Refuting the Halting Problem Diagonalization Argument

Every machine that halts in a reject state is a halting computation. At least two proofs ignore this when constructing Sipser's Figure 4.5. Because these two proofs ignore this when they insert machine D in Sipser's Figure 4.5 they do so incorrectly.

When machine D is inserted in both Figure 4.4 and Figure 4.5 correctly then the contradiction goes away. Since Sipser implicitly assumes that every blank entry of Figure 4.4 is a ~halt entry Figure 4.7a makes this explicit.

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$ ...	$\langle D \rangle$
M_1	accept	~halt	accept	~halt	reject
M_2	accept	accept	accept	accept	reject
M_3	~halt	~halt	~halt	~halt	accept
M_4	accept	accept	~halt	~halt	accept
...					
D	reject	reject	accept	accept	reject

Figure 4.7a (corrected figure 4.6, inserting D into figure 4.4)

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$ ...	$\langle D \rangle$
M_1	accept	reject	accept	reject	accept
M_2	accept	accept	accept	accept	accept
M_3	reject	reject	reject	reject	accept
M_4	accept	accept	reject	reject	accept
...					
D	accept	accept	accept	accept	accept

Figure 4.7b (corrected figure 4.6, inserting D into figure 4.5)

The following portions of pages 166-167 are directly relevant to the rebuttal.
Sipser, Michael 1997. Introduction to the Theory of Computation. Boston: PWS Publishing Company (165-167)

Where is the diagonalization in the proof of Theorem 4.9? It becomes apparent when you examine tables of behavior for TMs H and D. In these tables we list all TMs down the rows, M_1, M_2, ... and all their descriptions across the columns, $\langle M_1 \rangle$, $\langle M_2 \rangle$, ... The entries tell whether the machine in a given row accepts the input in a given column. The entry is *accept* if the machine accepts the input but is blank if it rejects or loops on that input. We made up the entries in the following figure to illustrate the idea.

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\cdots$
M_1	*accept*		*accept*		
M_2	*accept*	*accept*	*accept*	*accept*	
M_3					$\cdots$
M_4	*accept*	*accept*			
$\vdots$		$\vdots$			

FIGURE 4.4
Entry i, j is *accept* if M_i accepts $\langle M_j \rangle$

In the following figure the entries are the results of running H on inputs corresponding to Figure 4.4. So if M_3 does not accept input $\langle M_2 \rangle$, the entry for row M_3 and column $\langle M_2 \rangle$ is *reject* because H rejects input $\langle M_3, \langle M_2 \rangle \rangle$.

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\cdots$
M_1	*accept*	*reject*	*accept*	*reject*	
M_2	*accept*	*accept*	*accept*	*accept*	$\cdots$
M_3	*reject*	*reject*	*reject*	*reject*	
M_4	*accept*	*accept*	*reject*	*reject*	
$\vdots$		$\vdots$			

FIGURE 4.5
Entry i, j is the value of H on input $\langle M_i, \langle M_j \rangle \rangle$

In the following figure, we added D to Figure 4.5. By our assumption, H is a TM and so is D. Therefore it must occur on the list M_1, M_2, ... of all TMs. Note that D computes the opposite of the diagonal entries. The contradiction occurs at the point of the question mark where the entry must be the opposite of itself.

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\cdots$	$\langle D \rangle$	$\cdots$
M_1	*accept*	*reject*	*accept*	*reject*		*accept*	
M_2	*accept*	*accept*	*accept*	*accept*	$\cdots$	*accept*	$\cdots$
M_3	*reject*	*reject*	*reject*	*reject*		*reject*	
M_4	*accept*	*accept*	*reject*	*reject*		*accept*	
$\vdots$		$\vdots$			$\ddots$		
D	*reject*	*reject*	*accept*	*accept*		?	
$\vdots$		$\vdots$					$\ddots$

FIGURE 4.6
If D is in the figure, a contradiction occurs at "?"