Revisiting Dummett's Proof-Theoretic Justification Procedures

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Abstract: Dummett's justification procedures are revisited. They are used as background for the discussion of some conceptual and technical issues in proof-theoretic semantics, especially the role played by assumptions in proof-theoretic definitions of validity.

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1 The placeholder view of assumptions

In his contribution to the *Logica Yearbook 2007*, Schroeder-Heister (2008, § 3) pointed out some dogmas of proof-theoretic semantics. One of the dogmas was the *the primacy of the categorical over the hypothetical*, or, as it was latter called, the *placeholder view of assumptions*.

According to this dogma, hypothetical arguments, or arguments with open assumptions, should be reduced to closed arguments, or closed proofs (proofs from no assumptions). In other words, assumptions are considered to be placeholders for closed proofs. The proof-theoretic definitions of validity for arguments proposed by Prawitz (1971, 1973, 2006, 2014) are prominent examples of the placeholder view of assumptions.

1.1 The problem with *reduction ad absurdum* arguments

In intuitionistic logic, *reductio ad absurdum* can be used to obtain negative sentences, or refutations. In such arguments, a contradiction (which in natural deduction systems is usually represented by an absurdity constant) is deduced from a collection of assumptions which are thereby shown to be jointly contradictory, or incompatible.

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The task of explaining the validity of arguments that use *reductio ad absurdum* becomes problematic when assumptions are considered placeholders for closed proofs and validity is explained as a constructive function from closed proofs of the assumptions to closed proofs of the conclusion, because the explanation then needs to appeal to proofs of contradictions. These proofs do not need to be actual proofs, but must be at least possible or conceivable if the explanation is to be at all comprehensible. Whether proofs of contradictions can be conceived, or what does it mean to conceive such things, is one of the questions that the advocates of the placeholder view have to deal with.

In some sense, the conundrum with *reductio ad absurdum* is reminiscent of a problem that Prawitz (1971, § IV.1.1) already dealt with in his first attempt at defining a proof-theoretic notion of validity. There, the problem was the vacuous validation of implications with an unprovable antecedent. Prawitz's solution was to reformulate the semantic clause for implication so as to consider extensions of the underlying atomic system where the antecedent would be provable.[2] However, the problem becomes much more prominent when dealing with contradictions, because our intuition is that they are not supposed to be provable under any circumstances whatsoever.

1.2 The primacy of assertion

Walking side by side with the placeholder view of assumptions is what we can call *the primacy of assertion over other speech acts*. The rationale is that the speech act of assertion comes with a commitment on the part of the speaker to offer justifications for the asserted sentence and thus, in order to correctly assert the sentence, the speaker must be in possession of such justifications, or be able to produce them. In other words, in order to correctly assert a sentence, one needs to have a proof of the sentence.

From this picture emerges an approach to proof-theoretic semantics based on assertibility conditions, with proofs acting as justifications associated with assertions. Here, another dogma discussed by Schroeder-Heister (2008, § 3) comes into play: *the transmission view of consequence*. But, in contrast with semantics based on truth conditions, instead of truth, it is *correct assertibility* which is transmitted from premisses to conclusion in valid arguments. Or, if one prefers to talk about what makes an assertion correct, or

[2] Unfortunately, the amendment was still insufficient to avoid validation of classical inferences in the implication fragment (Sanz, Piecha, & Schroeder-Heister, 2014, § 4).

justified, one can say that consequence transmits *proof* instead of *truth*. As a result, the approach assumes a distinctively epistemological character.

However, more complications related to hypothetical reasoning surface: it seems counterintuitive, to say the least, to hold that a speaker engaged in a hypothetical argument is committed to the *assertion* of either the assumptions or the conclusion of the argument. As a matter of fact, the speaker may even reject them and, given the argument is valid, her reasoning remains unassailable. In particular, the point reappears with renewed force when considered in the context of arguments that use *reductio ad absurdum*, since it would commit us to the possibility of correctly asserting absurdities.

One can appeal to a concept of *conditional assertion* to try and salvage the approach from such objections while preserving an unified explanatory model based primarily on assertion and proof. Thus, the conclusions of hypothetical arguments are taken not to be asserted outright but only under certain conditions. That is, the conclusions of hypothetical arguments are *conditionally asserted*. In terms of speech acts, however, it is not at all clear whether conditional assertion constitutes any assertion at all.

It seems to me that trying to explain deductive validity in terms of assertions and proofs is misguided. I am not trying to deny that deductive reasoning has epistemic importance or that deductive reasoning transmits evidence, or justification, from the premisses to the conclusion. If there is a deductive relation between premisses and conclusion, then, of course, the correct assertion of the premisses would lead to the correct assertion of the conclusion and, similarly, if proof for the premisses are provided then a proof for the conclusion is obtained. Rather, I contend that to *explain* deductive validity by reducing it to this transmission effect is to put the cart before the horse and confuse the cause with its effects, the disease with its symptoms.

2 BHK vs Gentzen

Proof-theoretic notions of validity have often been inspired by a mixture of ideas involving the BHK interpretation of the logical constants and Gentzen's well-known remarks on the rules of natural deduction. In particular, the conception of validity underlying the placeholder view of assumptions is largely informed by the BHK interpretation of implication: an argument from A to B is valid if, and only if, every proof of A can be transformed into a proof of B. Yet, with its unqualified reference to proofs, this view is not immediately amenable to the recursive treatment required of semantic clauses and definitions (Prawitz, 2007, § 2.1). In this context, Gentzen's ideas are often developed into a notion of canonical proof in order to achieve recursiveness for an approach primarily based on the BHK interpretation.

On the other hand, the core of Gentzen's ideas are independent of the BHK interpretation. They are best represented by what became known as *proof-theoretic harmony*. Harmony, as a fundamental principle of natural deduction systems, applies equally well to deductions from assumptions as to the particular case of proofs (deductions from no assumptions). By appealing to harmony while at the same time avoiding the BHK interpretation and the placeholder view of assumptions, we can develop a more appropriate proof-theoretic notion of validity.

2.1 A more Gentzenian approach to validity

The introduction rules for a logical constant γ can be seen as an explanation of the *canonical use* of a sentence *as a conclusion* in a deductive argument (where, of course, γ is the sentence's main connective). This is achieved by exhibiting the conditions for obtaining a sentence $A \gamma B$ as a conclusion of an argument (where γ is a binary connective). In the paradigmatic case, these conditions are expressed in terms of the component sentences A and B.[3]

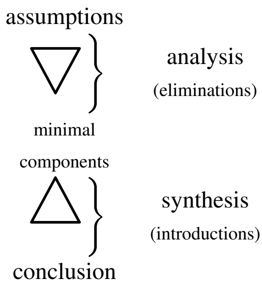
In an analogous manner, the elimination rules for a logical constant can be seen as an explanation of the *canonical use* of a sentence *as an assumption* in a deductive argument. This is accomplished by exhibiting the consequences that can be extracted from the sentence (as a major premiss of an elimination rule, possibly in the context of some minor premisses).

Thus, introduction and elimination rules stand for two distinct aspects of the deductive use of the logical constants. Harmony arises as a requirement of balance between those two aspects such that there is a equilibrium between what is required by the introduction rules and what consequences are extracted by the elimination rules. As a result, among other things, harmony guarantees that there is nothing to be gained by performing roundabout derivations where sentences are obtained by an introduction rule to be immediately after analysed by the corresponding elimination rule. Therefore, for a proper understanding of the deductive practice, it suffices to look

[3] Notice that the conditions do not necessarily need to be expressed in terms of closed proofs of A and B, but can be expressed in terms of assumptions A and B or of arguments for A and B which may depend on other assumptions.

at the collection of direct derivations, also known as normal derivations.

The normal derivations have a very perspicuous form (Prawitz, 1965, Chapter IV, § 2, Theorem 2). They are composed of (can be divided into) two parts: an analytic part, where the assumptions are analysed (destructed), and a synthetic part, where the conclusion is synthesized (constructed) from the components resulted from the analysis.[4]



The equilibrium between introductions and eliminations suggest that, if we were to *supplement* the assumptions on top through a process of inversion by backward application of introductions, we would arrive at the minimal components required for the synthetic part. And, similarly, if we were to *complement* the conclusion by forward application of eliminations, we would arrive at the minimal components resulted from the analysis of the assumptions. Accordingly, the harmonious inferential behaviour of the logical constants has sometimes been expressed by pointing out that introductions and eliminations can be, in some sense, obtained from one another by *inversion principles*.

Gentzen's investigations into logical deduction can thus supply the basic pieces for a proof-theoretical notion of logical validity for arguments based on the inferential meaning conferred on the logical constants by either their introduction rules or their elimination rules. In particular with respect to the problems discussed in the previous section, the Gentzenian approach has the advantage of giving proper heed to assumptions and being fairly independent from specific speech acts.[5]

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[4] In the general case, each of these parts can, of course, be empty.

[5] For instance, deductive arguments can be used to show someone who denies the conclusion that she has to deny at least one of the assumptions. They can also be used to explore the consequences of a conjecture. These applications of deductive arguments align very well with the Gentzenian approach, but none of them necessarily involves anyone making any assertions.

Gentzen's ideas suggest that, although a persistent dogma in much of the discussion around proof-theoretic semantics, the placeholder view of assumptions can be challenged from an authentic proof-theoretic perspective. In the next section, I revisit Dummett's justification procedures. I argue that, as a development of the Gentzenian approach just sketched, they afford a notion of proof-theoretic validity that incorporates assumptions in an essential way.[6]

I stay at the level of the core concepts, without going into rigorous definitions. Nonetheless, I hope that my explanations would be sufficient to give an overall idea of the relationship between the justification procedures (how they can be understood as emerging from a shared framework) and I also presume to have provided enough detail so that an interested and motivated reader would be able to attempt rigorous definitions of her own based on the approach outlined. In the last section, I discuss, to some extent, how Dummett's procedures perform with respect to adequacy to intuitionistic logic, especially in comparison with approaches that endorse the placeholder view of assumptions.

3 An overview of Dummett's approach

Dummett (1991, Chapter 11–13) proposed two proof-theoretic justification procedures for logical laws which amount to definitions of logical validity for arguments. The "verificationist" procedure defines validity of arguments on the basis of introduction rules for logical constants and the "pragmatist" procedure defines validity of arguments on the basis of elimination rules for logical constants.[7]

These proof-theoretic justification procedures play an important role in Dummett's philosophical anti-realist programme. They are central pieces of his very detailed and elaborate argument for rejection of classical logic in favor of intuitionistic logic.[8] In particular, Dummett (1975, 1991) has

[6]It is important to notice that, although the Dummettian approach that I advocate rejects the placeholder view of assumptions, other dogmatic characteristics, like the unidirectional and global character of the semantics, remain unchallenged (Schroeder-Heister, 2016, § 2.3 and 2.4).

[7]I adopt the characterizations "verificationist" and "pragmatist" from Dummett. However, without denying their existence, I do not imply with the adoption of the terminology any connections outside the domain of logical validity. Therefore, I refer to verificationism and pragmatism just as markers to distinguish between approaches to validity based on introduction rules and elimination rules, respectively.

[8]At the end of this argument, Dummett (1991, p. 300) writes: "We took notice of the

conjectured that proof-theoretic notions of validity would justify exactly intuitionistic logic.

Dummett's definitions of validity are based on canonical inference rules for the logical constants. These inference rules are thought to fix the meaning of the logical constants by displaying their canonical deductive use. They are, in Dummett's terminology, "self-justifying".

In contrast with some definitions found in the literature, Dummett's definitions are not based on semantic clauses for particular logical constants. Instead, he assumes that self-justifying rules are given. These self-justifying rules are introduction rules in the context of the verificationist procedure, and elimination rules in the context of the pragmatist procedure. In both procedures, the definitions are stated irrespective of the particular constants or rules provided. Therefore, Dummett's definitions can, at least in principle, be applied without modification to different logics by providing the appropriate self-justifying rules for the logical constants.

3.1 Core concepts

Both the verificationist and the pragmatist procedures can be seen as products of a basic common framework. The core notions of validity behind the justification procedures can be informally outlined as follows:

verificationism whenever the assumptions can be obtained in a canonical manner, the conclusion can also be obtained in a canonical manner

pragmatism any consequence that can be drawn in a canonical manner from the conclusion can also be drawn in a canonical manner from the assumptions.

The expression "canonical manner" is an allusion to *canonical arguments*. As usual in proof-theoretic notions of validity, canonical arguments are the main ingredients of the justification procedures. An important feature, however, is that Dummett's canonical arguments are *not closed proofs*,

problem what metalanguage is to be used in giving a semantic explanation of a logic to one whose logic is different. A metalanguage whose underlying logic is intuitionistic now appears a good candidate for the role, since its logical constants can be understood, and its logical laws acknowledged, without appeal to any semantic theory and with only a very general meaning-theoretical background. If that is not *the* right logic, at least it may serve as a medium by means of which to discuss other logics."

but instead *may depend on assumptions*. Consequently, when precisely formulated, the definitions of validity must take into account the assumptions on which the canonical arguments depend.

The canonical arguments are composed *primarily* of canonical inferences. However, they cannot be required to be *entirely* composed of canonical inferences. They must allow for the possibility of *subordinate subarguments*, that is, subarguments cultivated under the support of additional assumptions (Dummett, 1991, p. 260). These subordinate subarguments, when not already canonical arguments themselves, are *critical subarguments*. They are critical in the sense that the validity of the original canonical argument would recursively depend on their validity. This means, of course, that much care should be dispensed to guarantee that critical subarguments are of lower complexity than the original canonical argument.

In a verificationist context, critical subarguments are detected through the presence of assumption discharge. In a pragmatist context, they are detected through the presence of minor premisses. These signs indicate, in their respective contexts, when assumptions are being added.

Now, returning to our informal notions of validity, in the verificationist procedure, the means to evaluate the conditions under which the assumptions may be obtained in a canonical manner are provided by *supplementations*. They result from substitution of the assumptions with canonical arguments. In the pragmatist procedure, the means to evaluate what consequences can be drawn from the conclusion are provided by *complementations*. They result from substitution of the conclusion with canonical arguments.

verificationism	**pragmatism**
canonical arguments	canonical arguments
(primarily introductions)	(primarily eliminations)
critical subarguments	critical subarguments
(revealed by assumption discharges)	(revealed by minor premises)
supplementation	complementation
(assumptions canonically unfolded)	(conclusion canonically unfolded)

Instead of as substitution operations, one can see the processes of supplementation and complementation more dynamically. The process of supplementation can be seen as the repeated backward application of introduction rules from the assumptions, thus growing the argument upwards, which is why Dummett also refers to the verificationist procedure as the *upwards justification procedure*. Similarly, the process of complementation can be seen as the repeated forward application of elimination rules to the conclusion of the argument as a major premiss, thus growing the argument downwards, which is why Dummett also refers to the pragmatist procedure as the *downwards justification procedure*.

In order to appraise the validity of an argument from Γ to C, the verificationist procedure examines its supplementations and investigates whether a canonical argument for C can be attained under the same conditions. Since supplementations result from canonical arguments *for* Γ, they may depend on assumptions Δ (remember that canonical arguments may depend on assumptions). Then, the canonical argument for C may not depend on other assumptions besides Δ.

In an analogous manner, in order to appraise the validity of an argument from Γ to C, the pragmatist procedure examines the complementations and investigates whether a canonical argument for the conclusion of the complementation, say Z, can be attained under the same conditions. Because complementations result from canonical arguments *from* C (as assumption and major premiss of elimination), they may depend on additional assumptions Δ required by minor premisses. Then, the canonical argument for Z may not depend on other assumptions besides Γ, Δ.

Supplementation

$$\begin{array}{c} \Delta, [A] \\ \vdots \\ \dfrac{B}{A \to B} \\ \vdots \\ \boxed{\dfrac{\Gamma}{C}} \end{array}$$

Complementation

$$\begin{array}{c} \begin{array}{cc} \boxed{\dfrac{\Gamma}{C}}, & \Delta \\ \vdots & \vdots \\ A \to B & A \end{array} \\ \hline B \\ \vdots \\ Z \end{array}$$

The canonical arguments used to supplement or complement may have critical subarguments. In the figures above, I indicate the form of possible supplementations and complementations of an illustrative argument from Γ to C. In the supplementation, the subargument from Δ, A to B may be critical. In the complementation, the subargument for the minor premiss A

of $\to$E may be critical (for instance, if validly, but not canonically, obtained from assumptions in Δ).

3.2 An illustrative example

With the help of the concepts explained so far, let us try and show the validity of an argument according to the pragmatist procedure. Since I did not gave any rigorous definitions, our validation of the argument must be carried out in an informal and intuitive manner. Still, we adhere to the overall pragmatist approach and appeal exclusively to the elimination rules for the logical constants.

$$\frac{A \to (B \wedge C)}{(A \to B) \wedge (A \to C)}$$

According to the pragmatist procedure, the argument depicted above would be valid if whatever consequences can be drawn from the conclusion in a canonical manner, can also be drawn from the assumptions in a canonical manner. In order to see what can be extracted canonically from the conclusion, we complement:

$$\frac{\dfrac{\dfrac{A \to (B \wedge C)}{(A \to B) \wedge (A \to C)}}{A \to B} \quad A}{B} \qquad \frac{\dfrac{\dfrac{A \to (B \wedge C)}{(A \to B) \wedge (A \to C)}}{A \to C} \quad A}{C}$$

The complementations were obtained by application of elimination rules until we arrived at a simple schematic letter (each consequence being the major premiss for the next application of an elimination rule). Let us concede that there is no loss of generality when we supply the minor premiss of $\to$E with a simple assumption A. So, as we can see, we have two complementations, with conclusions B and C, respectively, and assumptions $A \to (B \wedge C)$ and A. In order to show validity, we must find canonical arguments from $A \to (B \wedge C)$ and A to B, and from $A \to (B \wedge C)$ and A to C.[9]

[9]The assumptions of the complementations happen to be the same in this case. In the general case, however, they have to be considered separately, e.g. each complementation has its own assumptions. In order to establish validity, we must show that the conclusion of the complementation can be obtained from the assumptions of the complementation, *for every complementation*.

$$\dfrac{\dfrac{A \to (B \wedge C) \qquad A}{B \wedge C}}{B} \qquad\qquad \dfrac{\dfrac{A \to (B \wedge C) \qquad A}{B \wedge C}}{C}$$

The example is a rather simple one (it does not involve, for instance, canonical arguments with critical subarguments). But it illustrates how a nontrivial argument (an argument whose *derivation* would need *both* eliminations and introductions) can be justified by appealing exclusively to the meaning conferred on the logical constants by their elimination rules.

4 Adequacy to intuitionistic logic

Sanz et al. (2014, § 4) showed that, in the fragment containing only implication, a strictly classical inference, Peirce's rule, is valid at the atomic level with respect to a proof-theoretic notion of validity proposed by Prawitz (1971, § IV.1.2). With implication already behaving classically at the atomic level, and assuming the validity of the usual rules for implication, conjunction and absurdity (but no disjunction), the result can be generalized to arbitrary complex sentences. As a consequence, we can have a set of logical constants powerful enough to account for all classically valid inferences in propositional logic (with the other constants being defined in terms of implication, conjunction, and absurdity).

In the context of Dummett's anti-realist philosophical programme, these kinds of results can have a very negative effect. They show that, contrary to what is intended, classical logic can be validated in a proof-theoretic setting, even when no classical principles are admitted in the semantics. Despite Dummett (1991, Chapter 15), there would be classical logic without bivalence (Sandqvist, 2009). Furthermore, the arguments used to establish these results can be expected to apply to any proof-theoretic notion based on the BHK view of implication and conservative extensions of production systems.

In contrast with Prawitz's early approach, Dummett's verificationist procedure does not adhere to the BHK view of implication as based on closed proofs, and is not conservative over extensions of production systems. Still, the verificationist procedure, as defined by Dummett (1991, p. 260), does not reject completely the placeholder view of assumptions, because the assumptions of canonical arguments are required to be atomic. As a result, from the point of view of a general approach to *logical* validity, too much emphasis is placed on the underlying atomic system and validity then ceases to be a schematic notion.[10]

One may be inclined to think that Dummett's approach based on atomic assumptions is basically equivalent to approaches based on production systems because atomic assumptions could be replaced by atomic axioms. However, the presupposition of monotonicity incorporated into production systems (by requiring that extensions of the production system be conservative) does not carry over to collections of assumptions. For instance, Goldfarb (2016, Counterexample 2) observed that, if the production system has no rules, the argument from $\varphi \to \psi$ to ψ is valid (where φ and ψ are atomic sentences). If the production system is extended with the rule φ/ψ, the argument ceases to be valid, which shows that verificationist validity is not monotonic over extensions of the production system.[11]

As a consequence, under the verificationist procedure with atomic assumptions, the atomic base cannot be correctly interpreted as expandable states of knowledge in the model of mathematics where, once a sentence is proved, it remains proved. Rather, conclusions cultivated under the support of some assumptions may not be available under other assumptions, and it is not intuitively compelling to restrict hypothetical arguments to conservatively extended collections of assumptions.

Since the arguments that show validation of Peirce's rule rely on the monotonicity of the atomic base, it is reasonable to expect that Dummett's verificationist procedure escapes validation of Peirce's rule, even in an atomic level. So, I would say that Dummett's procedure compares favourably in this respect with approaches based on closed proofs and conservative extensions of production systems.[12]

On the other hand, Dummett's pragmatist procedure incorporates as-

[10]I agree with Goldfarb's (2016, § 4) assessment that Dummett's strategy, inspired by the BHK interpretation, of defining validity for actual sentences and then generalizing this notion in order to achieve logical validity is problematic.

[11]Disregarding other problematic features of the verificationist procedure, I agree with Michael Dummett and Ofra Magidor that Counterexample 2 is not really worrisome (Goldfarb, 2016, Postscript). In this context, the absence of atomic rules are more correctly interpreted, not as the absence of knowledge about the inferential relations between atomic sentences, but instead as the knowledge that there are no inferential relations between atomic sentences.

[12]Goldfarb (2016, Counterexample 3) pointed out another problem with the verificationist procedure: the validation of the intuitionistically underivable rule of distribution of implication over disjunction. Since proof-theoretic approaches based on closed proofs and conservative extensions are also affected (Piecha, Sanz, & Schroeder-Heister, 2015, § 4), I suspect the roots of the problem here are more profound than the mere format of the atomic base, or the fact that assumptions are restricted to atomic sentences.

sumptions entirely (atomic and complex). As a matter of fact, in the pragmatist sense, the paradigmatic canonical argument is hypothetical. This is perfectly aligned with the view, formulated earlier, that the elimination rules express the deductive canonical use of sentences as assumptions. As a result, the pragmatist procedure is not subject to any of the objections raised in Section 1. It fits better with investigations grounded on other speech acts besides assertion and provides a more perceptive explanation of the meaning of conjectures and the validity of arguments that use *reductio ad absurdum*.

Furthermore, in contrast with the verificationist case, a little reflection shows that the pragmatist notion of validity is schematic. In particular, suppose some arbitrary atomic rules determining the inferential relationships between atomic sentences are supplied. These rules could be applied to the atomic conclusion of a canonical argument in order to extract further atomic consequences. Now, consider, for instance, our example in Section 3.2. It is easy to see that, even if B or C were complex, any further elimination or atomic rules applied in the complementations could be transferred to the canonical arguments without adding any assumptions with respect to the complementations.

Finally, I would like to suggest that the pragmatist procedure may offer a notion of validity more adequate with respect to intuitionistic logic. To this effect, I argue informally that atomic Peirce's rule is not valid according to the pragmatist procedure.

Claim 1 *Let φ and ψ be atomic sentences. Then, Peirce's rule*

$$\frac{(\varphi \to \psi) \to \varphi}{\varphi}$$

is not valid with respect to the pragmatist procedure.

Informal argument. The conclusion φ is already an atomic sentence, therefore there is nothing to complement. Now, we need to ask ourselves whether it is possible to obtain a valid canonical argument for φ from $(\varphi \to \psi) \to \varphi$. Any such valid canonical argument will have $(\varphi \to \psi) \to \varphi$ as major premiss of $\to$E.

$$\frac{(\varphi \to \psi) \to \varphi \qquad (\varphi \to \psi)}{\varphi}$$

At this point, since $(\varphi \to \psi)$ was not among the available assumptions, Peirce's rule would be valid only if we could validly deduce $(\varphi \to \psi)$ from no assumptions (a critical subargument). But this is not the case, because complementations of $(\varphi \rightarrow \psi)$ will have ψ as conclusion with only φ as assumption

$$\dfrac{\overline{(\varphi \rightarrow \psi)} \qquad \varphi}{\psi}$$

and there is no general way to obtain a canonical argument for ψ from φ unless, of course, in the particular cases where $\psi = \varphi$ or ψ can be extracted from φ by accepted atomic inferences. □

Since I favored a more conceptual discussion in detriment of technical developments, I am not able to produce here a more rigorous proof of this claim. But I still hope that my informal discussion and explanations deliver some evidence for the adequacy of the pragmatist procedure to intuitionistic logic.

Perhaps I should not end without a warning. While revisiting Dummett's procedures, my agenda was to advance what I think is a more appropriate approach to proof-theoretic validity. As a consequence, I may have presented the justification procedures under a perspective at odds with Dummett's own.[13] Albeit exegetical correctness were obviously not among my primary concerns, I do believe that Dummett's procedures supply the essential elements for an interesting proof-theoretic notion of validity, one that is free from the influence of the BHK interpretation and rejects completely the placeholder view of assumptions. I attempted, so to say, to separate the wheat from the chaff.

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[13]For instance, Dummett seems to be committed to the primacy of assertion and to endorse the BHK interpretation as the authoritative source on the meaning of the intuitionistic logical constants.

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