Individuals Enough For Classes

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Abstract
This paper builds on the system of David Lewis's "Parts of Classes" to provide a foundation for mathematics that arguably requires not only no distinctively mathematical ideological commitments (in the sense of Quine), but also no distinctively mathematical ontological commitments. Provided only that there are enough individual atoms, the devices of plural quantification and mereology can be employed to simulate quantification over classes, while at the same time allowing all of the atoms (and most of their fusions with which we are concerned) to be individuals (that is, urelements of classes). The final section of the paper canvasses some reasons to be committed to the required ontology for other than mathematical reasons.

Platonism about classes is a very attractive position, and the desire to see if we can capture the power of class theory with the minimum of metaphysical resources is also understandable, since the metaphysical commitment to mathematical objects can seem both mysterious and extravagant. There is also the independent interest in seeing what theoretical tasks can be achieved with which theoretical resources, both in order to understand our theoretical resources and to more deeply understand the connections (or possible connections) between our different commitments. One "reduction" of mathematical commitments has recently been offered which, if successful, explains the "ideology" of set or class theory in terms of less distinctively mathematical, and arguably more familiar terms. In the account of mathematics offered in the appendix of Lewis 1991 (co-written with John Burgess and A.P. Hazen) and again in Lewis 1993, the membership relation is structurally defined in terms of mereological relationships (i.e. part-whole relationships) and plural quantification. Since membership can be taken as the fundamental set-theoretic and class-theoretic relationship, a foundation for class theory (and thus, if a suitable form of structuralism is right, most or all of mathematics) is thus provided. All that needs to be added is an ontological postulate: that there are enough things (indeed, canonically, mereologically atomic (i.e. indivisible) things)[1]. Fortunately, saying that there is enough can be done without invoking class theory either: it too can be done in terms of mereology and plural quantification, and so it seems that class theory can be done "nominalistically", in Goodman's sense (Lewis 1993 p 17). Furthermore, the

[1] It may be possible to effect the megethological reconstruction of class theory without assuming that the requisite things are atomic: see Hazen 1996.

ontology, while 'quantitatively' vast, is no more than standard Platonic set and class theories require anyway—and so an attractive package is available.

The framework offered to us by Lewis, Burgess and Hazen (though it should be noted that Burgess and Hazen do not endorse the framework, and Hazen 1993 and 1997 indeed objects to it) does have an annoying drawback, and one which makes it seem as though the ghost of specifically mathematical commitments have not been entirely banished. The drawback is that most of the objects postulated cannot be considered to be class-theoretic individuals—they cannot be treated as ur-elements.[2] Structuralism muddies the water here, in ways I shall go on to explain, but one might be entitled to think that much of the ontology postulated was distinctively mathematical if the objects postulated could not all be considered as ur-elements, or as non-classes. Indeed, even with structuralism, Lewis's framework leaves us with "our lamentable ignorance of the whereabouts and character of the classes" (Lewis 1993 p 17). The framework cannot be as neutral about what it is that makes up so much of Reality as perhaps we would like, especially if we wish to rest class theory on as austere foundations as possible. This paper explores options for producing a modified account based on the resources of the appendix of Lewis 1991 (hereafter the Appendicital account), in which the entire ontology postulated can be satisfactorily mathematically treated as being composed of non-classes. In doing so, this paper more closely approaches the goal of doing mathematics without distinctively mathematical relationships nor distinctively mathematical ontology. I shall also discuss the option of combining the systems developed with a style of modalist account of class theory, and the advantages which might thereby be gained.

1. The State-of-Play in Lewis's System

Lewis 1991 begins with the suggestion that the relation of sub-set to set (and sub-class to class) is the relationship of part-to-whole: as the title suggests, classes do have parts, and those parts (according to the so-called Main Thesis, p 7) are all and only the sub-classes of that class. Given this, the only class-theoretic notion that remains to be

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[2] The reason for this is that there must be a proper class of sets, as well as all the proper classes which are literally beyond number, whereas there must be merely a set of ur-elements in Lewis's system (the proof of this is left as an exercise).

explained is the relation of an object to the unit class of that object (the class that has only that object as its member): for given unit classes, or singletons, and the relation of sub-classhood, the usual primitive of class-theoretic membership can be defined. (A class has an object as a member iff the singleton of that object is a part of that class).

In order to reconstruct set theory on the basis of this idea, Lewis employs several theoretical resources. The first is mereology, the formal treatment of the relation of part-to-whole. In particular, Lewis uses a mereological system powerful enough to admit Unrestricted Composition: that for any objects whatsoever (including, obviously, classes), there is a fusion or mereological sum of those objects. For most purposes, Lewis relies on an atomic mereology: a mereology which takes everything to be ultimately composed of partless things (mereological atoms). A terminological note is in order here: as well as the chemical sense of "atom", the word has two distinct technical senses in mereology and in set theory. In mereology, an "atom" is a partless thing (or a partless thing which itself is potentially part of other things, which comes to the same thing in the context of this paper, but may not in some theories of the part-whole relation). In set-theory, on the other hand, "atom" is sometimes used to mean "urelement"—a member of a set or class which is not itself a set or class. I will be using "atom" only in its mereological sense in this chapter.

As well as mereology, the account employs plural quantification. Plural quantification is quantified talk about several things at once which is not to be understood as singular quantifiers over collective entities (sets, groups, collections, or whatever). For reasons similar to those of Boolos 1984 (which the interested reader is referred to for a classic presentation of plural quantification), Lewis argues that plural quantification is an ordinary part of natural language, and not in any case to be understood as some form of singular quantification. That plural quantification makes sense in its own right is controversial, so Lewis's framework rests here on controversial foundations. Nevertheless, I shall take it for granted for the purposes of this paper that plural quantification is in order, and is ontologically innocent. Plural quantification, though similar to quantification over higher-order objects in some ways, is distinguishable formally at least from full second-order quantification in that there is no mechanism for plurally quantifying over "pluralities": while I may plurally quantify over trees (e.g. some trees surround my house), I cannot plurally quantify over pluralities of trees (e.g. some bunches of trees surround houses, other bunches do not). At that point, commitment to higher order entities must be invoked (in the example, bunches of trees as well as just trees). Plural quantifiers are analogous formally to monadic second-order quantifiers. In this paper I shall sometimes use upper-case letters for plural variables or as constant labels for certain pluralities.

Classes, for Lewis, are mereological sums, or aggregates, of singletons (unit sets). While any singletons form a class, not every class will have a singleton, on pain of naive paradoxes: some of the largest classes (the proper classes) lack singletons. Those classes which have singletons are sets, as in standard class theories (e.g. Gödel 1940). As well as classes, Lewis has a null set, which is not a class on his definition since it is not a unit class or union of unit classes. Lewis, for convenience, stipulates that the null set will be the fusion of non-classes (pp 10-15), though I shall not follow this stipulation, since it will not be convenient in the systems I discuss. The null set plus the classes which belong to classes comprise the sets, and the sets plus the proper classes are Lewis's mathematical universe. All that remains to be accounted for is the singleton relation (the relation between a unit-class and its member), and what all these atomic singletons might be.

Two alternative accounts of the relation between singleton classes and their members are offered in *Parts of Classes*. The official line presented in the body of the book is that the relation between singletons and their members is a primitive, internal, and perhaps somewhat mysterious one, which we should nevertheless postulate because of the utility of set theory. In the appendix, however (co-written with John Burgess and Allen Hazen), *Parts of Classes* discusses a different view of the singleton relation: various strategies for reducing the singleton relation into a complicated mereological connection are discussed. Of these, my favourite is the variety which employs the *Method of Double Images*[3]. I will not try to explain the detail here (the

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[3] (Lewis 1991, pp 121-127), where it is credited to Burgess. I prefer it to the Method of Extraneous Ordering (p127-133) proposed by Hazen (and the 'Hybrid Method' which partially relies on it) in this context, because I find the "Well Ordering Assumption" of Hazen's method to be less intuitive than the assumptions behind Burgess's method, though since both make assumptions of the same power this preference may merely be aesthetic. In any case, Burgess's method can be extended to cope with an atomless ontology better than Hazen's method (see Hazen 1997), and while I am not concerned with

reader is referred to Lewis 1991 or Lewis 1993), but the strategy is as follows: by means of mereology and plural quantification, one uses The Method of Double Images to be able to simulate quantification over pairs of objects for which an ordering is specified. (That is, the Method enables us to tell which of the pair is the first object and which is the second). Then[4] mathematical structuralism is invoked: the idea that in our talk of the singleton relation, we are not talking unequivocally about one given relation, but are really generalising about any relation that satisfies the constraints given by the axioms governing the singleton relation. So, according to structuralism, we are talking about all the relations which have the formal character of "the relation" apparently described by the axioms of set theory. Given structuralism and given that mereology and plural quantification have served up pluralities of pairs which are effectively ordered, one then takes talk about the singleton relation to be treated as a generalisation about all the pluralities of special pairs that satisfy the conditions laid down by the axioms. It is as if we have used mereology and plural quantification to give us all the classes of ordered pairs, and then applied a structuralist interpretation to the axioms so that claims about the singleton relation are true iff the claims are true for all classes of ordered pairs which conform to the axioms. The only difference is that instead of classes of ordered pairs, we have pairs which we discuss plurally and which are "ordered" by means of the Double Images device.

This method has one great advantage, and one evident drawback. The advantage is that it explains the singleton relation using only mereology and plural quantification (resources which Lewis argues we have in any case) — and while it does not completely remove the involvement of internal relations, the two internal relations which remain — the relation of part-to-whole and the relation between pluralities and the things that make them up — are presumably less mysterious and more everyday than the internal primitive singleton relation would have been. The evident drawback is that the Appendicital account of the truth of claims about singletons is not, it seems, what mathematicians actually think they are talking about

the details of such an extension in this paper, relying on Burgess's method provides a structure that needs less alteration when extended.

[4] Strictly speaking, between these two steps are further applications of the method of Double Images to provide pairs which can deal with "Gunk" — see Lewis 1991 pp 133-136 for details of this step.

when they talk about set-membership or the singleton relation — I believe it needs to be seen as a *reform* to mathematics. Lewis is reluctant to have metaphysicians tell mathematicians that they were wrong all along, and is not sure that how much weight his suspicion of postulated internal relations should be given, so his considered position in Lewis 1991 seems to be undecided between the view of the singleton relation given in the text and the sort provided in the appendix. However, by Lewis 1993, Lewis seems to have sided with the account of the Appendix, apparently having overcome his scruples about correcting mathematicians. Lewis 1993 also provides a proof that, given the basic constraints of his framework (including particularly some hypotheses expressed using mereology and plural quantification about the number of atoms in Reality, so called "megethological" claims[5]), there will be pairs which satisfy the constraints that a singleton relation must conform to.

2. How atoms can perform double duty

In this section I will outline how, given the resources of the Lewis-Burgess-Hazen framework of Lewis 1991 and Lewis 1993, a system can be constructed so that there is almost no distinctively mathematical ontology: certainly there is no need to quarantine a Large group of atoms from the individuals to ensure there are enough singletons. The system outlined in this section will then be discussed and modified in subsequent sections. The following sections rely on the technical machinery established in Lewis 1991, particularly the appendix to that book (hereafter it will be referred to simply as "the Appendix"), and shall make several of the same assumptions about class theory (e.g. that the relation of subclass to class is that of part to whole) but by and large I shall not duplicate that machinery here, but rather give references to the relevant postulates and proofs. In addition, I will concentrate for the most part on providing a class theory with the assumption that there are no atomless things (no *Gunk*, to use Lewis's term). This is because adding Gunk to the picture is somewhat complicated, and because the method for extending a treatment of the sort I will be discussing from a treatment of atoms alone to a treatment of atoms and Gunk has already been substantially provided in the Appendix, pp 133-136. However, I

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[5] "Megethology", Lewis's coinage, is used to refer to systems which use the combination of mereological devices and plural quantification to make claims about the "size" of the universe. The origin is "from *megethos* 'size' + *logos* 'doctrine'" (Lewis 1991 p 136).

shall provide some modifications to the procedure for accommodating Gunk needed for the system being constructed at the end of this piece. I will proceed with an analogue of the *Method of Double Images* (Lewis 1991 pp 121–127), for reasons mentioned in footnote three.

The idea behind the approach to be taken is that atoms must serve "double duty" — the very same atoms must be capable of functioning as individuals and as classes. The challenge is to distinguish atoms to be treated as individuals (or "in their individual aspect") and atoms to be treated as being singletons (or "in their singleton aspect"), using only the resources of mereology and plural quantification. After all, we have to produce a structure isomorphic to the traditional picture of sets, where an atom is an individual or a singleton once and for all, and it is unequivocally false to say of anything which is an individual that it has any members, and unequivocally true to say of any singleton that it has a member.

The method is surprisingly simple (or at least all that is required is a simple addition to the Method of Double Images). I will outline a proposed method of meeting the desiderata, and will then discuss what advantages and disadvantages such a system would produce.

Like Lewis, I will begin with Lewis's "framework", which includes: a system of principles governing plural quantification and mereology (including some principles analogous to set-theoretic principles, such as Choice and Replacement schemata), and in addition some constraints on the size of the universe which can be stated without employing set-theoretic language but which provide the system with power analogous to that of systems of set or class theory. I will begin by assuming hypotheses P, U and I about the size of the universe (Lewis 1991 p 93-94)[6]. Then various "principles of the framework" need to be affirmed. Lewis mentions principles which he will employ as needed, without attempting to systematise them or to derive them from a comprehensive unified basis. More work could be done in providing for a basis for a framework for the job of providing for class theory, but for now I am content to follow Lewis. Principles affirmed include two Choice schemata on pp. 71-72, two Replacement schemata on pp. 91-92, a Dedekind schema on p. 88, and some

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[6] The adoption of these hypotheses is also one of the many places in the technique which relies on plural quantification: if plural quantification is not in order, or requires set theory in order to make sense, then the project is in trouble very early.

principles about size affirmed on p. 90-91. Again, to save needless duplication, the reader is referred to Lewis 1991.

Next, designate some atom "n" (for null). Every class will contain n as a part. Let us also define some features of a relation (or quasi-relation) called the "Esingleton relation" relation (for in some ways this will be an ersatz singleton relation). I will speak to begin with as if Esingleton is a typical relation, and then provide a recasting of such talk in terms of mereology and plural quantification. Esingleton is a relation that holds between atoms and other objects —either atoms or fusions of atoms (since for the moment we are not concerned with Gunk). Esingleton is asymmetric, and let us call the atom in the first place of the relation the "Esingleton" of the object in the second place of the relation in each case of the relation. The Esingleton relation also obeys the following restrictions:

1. *Distinctness*[7] — no two objects share the same Esingleton
2. *Functionality* — nothing has more than one Esingleton
3. *Domain* — n has an Esingleton, any small fusion of atoms which does not overlap n has an Esingleton, any small fusion of Esingletons and n has an Esingleton, and nothing else has an Esingleton.
4. *Null Set* — n is not an Esingleton of anything.
5. *Atomicity* — all Esingletons are atoms.[8]

The first three of these constraints are at least roughly analogous to constraints Lewis places on the Singleton relation in his system (Lewis 1991 p. 95). Null set is distinctive in this system, but obvious enough given the intended treatment of n as the null class. Lewis's system also has an axiom called Induction (p. 96), which derives its usefulness from classes being distinct from individuals, and so is much less useful in the system I am to set up. Induction has two main uses in Lewis's system. One is to ensure that singletons are atomic (p. 96) — the counterpart of which is explicitly ensured by my Atomicity. The other thing which Induction is employed for is to ensure that the class theory he derives is well founded — that is, it has the equivalent

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[7] The names of these conditions bear some connection with the names of conditions on the singleton relation in *Parts of Classes* — they are often different from the ones stated there however, and the names are to suggest no more than similarity in function.

[8] Atomicity and Null Set might be thought to have been already covered by what was said about the Esingleton relation in the text. Their explicit addition to the list of axioms cannot hurt, especially since it is virtuous to explicitly state axioms where possible.

of an axiom of Foundation or Regularity (Lewis uses the German name for such an axiom: *Fundierung*). This is not an essential feature of set or class theories, but it does form part of the standard axiomatisations. It would be as well to ensure that the system I will develop will also display this feature. To ensure this I must add another constraint of the Esingleton relation:

> 6. *Foundation* — whenever there are some things, either at least one of them is not an object which is the fusion of at least one Esingleton with n and which contains no non-Esingletons besides n as parts, or one of them is such that none of the objects which have Esingletons among its parts are among those things.

Note that these theses are not enough to totally characterise the Esingleton relation — for example, they leave open whether there are any small fusions of atoms containing n which lack an Esingleton, since they leave open whether there are any atoms besides n which fail to be Esingletons. Nevertheless, they are enough of a partial characterisation to allow for a reconstruction of a notion of class which will have the familiar features (or most of them). These characterisations of classes and related terminology are only a first pass at the definitions that will eventually be adopted (structuralism has not yet been invoked, for example):

> A *class* is any fusion that exhaustively divides into n and Esingletons (n on its own is a class, but every class has n as a part). An object is a *member* of a class iff that object's Esingleton is a part of that class. A class is a *set* iff it is small.

Alternatively, we could define a set as a class which possesses a singleton, as it will turn out that given Domain and the definition of class and membership that this comes to the same thing. What to say about what individuals are is slightly tricky, as I will discuss further below. For now, assume that all fusions of atoms that are distinct from n are individuals.

This system provides an adequate interpretation for the axioms of class theory, and does so without any distinctively mathematical ontology (with the exception of n, at this stage). That is, aside from n, the other atoms are non-classes and fusions not containing n are also non-classes. The versions of the "standard axioms" of set theory which Lewis recovers in pp 100-107 can also be recovered by this system, once the definition of individuals is suitably tweaked—the proofs that the standard axioms can be recovered may be found in the first appendix of Nolan 2002, though as they are routine and in many respects similar to the recovery of the standard axioms in Lewis 1991 I will not include them here. The "standard axioms" that this system supports provide for models of ZF, with Choice and (if desired) ur-elements. Of course, given ZF+Choice, models of ZF without Choice can be constructed. As well as providing models for these popular set theories, the system presented and the system presented by Lewis are class theories. With a suitable plural comprehension axiom, the system delivered by Lewis and by this system is Quine-Morse class theory (see Fraenkel, Bar-Hillel and Levy 1973 p 138-142).[9] Megethological conditions can also be imposed to ensure that various large cardinal axioms hold (see Lewis 1991 p 137-139), or other favourites such as the Generalised Continuum Hypothesis (Lewis 1991 p 139), though the project of determining what megethological conditions might be needed to ensure other popular additions to set-theory is a project which is, to my knowledge, largely unexplored.

The system of Esingletons also retains many of the intuitions which Lewis mobilises to make his account of classes plausible in the first place (these intuitions, which he expresses in the form of theses, may be found on p 4 and p 7 of Lewis 1991). His First Thesis, that one class is part of another iff the first is a subclass of the second, is true in the system outlined, as is his Priority Thesis, that no class is part of any individual. Given the assumption that all fusions distinct from n are individuals, his Fusion thesis is retained as well (though I shall soon examine and adopt reasons why this assumption about individuals should be rejected, and so the Fusion thesis ought to be restricted). His Division Thesis is also satisfied in the form in which it is stated on p 7: Reality does "divide exhaustively into individuals and classes" (and indeed every object does, since all the atoms are either n or distinct from n). However, Lewis now thinks that his intended Division Thesis is not captured (or at least not unambiguously captured) by this formulation, and the intended formulation[10] (that everything is a class, an individual, or a fusion of classes and individuals), will not necessarily be, since some fusions of n and individuals may be neither. So in one form or another all of the basic intuitions he outlines in the start of Parts of Classes are

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[9] The proof that a suitably powerful axiom of impredicative comprehension is provided by these systems can also be found in the first appendix of Nolan 2002.

[10] See the note on p 208 in the version of "Mathematics is Megethology" printed in Lewis 1998.

vindicated by the above system. However, the above system does not satisfy his Main Thesis (that the parts of a class are all and only their subclasses). This might seem surprising in light of the fact that Lewis claims his Main Thesis follows from those four basic claims. It does so, however, only on his preferred understanding of the Division Thesis.

The system outlined, then, provides classes which will do the jobs demanded of them by mathematicians, since they satisfy the "standard axioms", allowing for models of standard set theories. It also has the advantage that, despite all atoms but n being individuals, no individual is also a class. More modifications need to be made before the theory will suit our other purposes, however. For one thing, the above system relies on the Esingleton relation, whereas we wish to have a system that relies only on mereology and plural quantification. The system as it stands also has some unusual features which would be nice to iron out. The special status of n, for example, is somewhat of a worry. It is a thing which is not part of any individual, and as the theory stands it seems that we are at least committed to one piece of mathematical ontology — the null set. It would be nice if we could dispense with distinctly mathematical ontology altogether. Another problem is that many individuals are playing double roles which are intuitively curious, for it seems intuitively curious that individuals are parts of, perhaps quite unrelated, classes. It would be unusual if, for example, a fusion of atoms was a lump of rock and also served, when fused to the null set, to be the set of all actual horses. The Method of Double Images will be used to provide a substitute for the Esingleton relation which relies on no more than mereology and plural quantification (plus some auxiliary assumptions), and mathematical structuralism will come to the rescue to solve the problems of the null set and dual roles of individual atoms.

The use of the Method of Double Images will be a relatively straightforward device to give some "ordered" pairs such that the first one of each is an Esingleton and the second an object. Use the Method to partition all of reality. Any R (see the Appendix, p 126) defined using the partition and which consists of b-pairs such that they obey the conditions set out for the "Esingleton relation" will do to define a substitute for the Esingleton relation. These Rs will be substitutes for the Esingleton relation in a manner almost completely analogous to the manner in which defined Rs satisfy the conditions to be singletons in the Appendix, so I will not dwell on how this is done here.

Instead of attempting the impossible task of selecting which R are the ones that corresponds with the *real* Esingleton relation, structuralism can provide a way to explain talk about classes — as *generalisations* about any Rs which meet the requirements (as in the Appendix, pp 140-141). In fact, structuralism should be taken further. There are at least an infinite number of choices which could be made about which atom is to be treated as the null set — which atom is to receive the honour of the name "n". In the absence of any distinctively mathematical ontology, selecting which atom is *really* n is as hopeless a task as determining which R are the real R in the absence of any distinctively mathematical relations. The candidates to jointly satisfy the conditions we have put on the Esingleton "relation" and the atom to be treated as n will be pluralities where all but one of them are b-pairs of the sort discussed in the previous paragraph and one of them is an atom — the atom which is to be treated as n relative to those b-pairs. Each plurality (let us call them Ps) which has as one of them an atom and the rest of them as b-pairs of the sort described will satisfy the formal conditions to be treated as if the atom is n and the plurality of b-pairs are taken to be the R which will do the work of the Esingleton relation. Relative to any given P, then, it will be possible to state which atoms are members of which classes, and which fusions are classes and which are individuals, and so on. A statement in set theory (or class theory) will, according to structuralism, be true iff it is something which is true for all P. The null set possesses a singleton, for example, because no matter which P we select, the atom designated to be n will have its image be the "second" member of a b-pair (and only one b-pair, from Functionality), and so the object which has its first image as a member of the b-pair will count as the Esingleton of the atom n relative to that P. There will be a fusion of that first member and the "n-atom", which will count as the singleton of n relative to taking the given P to give us the Esingleton "relation" and the null atom.

Lewis favours treating this structuralist generalisation as a form of supervaluation (see van Frassen 1966): in a case of indeterminacy (such as the case where there are multiple eligible candidates to function as the Rs), a sentence about the singleton relation (or its substitute) is true if it is true no matter which R might be selected to be the correct one, false if it is false no matter which R might be selected, and truthvalueless otherwise. So the claim that the atoms which constitute my coffee cup (if indeed it is atomic) include the null set will, for all we have said so far, be truthvalueless, since there are Ps such that the atom they would designate to be n is a part of my cup, but others (many more others) which would accord some other atom that honour. My talk of possible "selection" is metaphorical, of course, since the method does not rely on it being possible for one "candidate" rather than another to actually be picked out, but the method should be clear enough.

However, there are other related methods of dealing with generalisations, particularly with those cases in which a claim is neither true for all possible assignments ("supertrue") or false for all possible assignments ("superfalse"). Some variations consist of assigning those claims which are neither supertrue nor superfalse some extra truth-value(s) intermediate between truth and falsehood: but there seems to me no motivation to introduce such additional truthvalues in this case. More appealingly, there is the strategy of taking to be false all of those claims which are not true (i.e. supertrue). This would preserve bivalence, and would allow us to keep the attractive intuition that nothing more is needed for a truth-apt sentence to be false than that it fail to be true. (A "minimalism" about falsehood, I suppose). In this case, too, it would more naturally fit in with the notion that the claims to be interpreted structuralistically are implicitly general: for usually, when a condition which is asserted to hold universally is found to hold for some instances but not for others, we judge the assertion false, and not merely truth-valueless.

Adopting this strategy has odd results of its own, of course, which orthodox supervaluationism avoids. By declaring sentences false when an orthodox supervaluation would declare them truthvalueless leads to the odd result that sometimes both a sentence and its negation will be declared false: since when a sentence is true on some but not all assignments, its negation will have the same status. This result could also be avoided if one out of each pair of a sentence and its negation was declared "positive", and it was held that only positive sentences are false when not supertrue, while their negations, as per usual, were declared true just in case the "positive" sentence was false. Or, especially if one takes seriously the idea that apparently specific claims are disguised generalisations about structures, one might be happy to accept the consequence that a sentence and its apparent negation might both be false: since claims of the form "Everything which is such-and-such is F" and

"Everything which is such-and-such is not F" are contraries rather than contradictories, and are both false when some things are F and some are not. I shall be pursuing the policy of judging class-theoretic claims which are not supertrue to be false (so, for example, while it will be true that some atom is the null set it will be false, for any particular atom selected, that *this* atom is the null set) , though what policy one adopts here is not of great significance, since our concern shall mainly be with what is true for all "candidates". The results are to some extent counter-intuitive no matter which strategy is pursued, but this is part of the price for structuralism, and arguably not a great price once we are aware of the implicitly general nature of the claims. This is especially so if structuralism about sets and classes is to be seen as a reform to mathematics rather than merely extending or clarifying it.

One complication to our general strategy might be worthwhile in order to preserve more of our intuitions about what should be called individuals. If something had to be treated as an individual by all Ps before it was true that it was an individual, then nothing constituted from atoms could be an individual, since for each atom there will be an assignment which assigns that atom to be the null set, and assigns the other atoms in the object as Esingletons. A theory which held that nothing was an individual, but also that for each thing (or almost each thing), that thing was neither a class nor an individual, would be a paradoxical-sounding revision that differed quite sharply from our pre-theoretic intuitions, and would also make class theory unsuitable for many applications where it is assumed that certain objects are individuals. So a better way of assigning the title "individual" should be developed if possible.

Developing such a method is easy enough. Instead of saying that an object is an individual iff all Ps assign it the status of an individual, it would be better to say that an object is an individual iff it is not (super)true that it is a class (i.e. there is some P which does not assign it the status of a class). Then it would turn out that everything would be an individual, while it still being the case that there are enough classes to go around.

Another, more serious problem concerning individuals is that each non-atomic individual will, according to some P or other, have the null set as a part but not have all of its other atoms being Esingletons, and so will not count as being members of classes according to that P, and so not at all. This should also be avoided. One of the most convenient methods of avoiding it when dealing with a specific bunch of objects one wishes to treat as individuals is to generalise only about those Ps according to which all the objects in question are distinct from the null atom. Of course this will not work when we wish to treat everything (or even every small thing) as being both an individual a member of a class—or even when we wish to treat of things whose aggregates are all of (atomic) Reality.

So to this point we have only a partial solution to the problem of trying to provide for all of class theory without embracing specifically mathematical ontology. It is an improvement over the solution given by Lewis, in that his solution required that a *proper class* of atoms be distinct from the objects being considered, while the solution sketched above requires only *one* atom be distinct from the objects to be treated as individuals. Even the current solution can (and will) be improved, but before I move on to consider improvements, it will be worthwhile to consider some of the features of the current approach, and some features of approaches which seek to have individuals do the work of classes generally.

3. A Surprising Feature of the Account

One surprising feature of this system of employing individual atoms to serve also as Esingletons is that not every fusion of individuals can itself be an urelement of a class. Here is a proof: suppose that there were proper-class many atomic individuals, and any fusion which divided exhaustively into some or all of those atoms was a member of a class. By the principle of unrestricted composition, there would be a fusion of those atoms corresponding to each sub-class of the class of atomic individuals. For every P, if all of the fusions were members of classes, they would all have Esingletons — so the sub-classes of the proper class of individual atoms could be put in one-one correspondence with the Esingletons of individuals composed entirely of those atoms. Since the Esingletons are the members of the proper class of atomic individuals, the sub-classes of that class must be able to be put into one-one correspondence with the members of a sub-class of that class. This can be shown to be impossible, using a variation of the argument for Cantor's theorem:[11]

For any given P:

[11] Thanks to Greg Restall for pointing out that this sort of proof can be employed using plural quantification even when functions are unavailable.

Let the Fs be the pairs, each of which contains a sub-class of the proper class of individual atoms (call the plurality of these sub-classes the Xs) and an Esingleton of an individual which exhaustively divides into the members of one of the Xs (call these Esingletons the Es), such that each F has as one of its "members" the Esingleton of the individual formed by the fusion of the members of an X which is also one of that F. Consider the Esingletons which are those of the Es that are not one of any F such that they are a member of the X which is one of that F — Call these Esingletons the Ks. Consider the fusion of the Esingletons which are members of K. Is the Esingleton of that fusion (call it x) one of the Ks? Either way lies contradiction:

Assume that x is one of the Ks. Then it will be identical to none of the members of the X which it is paired with in an F (as that is what it is to be one of the Ks). The X which it is paired with, however, will be the class of all and only the atoms which are the Ks (by the definition of the Fs). So, in particular, one of them will be x (*ex hypothesi*). So x will not be identical to itself.

Assume that x is not one of the Ks. Then it will be identical to one of the members of the X which it is paired with. The X which it is paired with will be the class of all and only the atoms which are the Ks. But x is not one of the Ks, by hypothesis, and so will not be identical to itself.

Since this reasoning holds for every P, it follows that , given a proper class of individuals that are members of classes, not all the fusions of those individuals can be members of classes, on pain of contradiction.

For this proof to work, I need to employ a sense of "pair" according to which proper classes can be "members" of pairs. The standard set-theoretic method of taking pairs to be sets of various sorts will not therefore be adequate, since proper classes are never members of sets. The pairs cannot be pluralities of two either. Since the proof plurally quantifies over the pairs, the pairs cannot themselves be pluralities, since standard plural quantification (see for example Boolos 1984) is characterised by the lack of ability to plurally quantify over the pluralities themselves, but rather allows only plural quantification over the objects which make up those pluralities. It would be possible to construct systems that allowed for higher-order plural quantification over pluralities (as is discussed in Hazen 1997), and the proof could be construed as plural quantification over two-membered pluralities in such systems, but I shall not avail myself of such a device. Apart from my suspicion of such a device, higher-order plural quantification is not something which is part of the framework of Lewis's in which I am conducting the present investigation. Thus, I would prefer a device for producing "pairs" which did not rely on resources unavailable in the framework. Fortunately, there are resources available in the framework to provide pairs of any two objects, even proper classes: the Burgess method of producing pairs (b-pairs) and the Hazen method of producing pairs (h-pairs). Since I have been employing the Burgess method, let me do so here — and take the "pairs" in the above proof to be the b-pairs of sub-classes and Esingletons of the appropriate sorts, where the b-pairs are the b-pairs according to some arbitrary suitable x, y, z, X, Y and Z. With a suitable device for producing the "pairs" of the above proof, it goes through, and so establishes the conclusion that if there are proper-class many atomic individuals which belong to classes, then not every fusion of those individuals is itself an urelement of a class.

In light of this result, it may be worth refining our conception of "individuals". Our current definition is that any non-class is an individual. Given this definition, it follows that in this system not all individuals are members of classes. This conflicts with more normal usage of the term "individual" (for example in Fraenkel, Bar-Hillel and Levy, 1973, p 23) in set theory, which is that an individual is any member of a class that is not itself a class. Let us adopt this definition instead (keeping in mind that the definition is to be applied after generalisation rather than before). Thus, instead of saying that some individuals are not members of classes, we will say that some fusions of individuals are not themselves individuals. This might strike some as worrying, or even absurd. It is not as bad as it first appears, however.

Most of the fusions of proper-class many individual atoms will need to be denied the status of individuals themselves, and once singletons are denied to these entities (or most of these entities), the other fusions of individual atoms can all be admitted to be individuals without causing any trouble. This seems to be the obvious approach to take in the light of this result, and it ought not to be too surprising that the counter-intuitive results of this theory happen in connection to things having proper-class many parts: the paradoxes of set (and class) theory have been known (or suspected) to stem from some postulated things being "too big" since the time of Cantor.[12] The analogy is even closer if the relation of sub-class to class is taken to be

[12] Hallett (1984) is a good discussion of the connection between "size" and the paradoxes.

mereological: for just as a proper class has no singleton in virtue of having proper-class many singletons as parts, so a Large non-class lacks a singleton in virtue of having proper-class many non-classes as parts. The denial of singletons to all aggregates of individual atoms is of course consistent with unrestricted mereological composition: aggregates of all and any individuals do still exist according to this theory, it is just that some are not members of classes. Furthermore, if it is stipulated that, out of the fusions which exhaustively divide into individuals, only the fusions of proper-class many atoms may fail to have singletons, the "Restricted Fusion Thesis" that *the fusion of any set of individuals is itself an individual* follows from the current system. However, Lewis' full Fusion Thesis (that any fusion of individuals is itself an individual (1991 p7)) will need to be denied.

If one wishes to retain the full Fusion thesis, one will need to abandon the project of employing proper-class many individuals to serve as the ontology for class theory.

However, given plausible class-theoretic assumptions, the reason for this is not just that atoms are being pressed into serving double duty: the denial of the full Fusion Thesis is a consequence of the postulation of a proper class of individual atoms. This means that, given that one wishes to postulate a proper class of individuals, postulating the usual distinct class-theoretic ontology on top will not help to preserve the Fusion thesis, so the result has to be seen as a cost of postulating the individual atoms in the first place, not some further problem one becomes embroiled in when one tries to reduce classes away. However, the general proof that not all fusions of individuals can also be individuals if there is a proper-class of individual atoms is not as straightforward as the one given above when we countenance the *prima facie* option that there might be more classes than even proper-class many individual atoms (an option which was clearly ruled out when the classes were one-one correlated in the obvious fashion with Esingletons each of which was also an individual atom).

The above proof showed that the fusions of individual atoms could not be put in a one-one correspondence with those atoms. The additional class-theoretic assumption to be used in the general proof that the singletons cannot be put in one-one correspondence with the fusions of individuals atoms when those individual atoms form a proper class will consist of a proof (or direct assumption) that all proper classes are of the same size: they are all able to be put in a one-one correspondence with each other. Once this is established, it will follow that the singletons can be put into one-one correspondence with the individual atoms (given that the singletons obviously form a proper class), and so cannot be put in one-one correspondence with all of the fusions of those individual atoms: thus, (given Functionality) some of those fusions will lack singletons.

All that needs to be now maintained is that all proper classes are of the same size. Lewis is already committed to this (Lewis 1991 p 98). This claim is embraced by nearly all theorists who accept a theory of classes: von Neumann made a claim which is equivalent to this one of the axioms of his class theory (axiom IV2 of von Neumann 1925)[13]. The claim that all proper classes are of the same size is equivalent to the axiom of global choice[14] (as it is called in Fraenkel, Bar-Hillel & Levy 1973, p 133), which they argue, I think persuasively, is a quite reasonable assumption when dealing with classes, and which is taken by Rubin and Rubin 1973 to be the natural axiom of choice to accept in class theory (were one to embrace any axiom of choice). Global choice is by no means merely an artifact of using the same ontology to do the work of individuals and classes—it is a fundamental principle of class theory. While of course there is no inconsistency in denying it, and indeed denying global choice is even consistent with choice (See Felgner 1976 p 278), the fact that the full-blown Fusion Thesis when there is a proper class of individuals is equivalent to the denial of Global Choice makes the Fusion Thesis unattractive in its own right in such cases.

4. Utilising the Surprising Feature

Denial of the unrestricted Fusion thesis is unusual and perhaps somewhat regrettable, but since it need only apply to fusions of proper-class many atoms (which would not be the first time that size has led to intuitively odd results in class theory), and since it

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[13] As cited in Fraenkel, Bar-Hillel and Levy 1973 p 137, where they provide a recasting of the idea as axiom (*). (*) directly implies that if a class is not a set then there is a function which maps it onto the class of all sets: therefore, every proper class can be mapped onto the class of all sets, so any proper class can be mapped one-one to any other.

[14] see pp 84-85 of Rubin and Rubin, 1963, for a proof that all proper classes are equivalent in size given the Well Ordering Theorem for classes (i.e. that their WE 5S or their P 1S is equivalent to their WE 1S), and pp 86-89 for a proof that the Well Ordering Theorem is equivalent to Global Choice (their AC 1S which is Fraenkel, Bar-Hillel & Levy's Axiom VIII^{c}_{σ}).

is not an unintuitive consequence that holds solely because of the double duty objects are serving in the theory, but rather would arise in any theory postulating proper-class many individual atoms which had the axiom of Global Choice, this result does not I think damn this reductive project. In fact, once it is recognized that fusions of proper-class many individual atoms may not themselves be individuals, this fact can be utilised to iron out the problem mentioned on p 15. The bug, remember, was that we were unable to talk about too many things as individuals at once, for it had to be the case that there was at least one P such that all of the individuals being discussed were distinct from the atom designated as n by that P. This bug can be corrected by using one of the large fusions to fill the same sort of roll that the atom n played — or, to be more precise, having, for each equivalent of a P, one of the large fusions of atoms play the role which is played in the Ps by atoms playing the "n" role. This will make it possible to talk of the class of all individuals, as we shall see, in a way which the previous system outlined could not. To outline how this is to work, I will follow the procedure used earlier, of first talking as if there is only one plurality of things which satisfies the criteria for being treated as the basis for talk about classes, and then generalising over all structures that meet the postulated conditions to produce the structuralist version of the theory.

As before, the background for this system will be the Framework, with the various Schemae, affirmed principles, and megethological postulates about size which I mentioned on p 6. Begin by selecting a fusion of proper-class many atoms such that it is completely distinct from at least one other fusion of proper-class many atoms. Let this big fusion be the "Null thing" — symbolised N. As before, an Esingleton relation can be postulated. Distinctness, Functionality and Atomicity can remain unchanged, while Domain and Foundation only need to be altered slightly, as follows:

Domain — N has an Esingleton, any small fusion of atoms has an Esingleton, any fusion of a small fusion of Esingletons and N has an Esingleton and nothing else has a singleton.

Foundation — whenever there are some things, either at least one of them is not an object which is the fusion of at least one Esingleton with N and which contains no non-Esingletons besides parts of N (and fusions of parts of N with Esingletons) as parts, or one of them is such that none of the objects which have Esingletons among its parts are among those things.

There need be no axiom analogous to the previous axiom of Null Set, since N will not be atomic and so cannot be an Esingleton by Atomicity. In addition, add this principle:

Esingleton Distinctness: every Esingleton is distinct from N. (That is, none of the atoms (or, indeed, anything) which are parts of N are Esingletons).

Since there are at least proper-class many atoms distinct from N (from the definition of N), there are sure to be enough Esingletons to go around even with this condition.

Definitions much like those on p 8 can now be employed. A *class* will be any fusion that contains only N and Esingletons (N will be the null class, in the same way that n was). An object is a *member* of a class iff that object's Esingleton is a part of that class. A *set* can be defined as any class whose mereological difference from N is small. (The previous definition, that a set is a small class, is no longer adequate because all classes will be large in virtue of having N as a part). Let us say that an *individual* is any small object.

The "standard axioms" can be proved in this system (again, see appendix one of Nolan 1997). Again, Lewis's Main Thesis is falsified, but the First Thesis, Priority, a version of Division and the restricted Fusion thesis (see above, p 14) are all satisfied by this system.

To dispense with the "Esingleton relation", and to iron out problems with this system like many individuals being part of the Null Class, we can now modify this theory using the Method of Double Images in the same way that the initial account of classes presented in this piece was modified. Any R defined using the Method of Double Images which consists of b-pairs such that they obey the conditions set out for the "Esingleton relation" will do to provide a structure with b-pairs that can obey the conditions set out for the "Esingleton relation". Before the structure is entirely adequate, of course, some large fusion needs to be assigned the status of N (and in fact this must be done before it is determinate whether a plurality of b-pairs satisfies the Esingleton conditions, for they must not assign to anything an Esingleton which is a part of N). So, as before, we will use pluralities (call them the Ps again) such that one of each of them is a large fusion satisfying the conditions on N and the rest of each of them are diatoms generated using the Method of Double Images and obeying the conditions set out on Esingletons (treating the relevant large object as N). Then claims about classes can be taken to be generalisations about all the Ps satisfying those conditions: a claim about classes is true iff it is satisfied by all Ps, and false otherwise (though again, as on pp 10–11 different approaches can be taken to the "true of some-but-not-all" cases). Individuals can then be defined as any small object which not all Ps assign the status of a class — that is, any small object. If one wishes, one can allow that some large objects are individuals too, in which case extra conditions should be added to the Esingleton conditions to ensure that such objects are always assigned Esingletons and are never classes.

We now have a system that is nearly ideal. Every small object is an individual, and yet every small object is such that it has a singleton. There is a class of all individuals — indeed, there is a class (the union of all classes) which has every member of a class as a member of it. Furthermore, all of this is modelled without any distinctively mathematical ontology or ideology. There are of course some drawbacks. The structuralism yields some curious consequences: for example, it is true that there is some atom which when fused to the null set yields a class, but for each atom it is false that *that* atom, when fused to the null set, yields a class; and the null set is a fusion of individuals, but for any given individuals it is false that exactly those individuals when fused yield the null class; and so on. But this is a feature of structuralism in general, and is a consequence that is tolerable once it is understood what is really going on. There are some lurking entities which look strange — there are things which are fusions of individuals but which are not themselves individuals, and for most Ps there are fusions of classes and individuals which are themselves neither classes nor individuals (but this is falsified by the Ps in which Reality exhaustively divides into N (or rather the N candidate) and Esingletons (or rather the Esingleton candidates). But such objects only appear amongst the Large — and the Large are in an area where our intuitions must be modified away from our naive preconceptions in any case.

I have so far been setting to one side the question of what to say in the case in which Reality is not completely atomic, but also contains Gunk, but extending the systems offered requires little that is novel. The system set out on p 133-136 of the Appendix will be completely satisfactory when the fusion of Gunk is small. The only alteration that is required to the system is that the pairs which are to appear in the Ps should be *new pairs* (in the sense stated on p 135 of the Appendix), and each of the Ps should have three marker atoms. When there are proper-class many pieces of Gunk, the procedure becomes a little more involved, for the proof offered for the "Not-too-much-Gunk Hypothesis" on p 134 of the Appendix will then not work. In cases where there are as many pieces of Gunk as there are atoms, there will still not be a problem — The Not-too-much-Gunk hypothesis will still be satisfactory.

However, a problem might arise similar to the problem of fusions of proper-class many atoms if there is too much Gunk. If there are proper-class many distinct pieces of Gunk, then there will be more fusions of Gunk than there are members of a proper class. Part of the solution is the same as the one outlined before — it will turn out that not all large fusions of Gunk will be members of classes — i.e. not all will have singletons. This will require modifying Domain so that instead of talking about "any small fusion of atoms" it talks about "any small fusion". As before, we do not have to deny singletons to every Large piece of Gunk, so Domain could allow some certain specified Large fusions of Gunk have singletons as well. For instance, one may wish to declare that maximally connected pieces of Gunk have singletons.

But this will not solve the problem entirely. For the method for assigning codes to pieces of Gunk and objects which are part Gunk and part atomic requires that there be as many atoms as there are pieces of Gunk (see the Appendix, p 135). We need a modified method of assigning codes. Fortunately one will be possible, since we do not need codes for every piece of Gunk — only those pieces which we will desire to have singletons will need to be coded so that their codes can appear in the new pairs which will be found among the Ps. To define the Gs which will be used to define the required codes (see p 135) we need only replace the Not-too-much-Gunk Hypothesis with something like the

Not-too-much-relevant-Gunk Hypothesis: There are some things G such that each one of G is the fusion of a small atomless thing and exactly one atom; every small atomless thing is the maximal atomless part of exactly one of G; and no two of G have an atom as a common part.

Of course, the Not-too-much-relevant-Gunk Hypothesis can and should be modified if it is desired that some large pieces of Gunk are to have singletons — then instead of "small atomless thing" a phrase of the form "small atomless thing or large atomless thing such that.... (fill in desired condition here)" should be substituted. Care must be taken that not too many Large pieces of gunk are assigned codes, but this is a pitfall which is easy enough to avoid. Once the codes are assigned in this fashion, the procedure can be carried out as before. The Not-too-much-relevant-Gunk Hypothesis is still a mild restraint on the amount of Gunk there is — there cannot be more small fusions of Gunk than there are atoms, after all, but like the case of fusions of atoms this is not something peculiar to the reductive project being discussed in this piece — if all of the small fusions of Gunk are to have singletons, and if singletons are atomic, then obviously there cannot be more small fusions of Gunk than atoms regardless of whether the singletons are taken to be part of a Platonic ontology or reduced to a non-mathematical ontology of individual atoms.

Perhaps even more than this can be accommodated in a system with the broad outlines of a framework of the sort dealt with in this chapter or in Lewis 1991. Hazen 1997 points out that there is a formal result, Stone's theorem (see Hazen 1997 pp 246–247), which shows that any non-atomic mereology can be isomorphically mapped into an atomic mereology. As Hazen goes on to argue, such mappings could be constructed, pieces of gunk could play the roles which the Method of Double Images and the axioms of Esingleton reserve for atoms, and so Gunk could be fitted into a framework for class theory without needing to be associated with atoms in the manner of the Appendix. I say only perhaps: for the use of Stone's lemma involves the employment of powerful set-theoretic devices such as ultrafilters (Hazen 1997 p 246), and so Stone's theorem as it stands cannot be relied upon without circularity. Hazen suggests that there are extensions to the framework which will allow Stone's theorem to be employed without appeal to set-theoretic entities: Hazen's specific suggestion is higher-order plural quantification. I doubt if I have much useful to say at present either for or against extending the framework with such higher-order plural quantification—however, it does not strike me as having the harmlessness of standard plural quantification, and that I am suspicious that it is smuggling theoretical resources which are the equivalent of set theory back into the pre-set-theoretic framework. My expression of opinion here should not be taken as an argument, of course.

Alternatively, we could simply add as an assertion that the requisite pairings for the job exist (Hazen outlines what pairings are needed for a non-atomistic framework in Hazen 1997 pp 244–246). They seem fairly harmless, and their existence may even follow from suitably strong forms of Plural Comprehension, without a detour through a proof of an equivalent of Stone's theorem. More work needs to be done on the question of how well the existence of the necessary pairings is justified, and how Esingletons are to be characterised in a non-atomic system. Extensions to non-atomic frameworks provides an interesting technical problem if nothing else, and may well be worth exploring if only to give the framework a greater deal of generality.

Let me note as an aside that as well as extending the system in various ways to fine-tune its treatment of Gunk, there are other extensions which are possible and may perhaps even be desirable. The fact that not every Large fusion of individuals can itself be an individual does not by itself preclude *all* of the Large individuals from membership of classes, and with suitable modifications of Domain certain privileged Large individuals and/or some "proper classes" may be accorded the honour of a singleton. Again, not every proper class can be a member of a class on pain of Cantorian paradox, and as usual certain proper classes should be forbidden singletons so as to prevent paradoxes analogous to the Russell paradox or the Burali-Forti paradox. In the first case, one had better forbid a singleton to the class of all non-self-membered classes which themselves are members of some class, and in the second one had better take care that the class of ordinals cannot belong to a class of higher ordinality. Again, Domain would have to be modified, and whether such tinkering is worth the bother will often be contentious philosophically. Such options for expansion will not be examined further here—I leave them for those who have some specific use for them.

With the accommodation of Gunk, the reductive system outlined reaches its final form (or at least the final form it will assume in this paper).[15] It does as well as any Platonic rival, save from the oddities that it inherits from being structuralist and from postulating a proper class of individual atoms. However, it requires no specifically mathematical ideology, and no distinctively mathematical ontology either, except in the sense that, apart from the needs of the framework, we may not have postulated so many objects in the first place. However, there may be independent

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[15] As well as extensions mentioned throughout the paper, one task worth completing would be the proof of Existence and Categoricity theorems for the systems outlined in this paper, like those given for Lewis's system in Lewis 1993. Since my systems lack Lewis's axiom of Induction, the form of these proofs would need to differ from Lewis's: but similar results for particular specifications of the requisite class of ur-elements should be forthcoming.

reasons for postulating so many individuals: if so, adapting them to be employed as the ontology of mathematics as well would not mean that they were a distinctively mathematical theoretical cost. Independent reasons for postulating such a vast quantity of objects can be found in the philosophy of modality.

5. Modal Ontology

In order to account for necessity and possibility, an ontology of possible worlds and the possible objects they "contain" is often postulated. Sometimes, as in Lewis 1986, these worlds and their contents are concrete entities much like our cosmos and the paradigmatic individuals that inhabit it: but those that take possible worlds or possibilia to be abstracta of one sort or another (propositions, or uninstantiated states of affairs, or what-have-you) may also be committed to a great deal of ontology. Often only set-many possible worlds and possible objects are postulated: but this may be too restrictive an approach to the range of possibilities there are. I have argued elsewhere (Nolan 1996) that there should be no limit to the cardinality of objects in possible worlds, since, among other reasons, this is a result of the natural application of an intuitive principle of recombination: of what can co-exist with what. Those who believe in the existence of possibilia, whether concrete or abstract, have good reasons to postulate a proper class of them. Once so many possibilia are postulated, enough will be atomic to provide the ontology needed for the constructions of this paper.

Allowing for a proper class of possibilia in Lewis's case would also deal with several problems his particular system currently suffers. According to Lewis 1986, there are only a set of possible objects (see p 104) and yet, for the system of Lewis 1991 to work, there must be a proper-class many atoms alone. As a consequence, most of the things which exist must exist outside even his infinitely vast pluriverse of possible worlds. The first problem with this, which Lewis acknowledges, is that we have very little idea what all of these things are, and that the strategies of his book are very little help in shedding light on the nature of these objects (Lewis 1991 p 142, Lewis 1993 p 17). More serious, perhaps, for Lewis's theory of modality, is that these objects, not being in any possible world, are impossible objects—and it is less than ideal for a modal theory, especially a concretist modal theory, to admit the literal existence of impossible objects.[16] Accepting a proper class of possibilia would relieve Lewis of the embarrassment of commitment to the existence of objects which by his own lights do not possibly exist.

Even those who do not accept that merely possible objects (or individual abstract representations thereof) literally exist, may still have modal ontological commitments amounting to proper-class-many objects. Those who reject the literal existence of merely possible objects may still accept the existence of possible worlds, for example, and those who believe in possible worlds (whether as abstract representations, simples, uninstantiated properties or unactualised total states of affairs) may have good reason to think that there are more than set-many logical possibilities. Those that do will have proper-class many possible worlds to serve as the material needed. A third source of quantitatively large theoretical commitment lies in propositions. The number of logically possible propositions is very large indeed: those needed to fully describe every possibility, for example, may form a proper class (especially if there is no limit to the cardinality of objects which can co-exist according to a possible world). Again, if those propositions are atomic (or have atomic constituents) they may serve the required job of providing the objects needed.

The ontology of modality is not the only potential source of theoretical commitment to proper-class many individuals—Piercian continuities, for example, are another kind of theoretical postulate which, while not widely accepted, would seem to furnish the required ontology. An ontology of possible worlds or their contents does seem to be one of the most obvious sources of the needed atoms, however, since it offers us as many things as there might be, or at least as many things as ways things could be. Of course, it is not necessary to find an independent reason to postulate so many objects: the fact that they are needed for mathematics may well be thought reason enough, and standard Platonism postulates as many objects (Platonic sets and classes) specifically for that end. However, if there is independent reason to postulate a proper class of individuals, this paper offers mathematics which is close to cost-free. Instead of distinctively mathematical relations such as set-membership or singleton

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[16] Of course, Lewis is committed to the existence of impossible objects in any case, as the fusion of things that exist in different possible worlds itself does not exist in any possible world (Lewis, 1986, p 211). But at least these objects resolve into parts which are all possible—which cannot be said for the "completely impossible" mathematical atoms.

relations, or subset relations, it has the advantage inherited from Lewis's system that it can do the work of these relationships with mereology and plural quantification; and instead of needing to postulate ontology solely for the purpose of mathematics, the individuals needed for the reconstructions of class theory offered in this paper will already have been postulated for independent purposes, and so will not count as a specifically mathematical cost. As well as any technical interest this paper may have as an exercise in exploring possibilities for a megethological framework, the prospect of a Platonist foundation of mathematics without the need to incur any specifically mathematical metaphysical commitments offers an attractive reduction of mathematical commitments to commitment to individuals only.[17]

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