The Notion of Truth in Natural and Formal Languages

The purpose of this paper is to complete the RHS of Tarski's famous formula: $\forall x \; True(x) \leftrightarrow \varphi(x)$

For any natural (human) or formal (mathematical) language L we know that an expression X of language L is true if and only if there are expressions Γ of language L that connect X to known facts.

By extending the notion of a Well Formed Formula to include syntactically formalized rules for rejecting semantically incorrect expressions we recognize and reject expressions that evaluate to neither True nor False.

An axiom is a proposition regarded as self-evidently true without proof. Axioms are really nothing more than a set of expressions of language that have been assigned the semantic property of True. Axioms form the ultimate foundation of Truth-conditional semantics.

The natural language equivalent to an axiom in formal language is a {known fact}. Some expressions of natural language are simply defined to be True.

Example: “a cat is an animal”.
Formalized as: (cat ∈ animals) or (cat ◁ animal)
where ◁ is the [is_a_type_of] operator adapted from UML Inheritance relation.
The only reason that we know that “a cat is an animal” is that it is defined to be True.

Rudolf Carnap defined Meaning Postulates (1952) formalizing natural language semantics:
$(x)\ Bachelor(x) \rightarrow \sim Married(x)$

Let 'W' be a primitive predicate designating the relation Warmer. Then 'W' is transitive, irreflexive, and hence asymmetric in virtue of its meaning:
(a) $(x)(y)(z)\ W(x,y) \land W(y,z) \rightarrow W(x,z)$
(b) $(x) \quad \sim W(x,x)$
(c) $(x)(y) \quad W(x,y) \rightarrow \sim W(y,x)$

Mendelson 1.4 An Axiom System for the Propositional Calculus
A wf C is said to be a consequence in S of a set Γ of wfs if and only if there is a sequence B1, ..., Bk of wfs such that C is Bk and, for each i, either Bi is an axiom or Bi is in Γ, or Bi is a direct consequence by some rule of inference of some of the preceding wfs in the sequence. Such a sequence is called a proof (or deduction) of C from **Γ**. The members of **Γ** are called the hypotheses or premisses of the proof. We use **Γ** ⊢ C as an abbreviation for “C is a consequence of **Γ**”.

An unordered set of WFF on the LHS of ⊢ becomes a formal proof when it is arranged into an ordered sequence of connected rules-of-inference with the RHS of ⊢ as the last element of this ordered sequence.

When the ordered set of connected rules-of-inference begins with one or more axioms (WFF defined with the semantic property of True) then the result of the formal proof is Truth.

Here is the resulting generic Truth predicate:
$\forall L \forall X\ \text{True}(L, X) \leftrightarrow \exists \boldsymbol{\Gamma} \subseteq \text{Axioms}(L)\ \exists \boldsymbol{\Psi} \subseteq \text{WFF}(L)\ (\text{Sequence}(\boldsymbol{\Gamma},\boldsymbol{\Psi}) \vdash X)$

Above Truth predicate explained in English
For all L element of set Formal_Systems For all X element of set L
There exists a contiguous sequence of rules-of-inference (inference chain)
beginning with Axioms $\boldsymbol{\Gamma}$ of language L connected to a sequence of WFF $\boldsymbol{\Psi}$
of language L deriving WFF consequent X at the end of this contiguous sequence.

Generalizing Tarski's 1933 Formal Correctness formula to every formal system:
$\forall X\ \ \text{True}(X) \leftrightarrow \varphi(X)$
becomes
$\forall L \forall X\ \ \text{True}(L,X) \leftrightarrow \varphi(L,X)$

Material Adequacy
This means that the objects satisfying φ should be exactly the objects that we would intuitively count as being true sentences of L, and that this fact should be provable from the axioms of the metalanguage.

$\forall L \forall X\ \text{False}(L, X) \leftrightarrow \exists \boldsymbol{\Gamma} \subseteq \text{Axioms}(L)\ \exists \boldsymbol{\Psi} \subseteq \text{WFF}(L)\ (\text{Sequence}(\boldsymbol{\Gamma},\boldsymbol{\Psi}) \vdash \sim X)$

$\forall L \forall X\ \ \sim\text{True}(L, X) \leftrightarrow \sim\exists \boldsymbol{\Gamma} \subseteq \text{Axioms}(L)\ \exists \boldsymbol{\Psi} \subseteq \text{WFF}(L)\ (\text{Sequence}(\boldsymbol{\Gamma},\boldsymbol{\Psi}) \vdash X)$

To verify that an expression X of language L is True or False only requires a syntactic logical consequence inference chain (formal proof) from a sequence of Axioms followed by a sequence of WFF to the consequent of X or ~X. (Backward chaining reverses this order).

Predicate logic is augmented with an <assign alias name> operator.
LHS is assigned as an alias name for the RHS
LHS ≡ RHS
The LHS is logically equivalent to the RHS *only because*
the LHS is merely an alias name (place-holder) for the RHS
The <assign alias name> operator allows an expression to refer directly to itself.

When we formalize expressions of language such as the Liar Paradox using the above universal truth predicate, we can finally understand its semantic error.

"This sentence is not True."
$LP \equiv \forall L \in \text{Formal\_Systems}\ \sim\text{True}(L, LP)$

Expanded definition of above:
$LP \equiv \forall L \in \text{Formal\_Systems}\ \sim\exists \boldsymbol{\Gamma} \subseteq \text{Axioms}(L)\ \exists \boldsymbol{\Psi} \subseteq \text{WFF}(L)\ (\text{Sequence}(\boldsymbol{\Gamma},\boldsymbol{\Psi}) \vdash LP)$

For all L element of set Formal_Systems there does not exist a sequence of Axioms Γ of language L connected to a subsequent sequence of WFF Ψ of language L that proves this sentence.

Sentence (mathematical logic)
In mathematical logic, a sentence of a predicate logic is a Boolean-valued well-formed formula with no free variables. A sentence can be viewed as expressing a proposition, something that must be true or false. The restriction of having no free variables is needed to make sure that sentences can have concrete, fixed truth values: As the free variables of a (general) formula can range over several values, the truth value of such a formula may vary.

$LP \equiv \forall L \in Formal\_Systems \sim\exists \boldsymbol{\Gamma} \subseteq Axioms(L)\ \exists \boldsymbol{\Psi} \subseteq WFF(L)\ (Sequence(\boldsymbol{\Gamma},\boldsymbol{\Psi}) \vdash LP)$

Since neither the above expression nor its negation can be satisfied within any formal system, the above expression is neither True nor False, thus semantically incorrect.

> ON FORMALLY UNDECIDABLE PROPOSITIONS
> OF PRINCIPIA MATHEMATICA AND RELATED SYSTEMS I
> by Kurt Gödel Vienna
>
> The analogy between this result and Richard's antinomy leaps to the eye; there is also a close relationship with the "liar" antinomy,[14]
>
> 14 Every epistemological antinomy can likewise be used for a similar undecidability proof.

Since Kurt Gödel said that the Liar Paradox "can … be used for a similar undecidability proof."
The semantic error of the Liar Paradox equally applies to the 1931 Incompleteness Theorem.