Lógica modal para relaciones entre conjuntos


## Resumen

En este artículo, presentamos un sistema de lógica modal que permite representar relaciones entre conjuntos o clases de individuos definidos por una propiedad específica. Introducimos dos operadores modales, [a] y <a>, que se utilizan respectivamente para expresar "para todo A" y "existe un A". Tanto la sintaxis como la semántica del sistema tienen dos niveles que evitan el anidamiento del operador modal. La semántica se basa en una variante de la semántica de Kripke, en donde los operadores modales se indexan sobre fórmulas de lógica proposicional ("pre-fórmulas" en el trabajo). Además, presentamos un conjunto de axiomas y reglas que rigen el sistema, y demostramos que el sistema es correcto y completo con relación a los modelos de Kripke.

En la sección final del artículo, discutimos posibles trabajos futuros. Consideramos la posibilidad de combinar nuestro operador modal con otras modalidades, como necesidad o conocimiento. Además, como ejemplo de la utilidad de nuestro operador modal, analizamos brevemente la fórmula de Barcan adaptada de manera conveniente dentro del marco de nuestro sistema. En resumen, proponemos la combinación de nuestro operador modal con otros como una forma más simple y compacta, aunque con un menor poder expresivo, para abordar la lógica modal cuantificada.



**Palabras clave:** lógica modal cuantificada, silogismos, fórmula Barcan, consistencia, completud


Modal Logic for Relationships between Sets


## Abstract

In this article, we present a modal logic system that allows representing relationships between sets or classes of individuals defined by a specific property. We introduce two modal operators, [a] and <a>, which are used respectively to express "for all A" and "there exists an A". Both the syntax and semantics of the system have two levels that avoid the nesting of the modal operator. The semantics is based on a variant of Kripke semantics, where the modal operators are indexed over propositional logic formulas ("pre-formulas" in the paper). Furthermore, we present a set of axioms and rules that govern the system and we prove that the logic is correct and complete with respect to Kripke models.

In the final section of the article, we discuss potential future work. We consider the possibility of combining our operator with other modalities, such as necessity or knowledge. Additionally, as an example of the utility of our modal operator, we briefly analyze a conveniently adapted Barcan formula within the framework of our system. In summary, we propose combining our modal operator with other ones as a simpler, more compact, albeit less expressive way to address quantified modal logic.



**Keywords:** quantified modal logic, syllogisms, Barcan formula, soundness, completeness

## 1. Introduction

There are many ways of expressing relations between classes of beings in a formal language or, more precisely, relations of inclusion and intersection between sets. The most common way is the use of first-order logic, which allows expressing the inclusion of a certain set into another one or the existence of some individuals belonging to one set that also belong to another one. This is done by using universal and existential quantification. We might be interested just in simpler formulas expressed in a more concise notation. For example, [a]b and <a>b for "all A's are B" and "some A's are B", in which A and B are classes defined by some property, simple or complex (not being A, being A and B, being A and D or E, and so on). We might also want to express more elaborated assertions that relate these "syllogisms", such as "If all B's are C or D, then there is some D or C that is B".

In this work, we propose a new approach, by using a modal operator, which is indexed over the subject of these kinds of "syllogisms" of the type we have described before, allowing for a more intuitive and expressive representation of relations between classes of individuals. This indexing of modal operators to the subject brings about a natural way of reasoning about classes or sets defined by properties or predicates, capturing a sense of modality that is simpler than standard first-order logic and avoids some of its limitations, given that it only deals with sets, not with individuals.

The reader might be curious about the pertinence of this proposal. The notation may be simpler and more compact, but it appears to be just a shortened way of expressing formulas of first-order logic, leaving aside the modal nature of the operator. [a]b, for example, is slightly more readable than $\forall x\ (Ax \rightarrow Bx)$, but nothing more. The real reason for the usefulness of this proposal is that this modal operator allows very easy integration with other modal operators (alethic, doxastic, epistemic…), thus enabling the modeling of complex modalities with quantification in a unified modal framework. In other words, it is a way of dealing with quantified modal logic just by using different combinations of modal operators, instead of combining modal operators and quantifiers of first-order logic. The system is less expressive than quantified modal logic because we will not be able to deal with n-ary relations for n greater than 1, just with unary properties, but this simplicity is the basis of the convenience of the language: we will deal just with propositional modal logic all the time. This potential for combining the modal operators presents interesting opportunities for interdisciplinary research and opens new avenues for investigating modal reasoning in diverse fields, including the representation of necessity, knowledge, and belief. This approach is also expandable to more refined kinds of quantification, for example considering proportions ("60% of A's are also B"), yielding complex expressions that can be expressed concisely and rigorously. This work will mainly be devoted to introducing the modal operator, showing its syntax, semantics, and the completeness and soundness of a simple axiomatic system in a variation of the usual Kripke semantics. In the last part of the paper, we outline the possibilities of the combination of this modal class operator with other modal operators, and we make a very brief analysis of the

Barcan formula in our language. Being aware of the difficulties of the topic, we just sketch it, opening the door to future works in this direction.

A last remark about the name of this modal logic we are introducing. Since there are works in the field of modal set theory, mainly the work of Linnebo (2010; 2013), more recently with Hamkins (Hamkins and Linnebo 2022), we have opted to call our system "modal class logic" to avoid confusion. The name may seem a bit vague but calling it "modal set logic" would be misleading. Our scope and goals are more limited than the work of Linnebo, and we do not aim to develop a full system for modal set theory.

## 2. Syntax and Semantics

The language that is going to be introduced here, which we will call $L_C$, has two layers: pre-formulas and formulas. Although both follow the usual syntax for propositional logic, they differ in their meaning. Pre-formulas do not represent propositions, but rather classes or sets of entities defined by some property. Thus, the negation of a pre-formula represents the complement of the corresponding set, and the conjunction, the intersection of two sets defined for some properties.

*Definition 1. (Syntax of the pre-formulas.)* We will denote by p any element of a non-empty set of literals, *Lit*. We define the set of pre-formulas recursively as follows:

$$\pi ::= p \mid \neg\pi \mid (\pi\wedge\pi)$$

The definition of disjunction, tautology, and contradiction in terms of ¬ and ∧ is the usual one. We also follow the standard convention for the suppression of parenthesis.

*Definition 2. (Atomic modal formulas.)* Given two pre-formulas, α and β, then a formula of the type $<\alpha>\beta$ is said to be an atomic modal formula, where α is called the index of the operator.

*Definition 3. (Duality of the modal operators.)* We define $[\alpha]\beta$, the dual of the previous modal operator, as follows:

$$[\alpha]\beta \leftrightarrow \neg<\alpha>\neg\beta \quad \text{(Dual)}$$

These modal formulas will function in the same way that atomic letters do in propositional logic because they will be the basic elements of the formulas of our language. They mean "some A's are B" and "all A's are B", respectively. From these basic modal formulas, we can define the set of wff's of $L_C$.

*Definition 4. (Syntax of formulas.)* If α is an atomic modal formula, we define recursively the set of wff's of $L_C$ as follows:

$$\varphi ::= \alpha \mid \neg\varphi \mid (\varphi\wedge\varphi)$$

This system of two levels avoids nesting modal operators. For example, we can express "all dogs are mammals, and some dogs are white" as $[d]m \wedge <d>w$, or "if all dogs are mammals and all mammals are animals, then all dogs are animals" as $[d]m \wedge [m]a \rightarrow [d]a$. We cannot mix in the same level pre-formulas and formulas: "$d \wedge [d]a$", for example, would be a badly formed formula, with a meaningless translation into natural language as "dogs, and all dogs are animals".

The semantics will reflect this pattern of two levels over a standard Kripke model: instead of a single W, we will have two non-empty sets, $W_P$ and $W_M$. We define the following:

*Definition 5. (Kripke model for modal class logic.)* The model M for $L_C$ is a tuple $\langle W_P, W_M, R, v \rangle$, where $W_P$ and $W_M$ are non-empty sets, R is a subset of $W_M \times W_P$, and *v* is a valuation function from literals to subsets of $W_P$.

The semantics of pre-formulas is the standard semantics of propositional formulas. The semantics of the modal formulas of $L_C$ is a variation of the standard one for modal operators but restricting the scope of the accessible worlds. To do that, we must adjust the semantics for reflecting this. The idea of restricting the model has been widely used in logic. (Plaza 1989) introduced it for public announcements, and this has been successfully used in dynamic epistemic logic. The paper has been republished in (Plaza 2007).

The exact way of restricting the model may vary depending on the purpose of the restriction: update of available information or other change of the state, constraint to a narrower situation, etc. Often, a model requires restricting W, R, and v: $W_{|\varphi}$ is the subset of W of worlds that satisfy a certain formula φ, and $R_{|\varphi}$ and $v_{|\varphi}$ are defined similarly, yielding the model $M_{|\varphi} = \langle W_{|\varphi}, R_{|\varphi}, v_{|\varphi} \rangle$, as Plaza proposed. Works that deal with public announcements in dynamic epistemic logic, for example (van Ditmarsch 2003) or (van Ditmarsch, van der Hoek and Kooi 2007), follow this path.

Our work just restricts the accessibility relationship R, because we just want to limit the scope of the modal operators to worlds in $W_P$ that satisfy a certain pre-formula φ. Instead of stating "all accessible worlds satisfy q", we want to express "all accessible worlds that satisfy p, also satisfy q", that is, "all p's are q". We do this by applying the corresponding restriction to R.

*Definition 6. (Restriction of R.)* We define $R_{|\varphi}$ as follows:

$$R_{|\varphi} = \{(w, w') \text{ in } W_M \times W_P \mid M, w' \models \varphi\}$$

$R_{|\varphi}$ is thus the subset of R in which the second projection of R satisfies φ. We could similarly restrict $W_P$ and v to define a restricted model, but it is not necessary, and $W_M$ has to remain unchanged since it does not satisfy any pre-formula. Thus, we have opted for the minimal change, which only affects R. Now we define the following:

*Definition 7. (Restricted model.)* Given a pre-formula φ and a model $M = \langle W_P, W_M, R, v \rangle$, the restricted model $M_{|\varphi}$ is the tuple $\langle W_M, W_P, R_{|\varphi}, v \rangle$, where $R_{|\varphi}$ is as described above.

In our language, restricting R to consider just the accessible worlds that satisfy the index makes the latter similar to the subject of a classical syllogism. Wff's of the type $[a]b$ or $<a>b$ could be seen indeed as a kind of complex syllogism, but interpreted on a Kripke frame: if w satisfies "all A's are B" or "some A's are B", this means that we obtain respectively a universal or particular quantification, after restricting the scope to A. Thus, although $[a]b$ and $<\alpha>\beta$ respectively are equivalent to $\Box(a\rightarrow b)$ or $\Diamond (a\wedge b)$, they aim to express different ideas. For example, stating that w satisfies $<\alpha>\beta$ wants to express a slightly different thing from $\Diamond (a\wedge b)$: the first wff says that some accessible world satisfying a, also satisfies b, not merely that some accessible world satisfies the conjunction of a and b. In other words, $<a>b$ expresses that, restricting us to the accessible worlds that satisfy a, some of them satisfy b. The key point is the idea of restricting the scope of the accessibility relationships, which is marked by the index of the modal operator.

The semantics of the language works in two steps. In $W_{p,}$ we have a set of worlds that satisfy the wff's of propositional logic. In $W_M$, a set of worlds that satisfy modal formulas and their composition by using logical connectives.

*Definition 8. (Satisfaction in $W_P$.)* Semantics in $W_p$ is the standard Kripkean semantics for propositional logic. For any w in $W_P$, we define:

- Base case: $M, w \Vdash p$ if and only if $w\in v(p)$.
- Conjunction: $M, w \Vdash \varphi\wedge \psi$ if and only if $M, w \Vdash \varphi$ and $M, w \Vdash \psi$.
- Negation: $M, w \Vdash \neg\varphi$ if and only if it is not the case that $M,w \Vdash \varphi$

*Definition 9. (Satisfaction in $W_M$.)* For any world w in $W_M$, we define the following, where $\varphi$ and $\psi$ are pre-formulas, and $\alpha$ and $\beta$ are formulas:

- Base case: $M, w \Vdash <\varphi> \psi$ if and only if there is some $w'\in W_P$ such that $wR_{|\varphi}w'$ and $M_{|\varphi}, w' \Vdash \psi$.
- Conjunction: $M, w \Vdash \alpha \wedge \beta$ if and only if $M, w \Vdash \alpha$ and $M, w \Vdash \beta$.
- Negation: $M, w \Vdash \neg \alpha$ if and only if it is not the case that $M, w \Vdash \alpha$

A note for the curious reader: Strictly speaking, $W_M$ and $W_P$ could be the same set or intersect. They do not necessarily have to be disjoint but, since we cannot mix in a single wff pre-formulas and formulas, we can consider them as disjoint for all practical purposes.

## 3. Axiomatic System

*Definition 10. (Axiomatic system of $L_C$.)* Again, we have to consider that $L_C$ has two layers. Pre-formulas are propositional wff's, so it has as axioms all propositional tautologies, and it is closed under the rule of substitution and *modus ponens*. The layer of wff's of $L_C$ has these two familiar axiom schemes for every pre-formulas $\varphi$ and $\psi$:

- Axiom K-c: $[\varphi]\, \psi \wedge [\, \varphi]\, \psi \leftrightarrow [\, \varphi]\, (\psi\wedge\gamma)$

- Dual-c: $[\varphi]\, \psi \leftrightarrow \neg\langle\varphi\rangle\neg\, \psi$

Regarding the rules, the first one is a variation of the rule of generalization or necessitation. The second one is the novelty of this system:

- Taut-c: If $\psi$ is a tautology in propositional logic, then for all pre-formulas $\varphi$, $[\varphi]\, \psi$ is a tautology in $L_C$.
- Transf: If $(\varphi \wedge \psi) \rightarrow (\varphi' \wedge \psi')$ in propositional logic, then $\langle\varphi\rangle\, \psi \rightarrow \langle\varphi'\rangle\, \psi'$ in $L_C$.

Also, $L_C$ is closed under the rule of substitution and *modus ponens,* and it contains all tautologies in $L_C$. The modal class operator is a family of operators which are indexed over the set of all pre-formulas. Thus, axioms K-c and Dual-c are the usual axioms K and Dual, and the same for the rule of generalization for indexed modalities. Sometimes axiom K-c can be restated in equivalent forms, such as $\langle\varphi\rangle\, \psi \vee \langle\varphi\rangle\, \gamma \leftrightarrow \langle\varphi\rangle(\psi \vee \gamma)$.

Axioms K-c and Taut-C are defined just for formulas with the same index. The rule Transf is the basis of several theorems that allow combining wff's with different indexes and transforming a wff with an index into another one with a different index, given certain conditions. Let us see some theorems that can be obtained by the application of Transf (the proofs are very easy and most of them are almost immediate):

1. Swap: $\langle\varphi\rangle\psi \leftrightarrow \langle\psi\rangle\, \varphi$.
2. Translat: $\langle\varphi\rangle(\psi\wedge\gamma) \leftrightarrow \langle\varphi\wedge\psi\rangle\gamma$. Note that Swap is a particular case of this.
3. CombU-E: $[\varphi]\psi \wedge \langle\varphi\rangle\, \gamma \rightarrow \langle\varphi\rangle(\psi\wedge\gamma)$
4. Contrp: $[\varphi]\psi \leftrightarrow [\neg\psi]\, \neg\varphi$
5. IntCIndex and ElimCIndex: $[\varphi]\gamma \rightarrow [\varphi\wedge\psi]\, \gamma$, equivalent to $\langle\varphi\wedge\psi\rangle\, \gamma \rightarrow \langle\varphi\rangle\, \gamma$.
6. IntDIndex and ElimDIndex: $\langle\varphi\rangle\, \gamma \rightarrow \langle\varphi\vee\psi\rangle\, \gamma$, equivalent to $[\varphi\vee\psi]\, \gamma \rightarrow [\varphi]\, \gamma$.

Thus, Transf is the keystone of the modal system because it states how to operate with modal formulas whose indexes are different.

## 4. Soundness and Completeness

*Definition 11. (Modal logic.)* A logic is a set of wff's that contains all tautologies of the language and is closed under the rules of *modus ponens* and uniform substitution. A modal logic is a logic in which we have modal operators in the syntax of the language and axioms and rules for them.

The language $L_C$ has two layers, the first one corresponding to pre-formulas is just standard propositional logic. In this layer, all non-modal propositional tautologies belong to the set of pre-formulas. The set of wff's in the second layer is formed by modal wff's as basic elements. All modal formulas that are tautologies, such as [a]a, belong to the set of wff's of $L_C$, and propositional tautologies of the type [a]b v ¬[a]b, that is, tautologies akin to propositional tautologies in which the basic elements are modal formulas. This set is also closed under *modus ponens*, the rule of necessitation, and has a set of axioms that we have seen yet.

*Definition 12. (Normal modal logic.)* A system of modal logic is normal if, for every square operator $\Box_i$, the system contains axioms Dual, K and is closed under the rule of necessitation:

$$\Box_i (\varphi\wedge\psi) \leftrightarrow \Box_i \varphi\wedge\Box_i \psi \quad K_i$$

$$\Box_i \varphi \leftrightarrow \neg\Diamond\neg\varphi \quad Dual_i$$

If φ is a tautology in propositional logic, $\Box_i \varphi$ is a tautology      Rule of necessitation

We will make an abuse of the notation and we will call our logic system $L_C$, like the language. As long the context is clear, it should not be problematic.

*Proposition 13.* $L_C$ is a normal logic.

*Proof.* For every modal operator [α], axioms K and Dual are valid in $L_C$. Also, $L_C$ is closed under the rule of necessitation for every operator [a].

We might think that the rule Transf might need a special kind of Kripke frame. We are going to prove a surprising feature of this rule.

*Proposition 14.* Rule Transf is sound in any Kripke model.

*Proof.* Let us consider a Kripke model. Let us assume that $\alpha\wedge\beta\rightarrow \alpha'\wedge\beta'$ is a tautology and that M, $w \Vdash \langle\alpha\rangle \beta$. This means that there is some w' such that $wR_{|\alpha}w'$, and w' satisfies β, and therefore it satisfies $\alpha\wedge\beta$. Given the initial assumption, w' must satisfy $\alpha'\wedge\beta'$ too, so $M_{|\alpha'}, w' \Vdash \beta'$ and therefore M, $w \Vdash \langle a'\rangle \beta'$.

Transf does not add any extra feature to a Kripke model, although it only appears in a multimodal system like ours, in which modal operators are indexed over a set of wff's of propositional logic. Thus, we do not need to verify any additional properties of the model. The following work showing the soundness and completeness of $L_C$ on a Kripke model is a variation of the standard proof, adapted to the peculiarities of $L_C$.

*Definition 15. (Strong soundness.)* A logic is strongly sound on a Kripke model M if, for any set of wff's Γ and for all φ, if $M, \Gamma \vdash \varphi$ then $M, \Gamma \Vdash \varphi$.

*Definition 16. (Strong completeness.)* A logic is strongly complete on a Kripke model M if, for any set of wff's Γ and any wff φ, if M, Γ ╟ φ, then M, Γ ├ φ.

### 4.1 Soundness

The proof of the soundness of the rule Transf in a Kripke model has been proven a few paragraphs above. We just must study axiom K-c. Axiom scheme Dual-c is sound by definition.

*Theorem 17.* Axiom K-c is sound.

*Proof:* The proof for K-c is a slight variation of the proof for the standard axiom K: If some w∈W satisfies [α] φ and [α]ψ then, restricting the accessibility relationship only to those worlds in $W_P$ that satisfy α, all accessible worlds from w satisfy φ and ψ, and thus satisfy φ∧ψ, and therefore M, w╟ [α](φ∧ψ). The converse of the implication is similar, so we get as a result that M, w ╟ [α](φ∧ψ) iff M, w╟ [α] φ and M, w╟ [α]ψ.

*Corollary 18.* $L_C$ is strongly sound on a Kripke model.

*Proof.* It follows from the previous propositions.

### 4.2 Completeness

The adaptation of the construction of a Henkin canonical model to our modal is as follows: We will consider two canonical sets of sets instead of just one, one for pre-formulas, and the other one for formulas. The canonical accessibility relationship is from one to the other. An interesting alternative exercise would be performing this task via filtration, as in (Chagrov 1997). We begin with some basic definitions.

*Definition 19. (Deducibility.)* Given a set of formulas Γ we say that a formula ψ is deducible from Γ, stated as Γ ├ ψ, if ψ is a tautology or there is a subset of Γ, {$\varphi_1$∧...∧ $\varphi_n$}, such that it implies ψ.

*Definition 20. (Consistent and maximally consistent set.)* A set of wff's Γ is said to be consistent if it does not contain the contradiction. It is said *maximally consistent* set if it is consistent and all possible supersets of it are not consistent. In other words, if we add any new formula to it, the new set is inconsistent.

*Definition 21. (Strong completeness in Kripke frames.)* A logic L is strongly complete with respect to the class of Kripke models if every L-consistent set of wff's is satisfiable in some Kripke frame. This is equivalent to definition number 16, but from a different approach.

The following lemma is crucial for the proof of completeness, we just state it, since it is standard.

*Proposition 22. (Lindenbaum lemma.)* If Σ is an L-consistent set of wff's, there is a is L-maximally consistent Σ' such that $\Sigma \subseteq \Sigma'$.

The proof can be found in (Chellas 1980) or (Blackburn, de Rijke, and Venema 2001) for example. Its relevance is that, given a set Σ, we can always find a maximally consistent set Σ' containing it. This Σ' will allow us to create a canonical model for our system to prove its completeness. The basic idea of the proof of soundness is that we say that a certain formula is deducible from a set Σ of wff's iff it is contained in every maximal superset of it. Thus, being Σ 'any maximally consistent superset of Σ

$$\Sigma \vdash \varphi \text{ iff } \in\varphi \in \Sigma', \text{ for all } \Sigma' \text{ such that } \Sigma \subseteq \Sigma'$$

As we have two layers in the language, we will have two sets of maximally consistent sets: a set of MCS's for pre-formulas, and another set of MCS's for formulas. There are several notations for MCS's. Some of them use uppercase Greek letters for MCS's, such as Γ and Δ. Other authors prefer to denote them by w, w', and so on. We will follow this last notation but having in mind that these "worlds" are MCS's, which is the keystone of the proof.

*Definition 23. (Canonical model for modal class logic.)* The canonical model for our logic $L_C$ is the tuple $\langle W_M, W_P, R(g), v \rangle$, where:

- $W_P$ is the set of all maximally consistent sets of pre-formulas.

- $W_M$ is the set of all $L_C$-maximally consistent sets of wff's.

- R(φ) is called the canonical relationship, defined as a binary relationship in $W_M x W_P$ in this way: For $w \in W_M$ and $w' \in W_P$, wR(φ)w' iff for all φ, ψ, if φ∈w' and ψ∈w', then $< \varphi > \psi \in w$. As we can see, it is the expected canonical relationship, but restricting the model to the worlds that satisfy φ. The standard R can be seen as R(T), where T is the tautology.

- v is the canonical valuation.

Now we need the following lemmas, whose proofs are standard. The idea is simple, we need to assure that there is indeed an accessible world.

*Lemma 24.* For any $w \in W_M$ and any $w' \in W_P$, wR(φ)w' iff $[\varphi]\psi \in w$.

The following lemma assures us that there is indeed the required accessible world.

*Lemma 25. (Existence lemma.)* For any world $w \in W_M$, if $<\varphi>\psi \in w$, then there is some $w' \in W_P$ such that wR(φ)w' and ψ∈w'. In other words, if $<\varphi> \psi$ in w, then there is some $w' \in W_P$ such that wR(φ)w' and ψ ∈w'.

Now, we can define recursively satisfaction of wff's over MCS's:

*Proposition 26. (Truth lemma.)* Given a canonical model M, for each pre-formula or wff $\varphi$, we have that $M, w \Vdash \varphi$ iff $\varphi \in w$, being w in $W_P$ or $W_M$ depending on whether $\varphi$ is a pre-formula or a formula, respectively.

We begin with pre-formulas, by induction over connectives as usual, for any $w' \in W_P$:

- Base case: If p is an atomic letter and $w' \in W_P$, $M, w' \models p$ if and only if $p \in w'$, or equivalently, if $w' \in v(p)$.

- Negation: $M, w' \models \neg \varphi$ if and only if it is false that $M, w' \models \varphi$, that is, if $\varphi \notin w'$.

- Conjunction: $M, w' \models \varphi \wedge \psi$ if and only if $M, w' \models \varphi$ and $M, w' \models \psi$, that is, if $\varphi \in w'$ and $\psi \in w'$.

For wff's in $W_M$, we define the following for any $w \in W_M$:

- $M, w \Vdash \langle\varphi\rangle \psi$ iff there is some w' in $W_P$ such that $wR(\varphi)w'$ and $\psi \in w'$.

Negation and conjunction are defined in the same way that in pre-formulas. Consequently, we have the following:

*Corollary 27. (Canonical model theorem.)* The modal logic $L_C$ is strongly complete with respect to its canonical model.

*Proof:* The proof is just a slight variation of the standard one for normal logics and Kripke models. The canonical model we have defined is a slight variation of a Kripke model, and $L_C$ is a variation of normal logic. Its novelty is just being a multimodal logic indexed over the set of pre-formulas. Regarding the rule Transf, we saw that it is sound in any Kripke model, now we are going to see that we need it in our system to be complete. Let us suppose, as usual, that that $\alpha\wedge\beta \rightarrow \alpha'\wedge\beta'$. If a certain MCS w contains a wff $\langle\alpha\rangle\beta$, then it must contain $\langle\alpha'\rangle\beta'$ too. Otherwise, either w is not consistent ($\neg\langle\alpha'\rangle\beta'$ is in w) or it is not maximal (we still can add either $\neg\langle\alpha'\rangle\beta'$ or $\langle\alpha'\rangle\beta'$).

## 5. Discussion: Combination with other Modal Operators

We have introduced a modal logical system that, in its simplest version, expresses relations of inclusion or intersection between sets and has the same expressive power as a little fragment of first-order logic. If we leave the work here, it does not add anything but a more compact notation, as we said in the introduction: $\Box\, [p]q$ is slightly more readable than $\Box\, \forall x\, (Px \rightarrow Qx)$. The real reason for the development of this modal operator is that it is very easily combinable with other modal operators, so we can express formulas about the necessity, knowledge, change, and other modalities of these relations between sets without using modal first-order logic. As we are going to see soon, this is a different approach to quantified modal logic. Instead of using the combination of quantifiers and modal operators, we just use modal operators, because we deal with quantification with the modal class operator we have defined.

We could even contemplate a single world in $W_M$, thus making $L_C$ just another way of expressing the relationship between subsets of a given domain, $W_P$. More precisely, expressing a relationship between a set of sets that is isomorphic to the subsets of a given domain, whereas the domain and $W_P$ may not be isomorphic. What makes it interesting is that it opens the possibilities for multiple combinations. We will not discuss the possible combination of the operator with itself, since its meaning is not clear at all.

### 5.1. Modal Operators in the Pre-formulas and the Formulas

Let us assume a non-empty $W_M$. Its worlds could represent different states, moments, spatial scopes, mental states (beliefs, knowledge of agents), and hypotheses, among others. Combining the class modal operator [a]b with other modal operators can lead to a more expressive system that is easily implementable by adding alethic, epistemic, or any kind of modal operator in general. We will add an alethic operator in the following paragraphs. We can add it to pre-formulas, formulas, or both levels.

*5.1.1. Modal operators in the formulas.* We define an extended modal language that incorporates an alethic operator in the layer of formulas. It will range over the formulas, not over pre-formulas.

*Definition 28. (Syntax of the extended formulas.)* Wff's expressing the composition of modal formulas are defined as follows, where α is an atomic modal formula:

$$\varphi ::= \alpha \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi$$

So far, we had a model in which we only had accessibility relationships from worlds in $W_M$ to worlds in $W_P$. Now, we have defined a standard modality whose accessibility relationship is in $W_M$ x $W_M$. To avoid confusion, since R is being used for the accessibility $W_M$ x $W_P$, we will denote by $R_{\Box M}$ the accessibility of the alethic operator.

*Definition 29. (Semantics of the extended formulas.)* If φ is a formula, we add the following definition to the semantics of formulas, for a w in $W_M$:

- M, w $\models \Box\, \pi$ iff all $w_i \in W_M$ such that $wR_{\Box M}w_i$ hold that M, $w_i \models \pi$.

As we expect, axioms for the □ operator would be Dual, K, and T. The same situation will arise when studying modal operators in the pre-formulas, so we will not repeat this comment.

*Example.* "Necessarily all dogs are animals" can be adequately represented as □[d]a. If a certain world w in $W_M$ satisfies □[d]a, all w' in $W_M$ that are accessible from w hold that M, w' $\Vdash$ [d]a, including w itself, given that the alethic operator follows the reflexivity axiom.

The possibilities of combining in a more complex formula necessity, knowledge, belief, and other modalities are multiple. We leave this to the reader's imagination. Just a remark: Some operators are non-normal, and therefore Kripke semantics is not useful for them. We would require different semantics, for example, neighbourhood semantics, for the structure of $W_M$.

*5.1.2. Modal operators in the pre-formulas*. The accessibility relationship is now in $W_P$ x $W_P$, so here we have the interesting fact that pre-formulas can be modal ones. It is not difficult to state [a]□b as "All A's are necessarily B's" (or necessary B's). Other modalities would require fewer natural ways of expressing their meaning. For example, the combination with an epistemic operator in the pre-formula can produce formulas such as [a]$K_x$b, which has to be read as the tortuous expression "All A's are known by X as being B", or a similar one that expresses that, for every individual belonging to A, X knows that it belongs to B.

*Definition 30. (Syntax of the extended pre-formulas.)* Given a non-empty set *Lit*, we define the set of pre-formulas recursively as follows:

$$\pi ::= \ p \ \ | \neg\pi \ | \ (\pi \wedge \pi) \ | \ \Box\pi$$

We add the corresponding accessibility relationships in $W_P$x$W_P$, we will denote it by $R_{\Box P}$.

*Definition 31. (Semantics of the extended pre-formulas.)* If π is a pre-formula, we add the following definition to the semantics of pre-formulas, for a w in $W_P$:

- M, w $\models \Box$ π iff all w'∈$W_P$ such that w $R_{\Box P}$w' hold that M, w' $\models$ π.

*Remark. (Two notes about the structure of $W_M$ and $W_P$.)* We have not talked too much about the internal structure of $W_M$ or $W_P$. This is because there are many possibilities and levels of complexity. Here we will outline some basic lines of work. The simplest form, in which $W_M$ contains a single world, represents an actualist vision of the world: There is only a possible state of affairs, so we could even omit the reference to this unique world if necessary. If we develop a system in which we introduce alethic operators, it will be necessary to have several worlds and the corresponding accessibility relations between them. The same for epistemic operators, possibly a family of accessibilities for a set of agents. The most complex situation is that in which we mix different modalities: necessity/possibility, knowledge, etc. More options include dynamic logic and spatial or temporal modal logics which would allow expressing change or narrowing or widening of scope, spatial or temporal.

The structure of $W_P$ can be also complex if we allow modalities in pre-formulas, but there is a more relevant issue that must be clarified. Worlds do not represent individuals. If a certain world in $W_M$ satisfies "all dogs are animals", this does not mean that each accessible world in $W_P$ represents a certain dog existing in the world or state $W_M$. We only want that the worlds in $W_M$ reflect the relationship between the sets of beings in these worlds, not the cardinality of these sets. The accessibility relationship of the modal class operator from $W_M$ to $W_P$ is structured in a way that the whole model is bisimilar to another one in which each world in $W_M$ has access to a set of worlds with the corresponding number of entities satisfying each property among these that are specified by the pre-formulas of the language.

*5.1.3. Modalities in formulas and pre-formulas*. We may have modal operators both in the pre-formulas and the formulas. They do not interfere among themselves, for we have a set of modal operators in $W_M$ x $W_M$, other ones defined in $W_P$x$W_P$, and the class modal operator, the only one defined in $W_M$ x $W_P$. Inevitably, this leads us to quantified modal logic at last. Our modal system is a way to interpret quantified wff's in modal terms and, if we combine it with other kinds of modalities, it is evident that we are using a variation of quantified modal logic in which quantification is made by using a modal operator. We will just sketch the topic in the next subsection, which will be the object of more detailed research in future works.

## 5.2. Modal Class Logic and the Barcan Formula: a Simple Analysis

In this subsection we will just sketch a brief introduction to this topic, considering QML from the perspective of a particular kind of combination of modal operators, the modal class one we are studying here, and other modalities. QML endeavors to merge first-order logic and modal logic, a task that entails numerous complexities and issues beyond the scope of this work. There is an extensive body of literature on the subject. We recommend (Janssen-Lauret 2022) for an introduction to the topic. Our approach offers an advantage in terms of reduced complexity and offers a different perspective: the modal class operator does not explicitly quantify individuals but rather concerns itself with the inclusion of relationships between sets of individuals. We will see that this offers an interesting approach to Barcan formulas in terms of relations between sets, rather than in terms of quantification over individuals.

Let us recall the Barcan schema, which in its universal formulation is stated as $\forall x \; \Box \; \varphi(x) \rightarrow \Box \forall x \; \varphi(x)$. Since $L_C$ uses propositional logic and not first-order logic, we can adapt it to $[d]\Box \; \varphi \rightarrow \Box[d] \; \varphi$, for a given domain of individuals D. We are in a situation in which we have a necessity operator both in the formulas and the pre-formulas, as well as the modal class operator. If we add $[\varphi]\Box \; \varphi \rightarrow \Box[\; \varphi] \; \psi$ as an axiom scheme to the axioms of $L_C$, we will obtain a variation of what Linsky and Zalta (1994) call “the simplest quantified modal logic”, a variation whose peculiarity is the absence of quantifiers, which are substituted by the modal class operators. We may also assume the validity of the axiom of reflexivity for the modal operator, both in formulas and pre-formulas. It is not very important in what follows, though.

Let us assume a bounded domain, vampires, and state this sentence: “all vampires are necessarily immortal” (i.e., for vampires, the property of being immortal is necessary, and there is no possibility of not being immortal), which we write in $L_C$ as $[v] \; \Box i$. The Barcan formula is stated in this situation as $[v]\Box i \rightarrow \Box[v]i$. If we accept it, the conclusion is $\Box[v]i$, “necessarily all vampires are immortal”, there is no possible situation in which vampires could not be immortal. What is interesting now is to explore the semantics of this entailment, and its relationship to actualism. If some w in $W_M$ satisfies $[v]\Box i$, this means that all its accessible worlds w’ in $W_P$ that satisfy v also satisfy $\Box i$, and hence all worlds in $W_P$ that are accessible from these w’ satisfy i, including w’, because necessity is reflexive. Now, we see the semantic interpretation of the Barcan formula in $L_C$: we can move the alethic accessibility relationship from $W_P$ to $W_M$. If w satisfies $[v]\Box i$, then all worlds in $W_M$ that are accessible from w via the alethic operator also satisfy [v]i, w itself, and therefore w satisfies □[v]i. The quantification, which is made via the modal class operator, is not changed. The converse Barcan formula would do the opposite movement, as expected, from $W_M$ to $W_P$: We can move the alethic accessibility relationship from $W_M$ to $W_P$.

Regarding the relationship of this issue to actualism, we must recall that worlds in $W_M$ represent possible states of affairs. Having understood the previous paragraph, the connection is almost immediate. We cannot talk about individuals, for we are in a propositional language, but $L_C$ allows us to talk about sets of individuals and relationships among these sets. The Barcan formula, thus, means this in set-theoretical terms: if a set is a subset of the set of beings that necessarily have a certain property, then necessarily the first set is a subset of the set of beings that have that property. Not a classical formulation but, instead of talking about the impossibility of contingent individual objects (Hayaki 2006), the Barcan formula as we have stated in $L_C$ affirms the impossibility of contingent sets of objects. Let us conclude the discussion here, as it deserves a more thorough examination in future works. In this context, we have merely utilized it to exemplify certain semantic considerations about the combination of the modal class operator and the alethic operator.

## 6. Concluding Remarks and Future Work

This paper has introduced a modal logic that allows to express relationships between sets, a way of introducing quantification in a modal way. Although much work remains to be done, we have identified several promising avenues for future research. The most obvious direction is to further investigate the combination of this modal operator with other operators, such as alethic, epistemic, doxastic, temporal, and other kinds. Each kind of operator has its own peculiarities which would have to be carefully studied in its combination with the modal class operator. Additionally, we suggest exploring the quantification of proportions and probabilities using a metric over the accessibility relationship. This line of research may follow the path started with the works of Larsen and Skou (1991) and Fagin and Halpern (1988), which studied modal logic for probabilities. A very similar scheme can be used for proportions ("60% of A's are B"), which although conceptually different from probability, may be modelised using almost the same approach. These are not difficult tasks, and they would increase the expressive power of this operator greatly.

A last area of further research that deserves more attention might involve the exploration of various relationships between worlds or states. These worlds can encompass imaginary scenarios, broader or more restrictive possibilities, changes over time or space, or hypotheses. The use of modal operators in pre-formulas and formulas has only been briefly outlined, and we encourage research into their precise meaning. This is of special interest in the case of extensions of $L_C$ with different kinds of modal operators in both pre-formulas and formulas. This line of inquiry could lead to further exploration of related topics, including the discussion on Barcan formulas that we have just sketched in the last part of this work, and its ontological and epistemological consequences.

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