] □

[41] This proof is a modification of some results that Alex Pruss shared with me.

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