Sensation-growth equations for non-zero threshold sensation, evaluated using non-traditional, bounded Fechnerian integration, for Fechner's Law and for Ekman's Law, using 12 different Weber Fractions

Lance Nizami[1]
Independent Research Scholar
Palo Alto, CA 94306, USA

ABSTRACT

An ongoing mystery in sensory science is how sensation magnitude $F(I)$, such as loudness, increases with increasing stimulus intensity I. No credible, direct experimental measures exist. Nonetheless, $F(I)$ can be inferred algebraically. Differences in sensation have empirical (but non-quantifiable) minimum sizes called just-noticeable sensation differences, ΔF, which correspond to empirically-measurable just-noticeable intensity differences, ΔI. The ΔIs presumably cumulate from an empirical stimulus-detection threshold I_{th} up to the intensity of interest, I. Likewise, corresponding ΔFs cumulate from the sensation at the stimulus-detection threshold, $F(I_{th})$, up to $F(I)$. Regarding the ΔIs, however, it is unlikely that all of them will be known experimentally; the procedures are too lengthy. The customary approach, then, is to find ΔI at a few widely-spaced intensities, and then use those ΔIs to interpolate all ΔIs using some smooth continuous function. The most popular of those functions is Weber's Law, $\Delta I/I = K$. But that is often not even a credible *approximation* to the data. However, there are other equations for $\Delta I/I$. Any such equation for $\Delta I/I$ can be combined with any equation for ΔF, through calculus, to altogether obtain $F(I)$. Here, two assumptions for ΔF are considered: $\Delta F = B$ (Fechner's Law) and $(\Delta F/F) = g$ (Ekman's Law). The respective integrals involve lower bounds I_{th} and $F(I_{th})$. This stands in broad contrast to the literature, which heavily favors non-bounded integrals. We, hence, obtain 24 new, alternative equations for sensation magnitude $F(I)$ (12 equations for $(\Delta I/I) \times 2$ equations for ΔF).

1. INTRODUCTION

A sensory stimulus of physical intensity I, however that is measured, produces a sensation of magnitude $F(I)$. How that magnitude grows with intensity continues to be a major topic in sensory science, having been of keen interest to psychologists and philosophers for nearly a century and a half, and continuing to engross a broad constituency of readers. Nevertheless, determining $F(I)$ has proven far more contentious than might have been imagined. First, there are no credible empirical measures of sensation magnitude; the evidence against existing "sensory scaling" practices is voluminous, and need not be cited here. Second, there are conceptual restrictions on algebraic

[1] nizamii2@att.net

derivations of possible $F(I)$s. Those restrictions are what this paper will explore, for perception in human subjects.

Before proceeding, it is necessary to note that the literature contains a number of pervasive assumptions. The first (but typically unmentioned) assumption is based on intuition: namely, that sensation F is a monotonically increasing function of stimulus intensity I. Another typically unstated assumption is that, when inferring $F(I)$, we may ignore any changing aspect of sensation that is not caused primarily by intensity change, for example, a heard change in a sound-waveform's "pitch", primarily correlated with waveform frequency. Yet another assumption, based again upon intuition, is that differences in sensation have minimum sizes; that is, they cannot become infinitely small, which would represent infinite sensitivity to changes in stimulus intensity. A further assumption is that the actual minimal differences in sensation, the sensation chunks (so-to-speak), can be added together to produce the intensity-dependent sensation magnitude. This assumption follows Fechner[1,2], and represents the notion that sensation growth with intensity follows a *ratio scale.* As an example of a ratio scale, consider the following non-sensory example from Reber and Reber[3], the scale of measured *weight* (not its ensuing sensation). Whether measured in kilograms or in any other weight unit, weight has a true zero-point. That is, the scale of weight has no negative values, and the bottom of the weight scale is "zero", representing the lack of an object to weigh. Any two weights can form a meaningful ratio, and any such ratio has the same meaning under multiplication of numerator and denominator by a given positive constant (e.g., a weight ratio of ½ has the same meaning as one of 8/16, and a weight ratio of 2/1 has the same meaning as 6/3).

Quantities describable using ratio scales can be added together to make proportionately greater quantities. Besides sensation, physical intensity is one such quantity. For a positive integer j labeling an intensity I_j that evokes a corresponding sensation F_j, let us denote an empirical just-noticeable sensation difference $\Delta F\ at\ F_j$ that specifies a corresponding just-noticeable intensity difference $\Delta I\ at\ I_j$ as follows:

$$\Delta I\ at\ I_j = (\Delta I)_j \quad (1a), \qquad corresponding\ to\ \Delta F\ at\ F_j = (\Delta F)_j \quad (1b)$$

For any detectable stimulus there is an empirical stimulus-detection threshold $I_{th} = I_1$. Hence, the increments $\Delta(I_j)$ accumulate from $I_{th} = I_1$ where $\Delta(I_{th}) = \Delta(I_1)$, up to the desired intensity I_{m+1}, reached from the lesser intensity I_m by the increment $\Delta(I_m)$. Correspondingly, the sensation increments $(\Delta F)_j$ accumulate from the sensation at the stimulus-detection threshold I_{th}, $F(I_{th}) = F(I_1)$, to the sensation $F(I_{m+1})$, reached from $F(I_m)$ by the increment $\Delta F(I_m)$.

Of course, it is unlikely that the sizes of the intensity increments will be known experimentally for sufficient numbers of I_j to provide a convincing empirical curve of $(\Delta I)_j$ versus I_j; the laboratory procedures simply prove too lengthy. To obtain laboratory just-noticeable intensity changes, therefore, the usual approach is to find ΔI empirically at a few widely-spaced values of I, and then infer its value in-between, by using some smooth continuous equation to to approximate the course of the graphed data points $\{x, y\} = \{I_j, \Delta(I_j)\}$. Of the candidate equations, the one that has proven most popular was proposed by Weber[4]. Fechner[1] dubbed it Weber's Law, and described it as follows: "The magnitude of the stimulus increment must increase in precise proportion to the stimulus already present, in order to bring about an equal increase in sensation [i.e., a constant $\Delta F(I)$]" (Ref. 1, p. 54). Note Fechner's added condition: that $\Delta F(I)$ be constant, a relation now called Fechner's Law. That particular stipulation will be explored further below. Meanwhile, Weber's Law can be written as an equation by introducing a unitless constant, K:

$$\frac{\Delta I}{I} = K \quad (2)$$

Fechner[1] described a lot of evidence for Weber's Law, some of which (according to Ref. 1, p. 125) dates at least back to Bouguer[5], whose stimulus was candle-light. The quantity $\Delta I/I$ itself is called the Weber Fraction and continues to be employed into the 21st Century to quantify human discriminability in vision[6-10], in hearing[11-17], and in flavor[18-20] (formerly called "taste"). There has been special interest in the pressure senses (touch, vibration, weight)[10, 21-40]. The Weber Fraction has also been used to quantify sensory discriminability in animals (citations omitted). Yet, the notion of the Weber Fraction being a constant (for a given kind of stimulus), that is Weber's *Law*, has been considered debatable for decades[41,42], despite ongoing claims of re-confirmation. Indeed, measuring $\Delta I/I$ is confounded by the experimental context (see, for example, Ross[43]). Masin[42] summarized experimental data for various senses in man, and concluded that, in many cases, Weber's Law may not even be a credible *approximation* to the plot of discriminability change versus intensity. Certainly, systematic deviations from Weber's Law are easily found in the literature, for example in audition[44-47]. All told, a close fit of Weber's Law to the empirical change of the just-noticeable intensity difference may be the exception rather than the rule. However, as Masin[42] remarked, one might not realize this after reading some published analyses, including those that Masin cites.

2 THE MATH: FECHNERIAN INTEGRATION

There have long existed other equations for the Weber Fraction $\Delta I/I$, besides Weber's Law. Following Weber[4], Fechner (Ref. 2, pp. 16-41) listed various alternative equations for the Weber Fraction, and combined them with Fechner's Law through calculus, in order to obtain equations for the putative dependence of sensation magnitude on stimulus intensity, $F(I)$. We can explore an even greater range of choices than Fechner did. Let us first explicate the most-general case of the algebra.

To begin, let us assume that there exist smooth, continuous relations $\alpha(I)$ and $\beta(F)$ such that

$$(\Delta I)_j = \alpha(I_j) \quad \text{(3a)} \qquad \text{and} \qquad (\Delta F)_j = \beta(F_j) \quad (3b) \qquad for\ j \geq 1, j \in \mathbb{I}^+$$

Clearly $(\Delta I)_j/\alpha(I_j) = 1$. Let m be the cumulative number of just-noticeable intensity changes ΔI between I_1 (which equals I_{th}) and I_{m+1}. Hence

$$m = \sum_{j=1}^{j=m} \frac{(\Delta I)_j}{\alpha(I_j)} \qquad (4)$$

Likewise, $(\Delta F)_j/\beta(F_j) = 1$ and the cumulative number of sensation changes ΔF between $F(I_{th}) = F(I_1)$ and $F(I_{m+1})$ is

$$m = \sum_{j=1}^{j=m} \frac{(\Delta F)_j}{\beta(F_j)} \qquad (5)$$

We can express the sensation increment $(\Delta F)_j$ as $F\big(I_j + (\Delta I)_j\big) - F(I_j)$, allowing a Taylor series

$$(\Delta F)_j = F\big(I_j + (\Delta I)_j\big) - F(I_j) = \left(\frac{dF(I)}{dI}\right)_j (\Delta I)_j + \left(\frac{d^2F(I)}{dI^2}\right)_j {(\Delta I)_j}^2 + O\big({(\Delta I)_j}^3\big) \qquad (6)$$

where $O\big({(\Delta I)_j}^3\big)$ refers to terms of third-and-higher order. Substituting $(\Delta F)_j$ into the right-hand-side of Eq. (5),

$$\sum_{j=1}^{j=m} \frac{(\Delta F)_j}{\beta(F_j)} = \sum_{j=1}^{j=m} \left(\frac{(\Delta I)_j}{\beta(F_j)} \cdot \left(\frac{dF(I)}{dI}\right)_j\right) + \sum_{j=1}^{j=m} \left(\frac{{(\Delta I)_j}^2}{\beta(F_j)} \cdot \left(\frac{d^2F(I)}{dI^2}\right)_j\right) + \sum_{j=1}^{j=m} \frac{O\big({(\Delta I)_j}^3\big)}{\beta(F_j)} \qquad (7)$$

From Eqs. (4) and (5), and taking the limit in which each just-noticeable intensity difference becomes infinitely small, we have, for $I_1 = I_{th}$,

$$\lim_{(\Delta I)_j \to dI} \sum_{I_j, j=1}^{I_j, j=m} \frac{(\Delta I)_j}{\alpha(I_j)} = \lim_{(\Delta I)_j \to dI} \sum_{F_j, j=1}^{F_j, j=m} \frac{(\Delta F)_j}{\beta(F_j)}, \ \ hence \int_{I_1}^{I_{m+1}} \frac{dI}{\alpha(I)} = \lim_{(\Delta I)_j \to dI} \sum_{F_j, j=1}^{F_j, j=m} \frac{(\Delta F)_j}{\beta(F_j)} \quad (8)$$

Further, denoting $F_1 = F(I_1) = F(I_{th})$, we recognize that

$$\lim_{(\Delta I)_j \to dI} \sum_{I_j, F_j, \ j=1}^{I_j, F_j, \ j=m} \left(\frac{(\Delta I)_j}{\beta(F_j)} \cdot \left(\frac{dF(I)}{dI} \right)_j \right) = \int_{F_1}^{F_{m+1}} \frac{dF}{\beta(F)} \quad (9)$$

Altogether, from Eqs. (7), (8), and (9),

$$\int_{I_1}^{I_{m+1}} \frac{dI}{\alpha(I)} = \int_{F_1}^{F_{m+1}} \frac{dF}{\beta(F)} + \lim_{(\Delta I)_j \to dI} \sum_{I_j, F_j, \ j=1}^{I_j, F_j, \ j=m} \left(\frac{{(\Delta I)_j}^2}{\beta(F_j)} \cdot \left(\frac{d^2 F(I)}{dI^2} \right)_j \right)$$
$$+ \lim_{(\Delta I)_j \to dI} \sum_{I_j, F_j, \ j=1}^{I_j, F_j, \ j=m} \frac{O\left({(\Delta I)_j}^3\right)}{\beta(F_j)} \quad (10)$$

Note well that, despite the equals sign, Eq. (10) is already an approximation, due to the limit $(\Delta I)_j \to dI$; real just-noticeable intensity differences never tend towards the indefinitely small, which (for present convenience) might be defined as "below the resolution of contemporary instruments". Regardless, let us maintain an "equals" sign, as often found in the literature; to some degree of approximation, then, we have

$$\int_{I_1}^{I_{m+1}} \frac{dI}{\alpha(I)} = \int_{F_1}^{F_{m+1}} \frac{dF}{\beta(F)} \quad (11)$$

This will be referred to henceforth as "bounded Fechnerian integration". The Taylor series expansion in Eq. (10) will have some region of validity, which must be established separately for each $\alpha(I)$ and $\beta(F)$. The relevant method is available in advanced textbooks, but the present author has never seen it used in the psychophysics literature. It will not be employed here either, because other considerations become paramount, as will be explained.

3 UNBOUNDED INTEGRATION VERSUS BOUNDED INTEGRATION

Remarkably, Fechner[1] ignored the fact that Eq. (11) was approximate. He and his contemporaries also ignored the bounds of integration, namely $\{I_{th} = I_1, \ I_{m+1}\}$ and $\{F(I_{th}) = F(I_1), F(I_{m+1})\}$. Introducing the shorthand $\Delta I = \alpha(I)$ and $\Delta F = \beta(F)$, what Fechner used were *indefinite* integrals of the form

$$\int \frac{dI}{\Delta I} = \int \frac{dF}{\Delta F} \quad (12)$$

Let us call this the "Fechnerian indefinite integral". Using indefinite integration avoids a difficult problem, a problem described as follows. The true minimum of sensation is zero, which can only be *guaranteed* by a stimulus intensity of zero, i.e. by removing the stimulus. So far, so good, for quantities that follow ratio scales. But it has been suggested that for human hearing, at least, the stimulus-detection threshold I_{th} is empirically associated[48,49] with nonzero loudness, $F(I_{th}) \neq 0$. This result should not be surprising, given that stimulus-detection thresholds in psychophysics are defined statistically; loudnesses "near threshold", for example, are sometimes heard and sometimes not[50,51], and we should expect the same for the other senses. Certainly, there seems to be no literature to the contrary. *Stimulus-detection thresholds are based upon experiencing sensation*. Therefore, let us assume that in humans, at least, the threshold sensation $F(I_{th})$ is nonzero, bearing in mind that the assignment of a particular value for I_{th} would be arbitrary due to threshold's statistical nature. Note well that I_{th} itself has sometimes been imagined to represent a just-noticeable sensation change, i.e. the very first.

In order to preserve intensity as a ratio scale, I_{th} will be assumed small compared to the available range of intensity that can evoke a change of sensation, the organism's *dynamic range*[52,53]. Threshold intensity now becomes the "effective zero" of the intensity scale. Otherwise, with threshold determined statistically, the lowest detectable intensity could be infinitely small, making $I_{th} = 0$ the bottom of the scale of *detectable* intensities. An infinitely small threshold has been advocated elsewhere[54], but the evidence is hardly compelling[55,56]. As Hellman and Zwislocki (Ref. 57, p. 687) state for hearing, "The threshold of audibility is a natural boundary condition which cannot be eliminated". Regardless, the quantum nature of stimuli sets a non-zero lower limit to stimulus intensity. Given a non-zero detection threshold, the "effective zero" of the sensation scale will be presumed to be the non-zero "sensation at threshold", $F(I_{th})$, which will be assumed small compared to the range of sensation that corresponds to the organism's dynamic *sensation* range.

Assigning non-zero bounds to Eq. (12) yields

$$\int_{I_{th}}^{I} \frac{dI}{\Delta I} = \int_{F(I_{th})}^{F} \frac{dF}{\Delta F} \quad (13)$$

Let us assume that $1/\Delta I$ is finite, smooth, and continuous, having no zeroes (presently, let us avoid the limits $I \to 0$ and $I \to \infty$). Let us likewise postulate some finite, smooth, and continuous function $G(I)$ such that

$$\int \frac{dI}{\Delta I} = G(I) + constant, \quad (14a) \qquad \int_{I_{th}}^{I} \frac{dI}{\Delta I} = G(I) - G(I_{th}) \quad (14b)$$

Recall now Fechner's Law[1]: ΔF is unchanging with F. Let us write this as $\Delta F = B$, where B is a constant. Fechner's Law continues to be of interest from both experimental and theoretical viewpoints[58-62]. From Eqs. (13) and (14b), and assuming Fechner's Law, $\Delta F = B$, we obtain

$$G(I) - G(I_{th}) = \int_{F(I_{th})}^{F} \frac{dF}{B} \quad (15)$$

Altogether then,

$$F(I)_{Fechner} = B \cdot \big(G(I) - G(I_{th})\big) + F(I_{th}), \qquad for\ I \geq I_{th} \quad (16)$$

When $I = I_{th}$, then $F = F(I_{th})$, as appropriate. Equation (16) can be expressed in another form, which proves useful. The Weber Fraction $\Delta I/I$ can have many forms, and therefore so do $1/\Delta I$ and $G(I)$. Therefore, let $B \cdot G(I) = K_1 \mathcal{G}(I)$ where K_1 is a composite of other constants that may arise from the literature equations for the Weber Fraction, so that

$$F(I)_{Fechner} = K_1\mathcal{G}(I) + F(I_{th}) - K_1\mathcal{G}(I_{th}) \quad (17)$$

Having considered Fechner's Law, $\Delta F = B$, a historically-popular relation for the just-noticeable sensation difference ΔF corresponding to the just-noticeable intensity difference ΔI, let us now consider an alternative relation,

$$\frac{\Delta F}{F} = g \quad (18)$$

where g is a constant. Stevens[63] describes this as "Ekman's Law", after Ekman[64], but it was used by Plateau (Ref. 65, p. 384). Nonetheless, let us follow the trend in the literature, and call it Ekman's Law. Under Ekman's Law,

$$G(I) - G(I_{th}) = \int_{F(I_{th})}^{F} \frac{dF}{gF} \quad (19)$$

from which

$$F(I)_{Ekman} = F(I_{th})\frac{H(I)}{H(I_{th})}\ for\ I \geq I_{th} \quad (20) \qquad where\ \ H(I) = exp\big(g \cdot G(I)\big)$$

When $I = I_{th}$, then $F = F(I_{th})$, as appropriate. Equation (20) can be transformed into a more revelatory form under the substitution $g \cdot G(I) = K_2\mathcal{G}(I)$, from which $K_2 = K_1 g/B$ and $H(I) = exp\big(K_2\mathcal{G}(I)\big)$. Note well that the role of $F(I_{th})$ as a multiplier cannot be well-discerned from the way that sensation-growth equations derived from Ekman's Law are used in the experimental-psychophysics literature.

4 RELATIONS FOR THE JUST-NOTICEABLE INTENSITY DIFFERENCE, AND THE CONSEQUENT SENSATION-GROWTH EQUATIONS

Up to this point, we have specified no function for ΔI besides Weber's Law, $\Delta I = KI$. It is now time to consider a variety of functions, that are chosen from the psychophysics literature because of their apparent utility. Consider those of Mayer[66] (cited by Grüsser[67]), Aubert[68], Langer[69], Nutting[70], von Helmholtz[71], Hecht[72], Luce and Edwards[73], Krantz[74], and Pierrel-Sorrentino and Raslear[75]. Table 1 lists those Weber Fractions, $\Delta I/I$, along with the resulting algebraic components needed to concatenate $F(I)$. The left-hand column (the first column) lists the actual Weber Fractions. The second column identifies the source of each one, listed in a Footnote to the Table. The Footnote is necessary because the Weber Fractions are often incorrectly attributed in the literature, their sources sometimes not being stated at all; here, some care was taken to trace the sources. The Footnotes also offer some details about the resulting $F(I)$. The Table's third column shows the term K_1 (for Eq. 17) as a function of the constants that appear in the respective Weber Fractions. The fourth column expresses $\mathcal{G}(I)/B$. The fifth (right-most) column shows the limiting values of $\mathcal{G}(I_{th})/B$ as the stimulus-detection threshold I_{th} becomes infinitely small, as explained in Section 5, below.

For illustrative cases, consider first a relatively simple case, the so-called "near-miss to Weber's Law" (Table 1, row **c**). Then, consider a relatively complicated case, Hecht's relation (row **j**). Now, under the "near-miss"[65,66,76,77], evaluating Fechner's Law using Eq. (17) yields

$$F(I)_{Fechner} = \frac{B}{K\nu}(I^{\nu} - I_{th}^{\nu}) + F(I_{th}) \quad (21)$$

Evaluating Ekman's Law using Eq. (20) yields

$$F(I)_{Ekman} = F(I_{th}) \cdot exp\left[\frac{g}{K\nu}(I^{\nu} - I_{th}^{\nu})\right] \quad (22)$$

Note that these two equations are linked as

$$F(I)_{Ekman} = F(I_{th}) \cdot exp\left[\frac{g}{B}\left(F(I)_{Fechner} - F(I_{th})\right)\right] \quad (23)$$

This is a general relation of $F(I)_{Ekman}$ to $F(I)_{Fechner}$, that arises from Eqs. (16) and (20) under all of the present assumptions. Consider now Hecht's relation[72] for the Weber Fraction (Table 1, row **j**). Under Fechner's Law, we obtain

$$F(I)_{Fechner} = \frac{B}{K}\left[\frac{C^2}{2} ln\left(\frac{\sqrt{I} + \sqrt{I + C}}{\sqrt{I_{th}} + \sqrt{I_{th} + C}}\right) + \sqrt{I}\sqrt{I + C}\left(\sqrt{I}\sqrt{I + C} - \frac{C}{2} - I\right) - \sqrt{I_{th}}\sqrt{I_{th} + C}\left(\sqrt{I_{th}}\sqrt{I_{th} + C} - \frac{C}{2} - I_{th}\right)\right] + F(I_{th}) \quad (24)$$

$F(I)$ under Ekman's Law is then evaluated according to Eq. (23).

Consider now an unusual case, Aubert's relation[68] (row **b**). Under Fechner's Law,

$$F(I)_{Fechner} = \frac{B}{2K} \cdot \left((ln(I \cdot I_{th}))\, ln\left(\frac{I}{I_{th}}\right)\right) + F(I_{th}) \quad (25)$$

Observe the term $ln(I \cdot I_{th})$. Recall that, contrary to popular misconceptions, taking logarithms does not remove units. The term $ln(I \cdot I_{th})$ has units of *natural logarithm of* [*the intensity units*]^2. The latter, thanks to the properties of logarithms, is equivalent to *twice the natural logarithm of the intensity units*. Hence, Aubert's Weber Fraction, $(\Delta I/I) = K/ln\, I$, will not possess an expected property of Weber Fractions – namely, unitlessness – *unless* K has units of *natural logarithm of the intensity units*.

5 EXAMINING THE LOWER LIMITS OF THRESHOLD AND SENSATION

Recall the earlier assumptions that $1/\Delta I$ and $G(I)$ are finite, smooth, and continuous, having no zeroes or infinities. Consider what happens now if, at the statistically-determined stimulus-detection threshold I_{th}, the respective sensation becomes vanishingly small: $F(I_{th}) \to 0$. Looking at the $F(I)$ derived under Fechner's Law (Eq. 16), this would cause no concern, because $F(I_{th})$ is an added term. But when Ekman's Law underlies $F(I)$ (Eq. 20), $F(I_{th})$ is a *multiplicative* term, such that $F(I_{th}) \to 0$ mandates $F(I) \to 0$ for all I, rendering $F(I)$ meaningless. If any $F(I)$ derived under Ekman's Law are to be meaningful, then, we cannot have $F(I_{th}) \to 0$ as $I \to I_{th}$. (Note well that no *a priori* value has yet been stipulated for $F(I_{th})$.) This agrees with the notion from auditory psychophysics[48,49] that loudness is nonzero at the stimulus-detection threshold: $F(I_{th}) \neq 0$. On the other hand, if I_{th} is made vanishingly small, we would expect sensation to likewise become vanishingly small; once the stimulus is *effectively* removed, the evoked sensation should always disappear. Clearly, there is a limits conundrum here[50]. The conundrum was historically avoided by using unbounded Fechnerian integration, for example, by combining Weber's Law with Ekman's Law without lower bounds, resulting in the generic power function since known as Stevens' Law[63]. In contrast, row **a** in Table 1 yields the *bounded* Fechnerian version under Ekman's Law.

Consider now an infinitely low detection threshold, as advocated elsewhere[54]. That is, imagine a threshold that can still be determined as stimulus intensity becomes smaller and smaller. Bearing in mind that stimulus-detection thresholds are based upon experiencing sensation, this means that sensation itself *cannot* become vanishingly small. The stimulus must still be seen or heard, i.e. $F(I_{th})$ does not approach zero, even as $I_{th} \to 0$. This, in turn, prompts an examination of what happens to the other threshold-dependent component of the derived $F(I)$, namely $G(I_{th})$, as $I_{th} \to 0$. In theory, for the $F(I)$ derived under Fechner's Law, we would obtain $F(I) \to B \cdot G(I) + F(I_{th})$ as $\lim_{I_{th}\to 0} G(I_{th}) = 0$, whereas $F(I) \to -\infty$ as $\lim_{I_{th}\to 0} G(I_{th}) = \infty$, and $F(I) \to \infty$ as $\lim_{I_{th}\to 0} G(I_{th}) = -\infty$. For the $F(I)$ derived under Ekman's Law, we would obtain $F(I) \to F(I_{th}) \cdot H(I)$ as $\lim_{I_{th}\to 0} G(I_{th}) = 0$, whereas $F(I) \to 0$ as $\lim_{I_{th}\to 0} G(I_{th}) = \infty$ and $F(I) \to \infty$ as $\lim_{I_{th}\to 0} G(I_{th}) = -\infty$. In short, if $I_{th} \to 0$ and also $\lim_{I_{th}\to 0} G(I_{th}) = 0$, then there is a workable general form for sensation magnitude under either Fechner's Law or Ekman's Law.

Now, $\mathcal{G}(I_{th})/B = G(I_{th})/K_1$ (or equivalently, $\mathcal{G}(I_{th})/g = G(I_{th})/K_2$), so that examining $\mathcal{G}(I_{th})/B$ informs us about $G(I_{th})$. Let us now examine what actually happens to $G(I_{th})$ as $I_{th} \to 0$ for the $F(I)$ whose components $\mathcal{G}(I)/B$ are listed in Table 1. The outcomes $\lim_{I_{th}\to 0} \mathcal{G}(I_{th})/B$ are in the rightmost column of Table 1. For cases **a** and **b**, i.e. the Weber Fractions of Weber[4] and Aubert[68], the respective limits are $-\infty$ and ∞. Evidently, an infinitely low threshold does not result in meaningful sensation-growth functions for these cases. For case **c**, the "near-miss to Weber's Law", the limit is 0, allowing a viable $F(I)$. The limit is likewise 0 for case **k**, the Weber Fraction of Luce and Edwards[73], under the restriction $0 < b < 1$; however, for the restriction $b > 1$, the limit is $ln(-\infty)$, an impossibility. This limitation, that $b \ngtr 1$ if $I_{th} \to 0$, is not mentioned by Luce and Edwards.

For the eight other Weber Fractions dealt with in Table 1, $\mathcal{G}(I_{th})/B$ approaches a constant. This poses no problem, because as $I \to (I_{th})$ under either Fechner's Law or Ekman's Law we have $(\mathcal{G}(I)/B) \to (\mathcal{G}(I_{th})/B)$ and hence $F(I) \to F(I_{th})$.

In sum: neither an infinitely low stimulus-detection threshold, nor a sensation of zero at *any* stimulus-detection threshold, are compatible with the present approach to determining sensation growth.

6 SUMMARY

A sensory stimulus of physical intensity I produces a sensation $F(I)$. There are no credible empirical measures of $F(I)$, and equations for $F(I)$ are conceptually restricted. Some significant examples of those restrictions are explored in this manuscript, for perception in human subjects.

Differences in sensation have empirical minimum sizes, ΔF. Each ΔF defines a just-noticeable intensity difference, ΔI. Further, for any detectable stimulus there is an empirical stimulus-detection threshold I_{th}, from which the ΔI hypothetically cumulate upwards with increasing intensity I as the just-noticeable sensation differences ΔF hypothetically cumulate upwards from the threshold sensation, $F(I_{th})$. In contrast to the literature, $F(I_{th})$ is assumed to be non-zero. However, the empirical sizes of the intensity increments ΔI will not be known in sufficient numbers to provide a convincing empirical plot of ΔI versus I. The latter is therefore inferred from fewer data points, fitted to some smooth continuous function, usually expressed as $\Delta I/I$, the Weber Fraction. The most popular Weber Fraction is that which is constant, a relation called Weber's Law. However, systematic deviations from Weber's Law are the empirical norm. For those non-constant Weber Fractions, there have consequently existed alternative equations.

Any Weber Fraction can be combined, through calculus, with ΔF (expressed as an equation) in order to obtain sensation growth $F(I)$. This is Fechnerian integration. One equation for ΔF is Fechner's Law, $\Delta F = B$, where B is constant. Another is Ekman's Law, $(\Delta F/F) = g$, where g is constant. Traditionally, lower bounds such as the stimulus-detection threshold I_{th} and the corresponding sensation $F(I_{th})$ are ignored. That tradition of *unbounded* integration is replaced here by non-traditional, *bounded* integration. The foundational calculus remains straightforward. Here, twelve Weber Fractions were chosen, all having historical utility. The Weber Fractions were combined with either Fechner's Law or with Ekman's Law to derive the sensation growth, $F(I)$. The result is 24 reimagined sensation-growth equations as alternatives to the status quo. The $F(I)$s have unique features. Notably, the threshold sensation $F(I_{th})$ is always additive for $F(I)$ found using Fechner's Law, and always multiplicative for $F(I)$ found using Ekman's Law. The latter agrees with the notion that sensation at the stimulus-detection threshold is nonzero.

Hopefully, the sensation-growth equations will prove useful; as Weiss[82] (p. 432) notes, "surely a loudness scale is valuable to a manufacturer of audio equipment". Finally, the new equations are examined in the limit as the stimulus-detection threshold I_{th} approaches zero, a limit that is important but, remarkably, is usually ignored in the literature. It transpires that neither an infinitely low stimulus-detection threshold, nor a sensation of zero at *any* value deemed to be the stimulus-detection threshold, are compatible with the present approach to determining sensation growth. Such outdated assumptions need to be abandoned.

7 ACKNOWLEDGMENTS

The author thanks Dr. Claire S. Barnes PhD for many helpful suggestions.

8 REFERENCES

1. G.T. Fechner, *Element der Psychophysik* [*Elements of Psychophysics*, H. E. Adler, Transl.], Holt, New York, NY, USA (1860/1966).

2. G.T. Fechner, *In Sachen der Psychophysik*, Breitkopf & Härtel, Leipzig, Germany (1877).

3. A.S. Reber and E.S. Reber, *The Penguin Dictionary of Psychology*, Penguin Books, New York, NY, USA (2001).

4. E.H. Weber, *De Pulsu, Resorpitione, Auditu, et Tactu: Annotationes Anatomicae et Physiologicae* [*The Sense of Touch*, Transl.: Academic Press for Experimental Psychology Society], Academic Press, London, UK (1834/1978).

5. P. Bouguer, *Traité d'Optique sur la Gradation de la Lumière*, Lacaille, Paris, France (1760).

6. K. Shinomori, B.E. Schefrin, and J.S.Werner, "Age-related changes in wavelength discrimination", *J. Opt. Soc. Am. A Opt. Image Sci. Vis.*, 18, 310-318 (2001).

7. A.T. Smith and N.E. Scott-Samuel, "First-order and second-order signals combine to improve perceptual accuracy", *J. Opt. Soc. Am. A Opt. Image Sci. Vis.*, 18, 2267-2272 (2001).

8. D. Roberson, J.R. Hanley, and H. Pak, "Thresholds for color discrimination in English and Korean speakers", *Cognition*, 112, 482-487 (2009).

9. R.A.J.M. Bierings, M.H. de Boer, and N.M. Jansonius, "Visual performance as a function of luminance in glaucoma: the DeVries-Rose, Weber's, and Ferry-Porter's Law", *Invest. Ophthal. Vis. Sci.*, 59, 3416-3423 (2018).

10. B.-S. Hadad and S. Schwartz, "Perception in autism does not adhere to Weber's law", *eLife*, 2019;8:e42223, 1-15 (2019).

11. S.J. France, B.S. Rosner, P.C. Hansen, C. Calvin, J.B. Talcott, A.J. Richardson, and J.F. Stein, "Auditory frequency discrimination in adult developmental dyslexics", *Percept. Psychophys.*, 64, 169-179 (2002).

12. S.T. Neely, K.S. Schairer, and W. Jesteadt, “Estimates of cochlear compression from measurements of loudness growth”, in *Auditory Signal Processing: Physiology, Psychoacoustics, and Models, papers presented at the XIIIth International Symposium on Hearing (ISH),* Dourdan, France, August 24–29, 2003, editors D. Pressnitzer, A. de Cheveigné, S. McAdams, and L. Collet, Springer Verlag, New York, NY, USA, 50-59 (2005).

13. F. Pedrielli, E. Carletti, and C. Casazza, “Just noticeable differences of loudness and sharpness for earth moving machines”, in *Proceedings of the 155th Meeting of the Acoustical Society of America, 5th Forum Acusticum of the EA, 9^{e} Congrès Français d’Acoustique of the SFA, 2nd ASA-EAA Joint Conference*, Paris, France, Société Française d’Acoustique, 1231-1236 (2008).

14. M. Wojtczak and N.F. Viemeister, “Perception of suprathreshold amplitude modulation and intensity increments: Weber’s law revisited”, *J. Acoust. Soc. Am.*, 123, 2220-2236 (2008).

15. F.-G. Zeng, Q. Tang, and T. Lu, “Abnormal pitch perception produced by cochlear implant stimulation”, *PLoS ONE*, 9: e88662. doi:10.1371/ journal.pone.0088662 (2014).

16. M.R.D. Maslin, M. Taylor, C.J. Plack, and K.J. Munro, “Enhanced intensity discrimination in the intact ear of adults with unilateral deafness”, *JASA Expr. Lett.* EL408, *J. Acoust. Soc. Am.* 137 (2015). http://dx.doi.org/10.1121/1.4914945

17. E. Roverud and E.A. Strickland, “Exploring the source of the mid-level hump for intensity discrimination in quiet and the effects of noise”, *J. Acoust. Soc. Am.*, 137, 1318-1335 (2015).

18. L. Orellana-Escobedo, J.J. Ornelas-Paz, G.I. Olivas, J.A. Guerrero-Beltran, J. Jimenez-Castro, and D.R. Sepulveda, “Determination of absolute threshold and just noticeable difference in the sensory perception of pungency”, *J. Food Sci.*, 77, S135-S139 (2012).

19. S. Camacho, M. Dop, C. de Graaf, and M. Stieger, “Just noticeable differences and Weber fraction of oral thickness perception of model beverages”, *J. Food Sci.*, 80, 1583-1588 (2015).

20. L.-L. Zhang, L. Zhao, H.-Y. Wang, R.-C. Zhi, B.-L. Shi, and N. Xie, “Determination of recognition threshold and just noticeable difference in the sensory perception of pungency of *Zanthoxylum bangeanum*”, *Int. J. Food Prop.*, 19, 1044-1052 (2016).

21. M. Morioka, and M.J. Griffin, “Difference thresholds for intensity perception of whole-body vertical vibration: effect of frequency and magnitude”, *J. Acoust. Soc. Am.*, 107, 620-624 (2000).

22. M. Salada, P. Vishton, J.E. Colgate, and E. Frankel, “Two experiments on the perception of slip at the fingertip”, in *Proceedings of the 12th International Symposium on Haptic Interfaces for Virtual Environment and Teleoperator Systems (HAPTICS’04),* Chicago, IL, USA (2004).

23. M.E. Héroux and F. Tremblay, “Weight discrimination after anterior cruciate ligament injury: a pilot study”, *Arch. Phys. Med. Rehab.*, 86, 1362-1368 (2005).

24. B. Güçlü, “Deviation from Weber’s Law in the non-Pacinian I tactile channel: a psychophysical and simulation study of intensity discrimination”, *Neur. Comput.*, 19, 2638-2664 (2007).

25. E. Francisco, V. Tannan, Z. Zhang, J. Holden, and M. Tommerdahl, “Vibrotactile amplitude discrimination capacity parallels magnitude changes in somatosensory cortex and follows Weber’s Law”, *Exp. Brain Res.*, 191:49 (2008). https://doi.org/10.1007/s00221-008-1494-6

26. N. Gurari, K.J. Kuchenbecker, and A.M. Okamura, "Stiffness discrimination with visual and proprioceptive cues", in *Proceedings of the Third Joint Eurohaptics Conference and Symposium on Haptic Interfaces for Virtual Environment and Teleoperator Systems*, Salt Lake City, UT, USA (2009).

27. E. Samur, J.E. Colgate, and M.A. Peshkin, "Psychophysical evaluation of a variable friction tactile interface", in *Human Vision and Electronic Imaging XIV*, edited by B.E. Rogowitz and T.N. Pappas, *Proceedings of SPIE-IS&T Electronic Imaging, SPIE Vol. 7240*, 72400J (2009).

28. J.W. Sensinger, A.E. Schultz, and T.A. Kuiken, "Examination of force discrimination in human upper limb amputees with reinnervated limb sensation following peripheral nerve transfer", *IEEE Trans. Neur. Sys. Rehab. Eng.*, 17, 438–444 (2009).

29. W.M. Bergmann Tiest, A.C.L. Vrijling, and A.M.L. Kappers, "Haptic perception of viscosity", in *EuroHaptics 2010, Part I, LNCS 6191*, edited by A.M.L. Kappers et al., Springer-Verlag, Berlin, Germany, 29–34 (2010).

30. N.G. Forta, M.J. Griffin, and M. Morioka, "Difference thresholds for vibration of the foot: dependence on frequency and magnitude of vibration", *J. Sound Vib.*, 330, 805-815 (2011).

31. J.K. Holden, E.M. Francisco, Z. Zhang, C. Baric, and M. Tommerdahl, "An undergraduate laboratory exercise to study Weber's Law", *J. Undergrad. Neurosci. Ed. (JUNE)*, 9:A71-A74 (2011).

32. S. Merchel, M. Ercan Altinsoy, and M. Stamm, "Just-noticeable frequency differences for whole-body vibrations", in *Proceedings of InterNoise 2011*, Osaka, Japan (2011).

33. N.B. Debats, I. Kingma, P.J. Beek, and J.B.J. Smeets, "Moving the Weber fraction: the perceptual precision for moment of inertia increases with exploration force", *PLoS ONE*, 7: e42941, 1-9 (2012).

34. C. Hatzfeld and R. Werthschützky, "Just noticeable differences of low-intensity vibrotactile forces at the fingertip", in *Haptics: Perception, Devices, Mobility, and Communication, EuroHaptics 2012*, edited by P. Isokoski and J. Springare, *Lecture Notes in Computer Science, vol. 7283*, Springer, Berlin, Germany (2012).

35. S.B. Schorr, Z.F. Quek, R.Y. Romano, I. Nisky, W.R. Provancher, and A.M. Okamura, "Sensory substitution via cutaneous skin stretch feedback", in *Proceedings of the 2013 IEEE International Conference on Robotics and Automation (ICRA)*, Karlsruhe, Germany (2013).

36. A.L. Skinner, C. Kent, J.M. Rossiter, C.P. Benton, M.G. Groen, and J.M. Noyes, "On the edge: haptic discrimination of edge sharpness", *PloS ONE*, 8:e73283 (2013) (no page numbers).

37. M.Y. Tsalamlal, N. Ouarti, and M. Ammi, "Psychophysical study of air jet based tactile stimulation", in *Proceedings of the IEEE World Haptics Conference (WHC) 2013*, Daejeon, South Korea, 639-644 (2013).

38. S.B.S. Rao and T.L. Simpson, "Measurement of difference thresholds on the ocular surface", *Invest. Ophthal. Vis. Sci.*, 55, 1095-1100 (2014).

39. W. Fu, M.M. van Paassen, and M. Mulder, "On the relationship between the force jnd and the stiffness jnd in haptic perception", in *SAP'17: Proceedings of the ACM Symposium on Applied Perception*, Cottbus, Germany, Article No. 11 (2017).

40. D. Gueorguiev, E. Vezzoli, A. Mouraux, B. Lemaire-Semail, and J.-L. Thonnard, “The tactile perception of transient changes in friction”, *J. Roy. Soc. Interface* 14:20170641 (2017). http://dx.doi.org/10.1098/rsif.2017.0641

41. C.W. Savage, *The Measurement of Sensation,* University of California Press, Berkeley, CA, USA (1970).

42. S.C. Masin, “The (Weber’s) law that never was”, in *Proceedings of the 25th Annual Meeting of the International Society for Psychophysics*, Galway, Ireland, 441-446 (2009).

43. H.E. Ross, “On the possible relations between discriminability and apparent magnitude”, *Br. J. Math. Stat. Psychol.*, 50, 187-203 (1997).

44. L. Nizami, J.F. Reimer, and W. Jesteadt, “The intensity-difference limen for Gaussian-enveloped stimuli as a function of level: tones and broadband noise”, *J. Acoust. Soc. Am.*, 110, 2505-2515 (2001).

45. L. Nizami, J.F. Reimer, and W. Jesteadt, “The mid-level hump at 2 kHz”, *J. Acoust. Soc. Am.*, 112, 642-653 (2002).

46. L. Nizami, “Intensity-difference limens predicted from the click-evoked peripheral N_1: the mid-level hump and its implications for intensity encoding”, *Math. Biosci.*, 197, 15-34 (2005).

47. L. Nizami, “The intensity-difference limen for 6.5 kHz: an even more ‘severe departure from Weber’s law’”, *Percept. Psychophys.*, 68, 1107-1112 (2006).

48. S. Buus, H. Musch, and M. Florentine, “On loudness at threshold”, *J. Acoust. Soc. Am.*, 104, 399-410 (1998).

49. S. Buus and M. Florentine, “Growth of loudness in listeners with cochlear hearing losses: recruitment reconsidered”, *J. As. Res. Otolaryngol.*, 3, 120-139 (2002).

50. L. Nizami, “A qualitative model relating loudness to detection ‘near threshold’ and above, giving zero loudness at a ‘hard threshold’ and nonzero loudness at a ‘soft (psychometric) threshold’”, in *Abstracts of the 32nd Annual Midwinter Research Meeting of the Association for Research in Otolaryngology*, Baltimore, MD, USA, 77 (2009).

51. L. Nizami, “Flaws in loudness standard ANSI S3.4-2007: the derivation of nonzero sensation at absolute detection threshold is circular”, in *Proceedings of InterNoise 2015,* San Francisco, CA, USA, Paper 38, (2015).

52. L. Nizami, *On Auditory Dynamic Range*, Doctoral dissertation, Dept. of Psychology, University of Toronto, Toronto, Ontario, Canada (1999).

53. L. Nizami, “Dynamic range relations for auditory primary afferents”, *Hear. Res.*, 208, 26-46 (2005b).

54. J.A. Swets, “Is there a sensory threshold?”, *Science*, 134, 168-177 (1961).

55. L. Nizami, "Lütkenhöner's 'Intensity dependence of auditory responses': an instructional example in how not to do computational neurobiology", in *Advances in Computer Science and Engineering*, edited by M. Schmidt, InTech, Rijeka, Croatia, 391-406 (2011).

56. L. Nizami, "Mistakes and misrepresentations in Heil, Neubauer, and Irvine (2011): why their improved equation is not, and the root cause of their failure, and how the Nizami-Schneider equation already deals with that root cause", in *Proceedings of the World Congress on Engineering and Computer Science 2014 (Lecture Notes in Engineering and Computer Science, Volume 2, WCECS 2014)*, San Francisco (Berkeley), CA, USA, October 2014, 567-575 (2014).

57. R.P. Hellman and J.J. Zwislocki, "Some factors affecting the estimation of loudness", *J. Acoust. Soc. Am.*, 33, 687-694 (1961).

58. R.L. McBride, "Psychophysics: could Fechner's assumption be correct?", *Aus. J. Psychol.*, 35, 85-88 (1983).

59. L.E. Krueger, "Reconciling Fechner and Stevens: toward a unified psychophysical law", *Behav. Brain Sci.*, 12, 251-320 (1989).

60. K.O. Johnson, S.S. Hsiao, and T. Yoshioka, "Neural coding and the basic law of psychophysics", *The Neuroscientist*, 8, 111-121 (2002).

61. F.W. Nutter Jr. and P.D. Esker, "The role of psychophysics in phytopathology: The Weber-Fechner law revisited", *Euro. J. Plant Pathol.*, 114, 199-213 (2006).

62. E.N. Dzhafarov and H. Colonius, "The Fechnerian idea", *Am. J. Psychol.*, 124, 127-140 (2011).

63. S.S. Stevens, "A metric for the social consensus", *Science*, 151, 530-541 (1966).

64. G. Ekman, "Weber's law and related functions", *J. Psychol.*, 47, 343-352 (1959).

65. M.J. Plateau, "Sur la mesure des sensations physiques, et sur la loi qui lie l'intensité de ces sensations à l'intensité de la cause excitante", *Bull. de l'Acad. Roy. Sciences Lettres Beaux-Arts de Belg. 2me. Sér.*, 33, 376-388 (1872).

66. T. Mayer, "Experimente über die Schärfe des Gesichtsinnes", *Götting. Anz. gelehr. Sach.*, 20, 401–402 (1754).

67. O.-J. Grüsser, "The discovery of the psychophysical power law by Tobias Mayer in 1754 and the psychophysical hyperbolic law by Ewald Hering in 1874", *Behav. Brain Sci.*, 16, 142-144 (1993).

68. H. Aubert, *Physiologie der Netzhaut*, E. Morgenstern, Breslau, Germany (1865).

69. P. Langer, *Die Grundlagen der Psychophysik: eine kritische Untersuchung*, Hermann Dufft, Jena, Germany (1876).

70. P.G. Nutting, "The luminous equivalent of radiation. III. Subjective light. 12. Photometric sensibility", *Bull. Bur. Stand.*, 5, 285-308 (1908).

71. H. von Helmholtz, *Helmholtz's Treatise on Physiological Optics. Vol. II. The Sensations of Vision*, George Bansha Publishing Company for Optical Society of America, Menasha, WI, USA (1924).

72. S. Hecht, “A theory of visual intensity discrimination”, *J. Gen. Physiol.*, 18, 767-789 (1935).

73. R.D. Luce and W. Edwards, “The derivation of subjective scales from just noticeable differences”, *Psychol. Rev.*, 65, 222-237 (1958).

74. D.H. Krantz, “Integration of just-noticeable differences”, *J. Mat. Psychol.*, 8, 591-599 (1971).

75. R. Pierrel-Sorrentino and T.G. Raslear, “Loudness scaling in rats and chinchillas”, *J. Comp. Physiol. Psychol.*, 94, 757-766 (1980).

76. J.P. Guilford, “A generalized psychophysical law”, *Psychol. Rev.*, 39, 73-85 (1932).

77. W.J. McGill and J.P. Goldberg, “A study of the near-miss involving Weber’s law and pure-tone intensity discrimination”, *Percept. Psychophys.*, 4, 105-109 (1968).

78. R.R. Riesz, “The differential intensity sensitivity of the ear for pure tones”, *Physiol. Rev.*, 31, 867-875 (1928).

79. W.M. Hartmann, *Signals, Sound, and Sensation*, Springer, New York, NY, USA (1998).

80. J. Delboeuf, “Étude psychophysique. Recherches théoriques et expérimentales sur la mesure des sensations et spécialement des sensations de lumière et de fatigue”, *Mém. cour. autres mém. pub. par l’Acad. Roy. Sciences Lettres Beaux-Arts de Belg.*, 23, 1-115 (1873).

81. I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products*, Academic, Berkeley, CA, USA (1980).

82. D.J. Weiss, “The impossible dream of Fechner and Stevens”, *Perception*, 10, 431-434 (1981).

Table 1. Weber Fractions $\Delta I/I$ and the components $\mathcal{G}(I)$ of the respective derived equations for sensation growth: $F(I) = K_1\mathcal{G}(I) + F(I_{th}) - K_1\mathcal{G}(I_{th})$ when assuming $\Delta F = B$ (Fechner's Law), and $F(I) = F(I_{th}) \cdot H(I)/H(I_{th})$ when assuming $(\Delta F/F) = g$ (Ekman's Law), where $H(I) = exp\big(K_2 \cdot \mathcal{G}(I)\big)$ and $K_2 = K_1 g/B$. The constants K, B, C, c, K_1, and K_2 all exceed zero. The value of K need not be the same from one Weber Fraction to another; the same symbol is used merely for convenience.

Weber Fraction, $\frac{\Delta I}{I}$	Source (footnote)	K_1	$\frac{\mathcal{G}(I)}{B}$	$\lim_{I_{th}\to 0} \frac{\mathcal{G}(I_{th})}{B}$
K	**a**	$\frac{1}{K}$	$ln\,I$	$-\infty$
$\frac{K}{ln\,I}$	**b**	$\frac{1}{2K}$	$(ln\,I)^2$	∞
$KI^{-\nu}, 0 < \nu < 1$	**c**	$\frac{1}{K\nu}$	I^ν	0
$C + KI^{-\nu}, 0 < \nu < 1$	**d**	$\frac{1}{C\nu}$	$ln\left(I^\nu + \frac{K}{C}\right)$	$ln\left(\frac{K}{C}\right)$
$\frac{K(I + C)}{I}$	**e**	$\frac{1}{K}$	$ln(I + C)$	$ln\,C$
$\frac{K(I + c)(I + C)}{I}, C > c$	**f**	$\frac{1}{K(C - c)}$	$ln\left(\frac{I + c}{I + C}\right)$	$ln\left(\frac{c}{C}\right)$
$\frac{K(I^2 + C)}{I^2}$	**g**	$\frac{1}{2K}$	$ln(I^2 + C)$	$ln\,C$

$\frac{K(\sqrt{I}+C)^2}{I}$	**h**	$\frac{2}{K}$	$ln(\sqrt{I}+C)-\frac{\sqrt{I}}{\sqrt{I}+C}$	$ln\,C$
$\frac{(\sqrt{I}+C)^2}{I}-1$	**i**	$\frac{1}{C}$	$\sqrt{I}-\frac{C}{2}ln(2\sqrt{I}+C)$	$-\frac{C}{2}ln\,C$
$\frac{K}{I(\sqrt{I+C}-\sqrt{I})^2}$	**j**	$\frac{1}{K}$	$\frac{C^2}{2}ln(\sqrt{I}+\sqrt{I+C})+\sqrt{I}\sqrt{I+C}\left(\sqrt{I}\sqrt{I+C}-\frac{C}{2}-I\right)$	$\frac{C^2}{2}ln\sqrt{C}$
$KI^{b-1}-1, b\neq 1$	**k**	$\frac{1}{b-1}$	$ln\left(1-\frac{1}{KI^{b-1}}\right)$	$0, 0<b<1$ $ln(-\infty), b>1$
$\frac{K\left(\sqrt{I+C}+\frac{K}{4}\right)}{I}$	**l**	$\frac{2}{K}$	$\sqrt{I+C}+\frac{K}{4}\left(1-ln\left(\sqrt{I+C}+\frac{K}{4}\right)\right)$	$\sqrt{C}+\frac{K}{4}\left(1-ln\left(\sqrt{C}+\frac{K}{4}\right)\right)$

a. Weber[4]. **b**. Fechner (Ref. 2, p. 19), reproduced from Aubert (Ref. 68, p. 69). **c**. Uncertain provenance, but old; allegedly used by Mayer[66] for visual acuity (cited in Grüsser[67]). Fechner (Ref. 2, p. 21) notes that it can be derived from Plateau (Ref. 65, p. 384). It is also found in Guilford (Ref. 76, p. 79). In auditory research, it is often attributed to McGill and Goldberg[77]. **d**. Nutting (Ref. 70, p. 292), under the following changes from Nutting's notation to the present notation: "P_m" $= C$, "n" $= \nu$, and "$(1-P_m)I_0^n$" $= K$. Note well that a popular Weber fraction of Riesz[78], $\frac{\Delta I}{I}=S_\infty+(S_0-S_\infty)\left(\frac{I}{I_0}\right)^{-r}$ where $0<r<0.5$, is a variation on Nutting's (as acknowledged by Riesz). Riesz performed hearing experiments and used energy as his variable rather than intensity, but in that case, energy (per meter-squared of area) equals intensity multiplied by the stimulus duration (Ref. 79, p. 31); using stimuli of fixed duration, the duration divides out of the Weber fraction. **e**. Fechner (Ref. 2, p. 35), reproduced from Delboeuf (Ref. 80, pp. 21, 54). Delboeuf himself named von Helmholtz as the actual source, but provided no references; the equation can, in fact, be found in the later English translation of von Helmholtz (Ref. 71, p. 177). **f**. Fechner (Ref. 2, p. 17), credited there (incompletely) to von Helmholtz. The equation can be found in the later English translation of von Helmholtz (Ref. 71, p. 180). **g**. Fechner (Ref. 2, p. 41), reproduced from Langer (Ref. 69, p. 62). **h**. Hecht (Ref. 72, p. 772). This is equivalent to the equation $\frac{\Delta I}{I}=c\left(1+\sqrt{1/KI}\right)^2$ common to papers of Hecht and co-authors from that era. **i**. Pierrel-Sorrentino and Raslear (Ref. 75, p. 765), for the case n = 0.5. Also written as $\frac{\Delta I}{I}=\frac{C(2\sqrt{I}+C)}{I}$. **j**. Hecht (Ref. 72, p. 772). **k**. Luce and Edwards (Ref. 73, p. 228). The integral is solved through the substitution $t=I^{b-1}$, resulting in a well-known form of the integrand, found in Gradshteyn and Ryzhik[81]. **l**. Krantz (Ref. 74, p. 595).