Halting Problem Proof from Finite Strings to Final States

When we understand that every potential halt decider essentially derives a formal mathematical proof from its inputs to its final states previously undiscovered semantic details emerge. When-so-ever the potential halt decider cannot derive a formal proof from its input strings to its final states of Halts or Loops, undecidability has been decided.

A Halting Decidability Decider H might be defined as:
A David Hilbert formalist proof
(in language of Turing machine descriptions)
from the initial state of a Turing machine description H.q0 and
its finite string inputs Wm W to final states H.halts and H.loops
corresponding to the mathematical logic predicates: Halts() and Loops().

∃ H ∈ Turing_Machine_Descriptions
∀tm ∈ Turing_Machine_Descriptions
∀i ∈ Finite_Strings
H.Halts(tm, i) ∨ H.Loops(tm, i) H.Halting_Undecidable(tm, i)

When-so-ever Turing machine H determines that no finite sequence of state transitions would correspond to the mathematical logic predicate Halts(tm, i) or Loops(tm, i) it transitions to H.Undecidable.

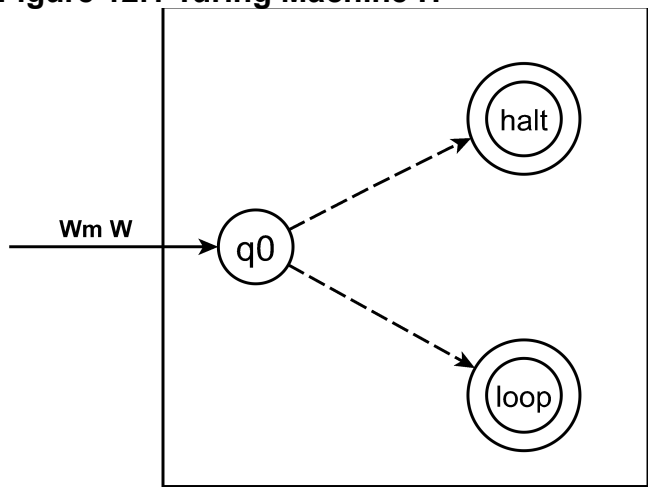
The formal proof involves tracing the sequence of state transitions of the input TMD as syntactic logical consequence inference steps in the formal language of Turing Machine Descriptions.

The proof would essentially be a hypothetical execution trace** of the state transition sequences of the input TMD. It cannot be an actual execution trace or the halt decider could have non-halting behavior. **Like step-by-step mode in a debugger.

If the first element of the input finite string pair is a correct TMD then this proof must necessarily proceed from the specified TMD start state through all of the state transitions of this TMD to the halting, looping or pathological behavior of this TMD.

The following has been adapted from material from the this book:
An Introduction to Formal Languages and Automata by Peter Linz 1990 pages 318-320
We begin our analysis by constructing a hypothetical halt decider: H.

Figure 12.1 Turing Machine H



The dashed lines proceeding from state (q0) are represented in the text definition as the asterisk ⊢* wildcard character. These conventions are used to encode unspecified state transition sequences.

Definition of Turing Machine H (state transition sequence)
H.q0 Wm W ⊢* H.halt // Wm is a TMD that would halt on its input W
H.q0 Wm W ⊢* H.loop // Wm is a TMD that would loop on its input W

The diagram and the state transition sequence indicate that H begins at its own start state H.q0 and is applied to finite string pair (Wm, W).

Then it proceeds through an unspecified set of state transitions to one of its final states.
H.halt is the final state of H indicating that Wm is a TMD would halt on input W.
H.loop is the final state of H indicating that Wm is a TMD would halt on input W.

In an attempt to provide a valid counter-example proving by contradiction that a Halt Decider cannot possibly exist for every possible TM / Input pair we create Turing Machine Ĥ by making the following changes to H:

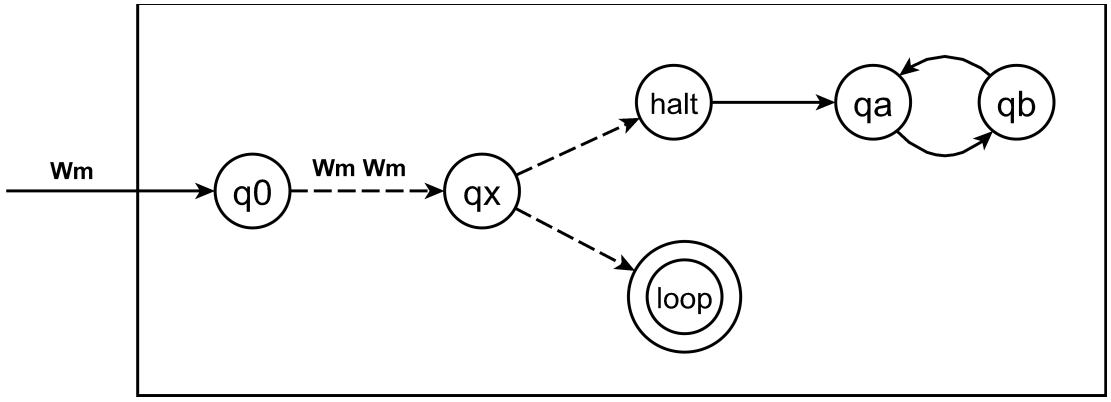
(1) Ĥ copies its input Wm a its Ĥ.q0 state and transitions to its Ĥ.qx state.

(2) Ĥ would begin to evaluate Wm Wm at its Ĥ.qx state in exactly the same way that H would begin to evaluate its Wm W input at its H.q0 state.

(3) States (qa) and (qb) are appended to existing final state ((halt)) such that any transition to state (halt) will cause Ĥ to loop.

Since Turing Machine Ĥ is created by adapting H, it would have exactly the same behavior at its Ĥ.qx state as H would have at its H.q0 state.

Figure 12.3 Turing Machine Ĥ



Definition of Turing Machine Ĥ (state transition sequence)
Ĥ.q0 Wm ⊢* Ĥ.qx Wm Wm ⊢* Ĥ.halt ∞
Ĥ.q0 Wm ⊢* Ĥ.qx Wm Wm ⊢* Ĥ.loop

If Turing Machine H is applied to Turing Machine descriptions [Ĥ] [Ĥ]
would H transition to H.y or H.n ?

H [Ĥ] [Ĥ2] // We append a "2" to the second Ĥ for clarity

Because H is performing a mathematical proof on finite strings [Ĥ] [Ĥ2] the required syntactic logical inference chain is simply the state transitions that [Ĥ] would make on its input [Ĥ2]. In simplest terms H is performing a step-by-step debug trace of [Ĥ].

Step-by-step debug trace of what [H] would do on its input [Ĥ][Ĥ2]
(01) H begins at its start state H.q0.
(02) H begins to evaluate what [Ĥ] would do on its input [Ĥ2].

Step-by-step debug trace of what [Ĥ] would do on its input [Ĥ2]
(03) [Ĥ] would begin at its start state [Ĥ].q0
(04) [Ĥ] would make a copy of its input [Ĥ2], we will call this [Ĥ3].
(05) [Ĥ] would transition to its state [Ĥ].qx.
(06) [Ĥ] would begin to evaluate what [Ĥ2] would do on its input [Ĥ3].

Step-by-step debug trace of what [Ĥ2] would do on its input [Ĥ3]
(07) [Ĥ2] would begin at its start state [Ĥ2].q0
(08) [Ĥ2] would make a copy of its input [Ĥ3], we will call this [Ĥ4].
(09) [Ĥ2] would transition to its state [Ĥ2].qx.
(10) [Ĥ2] would begin to evaluate what [Ĥ3] would do on its input [Ĥ4].

Can you see the infinite recursion?

H [Ĥ] [Ĥ] specifies an infinitely recursive evaluation sequence. Every HP proof by contradiction depends this same infinite recursion. What no one ever noticed before is that the debug trace by the halt decider must abort its evaluation long before it ever reaches the appended infinite loop.

Because of this every TMD in the infinitely recursive sequence is defined in terms of H each would reject the whole sequence as semantically incorrect before even beginning any halting evaluation, and transition to H.loop.

The following two paragraphs require the original Linz definition of H
H.q0 Wm W ⊢* H.qy // Wm is a TMD that would halt on its input W
H.q0 Wm W ⊢* H.qn // else

As further evidence that infinitely recursive evaluation sequence has been overlooked we only need to know that every TMD always requires its own TM / Input pair. For any finite sequence of input H could always decide halting.

If Ĥ did not copy its input and instead simply took two inputs then the Halting Decision would be easy. Ĥ would transition to its Ĥ.qy state because Ĥ2 would transition to its Ĥ2.qn state on null input. This would cause H to transition to its H.qn state deciding halting for [Ĥ] [Ĥ2].

If we assume that a halting decidability decider could use something like the Prolog predicate unify_with_occurs_check/2 to detect and report the infinite recursive evaluation sequence of H(Ĥ, Ĥ2), then an actual halt decider might be defined as follows:

What happens if H decides infinite recursion at the recursion depth of three?
Ĥ2 aborts Ĥ3 and transitions to Ĥ2.loop.
which causes Ĥ1 to transition to Ĥ1.halt and loop.
which causes Ĥ to transition to Ĥ.loop.
which causes H to transition to H.halt. // thus H has decided halting for (Ĥ, Ĥ2)

What happens if H decides infinite recursion at the recursion depth of two?
Ĥ1 aborts Ĥ2 and transitions to Ĥ1.loop.
which causes Ĥ to transition to Ĥ.halt and loop.
which causes H to transition to H.loop. // thus H has decided halting for (Ĥ, Ĥ2)

What happens if H decides infinite recursion at the recursion depth of one?
Ĥ aborts Ĥ2 and transitions to Ĥ.loop.
which causes H to transition to H.halt. // thus H has decided halting for (Ĥ, Ĥ2)

Turing Machine Description Language
(Generalized from: TM, The Turing Machine Interpreter by David S. Woodruff)
The following is space delimited 'quintuples', each one of which is a five-symbol Turing Machine instruction. It is written in AWK regular expression syntax:

1) Initial State
2) Current Tape Symbol
3) Next State
4) Write Tape Symbol
5) Move Tape Head Left or Right

```
--1--       --2--      --3--       --4--      --5--
[0-9]+ [\x20-\x7e] [0-9]+ [\x20-\x7e] [LR]
```

If we make a fixed number of (hexadecimal) states then The UTM could simply hold a sorted list of std::strings. A sorted list would allow state transitions to occur by binary search. This would allow a very simple DFA based syntax checker.

The tape could be constructed using a std::vector. Binary Zero end markers would be in the tape alphabet and not the input alphabet. These trigger memory reallocation to enlarge the tape.

Clocksin and Mellish 2003, Programming in Prolog Using the ISO Standard Fifth Edition Chapter 10 The Relation of Prolog to Logic, page 254.

> Finally, a note about how Prolog matching sometimes differs from the unification used in Resolution. Most Prolog systems will allow you to satisfy goals like:
>
> **equal(X, X).**
>
> **?- equal(foo(Y), Y).**
>
> that is, they will allow you to match a term against an uninstantiated subterm of itself. In this example, foo(Y) is matched against Y, which appears within it. As a result, Y will stand for foo(Y), which is foo(foo(Y)) (because of what Y stands for), which is foo(foo(foo(Y))), and so on. So Y ends up standing for some kind of infinite structure. Note that, whereas they may allow you to construct something like this, most Prolog systems will not be able to write it out at the end. According to the formal definition of Unification, this kind of “infinite term” should never come to exist. Thus Prolog systems that allow a term to match an uninstantiated subterm of itself do not act correctly as Resolution theorem provers. In order to make them do so, we would have to add a check that a variable cannot be instantiated to something containing itself. Such a check, an occurs check, would be straightforward to implement, but would slow down the execution of Prolog programs considerably. Since it would only affect very few programs, most implementors have simply left it out [1].
>
> [1] The Prolog standard states that the result is undefined if a Prolog system attempts to match a term against an uninstantiated subterm of itself, which means that programs which cause this to happen will not be portable. A portable program should ensure that wherever an occurs check might be applicable the built-in predicate unify_with_occurs_check/2 is used explicitly instead of the normal unification operation of the Prolog implementation. As its name suggests, this predicate acts like =/2 except that it fails if an occurs check detects an illegal attempt to instantiate a variable.