Refuting the Sipser Halting Problem Diagonalization Argument

Every machine that halts in a reject state is a halting computation. When machine D is inserted into Figure 4.5 deriving Figure 4.6 the fact that a reject state is a halting computation is ignored. This makes the values at $\langle D,\langle M_1\rangle\rangle$ and $\langle D,\langle M_1\rangle\rangle$ in Figure 4.6 incorrect. When machine D is inserted into both Figure 4.4 and Figure 4.5 correctly (figures 4.4b and 4.5a respectively) the contradiction is eliminated.

	$\langle M_1\rangle$	$\langle M_2\rangle$	$\langle M_3\rangle$	$\langle M_4\rangle$ ...
M_1	accept		accept	
M_2	accept	accept	accept	accept
M_3				
M_4	accept	accept		
...				

Original Figure 4.4

	$\langle M_1\rangle$	$\langle M_2\rangle$	$\langle M_3\rangle$	$\langle M_4\rangle$ ...
M_1	accept	~halt	accept	~halt
M_2	accept	accept	accept	accept
M_3	~halt	~halt	~halt	~halt
M_4	accept	accept	~halt	~halt
...				

Figure 4.4a (converted from Figure 4.4 making ~halt assumption explicit)

	$\langle M_1\rangle$	$\langle M_2\rangle$	$\langle M_3\rangle$	$\langle M_4\rangle$ ...
M_1	accept	reject	accept	reject
M_2	accept	accept	accept	accept
M_3	reject	reject	reject	reject
M_4	accept	accept	reject	reject
...				

Original Figure 4.5 (underlining added)

	$\langle M_1\rangle$	$\langle M_2\rangle$	$\langle M_3\rangle$	$\langle M_4\rangle$ ...	$\langle D\rangle$ ...
M_1	accept	~halt	accept	~halt	DC
M_2	accept	accept	accept	accept	DC
M_3	~halt	~halt	~halt	~halt	DC
M_4	accept	accept	~halt	~halt	DC
...					
D	reject	reject	accept	accept	reject
...					

Figure 4.4b (Insert D into Figure 4.4a)

	$\langle M_1\rangle$	$\langle M_2\rangle$	$\langle M_3\rangle$	$\langle M_4\rangle$ ...	$\langle D\rangle$ ...
M_1	accept	reject	accept	reject	DC
M_2	accept	accept	accept	accept	DC
M_3	reject	reject	reject	reject	DC
M_4	accept	accept	reject	reject	DC
...					
D	accept	accept	accept	accept	accept
...					

Figure 4.5a (Insert D into Figure 4.5)

The above refutation of the Sipser diagonalization proof applies to all halting problem diagonalization proofs. Sipser was chosen as a widely available and very clear proof.

The Sipser proof was the basis for this superb lecture by
Professor Dan Gusfield of UC Davis:

L15: Proof by Diagonalization that ATM (Halting Problem) is Not Decidable
Dec 12, 2012 https://www.youtube.com/watch?v=jM6osxSX9GA

The following portions of pages 166-167 are directly relevant to the rebuttal.
Sipser, Michael 1997. Introduction to the Theory of Computation. Boston: PWS Publishing Company (165-167)

Where is the diagonalization in the proof of Theorem 4.9? It becomes apparent when you examine tables of behavior for TMs H and D. In these tables we list all TMs down the rows, M_1, M_2, ... and all their descriptions across the columns, $\langle M_1 \rangle$, $\langle M_2 \rangle$, ... The entries tell whether the machine in a given row accepts the input in a given column. The entry is *accept* if the machine accepts the input but is blank if it rejects or loops on that input. We made up the entries in the following figure to illustrate the idea.

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\cdots$
M_1	*accept*		*accept*		
M_2	*accept*	*accept*	*accept*	*accept*	
M_3					$\cdots$
M_4	*accept*	*accept*			
$\vdots$		$\vdots$			

FIGURE 4.4
Entry i, j is *accept* if M_i accepts $\langle M_j \rangle$

In the following figure the entries are the results of running H on inputs corresponding to Figure 4.4. So if M_3 does not accept input $\langle M_2 \rangle$, the entry for row M_3 and column $\langle M_2 \rangle$ is *reject* because H rejects input $\langle M_3, \langle M_2 \rangle \rangle$.

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\cdots$
M_1	*accept*	*reject*	*accept*	*reject*	
M_2	*accept*	*accept*	*accept*	*accept*	$\cdots$
M_3	*reject*	*reject*	*reject*	*reject*	
M_4	*accept*	*accept*	*reject*	*reject*	
$\vdots$		$\vdots$			

FIGURE 4.5
Entry i, j is the value of H on input $\langle M_i, \langle M_j \rangle \rangle$

In the following figure, we added D to Figure 4.5. By our assumption, H is a TM and so is D. Therefore it must occur on the list M_1, M_2, ... of all TMs. Note that D computes the opposite of the diagonal entries. The contradiction occurs at the point of the question mark where the entry must be the opposite of itself.

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\cdots$	$\langle D \rangle$	$\cdots$
M_1	*accept*	*reject*	*accept*	*reject*		*accept*	
M_2	*accept*	*accept*	*accept*	*accept*	$\cdots$	*accept*	$\cdots$
M_3	*reject*	*reject*	*reject*	*reject*		*reject*	
M_4	*accept*	*accept*	*reject*	*reject*		*accept*	
$\vdots$		$\vdots$			$\ddots$		
D	*reject*	*reject*	*accept*	*accept*		?	
$\vdots$		$\vdots$					$\ddots$

FIGURE 4.6
If D is in the figure, a contradiction occurs at "?"