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Non-classical, *bounded* Fechnerian integration for loudness: contrary to Luce and Edwards, initial loudness-difference-size stipulations are only recouped for linear loudness growth

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ABSTRACT

A major question in sensory science is how a sensation of magnitude F (such as loudness) depends upon a sensory stimulus of physical intensity I (such as a sound-pressure-wave of a particular root-mean-square sound-pressure-level). An empirical just-noticeable sensation difference $(\Delta F)_j$ at F_j specifies a just-noticeable intensity difference $(\Delta I)_j$ at I_j. Intensity differences accumulate from a stimulus-detection threshold I_{th} up to a desired intensity I. Likewise, the corresponding sensation differences are classically presumed to accumulate, accumulating up to $F(I)$ from $F(I_{th})$, a non-zero sensation (as suggested by hearing studies) at I_{th}. Consequently, sensation growth $F(I)$ can be obtained through Fechnerian integration. Therein, empirically-based relations for the Weber Fraction, $\Delta I/I$, are individually combined with either Fechner's Law $\Delta F = B$ or Ekman's Law $(\Delta F/F) = g$; the number of cumulated steps in I is equated to the number of cumulated steps in F, and an infinite series ensues, whose higher-order terms are ignored. Likewise classically ignored are the integration bounds I_{th} and $F(I_{th})$. Here, we deny orthodoxy by including those bounds, allowing hypothetical sensation-growth equations for which the differential-relations $\Delta F(I) = F(I+\Delta I) - F(I)$ or $(\Delta F(I)/F(I)) = \big(F(I+\Delta I) - F(I)\big)/F(I)$ do indeed return either B or g, for *linear* growth of sensation F with intensity I. Also, 24 sensation-growth equations $F(I)$, which had already been derived by the author likewise using bounded Fechnerian integration (12 equations for the Weber Fraction $(\Delta I/I)$, each combined with either Fechner's Law or with Ekman's Law), are scrutinized for whether their differential-relations return either B or g respectively, particularly in the limits $(\Delta I/I) \ll 1$ and the even-more-extreme limit $(\Delta I/I) \to 0$, both of which seem unexplored in the literature. Finally, some relevant claims made by Luce and Edwards (1958) are examined under *bounded* Fechnerian integration: namely, that three popular forms of the Weber Fraction, when combined with Fechner's Law, produce sensation-growth equations that subsequently return the selfsame Fechner's Law. Luce and Edwards (1958) prove to be wrong.

1 Introduction

This article is a critical re-evaluation of a foundational theory of mathematical psychophysics. It focuses on derivations, without comparison to measurement (which is not germane to the task).

A sensory stimulus of physical intensity I, however I is measured, produces a sensation of magnitude $F(I)$. How sensation magnitude grows with intensity continues to be a major topic in sensory science. Classically [1-20], it has been assumed that the actual minimal *differences* in sensation, the sensation "chunks" (so-to-speak), can be added together to produce the sensation magnitude. This exemplifies *ratio scaling*, in which quantities can be added together to make proportionately greater quantities. (Physical intensity is one such quantity.) These notions can be expressed algebraically. For a positive integer j labeling an intensity I_j that evokes a corresponding sensation F_j, consider an empirical just-noticeable sensation difference $\Delta F\ at\ F_j$ that corresponds to a just-noticeable intensity difference $\Delta I\ at\ I_j$:

$\Delta I\ at\ I_j\ is\ called\ (\Delta I)_j \quad (1a)$,
$corresponding\ \ to\ \ \Delta F\ at\ F_j, called\ (\Delta F)_j \quad (1b)$

Any detectable stimulus has an experimentally-determined stimulus-detection *threshold*, here denoted $I_{th} = I_1$. Now imagine m successive increments $\Delta\left(I_j\right)$. They accumulate from $I_{th} = I_1$, where $\Delta(I_{th}) = \Delta(I_1)$, up to a desired intensity I_{m+1}, reached from the lesser intensity I_m by the increment $\Delta(I_m)$. Correspondingly, the sensation increments $(\Delta F)_j$ accumulate from the sensation at the stimulus-detection threshold I_{th}, namely $F(I_{th}) = F(I_1)$, up to the sensation $F(I_{m+1})$, reached from $F(I_m)$ by the increment $\Delta F(I_m)$.

Let us assume that there exist smooth, continuous relations $\alpha(I)$ and $\beta(F)$ such that

$$(\Delta I)_j = \alpha\left(I_j\right) \quad (2\text{a}) \quad and \quad (\Delta F)_j = \beta\left(F_j\right) \quad (2b)$$
$$for\ j \geq 1, j \in \mathbb{I}^+$$

Several well-known steps (omitted for the sake of brevity) lead to

$$\int_{I_1}^{I_{m+1}} \frac{dI}{\alpha(I)}$$
$$= \int_{F_1}^{F_{m+1}} \frac{dF}{\beta(F)}$$
$$+ \lim_{(\Delta I)_j \to dI} \sum_{I_j,\, F_j,\ \ j=1}^{I_j,\, F_j,\ \ j=m} \left(\frac{{(\Delta I)_j}^2}{\beta\left(F_j\right)} \cdot \left(\frac{d^2 F(I)}{dI^2} \right)_j \right)$$
$$+ \lim_{(\Delta I)_j \to dI} \sum_{I_j,\, F_j,\ \ j=1}^{I_j,\, F_j,\ \ j=m} \frac{O\left({(\Delta I)_j}^3\right)}{\beta\left(F_j\right)} \quad (3)$$

where O represents "order of". To a first approximation, then, and replacing "approximately" by "equals",

$$\int_{I_1}^{I_{m+1}} \frac{dI}{\alpha(I)} = \int_{F_1}^{F_{m+1}} \frac{dF}{\beta(F)} \qquad (4)$$

Equation (4) represents "bounded Fechnerian integration". We therefore depart from the literature, in which the bounds are typically omitted from the calculus. Of course, the Taylor series expansion in Eq. (3) will have some region of validity, which must be established separately for each $\alpha(I)$ and $\beta(F)$. The method appears in advanced textbooks, but will not be elaborated here, because the present author has never seen it used in the psychophysics literature.

2 General equations for sensation growth, from bounded Fechnerian integration under Fechner's Law or Ekman's Law

Remarkably, the "father" of psychophysics, Gustav Fechner [1], ignored the approximate nature of Eq. (4). Indeed, he and his contemporaries also ignored the bounds of integration, $\{I_1,\ I_{m+1}\}$ and $\{F(I_1),\ F_{m+1}\}$. That is, what Fechner and later investigators used was "Fechnerian indefinite integration", as follows for the smooth continuous functions $\alpha(I) = \Delta I$ and $\beta(F) = \Delta F$:

$$\int \frac{dI}{\Delta I} = \int \frac{dF}{\Delta F} \quad (5)$$

The solutions of the integrals lacking bounds are vague. But *definite* integrals can be found if we can specify a lower bound for the stimulus intensity. For human hearing, the stimulus-detection threshold I_{th}, $I_{th} \neq 0$, is empirically associated with nonzero loudness, $F(I_{th}) \neq 0$ [3-4]. This result should be no surprise; stimulus-detection thresholds are defined statistically in psychophysics. That is, loudness "near threshold", for example, sometimes occurs and sometimes does not [5-6]. The same should apply to the other senses. After all, a threshold cannot be defined in terms of what is *not* detected! Let us hence assume that in humans, at least, $F(I_{th}) \neq 0$.

For the sensation magnitude $F(I_1) = F(I_{th})$ evoked at the stimulus-detection threshold $I_1 = I_{th}$, Eq. (4) can be written as

$$\int_{I_{th}}^{I} \frac{dI}{\Delta I} \ = \int_{F(I_{th})}^{F} \frac{dF}{\Delta F} \quad (6)$$

Let us assume that $1/\Delta I$ is finite, smooth, and continuous, having no zeroes (let us avoid the limits $I \to 0$ and $I \to \infty$). Let us likewise postulate some finite, smooth, and continuous function $G(I)$:

$$\int \frac{dI}{\Delta I} = G(I) + some\ constant, \quad (7a)$$

$$\int_{I_{th}}^{I} \frac{dI}{\Delta I} = G(I) - G(I_{th}) \quad (7b)$$

Now introduce a classical assumption about just-noticeable sensation differences, namely, that they are constant. This is Fechner's Law, $\Delta F = B$. From Eqs. (6) and (7b),

$$G(I) - G(I_{th}) = \int_{F(I_{th})}^{F} \frac{dF}{B} \quad (8)$$

Evaluating, we obtain

$$F(I) = B \cdot \big(G(I) - G(I_{th})\big) + F(I_{th}) \quad (9)$$

for $I \geq I_{th}$. When $I = I_{th}$, then $F = F(I_{th})$ as required. Equation (9) can be expressed in a particularly useful form, as follows. Note that the quantity $\Delta I/I$ will arise in these calculations. It is called the Weber Fraction, and it takes many forms proposed from data. Therefore, $1/\Delta I$ and $G(I)$ do also. Let $B \cdot G(I) = K_1 \mathcal{G}(I)$, where K_1 is a composite of other constants which may arise from the Weber Fraction. From Eq. (9),

$$F(I) = K_1 \mathcal{G}(I) + F(I_{th}) - K_1 \mathcal{G}(I_{th}) \quad (10)$$

Now consider, for the just-noticeable sensation difference ΔF, an alternative to Fechner's Law, *viz.*,

$$\frac{\Delta F}{F} = g \quad (11)$$

Stevens [7] describes this as "Ekman's Law", after Ekman [8]. However, it was used by Plateau ([9], p. 384); indeed, it may pre-date Plateau. Nonetheless, let us call it Ekman's Law. Under it,

$$G(I) - G(I_{th}) = \int_{F(I_{th})}^{F} \frac{dF}{gF} \quad (12)$$

from which

$$F(I) = F(I_{th}) \frac{H(I)}{H(I_{th})} \quad (13a)$$

$$where\ H(I) = exp\big(g \cdot G(I)\big) \quad (13b)$$

for $I \geq I_{th}$. When $I = I_{th}$, then $F = F(I_{th})$ as required. Equation (13a) can be transformed into a more revelatory form under the substitution $g \cdot G(I) = K_2 \mathcal{G}(I)$, from which $K_2 = K_1 g/B$ and $H(I) = exp\big(K_2 \mathcal{G}(I)\big)$.

3 Sensation-growth equations for which the sensation difference would return Fechner's Law or Ekman's Law

Fechnerian integration is inexact (Eq. 3). We might ask whether any $F(I)$ obtained under either Fechner's Law $\Delta F = B$ or Ekman's Law $(\Delta F/F) = g$ actually returns those respective Laws. We can answer for Fechner's Law, by noting whether $[\Delta F(I)]_{Fechner} = F(I + \Delta I) - F(I)$ returns B. We merely substitute into the derived $F(I)$ whatever ΔI had been originally assumed when deriving that $F(I)$. Similarly, for Ekman's Law, we evaluate whether

$$\left[\frac{\Delta F(I)}{F(I)}\right]_{Ekman} = \frac{F(I + \Delta I) - F(I)}{F(I)} \quad (14)$$

returns the value g. That is, for Fechner's Law,

$$[\Delta F(I)]_{Fechner} = F(I + \Delta I) - F(I)$$

$$= B \cdot \big(G(I + \Delta I) - G(I)\big)$$
$$= K_1 \cdot \big(\mathcal{G}(I + \Delta I) - \mathcal{G}(I)\big) \quad (15)$$

Equations (13) and (14) allow $[\Delta F(I)/F(I)]_{Ekman}$:

$$\left[\frac{\Delta F(I)}{F(I)}\right]_{Ekman} = \frac{H(I + \Delta I)}{H(I)} - 1 \quad (16)$$

where $H(I) = exp\big(g \cdot G(I)\big) = exp\big(K_2 \mathcal{G}(I)\big)$, hence

$$\left[\frac{\Delta F(I)}{F(I)}\right]_{Ekman}$$

$$= \frac{exp\ \big(g \cdot G(I + \Delta I)\big)}{exp\ \big(g \cdot G(I)\big)} - 1$$
$$= -1 + exp\left[K_2 \cdot \big(\mathcal{G}(I + \Delta I) - \mathcal{G}(I)\big)\right] \quad (17)$$

From Eq. (15), and recalling that $K_2 = K_1 g/B$, Eq. (17) can be written

$$\left[\frac{\Delta F(I)}{F(I)}\right]_{Ekman}$$

$$= -1 + exp\left(\frac{K_2}{K_1} \cdot [\Delta F(I)]_{Fechner}\right)$$
$$= -1 + exp\left(\frac{g}{B} \cdot [\Delta F(I)]_{Fechner}\right) \quad (18)$$

That is, if we know $[\Delta F(I)]_{Fechner}$, then we know $[\Delta F(I)/F(I)]_{Ekman}$. To find $[\Delta F(I)]_{Fechner}$, note that Eq. (15) and Eq. (7b) altogether yield

$$[\Delta F(I)]_{Fechner}$$

$$= B \cdot \left(\int_{I_{th}}^{I+\Delta I} \frac{dI}{\Delta I} - \int_{I_{th}}^{I} \frac{dI}{\Delta I}\right) = B \cdot \int_{I}^{I+\Delta I} \frac{dI}{\Delta I} \quad (19)$$

For Eq. (19) to yield $[\Delta F(I)]_{Fechner} = B$, the term ΔI must be $constant$. The respective Weber Fraction is then $(\Delta I/I) = constant/I$. This Weber Fraction is not evident in the literature, but would allow a linear growth in $F(I)$ under Fechner's Law $\Delta F = B$; that is, from Eqs. (7b) and (9), we obtain $F(I) = F(I_{th}) + (B/constant) \cdot (I - I_{th})$. Conversely, Ekman's Law $(\Delta F/F) = g$ would yield $F(I) = F(I_{th})\, exp\big((g/constant) \cdot (I - I_{th})\big)$.

Now consider what possible $F(I)$s would return Ekman's Law itself. Combining Eqs. (18) and (19),

$$\left[\frac{\Delta F(I)}{F(I)}\right]_{Ekman}$$
$$= -1 + \ exp\left(g \cdot \int_{I}^{I+\Delta I} \frac{dI}{\Delta I}\right) \quad (20)$$

Replacing the left-hand-side of the equation by g (Eq. 11), and solving its right-hand-side using Eq. (7b), we obtain

$$G(I + \Delta I) - G(I) = (ln(g + 1))/g \quad (21)$$

Recall that ΔI is presumed to be a function of intensity (Eq. 2). This makes Eq. (21) a *functional*, that is, a function of functions. For this particular functional, the logarithmic term $ln(g + 1)$ suggests a logarithmic solution for $G(I)$. Substituting a potential solution $G(I) = (1/g) ln\big(C(I + c)\big) + constant$ into Eq. (21), where C and c are constants, yields $\Delta I = g(I + c)$. The latter should satisfy Eq. (7b) for the aforementioned potential logarithmic solution $G(I)$. In fact, from Eq. (7b), it yields

$$\frac{1}{g} ln \frac{I + c}{I_{th} + c} = G(I) - G(I_{th}) \quad (22)$$

which is correct for the potential solution $G(I)$. Hence, from Eq. (13),

$$F(I) = F(I_{th}) \frac{I + c}{I_{th} + c} \quad (23)$$

This is a linear $F(I)$. Conversely, Fechner's Law $\Delta F = B$ combined with $\Delta I = g(I + c)$ gives

$$F(I) = F(I_{th}) + (B/g)\ ln \frac{I + c}{I_{th} + c} \quad (24)$$

The relation $\Delta I = g(I + c)$ can be expressed as a Weber Fraction, $(\Delta I/I) = g(I + c)/I$. The latter transpires to be a special case (thanks to g) of a Weber Fraction credited to Hermann von Helmholtz ([10], p. 177). Indeed, there are at least *two* particular cases of it in the literature, as will be discussed.

To summarize: bounded Fechnerian integration using either Fechner's Law $\Delta F = B$ or Ekman's Law

$(\Delta F/F) = g$ will only return the respective Fechner's Law or Ekman's Law when sensation magnitude $F(I)$ is a linear function of intensity. All other $F(I)$s obtained under bounded Fechnerian integration will be inaccurate. But the bounds represent real-world conditions. The literature shows remarkably little interest in these matters, but it is vast. Furthermore, the Weber Fraction that returns Fechner's Law is $(\Delta I/I) = constant/I$, a Weber Fraction not used in the literature. In contrast, the Weber Fraction that returns Ekman's Law is $(\Delta I/I) = g(I + c)/I$, a specialized form of a Weber Fraction that has been in print for some time.

All this is not to say that Fechnerian integration has not been studied; indeed, Luce and Edwards [11], amongst many others, have examined *non-bounded* Fechnerian integration. As shown below, such non-bounded integration can produce different results from the concern of the present paper, which is *bounded* Fechnerian integration.

4 The sensation difference for sensation-growth equations derived under various Weber Fractions

There are many published equations for the Weber Fraction $\Delta I/I$. The present author has assembled some of them into a Table, presented elsewhere for the sake of space (InterNoise 2020 e-Congress, Seoul, S. Korea), along with the sensation-growth functions $F(I)$ that can be derived from them under Fechner's Law $\Delta F = B$ with bounded Fechnerian integration (and, by implication, the $F(I)$s that arise under Ekman's Law $(\Delta F/F) = g$). Here, all of the respective $F(I)$s will be evaluated for whether they respectively return $[\Delta F(I)]_{Fechner} = B$ or $[\Delta F(I)/F(I)]_{Ekman} = g$.

Table 1 shows the $\Delta F(I)$ and $\Delta F(I)/F(I)$, after first reiterating the Weber Fractions $\Delta I/I$, and then shows the respective evaluated $[\Delta F(I)]_{Fechner} = BK_1(\Delta \mathcal{G}(I)/B)$ (from Eq. 15) for the $F(I)$ derived under Fechner's Law. From the latter, the $[\Delta F(I)/F(I)]_{Ekman}$ for Ekman's Law can be inferred, according to Eq. (18).

Findings: none of the $F(I)$s obtained under Fechner's Law returns $[\Delta F(I)]_{Fechner} = B$. Indeed, the only cases that returned constants at all, rather than equations, are case **a** (Weber's Law) and case **e** (Delboeuf [19]), in which an unspecified constant is added to Weber's Law. For both these cases, $[\Delta F(I)/F(I)]_{Ekman}$ is also constant.

5 The returned sensation difference in the limit that the Weber Fraction is much less than unity

Given that the $[\Delta F(I)]_{Fechner}$ or $[\Delta F(I)/F(I)]_{Ekman}$ evaluated from $F(I)$ may transpire to be a function of the independent variable I (Table 1) rather than a constant, and given that I is related to ΔI through the Weber Fraction $\Delta I/I$, then altogether the functions in question can be written as functions of the Weber Fraction (see Table 1, third column). This facilitates the examination of two useful limits. First, note that, empirically, the value of the Weber Fraction for human subjects can be well below unity for broad intensity ranges centered on relatively moderate stimulus intensities (for general empirical results, see [24-28]; for hearing, see the reviews in [29-31]; for vision, see the review in [32]; and for cutaneous pressure, see the review in [33]). (Of course, the Weber Fraction's magnitude depends upon the intensity units [34, 35].) Let us examine $[\Delta F(I)]_{Fechner}$ and $[\Delta F(I)/F(I)]_{Ekman}$ in "the limit of the limit of" $(\Delta I/I) < 1$, namely, $(\Delta I/I) <\, < 1$. Table 2 lists the results for $[\Delta F(I)]_{Fechner}$, from which the results for $[\Delta F(I)/F(I)]_{Ekman}$ can be inferred. Cases **a**, **c**, **e**, and **f** in Table 2 return a limit of $[\Delta F(I)]_{Fechner} \to B$, from which therefore $[\Delta F(I)/F(I)]_{Ekman} \to -1 + exp(g)$. For the left-hand term (the differential) of the latter, substituting g gives $g + 1 \to exp(g)$, a well-known general approximation for $exp(g)$ in the limit $0 \leq g < 0.5$, which improves as $g \to 0$. In other words, returning Fechner's Law in a particular limit does not guarantee a return of Ekman's Law in that same limit; further limits are required. In contrast to cases **a**, **c**, **e**, and **f**, consider case **d**, Nutting's [18] Weber Fraction, which returns a constant that only approaches B if $vC < 1$ *and* $vC \to 0$ (see below).

The limit $(\Delta I/I) << 1$ produces equations (not constants) for the other Weber Fractions of Table 1.

A second limit worth considering is the even more extreme limit $(\Delta I/I) \to 0$. From the very beginnings of psychophysics, there has been an intuitive stipulation that sensation magnitude depends *monotonically* upon stimulus intensity. Hence, the sensation increment ΔF should shrink in tandem with the corresponding just-noticeable intensity difference ΔI; that is, as $\Delta I \to 0$ (hence $(\Delta I/I) \to 0$), we should see $\Delta F(I) \to 0$. And, indeed, $[\Delta F(I)]_{Fechner}$ and $[\Delta F(I)/F(I)]_{Ekman}$ both approach zero as $(\Delta I/I) \to 0$ for the cases **a**, **b**, **c**, **e**, **f**, **h**, **i**, and **j** in Table 1. For cases **g** (Langer [20]) and **l** (Krantz [23]), however, the differentials only approach zero if $K \to 0$ in the respective Weber Fractions (see the footnotes to Table 2). Of course, $K \to 0$ is the only way that $(\Delta I/I) \to 0$ in Weber's Law, $(\Delta I/I) = K$, the most basic Weber Fraction.

For cases **d** (Nutting [18]) and **k** (Luce & Edwards [11]), $[\Delta F(I)]_{Fechner}$ initially seems to approach $\frac{0}{0}$. But $(\Delta I/I)$ can be treated as a variable in a series expansion. Hence, for case **d**, the limit becomes $(B/\nu C)\, ln(1+\nu C)$, which results in $\frac{0}{0}$ *unless* $\nu C < 1$ and $\nu C \to 0$; then, $[\Delta F(I)]_{Fechner} \to B$ (see footnotes to Table 2). For case **k** (Luce & Edwards [11]), $[\Delta F(I)]_{Fechner}$ can only reach zero if the exponent b in the Weber Fraction equals unity (see footnotes to Table 2), which is forbidden for this particular Weber Fraction. This contradiction hypothetically eliminates the Nutting [18] and Luce and Edwards [11] Weber Fractions from contention in devising $F(I)$. In this light, the failure of the Luce and Edwards [11] Weber Fraction should not be surprising, as it is a specialized version of the Nutting [18] Weber Fraction, although Luce and Edwards do not say so.

6 Luce and Edwards: "Fechner's Law is returned by the Weber-Fechner Law"

Combining Weber's Law $(\Delta I/I) = K$ and Fechner's Law $\Delta F = B$ through bounded Fechnerian integration yields the so-called Weber-Fechner Law, represented in Table 1 by row **a**:

$$F(I) = F(I_{th}) + \frac{B}{K} ln \frac{I}{I_{th}} \quad (25)$$

This equation does not return $\Delta F = B$. In fact, $[\Delta F(I)]_{Fechner} = (B/K)\, ln(1+K)$ (Table 1, row **a**). This contradiction was noted by Dzhafarov and Colonius ([36], p. 132; after [37]). Remarkably, however, Luce and Edwards ([11], p. 225) state the opposite, for their typical, *non-bounded* Fechnerian integration: "Only for a very few Weber functions – some pathological ones, Weber's law, and its generalization $\Delta x = kx + c$ – does the "mathematical auxiliary principle" [Fechner's term for Fechnerian integration] yield a Fechner function [i.e., $F(I)$] with equal jnd's [*sic*]". Note once again that Weber's Law refers to the constant Weber Fraction, case **a** in Tables 1 and 2. The phrase "equal jnd's" refers to Fechner's Law $\Delta F = B$. "Pathological" was not explained, but might refer to the finding (Section 3, above) that $\Delta I = constant$ yields $[\Delta F(I)]_{Fechner} = B$.

Algebra-wise, Luce and Edwards' x is the present I. Luce and Edwards set $B = 1$ in $\Delta F = B$, purely for algebraic convenience; this particular convenience was later used by others, for example implicitly in Eq. (5) of Kostal and Lansky [38]. But to continue: for Weber's Law, Luce and Edwards ([11], p. 227) set $I + \Delta I = kI,\ k > 1$, which amounts to $(\Delta I/I) = k - 1,\ k > 1$. Luce and Edwards then note that substituting $F(I) = \log I / \log k$ (where the base of the logarithm is irrelevant) into $\Delta F(I) = F(I + \Delta I) - F(I)$ returns their assumption that $\Delta F = 1$.

Let us scrutinize the Luce and Edwards results in the context of bounded Fechnerian integration, first by composing the Luce and Edwards algebra in the present notation. The condition $(\Delta I/I) = k - 1$, $k > 1$ can be written as $(\Delta I/I) = K,\ K > 0$. This K is the same as the K used to derive Eq. (25). Further, in order to have the present-day $\Delta F = B$ as an underlying assumption of $F(I)$, we must introduce a multiplier B into the Luce and Edwards algebra. The Luce and Edwards sensation-growth equation now becomes $F(I) = B \log I / \log(1 + K)$ for $(\Delta I/I) =$

$K,\ K > 0$. Using base e for the logarithms, the Luce and Edwards equation is equivalent to

$$F(I) = \frac{B}{ln(1+K)} lnI_{th} + \frac{B}{ln(1+K)} ln\frac{I}{I_{th}} \quad (26)$$

Compare this now to Eq. (25): this obliges $K = ln(1+K)$, under which $[\Delta F(I)]_{Fechner} = (B/K)\, ln(1+K)$ (Table 1, row **a**) devolves to $[\Delta F(I)]_{Fechner} = B$. In other words, *except* for any cases actually satisfying $K = ln(1+K)$, Luce and Edwards [11] have *not* proven that Fechner's Law is returned by the sensation-magnitude equation that results from Fechnerian integration of Weber's Law combined with Fechner's Law. Elsas [37] realized this nearly a century earlier (according to [39], p. 716).

An obvious solution to $K = ln(1+K)$ is $K = 0$, i.e. $\Delta I = 0$. But this defeats the purpose of the exercise. However, for $0 < K \leq 1$, a Taylor's series provides the approximation $ln(1+K) \approx K - (K^2/2) + (K^3/3) - \cdots$, such that $ln(1+K) \to K$ as $K \to 0$, thus giving the limiting case $K \approx K$ for "very small" Weber Fractions $(\Delta I/I) = K$. Thus, the Weber-Fechner Law returns $\Delta F = B$ only in the limits $0 < K \leq 1$, $K \to 0$. Indeed, Weber's Law $(\Delta I/I) = K$ might yield $K \leq 1$ when fitted to the mid-intensity range of much discriminability data. But this K might remain too large to satisfy the limiting condition.

Luce and Edwards [11] did not examine cases where $F(I)$ was derived under Ekman's Law, $(\Delta F/F) = g$. For example, when Ekman's Law is combined with Weber's Law $(\Delta I/I) = K$ through bounded Fechnerian integration, the result is the "bounded version" of Stevens' well-proselytized Power Law:

$$F(I) = F(I_{th}) \cdot \left(\frac{I}{I_{th}}\right)^{\frac{g}{K}} \quad (27)$$

The term $n = \frac{g}{K}$ is the Stevens Exponent. From $[\Delta F(I)]_{Fechner} = (B/K)\, ln(1+K)$ we obtain $[\Delta F(I)/F(I)]_{Ekman} = -1 + (1+K)^{g/K}$ (after Eq. 18). Now letting $g = K$, corresponding to $\Delta I = gI$, produces $[\Delta F(I)/F(I)]_{Ekman} = g$. In other words, Ekman's Law is returned when the Stevens Exponent is unity. This confirms the linear growth seen in Eq. (23) under the Weber Fraction $\Delta I = g(I + c)$, $c = 0$.

7 Luce and Edwards: "Fechner's Law is returned by the generalized Weber's Law"

The Weber Fraction $\Delta I = g(I + c)$ is a case of $(\Delta I/I) = K(I + C)/I,\ K, C > 0$. This particular Weber Fraction is dealt with in row **e** of Tables 1 and 2. Its obvious difference from Weber's Law $\Delta I = KI$ is the addition of a second unknown, C, which (in one explanation) "may represent the amount of sensory noise that exists" when $I = 0$ ([26], p. 6). This "generalized Weber's Law" ([40], pp. 36, 218) has "long been recognized as a better approximation to experimental data" than Weber's Law alone ([8], p. 347).

Note that $\Delta I/I = K(I + C)/I$ was used by Luce and Edwards [11], under their own notation, as the "generalization $\Delta x = kx + c$" ([11], p. 225). Luce and Edwards also noted ([11], p. 229) that a particular equation that uses the term $kI + c$, namely $F(I) = \log(kI + c)/\log(1 + k)$, satisfies their simplification $\Delta F = 1$. However, for compliance with the present notation, we must introduce the multiplier B. Let us also use natural logarithms. Altogether, we have $F(I) = B\, ln\big(K(I + C)\big)/ln(1 + K)$. This resembles the sensation-growth equation that arises under $(\Delta I/I) = K(I + C)/I$ and $\Delta F = B$ in bounded Fechnerian integration,

$$F(I) = F(I_{th}) + (B/K) ln\frac{I + C}{I_{th} + C} \quad (28)$$

Luce and Edwards' own equation is therefore equivalent to

$$F(I) = \frac{B}{ln(1+K)} ln\big(K(I_{th} + C)\big) + \frac{B}{ln(1+K)} ln\frac{I + C}{I_{th} + C} \quad (29)$$

Comparing Eq. (29) to Eq. (28) yields $K = ln(1 + K)$. The term $ln(1 + K)$ appears in row **e** of Table 1, for the Weber Fraction $\Delta I/I = K(I+C)/I$. In short, we have arrived at the same situation described above for Weber's Law, $(\Delta I/I) = K$: Luce and Edwards have, as above, found a special case that satisfies bounded Fechnerian integration only for $K = ln(1+K)$. Krantz ([23], p. 592) inadvertently emphasizes this point by reiterating Luce and Edwards' [11] findings for the generalized Weber's Law.

Luce and Edwards [11] did not examine cases where $F(I)$ was derived under Ekman's Law, $(\Delta F/F) = g$. Combining the latter with $\Delta I/I = K(I+C)/I$ through bounded Fechnerian integration yields

$$F(I) = F(I_{th}) \cdot \left(\frac{I+C}{I_{th}+C}\right)^{\frac{g}{K}} \quad (30)$$

Following the same logic as in Section 6 above, we obtain $[\Delta F(I)/F(I)]_{Ekman} = g$ for $\Delta I = g(I+C)$, a very unique intensity-difference.

8 Luce and Edwards: "Fechner's Law is returned by a power function for the intensity difference"

Luce and Edwards [11] discuss the Weber Fraction that is inherently defined by $x + \Delta x = ax^b, b \neq 1$ (Luce & Edwards [11], p. 228), claiming that it returns their stipulation $\Delta F = 1$. Let us examine their claim by first converting the Luce and Edwards algebra to the present notation, replacing 1 by B and x by I, and assuming natural logarithms. The Luce and Edwards Weber Fraction is $(\Delta I/I) = KI^{b-1} - 1$, $b \neq 1$. Combining this with $\Delta F = B$ under *non-bounded* Fechnerian integration ([11], p. 228) gives

$$F(I) = B\,\frac{ln\left(ln\left(K^{\frac{1}{b-1}} \cdot I\right)\right)}{ln\, b} \quad (31)$$

The logarithm in the denominator implies that $b > 0$. Oddly, Luce and Edwards state only that $b \neq 1$. Regardless, evaluating $\Delta F(I)$ does return $\Delta F(I) = B$, as required for $F(I)$ to be an exact solution of the non-bounded Fechnerian integral. The *bounded* Fechnerian integral for $(\Delta I/I) = KI^{b-1} - 1$ combined with $\Delta F = B$ yields

$$F(I) = \frac{B}{b-1}\, ln\, \frac{\left(1 - \frac{1}{KI^{b-1}}\right)}{\left(1 - \frac{1}{KI_{th}{}^{b-1}}\right)} + F(I_{th}) \quad (32)$$

Comparison to Eq. (31) illuminates the different outcomes of bounded versus non-bounded integration. From Eq. (32),

$$[\Delta F(I)]_{Fechner}$$

$$= \frac{B}{b-1}\, ln\left(\frac{(KI^{b-1})^b - 1}{(KI^{b-1})^{b-1}(KI^{b-1}-1)}\right) \quad (33)$$

(row **k** of Table 1). This is not *B*. Are there any circumstances in which $[\Delta F(I)]_{Fechner} \to B$ for Eq. (32)? As mentioned above, often $(\Delta I/I) < 1$ for the broad middle of the detectable range of many psychophysical stimuli. Also, we may restrain the range of values for K, by assuming that the Weber Fraction very crudely follows Weber's Law, $(\Delta I/I) = K$; if indeed $(\Delta I/I) < 1$ then $0 < K < 1$. By convention, however, $\Delta I > 0$; this necessitates $(\Delta I/I) > 0$. Therefore, if $(\Delta I/I) = KI^{b-1} - 1$ (after Luce and Edwards), then $K > I^{1-b}, b \neq 1$. So we have assembled some restraints on K.

Let us now assume that $0 < I < 1$, and that, for the sake of argument, $b > 1$. If so, then the requirement $K > I^{1-b}$ implies $K > 1$. But this is incompatible with the presumption that $0 < K < 1$. We might now ask whether the Weber Fraction of Luce and Edwards could have *any* value for b that is compatible with the empirical possibility that $0 < (\Delta I/I) < 1$ at relatively moderate stimulus levels. This condition, when combined with Luce and Edwards' $(\Delta I/I) = KI^{b-1} - 1, b \neq 1$, altogether implies that $\left(1 - (ln\,(K)\,/ln\,I)\right) < b < \left(1 - (ln\,(K/2)\,/ln\,I)\right)$, which is equivalent to $0 < \left(b - 1 + (ln\,(K)\,/ln\,I)\right) < (ln\;2\,/ln\,I)$. Now, consider some evaluated practical cases of these restrictions. If $I = 0.01$ and $K = 0.4$, then $0.801 < b < 0.650$, an absurdity. Similarly, if $I = 0.01$ and $K = 0.04$, then $0.301 < b < 0.150$, another absurdity, and if $I =$

0.001 and $K = 0.4$, then $0.867 < b < 0.767$, yet another absurdity.

Altogether, scrutiny of the Luce and Edwards [11] Weber Fraction $(\Delta I/I) = KI^{b-1} - 1,\ b \neq 1$ reveals that it is not compatible with empirical discriminability that obeys $0 < (\Delta I/I) < 1$.

9 Conclusions

Equations for sensation magnitude as a function of stimulus intensity, $F(I)$, can be derived through Fechnerian integration, given stipulations that specify how just-noticeable sensation differences ΔF relate to sensation magnitude, and how the corresponding just-noticeable intensity differences ΔI relate to stimulus intensity. But Fechnerian integration involves truncating a series of equations, therefore being inexact. Furthermore, it is habitually done without specified bounds, adding further uncertainty. Here, the Fechnerian integration is bounded, following the concept that sensation cannot be zero at the stimulus-detection threshold. Insights emerge that are absent from the literature.

Traditionally, sensation differences are stipulated to follow either Fechner's Law $\Delta F = B$ or Ekman's Law $(\Delta F/F) = g$. It transpires that these same laws will only be recouped from the derived $F(I)$ (i.e., Fechnerian integration is exact) when sensation magnitude is a linear function of intensity; any other sensation-growth equations will be inaccurate. But the respective Weber Fractions $\Delta I/I$ that return Fechner's Law and Ekman's Law are not amongst the plethora found useful in the literature.

A Table of published Weber Fractions is presented here, along with the differentials $\Delta F(I)$ of the sensation-growth functions $F(I)$ that can be derived from the respective Weber Fractions under Fechner's Law $\Delta F = B$ with bounded Fechnerian integration (and, by implication, the $F(I)$s that arise under Ekman's Law $(\Delta F/F) = g$). None of the differentials $\Delta F(I)$ and $\Delta F(I)/F(I)$ respectively yield Fechner's Law or Ekman's Law. The differentials are then examined for the limit $(\Delta I/I) \ll 1$, and also in the limit $(\Delta I/I) \to 0$, in which the sensation difference $\Delta F(I)$ should approach zero. In the limit $(\Delta I/I) \ll 1$, Fechner's Law (but not necessarily Ekman's Law) was returned by some Weber Fractions. In the limit $(\Delta I/I) \to 0$, the sensation difference fails to approach zero for a Weber Fraction of Nutting, and for a specialized Weber Fraction from Luce and Edwards [11].

Luce and Edwards' [11] well-cited paper "The Derivation of Subjective Scales from Just Noticeable Differences" claims that Fechner's Law is returned for three Weber Fractions: Weber's Law (i.e., a constant Weber Fraction), the more elaborate "generalized Weber's Law", and a power function minus a constant. All three claims prove wrong for sensation-growth equations derived through bounded Fechnerian integration.

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References

[1] G.T. Fechner, *Element der Psychophysik* [*Elements of Psychophysics*, H. E. Adler, Transl.], New York, NY, USA: Holt (1860/1966).

[2] G.T. Fechner, *In Sachen der Psychophysik*, Leipzig, Germany: Breitkopf & Härtel (1877).

[3] S. Buus, H. Musch, & M. Florentine, "On Loudness at Threshold," *Journal of the Acoustical Society of America* vol. 104, 399-410 (1998).

[4] S. Buus & M. Florentine, "Growth of Loudness in Listeners with Cochlear Hearing Losses: Recruitment Reconsidered," *Journal of the Association for Research in Otolaryngology* vol. 3, pp. 120-139 (2002).

[5] L. Nizami, "A Qualitative Model Relating Loudness to Detection 'Near Threshold' and Above, Giving Zero Loudness at a 'Hard Threshold' and Nonzero Loudness at a 'Soft (Psychometric) Threshold'," in *Abstracts of the 32nd Annual Midwinter Research Meeting of the Association for Research in*

Otolaryngology, Baltimore, MD, USA, p. 77 (2009).

[6] L. Nizami, "Flaws in Loudness Standard ANSI S3.4-2007: the Derivation of Nonzero Sensation at Absolute Detection Threshold is Circular," Paper 38, *Proceedings of InterNoise 2015*, San Francisco, CA, USA (2015).

[7] S.S. Stevens, "A Metric for the Social Consensus," *Science* vol. 151, pp. 530-541 (1966).

[8] G. Ekman, "Weber's Law and Related Functions," *Journal of Psychology* vol. 47, pp. 343-352 (1959).

[9] M.J. Plateau, "Sur la Mesure des Sensations Physiques, et sur la Loi qui lie l'Intensité de ces Sensations à l'Intensité de la Cause Excitante," *Bulletins de l'Académie Royale des Sciences, des Lettres et des Beaux-Arts de Belgique 2me. Sér*. vol. 33, pp. 376-388 (1872).

[10] H. von Helmholtz, *Helmholtz's Treatise on Physiological Optics. Vol. II. The Sensations of Vision*, Menasha, WI, USA: George Bansha Publishing Company for Optical Society of America (1924).

[11] R.D. Luce & W. Edwards, "The Derivation of Subjective Scales from Just Noticeable Differences," *Psychological Review* vol. 65, pp. 222-237 (1958).

[12] E.H. Weber, *De Pulsu, Resorpitione, Auditu, et Tactu: Annotationes Anatomicae et Physiologicae* [*The Sense of Touch*, Transl.: Academic Press for Experimental Psychology Society], London, UK: Academic Press (1834/1978).

[13] H. Aubert, *Physiologie der Netzhaut*, Breslau, Germany: E. Morgenstern (1865).

[14] T. Mayer, "Experimente über die Schärfe des Gesichtsinnes," *Göttingische Anzeigen von gelehrten Sachen* vol. 20, pp. 401-402 (1754).

[15] O.-J. Grüsser, "The Discovery of the Psychophysical Power Law by Tobias Mayer in 1754 and the Psychophysical Hyperbolic Law by Ewald Hering in 1874," *Behavioral & Brain Sciences* vol. 16, pp. 142-144 (1993).

[16] J.P. Guilford, "A Generalized Psychophysical Law," *Psychological Review* vol. 39, pp. 73-85 (1932).

[17] W.J. McGill & J.P. Goldberg, "A Study of the Near-Miss involving Weber's Law and Pure-Tone Intensity Discrimination," *Perception & Psychophysics* vol. 4, pp. 105-109 (1968).

[18] P.G. Nutting, "The Luminous Equivalent of Radiation. III. Subjective Light. 12. Photometric Sensibility," *Bulletin of the Bureau of Standards* vol. 5, pp. 285-308 (1908).

[19] J. Delboeuf, "Étude Psychophysique. Recherches Théoriques et Expérimentales sur la Mesure des Sensations et Spécialement des Sensations de Lumière et de Fatigue," *Mémoires Couronnés et Autres Mémoires Publiés par l'Academie Royale des Sciences, des Lettres et des Beaux-Arts de Belgique* vol. 23, pp. 1-115 (1873).

[20] P. Langer, *Die Grundlagen der Psychophysik: eine kritische Untersuchung*, Jena, Germany: Hermann Dufft (1876).

[21] S. Hecht, "A Theory of Visual Intensity Discrimination," *Journal of General Physiology* vol. 18, pp. 767-789 (1935).

[22] R. Pierrel-Sorrentino & T.G. Raslear, "Loudness Scaling in Rats and Chinchillas," *Journal of Comparative & Physiological Psychology* vol. 94, pp. 757-766 (1980).

[23] D.H. Krantz, “Integration of Just-Noticeable Differences,” *Journal of Mathematical Psychology* vol. 8, pp. 591-599 (1971).

[24] S. Hecht, “The Visual Discrimination of Intensity and the Weber-Fechner Law,” *Journal of General Physiology* vol. 7, pp. 235-267 (1924).

[25] R.D. Luce & E. Galanter, “Discrimination,” in R. D. Luce, R. R. Bush, & E. Galanter (Eds.), *Handbook of Mathematical Psychology, Vol. 1*, New York, NY, USA: Wiley, pp. 191-243 (1963).

[26] G.A. Gescheider, *Psychophysics: Method, Theory, and Application*, Hillsdale, NJ, USA: Erlbaum (1985).

[27] W.M. Hartmann, *Signals, Sound, and Sensation*, New York, NY, USA: Springer (1998).

[28] S.C. Masin, “The (Weber’s) Law that Never Was,” in *Proceedings of the 25th Annual Meeting of the International Society for Psychophysics, Galway, Ireland*, pp. 441-446 (2009).

[29] B. Scharf & S. Buus, “Audition I: Stimulus, Physiology, Thresholds,” in K.R. Boff, L. Kaufman, & J.P. Thomas (Eds.), *Handbook of Perception and Human Performance, Vol. 1: Sensory Processes and Perception* (Chapter 14, pp. 1-71), Toronto, Ontario, Canada: Wiley (1986).

[30] D.M. Green, “Auditory Intensity Discrimination,” in W.A. Yost, A.N. Popper, & R.R. Fay (Eds.), *Human Psychophysics*, Santa Clara, CA, USA: Springer, pp. 13-55 (1993).

[31] J.J. Zwislocki, *Sensory Neuroscience: Four Laws of Psychophysics*, New York, NY, USA: Springer (2009).

[32] D.C. Hood & M.A. Finkelstein, “Sensitivity to Light,” in K.R. Boff, L. Kaufman, & J.P. Thomas (Eds.), *Handbook of Perception and Human Performance, Vol. 1: Sensory Processes and Perception* (Chapter 5, pp. 1-66), Toronto, Ontario, Canada: Wiley (1986).

[33] C.E. Sherrick & R.W. Cholewiak, “Cutaneous Sensitivity,” in K.R. Boff, L. Kaufman, & J.P. Thomas (Eds.), *Handbook of Perception and Human Performance, Vol. 1: Sensory Processes and Perception* (Chapter 12, pp. 1-58), Toronto, Ontario, Canada: Wiley (1986).

[34] D.J. Weiss, “The Impossible Dream of Fechner and Stevens,” *Perception* vol. 10, pp. 431-434 (1981).

[35] A.K. Myers, “Psychophysical Scaling and Scales of Physical Stimulus Measurement,” *Psychological Bulletin* vol. 92, pp. 203-214 (1982).

[36] E.N. Dzhafarov & H. Colonius, “The Fechnerian Idea,” *American Journal of Psychology* vol. 124, pp. 127-140 (2011).

[37] A. Elsas, *Ueber die Psychophysik. Physikalische und erkenntnistheoretische Betrachtungen* [*On Psychophysics. Physical and Epistemological Studies*], Marburg, Germany: Elwert (1886).

[38] L. Kostal & P. Lansky, “Coding Accuracy on the Psychophysical Scale,” *Scientific Reports* vol. 6:23810, pp. 1-6 (2016) DOI: 10.1038/srep23810

[39] E.N. Dzhafarov & H. Colonius, “Multidimensional Fechnerian Scaling: Basics,” *Journal of Mathematical Psychology* vol. 45, pp. 670-719 (2001).

[40] D. Laming, *The Measurement of Sensation*, New York, NY, USA: Oxford University Press (1997).

Table 1. The Weber Fraction $\Delta I/I$, and the resultant $\Delta\mathcal{G}(I)/B$ of the evaluated sensation difference $[\Delta F(I)]_{Fechner} = BK_1(\Delta\mathcal{G}(I)/B)$ for the sensation magnitude $F(I) = K_1\mathcal{G}(I) + F(I_{th}) - K_1\mathcal{G}(I_{th})$. The $F(I)$ arises from combining the Weber Fraction with Fechner's Law, $\Delta F = B$, under bounded Fechnerian integration. From $[\Delta F(I)]_{Fechner}$ we obtain $[\Delta F(I)/F(I)]_{Ekman} = -1 + exp\left((K_2/K_1)\cdot[\Delta F(I)]_{Fechner}\right)$. The footnotes list K_1; recall that $K_2 = K_1 g/B$ (text). The terms K, B, C, c, K_1, and K_2 all exceed zero. The value of K need not be the same from one Weber Fraction to another; the same symbol is used merely for convenience.

Weber Fraction, $\frac{\Delta I}{I}$	Source (footnote)	$\frac{\Delta\mathcal{G}(I)}{B}$ for $F(I)$ obtained under Fechner's Law
K	**a**	$ln(1+K) = ln\left(1+\frac{\Delta I}{I}\right)$
$\frac{K}{ln\,I}$	**b**	$ln\left(1+\frac{K}{ln\,I}\right)ln\left(I^2\left(1+\frac{K}{ln\,I}\right)\right) = ln\left(1+\frac{\Delta I}{I}\right)ln\left(I^2\left(1+\frac{\Delta I}{I}\right)\right)$
$KI^{-\nu}, 0<\nu<1$	**c**	$I^\nu((1+KI^{-\nu})^\nu - 1) = I^\nu\left(\left(1+\frac{\Delta I}{I}\right)^\nu - 1\right)$
$C + KI^{-\nu}, 0<\nu<1$	**d**	$ln\left(\frac{I^\nu\left(1+C\left(1+\frac{K}{C}I^{-\nu}\right)\right)^\nu + \frac{K}{C}}{I^\nu + \frac{K}{C}}\right) = ln\left(\frac{C\left(1+\frac{\Delta I}{I}\right)^\nu + \frac{\Delta I}{I} - C}{\frac{\Delta I}{I}}\right)$
$\frac{K(I+C)}{I}$	**e**	$ln(1+K) = ln\left(1+\left(\frac{\Delta I}{I}\cdot\frac{I}{I+C}\right)\right)$
$\frac{K(I+c)(I+C)}{I}, C>c$	**f**	$ln\left(\frac{1+K(I+C)}{1+K(I+c)}\right) = ln\left(\frac{1+\left(\frac{\Delta I}{I}\cdot\frac{I}{I+c}\right)}{1+\left(\frac{\Delta I}{I}\cdot\frac{I}{I+C}\right)}\right)$
$\frac{K(I^2+C)}{I^2}$	**g**	$ln\left(\frac{K^2(I^2+C)}{I^2} + 2K + 1\right) = ln\left(\left(1+\frac{\Delta I}{I}\right) + 2K\right)$
$\frac{K(\sqrt{I}+C)^2}{I}$	**h**	$ln\left(\frac{\sqrt{I+K(\sqrt{I}+C)^2}+C}{\sqrt{I}+C}\right) + \frac{\sqrt{I}}{\sqrt{I}+C} - \frac{\sqrt{I+K(\sqrt{I}+C)^2}}{\sqrt{I+K(\sqrt{I}+C)^2}+C}$ $= ln\left(\frac{\sqrt{I\left(1+\frac{\Delta I}{I}\right)}+C}{\sqrt{I}+C}\right) + \frac{\sqrt{I}}{\sqrt{I}+C} - \frac{\sqrt{I\left(1+\frac{\Delta I}{I}\right)}}{\sqrt{I\left(1+\frac{\Delta I}{I}\right)}+C}$

$\frac{(\sqrt{I}+C)^2}{I}-1 = \frac{C(2\sqrt{I}+C)}{I}$	**i**	$\sqrt{I+C(2\sqrt{I}+C)}-\sqrt{I}-\frac{C}{2}ln\left(\frac{2\sqrt{I+C(2\sqrt{I}+C)}+C}{2\sqrt{I}+C}\right) = \sqrt{I\left(1+\frac{\Delta I}{I}\right)}-\sqrt{I}-\frac{C}{2}ln\left(\frac{2\sqrt{I\left(1+\frac{\Delta I}{I}\right)}+C}{2\sqrt{I}+C}\right)$
$\frac{K}{I(\sqrt{I+C}-\sqrt{I})^2}$	**j**	$\frac{C^2}{2}ln\left(\frac{\sqrt{I+\frac{K}{(\sqrt{I+C}-\sqrt{I})^2}}+\sqrt{I+\frac{K}{(\sqrt{I+C}-\sqrt{I})^2}+C}}{\sqrt{I}+\sqrt{I+C}}\right) + \sqrt{I+\frac{K}{(\sqrt{I+C}-\sqrt{I})^2}}\sqrt{I+\frac{K}{(\sqrt{I+C}-\sqrt{I})^2}+C}\left[\sqrt{I+\frac{K}{(\sqrt{I+C}-\sqrt{I})^2}}\sqrt{I+\frac{K}{(\sqrt{I+C}-\sqrt{I})^2}+C}-\frac{C}{2}-\left(I+\frac{K}{(\sqrt{I+C}-\sqrt{I})^2}\right)\right] -\sqrt{I}\sqrt{I+C}\left(\sqrt{I}\sqrt{I+C}-\frac{C}{2}-I\right)$ $=\frac{C^2}{2}ln\left(\frac{\sqrt{I\left(1+\frac{\Delta I}{I}\right)}+\sqrt{I\left(1+\frac{\Delta I}{I}\right)+C}}{\sqrt{I}+\sqrt{I+C}}\right) + \sqrt{I\left(1+\frac{\Delta I}{I}\right)}\sqrt{I\left(1+\frac{\Delta I}{I}\right)+C}\left[\sqrt{I\left(1+\frac{\Delta I}{I}\right)}\sqrt{I\left(1+\frac{\Delta I}{I}\right)+C}-\frac{C}{2}-\left(I\left(1+\frac{\Delta I}{I}\right)\right)\right] -\sqrt{I}\sqrt{I+C}\left(\sqrt{I}\sqrt{I+C}-\frac{C}{2}-I\right)$
$KI^{b-1}-1, b\neq 1$	**k**	$ln\left(\frac{(KI^{b-1})^b-1}{(KI^{b-1})^{b-1}(KI^{b-1}-1)}\right) = ln\left(\frac{\left(1+\frac{\Delta I}{I}\right)^b-1}{\left(1+\frac{\Delta I}{I}\right)^{b-1}\cdot\frac{\Delta I}{I}}\right)$
$\frac{K\left(\sqrt{I+C}+\frac{K}{4}\right)}{I}$	**l**	$\frac{K}{2}\left(1-\frac{1}{2}ln\left(\frac{\sqrt{I+C}+\frac{K}{4}+\frac{K}{2}}{\sqrt{I+C}+\frac{K}{4}}\right)\right) = \frac{K}{2}\left(1-\frac{1}{2}ln\left(1+\frac{\frac{K}{2}}{\frac{\Delta I}{I}\cdot\frac{I}{K}}\right)\right)$

a. Weber [12]. $K_1 = \frac{1}{K}$. **b**. Aubert ([13], p. 69), reproduced in Fechner ([2], p. 19). $K_1 = \frac{1}{2K}$. **c**. Uncertain provenance, but old; allegedly used by Mayer [14] for visual acuity (cited in [15]). Fechner ([2], p. 21) notes that it can be derived from Plateau ([9], p. 384). It is also found in Guilford ([16], p. 79). In auditory research, it is often attributed to McGill and Goldberg [17]. $K_1 = \frac{1}{K\nu}$. **d**. Nutting ([18], p. 292). $K_1 = \frac{1}{\nu C}$. **e**. Fechner ([2], p. 35), reproduced from Delboeuf ([19], pp. 21, 54). Delboeuf himself named von Helmholtz as the actual source, but provided no references; the equation can, in fact, be found in the later English translation of von Helmholtz ([10], p. 177). $K_1 = \frac{1}{K}$. **f**. Fechner ([2], p. 17), credited there (incompletely) to von Helmholtz. The equation can, in fact, be found in the later English translation of von Helmholtz ([10], p. 180). $K_1 = \frac{1}{K(C-c)}$.

g. Fechner ([2], p. 41), reproduced from Langer ([20], p. 62). $K_1 = \frac{1}{2K}$. **h**. Hecht ([21], p. 772). $K_1 = \frac{2}{K}$. **i**. Pierrel-Sorrentino and Raslear ([22], p. 765), for the case n = 0.5. $K_1 = \frac{1}{C}$. **j**. Hecht ([21], p. 772). $K_1 = \frac{1}{K}$. **k**. Luce and Edwards ([11], p. 228). $K_1 = \frac{1}{b-1}$. **l**. Krantz ([23], p. 595). $K_1 = \frac{2}{K}$.

Table 2. Evaluation of $[\Delta F(I)]_{Fechner} = BK_1(\Delta\mathcal{G}(I)/B)$ and $[\Delta F(I)/F(I)]_{Ekman} = -1 + exp\left((K_2/K_1)\cdot[\Delta F(I)]_{Fechner}\right)$ in the limit in which the Weber Fraction ($\Delta I/I$) is much less than unity, for which $\lim_{(\Delta I/I)\ll 1}[\Delta F(I)/F(I)]_{Ekman} = -1 + exp\left((K_2/K_1)\cdot \lim_{(\Delta I/I)\ll 1}[\Delta F(I)]_{Fechner}\right)$. Each "note" refers to the corresponding source listed in Table 1.

Weber fraction, $\Delta I/I$	Notes	$\lim_{(\Delta I/I)\ll 1}[\Delta F(I)]_{Fechner}$
K	**a**	B
$\frac{K}{ln\, I}$	**b**	$\frac{B}{2K}\left(2K + \left(\frac{K}{ln\, I}\right)^2\right)$
$KI^{-\nu}, 0 < \nu < 1$	**c**	B
$C + KI^{-\nu}, 0 < \nu < 1$	**d**	$\frac{B}{\nu C} ln(1 + \nu C)$
$\frac{K(I + C)}{I}$	**e**	B
$\frac{K(I + c)(I + C)}{I}, C > c$	**f**	B
$\frac{K(I^2 + C)}{I^2}$	**g**	$\frac{B}{2K}\left(\frac{K^2(I^2 + C)}{I^2} + 2K + 1\right)$
$\frac{K\left(\sqrt{I} + C\right)^2}{I}$	**h**	$\frac{2B}{K}\left[ln\left(1 + \frac{\sqrt{I}}{C}\left(1 + \frac{K\left(\sqrt{I}+C\right)^2}{2I}\right)\right) - ln\left(\frac{\sqrt{I}+C}{C}\right) + \frac{\sqrt{I}}{\sqrt{I}+C} - \frac{\sqrt{I}\left(1 + \frac{K\left(\sqrt{I}+C\right)^2}{2I}\right)}{\sqrt{I}\left(1 + \frac{K\left(\sqrt{I}+C\right)^2}{2I}\right) + C}\right]$
$\frac{\left(\sqrt{I} + C\right)^2}{I} - 1$	**i**	$\frac{B}{C}\left(C + \frac{C^2}{2\sqrt{I}} - \frac{C}{2} ln\left(1 + \frac{C}{\sqrt{I}}\right)\right)$

$\frac{K}{I\left(\sqrt{I+C}-\sqrt{I}\right)^2}$	**j**	$\frac{B}{K}\left[\frac{C^2}{2} ln\left(\frac{\sqrt{I}\left(1+\frac{K}{2I\left(\sqrt{I+C}-\sqrt{I}\right)^2}\right)+\sqrt{I+C}}{\sqrt{I}+\sqrt{I+C}}\right) + \sqrt{I}\sqrt{I+C}\left(\frac{K\sqrt{I}\sqrt{I+C}}{I\left(\sqrt{I+C}-\sqrt{I}\right)^2}\left(1+\frac{K}{4I\left(\sqrt{I+C}-\sqrt{I}\right)^2}\right) - \frac{K}{I\left(\sqrt{I+C}-\sqrt{I}\right)^2}\left(\frac{C}{4}+I\left(1+\frac{K}{2I\left(\sqrt{I+C}-\sqrt{I}\right)^2}\right)+\frac{I}{2}\right)\right)\right]$
$KI^{b-1}-1, b \neq 1$	**k**	$\frac{B}{b-1} ln\left(\frac{b}{1+(b-1)(KI^{b-1}-1)}\right)$
$\frac{K\left(\sqrt{I+C}+\frac{K}{4}\right)}{I}$	**l**	$B\left(1-\frac{1}{2} ln\left(1+\frac{\frac{K}{2}}{\sqrt{I+C}+\frac{K}{4}}\right)\right)$

a. Uses $ln(1+x) \cong x$ for $-1 < x \leq 1$. **b**. Uses $ln(1+x) \cong x$ for $-1 < x \leq 1$. **c**. Uses $(1+x)^n \cong 1+nx$ for $-1 < x \leq 1$. **d**. At first glance, it appears that $\frac{\Delta \mathcal{G}(I)}{B} \to \frac{0}{0}$, but using $(1+x)^n \cong 1+nx$ for $-1 < x \leq 1$, an approximation that improves as $x \to 0$, we obtain

$$\lim_{\frac{\Delta I}{I}\to 0} [\Delta F(I)]_{Fechner} = \lim_{\frac{\Delta I}{I}\to 0} \frac{B}{\nu C} ln\left(\frac{C\left(1+\frac{\Delta I}{I}\right)^{\nu}+\frac{\Delta I}{I}-C}{\frac{\Delta I}{I}}\right) \cong \lim_{\frac{\Delta I}{I}\to 0} \frac{B}{\nu C} ln(1+\nu C)$$

using $(1+x)^n \cong 1+nx$ for $-1 < x \leq 1$ such that $\left(1+\frac{\Delta I}{I}\right)^{\nu} \cong 1+\nu\frac{\Delta I}{I}$. Now if $\nu C < 1$, then $\frac{B}{\nu C} ln(1+\nu C) \cong B$ using $ln(1+x) \cong x$ for $-1 < x \leq 1$, an approximation that improves as $x \to 0$. **e**. Uses $ln(1+x) \cong x$ for $-1 < x \leq 1$. **f**. Uses $ln(1+x) \cong x$ for $-1 < x \leq 1$. **g**. Uses $ln(1+x) \cong x$ for $-1 < x \leq 1$. At first glance, it seems that $\frac{\Delta \mathcal{G}(I)}{B} \nrightarrow 0$ as $\frac{\Delta I}{I} \to 0$, but the latter means that $\frac{K(I^2+C)}{I^2} \to 0$, which is achieved for any given I only if $K \to 0$, in which case $\frac{\Delta \mathcal{G}(I)}{B} \to 0$. **h**. Uses $(1+x)^n \cong 1+nx$ for $n=\frac{1}{2}$ for $-1 < x \leq 1$. **i**. Uses $(1+x)^n \cong 1+nx$ for $n=\frac{1}{2}$ for $-1 < x \leq 1$. **j**. Uses $(1+x)^n \cong 1+nx$ for $n=\frac{1}{2}$ for $-1 < x \leq 1$, and additionally assumes that $C \gg \frac{\Delta I}{I}$. **k**. The limits of $\frac{\Delta \mathcal{G}(I)}{B}$ as $\frac{\Delta I}{I} \to 0$ are difficult to discern, but using $(1+x)^n \cong 1+nx$ for $-1 < x \leq 1$, an approximation that improves as $x \to 0$, we obtain

$$\lim_{\frac{\Delta I}{I}\to 0} [\Delta F(I)]_{Fechner} = \lim_{\frac{\Delta I}{I}\to 0} \frac{B}{b-1} ln\left(\frac{\left(1+\frac{\Delta I}{I}\right)^{b}-1}{\left(1+\frac{\Delta I}{I}\right)^{b-1}\cdot\frac{\Delta I}{I}}\right) \cong \lim_{\frac{\Delta I}{I}\to 0} \frac{B}{b-1} ln\left(\frac{b}{1+(b-1)\frac{\Delta I}{I}}\right) = \frac{B}{b-1} ln(b)$$

which reaches zero only when $b=1$, which is forbidden in the specification of this particular Weber Fraction. **l**. At first, it seems that $\frac{\Delta \mathcal{G}(I)}{B} \nrightarrow 0$ as $\frac{\Delta I}{I} \to 0$, but the latter means that $\frac{K\left(\sqrt{I+C}+\frac{K}{4}\right)}{I} \to 0$, which is achieved for any given I only if $K \to 0$, in which case $\frac{\Delta \mathcal{G}(I)}{B} \to 0$.