The Notion of Truth in Natural and Formal Languages

The purpose of this paper is to complete the RHS of Tarski's famous formula: $\forall x\ True(x) \leftrightarrow \varphi(x)$

For any natural (human) or formal (mathematical) language L we know that an expression X of language L is true if and only if there are expressions Γ of language L that connect X to known facts.

By extending the notion of a Well Formed Formula to include syntactically formalized rules for rejecting semantically incorrect expressions we recognize and reject expressions that evaluate to neither True nor False.

An axiom is a proposition regarded as self-evidently true without proof. Axioms are really nothing more than a set of expressions of language that have been assigned the semantic property of True. Axioms form the ultimate foundation of Truth-conditional semantics.

The only way that we know that the type of animal named {cat} is not a type of {ice cream cone} is the axioms of natural language that define the semantic meaning of these English concepts.

"A cat is an animal": Formalized as: $Cat \subset Animal$ is essentially an expression of language that has been assigned the semantic property of True. The set of these expressions define the axioms of natural language.

Rudolf Carnap defined Meaning Postulates (1952) formalizing natural language semantics:
$(x)\ Bachelor(x) \rightarrow {\sim}Married(x)$

Let 'W' be a primitive predicate designating the relation Warmer. Then 'W' is transitive, irreflexive, and hence asymmetric in virtue of its meaning:
(a) $(x)(y)(z)\ W(x,y) \land W(y,z) \rightarrow W(x,z)$
(b) $(x)\quad {\sim}W(x,x)$
(c) $(x)(y)\quad W(x,y) \rightarrow {\sim}W(y,x)$

Mendelson 1.4 An Axiom System for the Propositional Calculus
A wf C is said to be a consequence in S of a set Γ of wfs if and only if there is a sequence B1, ..., Bk of wfs such that C is Bk and, for each i, either Bi is an axiom or Bi is in Γ, or Bi is a direct consequence by some rule of inference of some of the preceding wfs in the sequence. Such a sequence is called a proof (or deduction) of C from **Γ**. The members of **Γ** are called the hypotheses or premisses of the proof. We use $\Gamma \vdash C$ as an abbreviation for "C is a consequence of **Γ**".

An unordered set of WFF on the LHS of ⊢ becomes a formal proof when it is arranged into an ordered sequence of connected rules-of-inference with the RHS of ⊢ as the last element of this ordered sequence.

When the ordered set of connected rules-of-inference begins with one or more axioms (WFF defined with the semantic property of True) then the result of the formal proof is Truth.

The natural language equivalent to an axiom in formal language is a {known fact}. Some expressions of natural language are simply defined to be true.

Example: "a cat is an animal".
Formalized as: (cat $\in$ animals) or (cat $\lhd$ animal)
where $\lhd$ is the [is_a_type_of] operator adapted from UML Inheritance relation.

Generalizing Tarski's 1933 Formal Correctness formula to every formal system:
$\forall X \ \text{True}(X) \leftrightarrow \varphi(X)$
becomes
$\forall L \forall X \ \text{True}(L,X) \leftrightarrow \varphi(L,X)$

Material Adequacy
This means that the objects satisfying φ should be exactly the objects that we would intuitively count as being true sentences of L, and that this fact should be provable from the axioms of the metalanguage.

$\forall L \forall X \ \text{True}(L,X) \leftrightarrow \varphi(L,X)$
Completing the RHS of this formula such that Material Adequacy is also satisfied:
$\varphi(L,X)$
becomes
$\exists \boldsymbol{\Gamma} \subseteq \text{Axioms}(L) \ \exists \boldsymbol{\Psi} \subseteq \text{WFF}(L) \ (\text{Sequence}(\boldsymbol{\Gamma},\boldsymbol{\Psi}) \vdash X)$
Sequence($\boldsymbol{\Gamma},\boldsymbol{\Psi}$) indicates a sequence of Axioms followed by a sequence of WFF

$\forall L \forall X \ \text{True}(L, X) \leftrightarrow \exists \boldsymbol{\Gamma} \subseteq \text{Axioms}(L) \ \exists \boldsymbol{\Psi} \subseteq \text{WFF}(L) \ (\text{Sequence}(\boldsymbol{\Gamma},\boldsymbol{\Psi}) \vdash X)$

For all L element of set Formal_Systems For all X element of set L
There exists a contiguous sequence (inference chain) beginning with Axioms Γ of language L followed by a sequence of connected rules-of-inference WFF Ψ of language L deriving WFF consequent X at the end of this connected sequence.

$\forall L \forall X \ \text{False}(L, X) \leftrightarrow \exists \boldsymbol{\Gamma} \subseteq \text{Axioms}(L) \ \exists \boldsymbol{\Psi} \subseteq \text{WFF}(L) \ (\text{Sequence}(\boldsymbol{\Gamma},\boldsymbol{\Psi}) \vdash {\sim}X)$

$\forall L \forall X \ {\sim}\text{True}(L, X) \leftrightarrow {\sim}\exists \boldsymbol{\Gamma} \subseteq \text{Axioms}(L) \ \exists \boldsymbol{\Psi} \subseteq \text{WFF}(L) \ (\text{Sequence}(\boldsymbol{\Gamma},\boldsymbol{\Psi}) \vdash X)$

To verify that an expression X of language L is True or False only requires a syntactic logical consequence inference chain (formal proof) from a sequence of Axioms followed by a sequence of WFF to the consequent of X or ~X. (Backward chaining reverses this order).

Predicate logic is augmented with an <assign alias name> operator.
LHS is assigned as an alias name for the RHS
LHS $\equiv$ RHS
The LHS is logically equivalent to the RHS *only because*
the LHS is merely an alias name (place-holder) for the RHS

The <assign alias name> operator allows an expression to refer directly to itself.

"This sentence is not True."
$LP \equiv \forall L \in Formal\_Systems \sim True(L, LP)$

Expanded definition of above:
$LP \equiv \forall L \in Formal\_Systems \sim \exists \boldsymbol{\Gamma} \subseteq Axioms(L)\ \exists \boldsymbol{\Psi} \subseteq WFF(L)\ (Sequence(\boldsymbol{\Gamma}, \boldsymbol{\Psi}) \vdash LP)$

For all L element of set Formal_Systems there does not exist a sequence of Axioms Γ of language L preceding a sequence of WFF Ψ of language L that proves LP.

Sentence (mathematical logic)
In mathematical logic, a sentence of a predicate logic is a Boolean-valued well-formed formula with no free variables. A sentence can be viewed as expressing a proposition, something that must be true or false. The restriction of having no free variables is needed to make sure that sentences can have concrete, fixed truth values: As the free variables of a (general) formula can range over several values, the truth value of such a formula may vary.

$LP \equiv \forall L \in Formal\_Systems \sim \exists \boldsymbol{\Gamma} \subseteq Axioms(L)\ \exists \boldsymbol{\Psi} \subseteq WFF(L)\ (Sequence(\boldsymbol{\Gamma}, \boldsymbol{\Psi}) \vdash LP)$

Since neither the above expression nor its negation can be satisfied within any formal system, the above expression is neither True nor False, thus semantically incorrect.