Modal twist-structures over residuated lattices

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Abstract

We introduce a class of algebras, called twist-structures, whose members are built as special squares of an arbitrary residuated lattice. We show how our construction relates to and encompasses results obtained by several authors on the algebraic semantics of non-classical logics. We define a logic that corresponds to our twist-structures and show how to expand it with modal operators, obtaining a paraconsistent many-valued modal logic that generalizes existing work on modal expansions of both Belnap–Dunn logic and paraconsistent Nelson logic.

Keywords: Twist-structure, paraconsistent modal logic, Nelson logic, many-valued logic, bilattice, residuated lattice.

1 Introduction and motivation

One of the latest and most challenging trends of research in non-classical logics is the attempt to combine different non-classical approaches together, for instance many-valued and modal logic [8, 9]. Such an interaction offers the advantage of dealing with modal notions like belief, knowledge, obligations, in connection with other aspects of reasoning that can be best handled using many-valued logics, for instance vagueness [5, 13] and inconsistency. If the ultimate aim is to model human reasoning, it is obvious that all these aspects have to be dealt with at the same time, therefore such study is especially interesting from the point of view of Theoretical Computer Science and Artificial Intelligence.

One of the best-known logical systems proposed for handling inconsistent and also partial information is the Belnap–Dunn logic [2, 3, 7]. This logic is based on four truth values, which can be thought of as the two classical ones plus two additional values meant to represent, respectively, lack of information and inconsistency (see the famous interpretation proposed by Belnap [2]). Such a simple approach, later on generalized by Ginsberg [12] with the notion of *bilattice*, proved to be very flexible and has been widely applied in different areas of Computer Science.

In [17] Odintsov and Wansing proposed a modal version of the Belnap–Dunn logic that aims at extending Belnap's treatment of partiality and inconsistency to a modal setting. In [18] this approach is taken a step further, introducing a modal version of paraconsistent Nelson logic [1] that can also be regarded as a generalization of Odintsov and Wansing's.

In the present study we adopt an even more general approach, introducing a class of algebras that can be used as a semantics for paraconsistent and many-valued modal logics, encompassing as

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special cases both the work done in [17] and in [18]. In our attempt to be as general as possible, it is our hope to lay a theoretical framework that can be used for any future study of paraconsistent modal logics.

We will first introduce through a concrete construction the algebraic structures we are interested in, then we will show that this class of algebras can be abstractly presented as a variety and we will discuss its logical counterpart.

The construction that we use is a generalization of the one known in the literature as *twist-structure*, whose importance has been growing in recent years within the study of algebras related to non-classical logics (see for instance [6, 11, 16, 19]). We believe that the generalization that we present here, even if we restrict our attention to the non-modal language, may have an independent interest for algebraists and logicians working on twist-structures and we hope that our work will encourage them to further explore the potential of this construction.

This article is organized as follows. In Section 2 we introduce our generalized twist-structure construction, by which we define algebras that are special second powers of residuated lattices. The algebraic structures thus obtained have an involutive lattice reduct together with two basic implication operations; for now there are no modal operators. We state some interesting properties of these algebras and we explain in what sense our construction can be seen as a generalization of existing work on twist-structures. In Section 3 we introduce an abstract equational presentation for our twist-structures and prove that the algebras obtained through the concrete construction of Section 2 are precisely those that satisfy our equations. In Section 4 we show how, starting with a residuated lattice that has one or more modal operators, it is possible to add modal operators on the associated twist-structure. We show that the class of modal twist-structures thus obtained can also be abstractly presented as a variety. Section 5 contains an interesting result from a universal algebraic point of view: namely, that the congruences of any (modal) twist-structure are isomorphic to those of the associated residuated lattice (with modal operators). This is also a generalization of a result that is known, for instance, for twist-structures defined over generalized Heyting algebras (those considered in [16]). Finally, in Section 6 we associate two logics to, respectively, non-modal and modal twist-structures, and we prove that our logics are algebraizable (therefore strongly complete) with respect to the corresponding algebraic structures.

2 The twist-structure construction

As mentioned above, we start with a residuated lattice $\mathbf{L}$ to build a new algebra whose carrier set is precisely the Cartesian product $L \times L$ and whose operations are defined in the first component as in a direct product but are somehow twisted in the second component. The idea behind our choice for a starting point is that residuated lattices are very general algebraic structures (encompassing Boolean algebras, Heyting algebras, fuzzy logics algebras, etc.) whose algebraic language has nevertheless a straightforward logical interpretation (conjunction, disjunction, negation, implication).

By a *residuated lattice* we mean here an algebra $\mathbf{L} = \langle L, \sqcap, \sqcup, \cdot, \backslash, /, 1\rangle$ such that $\langle L, \cdot, 1\rangle$ is a monoid, $\langle L, \sqcap, \sqcup\rangle$ is a lattice with associated order $\sqsubseteq$ and the following residuation properties hold: for all $a, b, c \in L$,

$$\text{(R)} \qquad a \cdot b \sqsubseteq c \quad \text{iff} \quad b \sqsubseteq a\backslash c \quad \text{iff} \quad a \sqsubseteq c/b.$$

DEFINITION 2.1
Let $\mathbf{L} = \langle L, \sqcap, \sqcup, \cdot, \backslash, /, 1\rangle$ be a residuated lattice. The *full twist-structure over* $\mathbf{L}$ is the algebra $\mathbf{L}^{\bowtie} = \langle L \times L, \wedge, \vee, \supset, \subset, \neg, \langle 1,1\rangle\rangle$ with operations defined, for all $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in L \times L$, as follows:

$$\begin{aligned}
\langle a_1, a_2\rangle \wedge \langle b_1, b_2\rangle &:= \langle a_1 \sqcap b_1, a_2 \sqcup b_2\rangle \\
\langle a_1, a_2\rangle \vee \langle b_1, b_2\rangle &:= \langle a_1 \sqcup b_1, a_2 \sqcap b_2\rangle \\
\langle a_1, a_2\rangle \supset \langle b_1, b_2\rangle &:= \langle a_1 \backslash b_1, b_2 \cdot a_1\rangle \\
\langle a_1, a_2\rangle \subset \langle b_1, b_2\rangle &:= \langle a_1 / b_1, b_1 \cdot a_2\rangle \\
\neg \langle a_1, a_2\rangle &:= \langle a_2, a_1\rangle
\end{aligned}$$

A *twist-structure over* $\mathbf{L}$ is an arbitrary subalgebra $\mathbf{A}$ (with respect to the language $\{\wedge, \vee, \supset, \subset, \neg, \langle 1,1\rangle\}$, where $\langle 1,1\rangle$ is considered as a 0-ary operation) of the full twist-structure $\mathbf{L}^{\bowtie}$ such that $\pi_1(A) = L$, where $\pi_1(A) = \{a_1 \in L : \langle a_1, a_2\rangle \in A\}$. We write $\mathbf{A} \leq \mathbf{L}^{\bowtie}$ to mean that $\mathbf{A}$ is a twist-structure over $\mathbf{L}$.

As mentioned above, we notice that the name 'twist-structure' refers to the fact that the first component of each binary operation is defined as in a direct product, while the second one is twisted in some way. Let us also observe that the technical condition that $\pi_1(A) = L$ is meant to ensure that the relation between a twist-structure and its associated residuated lattice is in some sense a canonical one. This will be made precise later, when we will start with an abstractly defined class of algebras and will prove that they coincide with the twist-structures defined above.

From Definition 2.1 it follows immediately that the reduct $\langle A, \wedge, \vee\rangle$ of any twist structure $\mathbf{A} \leq \mathbf{L}^{\bowtie}$ is a lattice whose order $\leq$ is given by

$$\langle a_1, a_2\rangle \leq \langle b_1, b_2\rangle \quad \text{iff} \quad a_1 \sqsubseteq b_1 \text{ and } b_2 \sqsubseteq a_2$$

for all $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in L \times L$. Moreover, the unary operation $\neg$ (that we think of as a *negation*) is involutive and order-reversing, i.e. it holds that $\neg\neg\langle a_1, a_2\rangle = \langle a_1, a_2\rangle$ and

$$\langle a_1, a_2\rangle \leq \langle b_1, b_2\rangle \quad \text{iff} \quad \neg\langle b_1, b_2\rangle \leq \neg\langle a_1, a_2\rangle.$$

Thus, the reduct $\langle A, \wedge, \vee, \neg\rangle$ is a structure that is sometimes referred to in the literature as an *involutive lattice* [19].

We introduce three derived operations in any $\mathbf{A} \leq \mathbf{L}^{\bowtie}$ defined, for all $a, b \in L \times L$, as follows:

$$\begin{aligned}
a \to b &:= (a \supset b) \wedge (\neg a \subset \neg b) \\
a \leftarrow b &:= \neg a \to \neg b \\
a * b &:= \neg(b \to \neg a).
\end{aligned}$$

As the notation suggests, one should think of all four operations $\supset, \subset, \to, \leftarrow$ as different kinds of implications, while the $*$ operation can be regarded as a second type of conjunction besides the lattice meet, the one that is sometimes called *fusion* or *strong conjunction* in the literature on substructural logics (more on this below).

Applying the definitions, we have that, for all $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in L_1 \times L_2$,

$$\begin{aligned}
\langle a_1, a_2\rangle \to \langle b_1, b_2\rangle &= \langle (a_1 \backslash b_1) \sqcap (a_2 / b_2), b_2 \cdot a_1\rangle \\
\langle a_1, a_2\rangle \leftarrow \langle b_1, b_2\rangle &= \langle (a_1 / b_1) \sqcap (a_2 \backslash b_2), b_1 \cdot a_2\rangle \\
\langle a_1, a_2\rangle * \langle b_1, b_2\rangle &= \langle a_1 \cdot b_1, (b_2 / a_1) \sqcap (b_1 \backslash a_2)\rangle.
\end{aligned}$$

Notice that all the operations of the twist-structure except $\neg, \rightarrow$ and $\leftarrow$ are defined, in the first component, exactly as in a direct product (if we identify $\langle \wedge, \vee, \supset, \subset, * \rangle$ with $\langle \sqcap, \sqcup, \backslash, /, \cdot \rangle$, respectively). This will turn out to be a key feature in our approach to the study of these structures.

The construction described in Definition 2.1 was inspired by the one introduced in [14]; see also [19, Theorem 3.3] and [6, Theorem 2.3]. We are going to discuss the relationship with these constructions below.

The following result also follows immediately from [14, Proposition 2.2].

PROPOSITION 2.2
Let $\mathbf{A} \leq \mathbf{L}^{\bowtie}$ be a twist-structure over a residuated lattice $\mathbf{L}$. Then $\langle A, * \rangle$ is a semigroup (i.e. the operation $*$ is associative) and, for all $a, b, c \in A$,

$$a * b \leq c \quad \text{iff} \quad b \leq a \rightarrow c \quad \text{iff} \quad a \leq c \leftarrow b.$$

Thus, in any twist-structure there are two residuated pairs and we see that the operation $*$ plays a similar role to the monoidal operation (fusion) in residuated lattices. Notice, however, that $\langle A, *, \langle 1, 1 \rangle \rangle$ need not be a monoid, because $\langle 1, 1 \rangle$ may not be the unit element. In fact, applying the definitions, we have, for instance

$$\langle a_1, a_2 \rangle * \langle 1, 1 \rangle = \langle a_1 \cdot 1, (1/a_1) \sqcap (1 \backslash a_2) \rangle = \langle a_1, (1/a_1) \sqcap a_2 \rangle.$$

Clearly, the equality $(1/a_1) \sqcap a_2 = a_2$ need not be satisfied in an arbitrary residuated lattice. In fact, using this equality it is easy to check that $\langle A, *, \langle 1, 1 \rangle \rangle$ *is a monoid if and only if* 1 *is the maximum element of the lattice order of* $\mathbf{L}$.

We are now going to state some interesting properties of twist-structures that will be used in the next section to provide an equational axiomatization of this class of algebras, thus proving that they form a variety.

Let $\mathbf{A} \leq \mathbf{L}^{\bowtie}$ be a twist-structure over $\mathbf{L}$. Let us define, for all $a \in A$,

$$a' := \neg (a \supset \langle 1, 1 \rangle).$$

Letting $a = \langle a_1, a_2 \rangle$ and applying the definitions, we have that

$$\langle a_1, a_2 \rangle' = \langle a_1, a_1 \backslash 1 \rangle.$$

We can then consider the set $A' \subseteq A$ defined as follows:

$$A' := \{ \langle a_1, a_2 \rangle' : \langle a_1, a_2 \rangle \in A \} = \{ \langle a_1, a_1 \backslash 1 \rangle : a_1 \in L \}.$$

For our purposes, the key feature of the function $' : A \rightarrow A'$ is that it deletes the second component of each element of A. Notice also that $\pi_1(A') = \pi_1(A)$. In fact, if $\mathbf{A}$ and $\mathbf{B}$ are twist-structures over the same $\mathbf{L}$ (thus, $\pi_1(A) = \pi_1(B) = L$), then $A' = B'$.

It is also easy to check that $\langle a_1, a_2 \rangle' \leq \langle b_1, b_2 \rangle'$ holds if and only if $a_1 \sqsubseteq b_1$. Thus, $\langle a_1, a_2 \rangle' = \langle b_1, b_2 \rangle'$ if and only if $a_1 = b_1$. Note also that $\langle 1, 1 \rangle$ is a fixed point of this function and that $\langle a_1, a_2 \rangle'' = \langle a_1, a_2 \rangle'$.

It is possible to endow the set A' with algebraic operations, which will allow us to view A' as the carrier set of an algebra, as follows. For any operation of the twist-structure $\circ \in \{ \wedge, \vee, *, \supset, \subset \}$, let the operation $\circ'$ be defined, for all $a, b \in A$, as

$$a \circ' b := (a \circ b)'.$$

Notice that, for any operation $\circ \in \{\wedge, \vee, *, \supset, \subset\}$, the following holds:

$$(a \circ b)' = (a' \circ b')'.$$

The previous equality reflects the fact that deleting the first component twice is the same as deleting it just once.

Now we can consider the algebra $\mathbf{A}' = \langle A', \wedge', \vee', *', \supset', \subset', \langle 1,1\rangle' \rangle$. Let us define a map $h\colon L \to A$, for all $a_1 \in L$, as $h(a_1) = \langle a_1, a_1 \backslash 1\rangle$. Obviously h is bijective. Moreover, for each operation $\circ \in \{\wedge, \vee, *, \supset, \subset\}$, it holds that $h(a_1 \circ b_1) = h(a_1) \circ' h(b_1)$ for all $a_1, b_1 \in L$. Thus h is an isomorphism between $\mathbf{L}$ and $\mathbf{A}'$. Hence, we also have, in particular, that $\mathbf{A}'$ is a residuated lattice.

Proposition 2.3
The map h defined by $h(a_1) = \langle a_1, a_1 \backslash 1\rangle$ is a residuated lattice isomorphism from $\mathbf{L}$ to $\mathbf{A}'$ for every twist-structure $\mathbf{A}$ over $\mathbf{L}$.

The above proposition implies the following. Let us take a term φ in the language of residuated lattices $\langle \sqcap, \sqcup, \cdot, \backslash, /, 1\rangle$. We can associate to φ another term φ' in the language $\langle \wedge', \vee', *', \supset', \subset', \langle 1,1\rangle' \rangle$ obtained by replacing any occurrence of $\sqcap$ within φ with $\wedge'$, any occurrence of $\sqcup$ with $\vee'$ etc. It is then obvious that if a residuated lattice $\mathbf{L}$ satisfies the equation $\varphi \approx \psi$ for terms φ, ψ, then the residuated lattice $\mathbf{A}'$ will satisfy the equation $\varphi' \approx \psi'$.

A similar result has been shown in [19, Corollaries 3.5 and 3.6] for integral residuated lattices, where a map ε is defined by $\varepsilon(a_1) = \langle a_1, 1\rangle$. It is easy to check that, when $\mathbf{L}$ is integral, we have that $h(a_1) = \varepsilon(a_1)$ (see also the definition of $\hat{\mathbf{L}}^\star$ in [19, Corollary 3.5]). Hence, the isomorphism established in Proposition 2.3 allows us to obtain the results of [19] as special cases.

If $\mathbf{L}$ is a generalized Heyting algebra, then by our construction we obtain exactly the $eN4$-lattices considered in [6]. Note also that the equalities

$$x \supset y = (x \wedge \langle 1,1\rangle) \to y$$

and

$$x \subset y = \neg x \to (\neg y \vee \langle 1,1\rangle)$$

hold in a twist-structure if and only if the underlying residuated lattice $\mathbf{L}$ is integral (i.e. if and only if 1 is the maximum of the lattice order of $\mathbf{L}$). This means, in particular, that in the twist-structure it is possible to define both operations $\supset$ and $\subset$ by using $\to$ (cf. [6, Definition 4.5]).

It is also obvious by construction that our twist-structures of Definition 2.1 are precisely the $\{\wedge, \vee, \supset, \subset, \neg, \langle 1,1\rangle\}$-subreducts of the 'residuated bilattices' introduced in [14].

3 $\mathcal{T}$-lattices

In this section we introduce a variety of algebras that will be proven to coincide, up to isomorphism, with the class of twist-structures considered in the previous section. We call the members of this variety $\mathcal{T}$-lattices, the letter $\mathcal{T}$ being meant to remind the reader that these algebras correspond to the above-mentioned twist-structures.

We deal with algebras $\mathbf{A} = \langle A, \wedge, \vee, \supset, \subset, \neg, e\rangle$ of type $\langle 2,2,2,2,1,0\rangle$, adopting the following conventions: for all $a, b \in A$,

- $a \to b := (a \supset b) \wedge (\neg a \subset \neg b)$

- $a \leftarrow b := \neg a \to \neg b$
- $a * b := \neg(b \to \neg a)$
- $a' := \neg(a \supset e)$
- $A' := \{a' : a \in A\}$
- $a \circ' b := (a \circ b)'$ for any operation $\circ \in \{\wedge, \vee, *, \supset, \subset, e\}$
- $\mathbf{A}' := \langle A', \wedge', \vee', *', \supset', \subset', e' \rangle$.

It is obvious that the above notation is consistent with the one used in Section 2 and it is indeed intended to be suggestive of the way in which we will prove the announced correspondence result.

Definition 3.1
A $\mathcal{T}$-*lattice* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \supset, \subset, \neg, e \rangle$ such that:

(i) the reduct $\langle A, \wedge, \vee, \neg \rangle$ is an involutive lattice, i.e. a lattice (with associated order $\leq$) equipped with a unary operation (called *negation*) that is involutive and satisfies De Morgan laws, i.e. such that $\neg\neg a = a$, $\neg(a \vee b) = \neg a \wedge \neg b$ and $\neg(a \wedge b) = \neg a \vee \neg b$,
(ii) the algebra $\mathbf{A}' = \langle A', \wedge', \vee', *', \supset', \subset', e' \rangle$ is a residuated lattice,
(iii) $(a \circ b)' = (a' \circ b')'$ for any operation $\circ \in \{\wedge, \vee, \supset, \subset, *\}$,
(iv) for all $a, b, c \in A$:

$$((a \wedge b) \to b) \supset ((a \wedge b) \to b) \leq (a \wedge b) \to b \tag{E1}$$
$$a \supset b \leq (a \wedge c) \supset b \tag{E2}$$
$$((a \supset a) \wedge (b \supset b)) \supset c \leq c \tag{E3}$$
$$a \leq (((b \supset e) \to (a \supset e)) \wedge (\neg(\neg b \supset e) \to \neg(\neg a \supset e))) \supset b \tag{E4}$$
$$e = \neg e \tag{E5}$$
$$(\neg(a \supset b))' = (\neg(\neg a \subset \neg b))' = (\neg(a \to b))' \tag{E6}$$

We denote by $\mathcal{T}$-Lat the class of $\mathcal{T}$-lattices.

At first sight it may not be easy to understand the role and intuitive meaning of axioms (E1) to (E6). These will be clarified in the proof of Proposition 3.3, and their sufficiency for obtaining our Theorem 3.6 will be shown both in the proof of the theorem and in that of Lemma 3.4.

Notice that all the conditions of Definition 3.1 can be expressed by equations. This obviously holds for (i), (iii) and (iv). As to (ii), recall that residuated lattices are an equational class [10, Theorem 2.7]. Therefore condition (ii) amounts to requiring that all elements of the algebra $\mathbf{A}'$ satisfy the equalities that axiomatize residuated lattices. If $\varphi \approx \psi$ is one of such equalities in the language of residuated lattices, we consider its translation $\varphi' \approx \psi'$ defined as in the previous section. That is, to any term φ in the language of residuated lattices $\langle \sqcap, \sqcup, \cdot, \backslash, /, 1 \rangle$ we associate the term φ' in the language $\langle \wedge', \vee', *', \supset', \subset', e' \rangle$ obtained by replacing any occurrence of $\sqcap$ with $\wedge'$, any occurrence of $\sqcup$ with $\vee'$ etc. Then, we replace any variable x occurring in φ' and ψ' by the term $x' := \neg(x \supset e)$. Denote by $\varphi^{\bullet}, \psi^{\bullet}$ the terms (which are obviously in the language of $\mathcal{T}$-lattices) obtained in this way. Then we have a new equation $\varphi^{\bullet} \approx \psi^{\bullet}$ that captures exactly the property of residuated lattices expressed by our original $\varphi \approx \psi$.

As an example, consider the equation $1 \backslash x \approx x$. First we replace $\langle \backslash, 1 \rangle$ by, respectively, $\langle \supset', e' \rangle$ and obtain the equation $e' \supset' x \approx x$, which is really a shorthand for

$$\neg(\neg(e \supset e) \supset x) \supset e) \approx x.$$

Then by replacing x by x', we have $e' \supset' x' \approx x'$, which is a shorthand for

$$\neg(\neg(e \supset e) \supset \neg(x \supset e)) \supset e) \approx \neg(x \supset e). \tag{3.1}$$

We see thus that the requirement that a $\mathcal{T}$-lattice $\mathbf{A}$ satisfy (3.1) is equivalent to requiring that the algebra $\mathbf{A}' = \langle A', \wedge', \vee', *', \supset', \subset', e' \rangle$, viewed as a residuated lattice, should satisfy the equation $e' \supset' x \approx x$. Therefore, we have that

Proposition 3.2
The class of $\mathcal{T}$-lattices is a variety.

From Definition 3.1 (ii) it follows that, for any $\mathcal{T}$-lattice $\mathbf{A}$, the algebra $(\mathbf{A}')^{\bowtie}$ is a full twist-structure. We are going to show that $\mathbf{A}$ is in fact embeddable into $(\mathbf{A}')^{\bowtie}$, thus proving that any $\mathcal{T}$-lattice $\mathbf{A}$ is isomorphic to a twist-structure over $\mathbf{A}'$.

We start by checking that any twist-structure is indeed a $\mathcal{T}$-lattice.

Proposition 3.3
Every twist-structure $\mathbf{A} \leq \mathbf{L}^{\bowtie}$ is a $\mathcal{T}$-lattice.

Proof. We have to check that the conditions of Definition 3.1 are satisfied. The first item is easily proved (it also follows from known results on bilattices and twist-structures: see for instance [15]). The second and third, as we have observed in the previous section, are also easy. We only check the fourth.

(E1). Notice that the condition $\langle a_1, a_2 \rangle \supset \langle a_1, a_2 \rangle \leq \langle a_1, a_2 \rangle$ is equivalent to $1 \sqsubseteq a_1$. In fact, the former amounts to

$$\langle a_1, a_2 \rangle \supset \langle a_1, a_2 \rangle = \langle a_1 \backslash a_1, a_2 \cdot a_1 \rangle \leq \langle a_1, a_2 \rangle$$

i.e., $a_1 \backslash a_1 \sqsubseteq a_1$ and $a_2 \sqsubseteq a_2 \cdot a_1$. In any residuated lattice it holds that $1 \sqsubseteq a_1 \backslash a_1$, hence we have that $1 \sqsubseteq a_1$. We have then to prove that the first component of $(\langle a_1, a_2 \rangle \wedge \langle b_1, b_2 \rangle) \to \langle b_1, b_2 \rangle$ is greater than 1. Applying the definitions, we have that such component is $((a_1 \sqcap b_1) \backslash b_1) \sqcap ((a_2 \sqcup b_2)/b_2)$. Both members of this last conjunction are greater than 1 in any residuated lattice, therefore we reach the desired result.

(E2). We have

$$((\langle a_1, a_2 \rangle \wedge \langle c_1, c_2 \rangle) \supset \langle b_1, b_2 \rangle) = \langle (a_1 \sqcap c_1) \backslash b_1,\ b_2 \cdot (a_1 \sqcap c_1) \rangle.$$

Therefore we have to check that $a_1 \backslash b_1 \sqsubseteq (a_1 \sqcap c_1) \backslash b_1$ and $b_2 \cdot (a_1 \sqcap c_1) \sqsubseteq b_2 \cdot a_1$. Both inequalities are true in any residuated lattice [10, Lemma 2.6].

(E3). For evaluating

$$(\langle a_1, a_2 \rangle \supset \langle a_1, a_2 \rangle) \wedge (\langle b_1, b_2 \rangle \supset \langle b_1, b_2 \rangle) \supset \langle c_1, c_2 \rangle$$

we only need to compute the first component of

$$(\langle a_1, a_2 \rangle \supset \langle a_1, a_2 \rangle) \wedge (\langle b_1, b_2 \rangle \supset \langle b_1, b_2 \rangle)$$

which is $(a_1 \backslash a_1) \sqcap (b_1 \backslash b_1)$. We have then to prove that

$$\langle ((a_1 \backslash a_1) \sqcap (b_1 \backslash b_1)) \backslash c_1,\ c_2 \cdot ((a_1 \backslash a_1) \sqcap (b_1 \backslash b_1)) \rangle \leq \langle c_1, c_2 \rangle$$

i.e., $((a_1\backslash a_1)\sqcap(b_1\backslash b_1))\backslash c_1 \sqsubseteq c_1$ and $c_2 \sqsubseteq c_2\cdot((a_1\backslash a_1)\sqcap(b_1\backslash b_1))$. Both inequalities are true in residuated lattices, since $1\sqsubseteq(a_1\backslash a_1)\sqcap(b_1\backslash b_1)$, which implies that $((a_1\backslash a_1)\sqcap(b_1\backslash b_1))\backslash c_1 \sqsubseteq 1\backslash c_1 = c_1$ and $c_2 = c_2\cdot 1 \sqsubseteq c_2\cdot((a_1\backslash a_1)\sqcap(b_1\backslash b_1))$.

(E4). Let us use the following abbreviations:

$$\langle x_1,x_2\rangle := (\langle b_1,b_2\rangle \supset \langle 1,1\rangle) \to (\langle a_1,a_2\rangle \supset \langle 1,1\rangle)$$
$$\langle y_1,y_2\rangle := \neg(\neg\langle b_1,b_2\rangle \supset \langle 1,1\rangle) \to \neg(\neg\langle a_1,a_2\rangle \supset \langle 1,1\rangle).$$

We have to prove that $\langle a_1,a_2\rangle \le (\langle x_1,x_2\rangle \wedge \langle y_1,y_2\rangle) \supset \langle b_1,b_2\rangle$. We have

$$\begin{aligned}(\langle x_1,x_2\rangle \wedge \langle y_1,y_2\rangle) \supset \langle b_1,b_2\rangle &= \langle x_1\sqcap y_1, x_2\sqcup y_2\rangle \supset \langle b_1,b_2\rangle\\ &= \langle (x_1\sqcap y_1)\backslash b_1, b_2\cdot(x_1\sqcap y_1)\rangle.\end{aligned}$$

Thus, we need to check that

$$a_1 \sqsubseteq (x_1\sqcap y_1)\backslash b_1 \quad \text{and} \quad b_2\cdot(x_1\sqcap y_1)\sqsubseteq a_2.$$

We have

$$\begin{aligned}\langle x_1,x_2\rangle &= \langle b_1\backslash 1, 1\cdot b_1\rangle \to \langle a_1\backslash 1, 1\cdot a_1\rangle\\ &= \langle b_1\backslash 1, b_1\rangle \to \langle a_1\backslash 1, a_1\rangle\\ &= \langle (b_1\backslash 1)\backslash(a_1\backslash 1)\sqcap(b_1/a_1), a_1\cdot(b_1\backslash 1)\rangle\end{aligned}$$

$$\begin{aligned}\langle y_1,y_1\rangle &= \neg(\langle b_2,b_1\rangle \supset \langle 1,1\rangle) \to \neg(\langle a_2,a_1\rangle \supset \langle 1,1\rangle)\\ &= \neg\langle b_2\backslash 1, 1\cdot b_2\rangle \to \neg\langle a_2\backslash 1, 1\cdot a_2\rangle\\ &= \langle b_2, b_2\backslash 1\rangle \to \langle a_2, a_2\backslash 1\rangle\\ &= \langle (b_2\backslash a_2)\sqcap((b_2\backslash 1)/(a_2\backslash 1)), (a_2\backslash 1)\cdot b_2\rangle.\end{aligned}$$

Therefore,

$$x_1\sqcap y_1 = (b_1\backslash 1)\backslash(a_1\backslash 1)\sqcap(b_1/a_1)\sqcap(b_2\backslash a_2)\sqcap((b_2\backslash 1)/(a_2\backslash 1)).$$

Notice that $a_1\sqsubseteq(b_1/a_1)\backslash b_1$ because, by residuation, this is equivalent to $(b_1/a_1)\cdot a_1\sqsubseteq b_1$, which is equivalent to $b_1/a_1\sqsubseteq b_1/a_1$. Obviously $x_1\sqcap y_1\sqsubseteq b_1/a_1$, therefore $(b_1/a_1)\backslash b_1\sqsubseteq(x_1\sqcap y_1)\backslash b_1$. Thus, we have

$$a_1\sqsubseteq(b_1/a_1)\backslash b_1\sqsubseteq(x_1\sqcap y_1)\backslash b_1$$

as desired. Similarly, it holds that $b_2\cdot(b_2\backslash a_2)\sqsubseteq a_2$ because, by residuation, the latter is equivalent to $b_2\backslash a_2\sqsubseteq b_2\backslash a_2$. Since $x_1\sqcap y_1\sqsubseteq b_2\backslash a_2$, we obtain

$$b_2\cdot(x_1\sqcap y_1)\sqsubseteq b_2\cdot(b_2\backslash a_2)\sqsubseteq a_2.$$

(E5). Immediate.

(E6). Also easy. It is sufficient to notice that, according to the definitions of the operations $\supset, \subset$ and $\to$ in twist-structures, the second component of $\langle a_1,a_2\rangle\supset\langle b_1,b_2\rangle$ coincides with the second component of $\neg\langle a_1,a_2\rangle\subset\neg\langle b_1,b_2\rangle$ and of $\langle a_1,a_2\rangle\to\langle b_1,b_2\rangle$. ∎

Our next aim is to prove the converse of Proposition 3.3, i.e. that any $\mathcal{T}$-lattice is indeed isomorphic to a twist-structure over a residuated lattice. In order to prove this result, we will need the following lemma.

LEMMA 3.4
Let $\mathbf{A}$ be a $\mathcal{T}$-lattice and θ a congruence of $\mathbf{A}$. Then, for all a,b,

$$\langle a,b\rangle \in\theta \qquad \text{if and only if} \qquad \langle a',b'\rangle, \langle(\neg a)',(\neg b)'\rangle \in\theta.$$

PROOF. Given that $x'=\neg(x\supset e)$ is a term in the language of $\mathcal{T}$-lattices, it is easy to see that $\langle a,b\rangle\in\theta$ implies $\langle a',b'\rangle, \langle(\neg a)',(\neg b)'\rangle\in\theta$. To prove the converse, assume $\langle a',b'\rangle, \langle(\neg a)',(\neg b)'\rangle\in\theta$. From the former assumption we obtain $\langle\neg(a'),\neg(b')\rangle\in\theta$, from which we have $\langle\neg(b')\to\neg(a'),\neg(b')\to\neg(b')\rangle\in\theta$. Similarly, from the latter assumption we can obtain $\langle(\neg b)'\to(\neg a)',(\neg b)'\to(\neg b)'\rangle\in\theta$. Now we apply meet to obtain

$$\langle(\neg(b')\to\neg(a'))\wedge((\neg b)'\to(\neg a)'),(\neg(b')\to\neg(b'))\wedge(\neg b)'\to(\neg b)'\rangle\in\theta$$

In order to use a more compact notation, let us set $x:=\neg(b')\to\neg(b')$ and $y:=(\neg b)'\to(\neg b)'$. We can then rewrite the above as

$$\langle(\neg(b')\to\neg(a'))\wedge((\neg b)'\to(\neg a)'),\ x\wedge y\rangle\in\theta$$

from which we can obtain

$$\langle((\neg(b')\to\neg(a'))\wedge((\neg b)'\to(\neg a)'))\supset b,\ (x\wedge y)\supset b\rangle\in\theta.$$

Concerning the first element of the pair we have obtained, notice that

$$a\leq((\neg(b')\to\neg(a'))\wedge((\neg b)'\to(\neg a)'))\supset b$$

is an instance of (E4). Concerning the second element, notice that $x=\neg(b')\to\neg(b')=(\neg(b')\wedge\neg(b'))\to\neg(b')$, as the latter equality holds in any lattice. Hence, we see that $x\supset x\leq x$ is an instance of (E1). Similarly, we have that $y\supset y\leq y$. It follows then that $(x\supset x)\wedge(y\supset y)\leq x\wedge y$. Applying (E2), we have

$$(x\wedge y)\supset b\leq((x\supset x)\wedge(y\supset y))\supset b.$$

By (E3), we have $((x\supset x)\wedge(y\supset y))\supset b\leq b$, so we obtain $(x\wedge y)\supset b\leq b$. Thus, we have the following:

$$a\leq((\neg(b')\to\neg(a'))\wedge((\neg b)'\to(\neg a)'))\supset b\ \theta\ (x\wedge y)\supset b\leq b.$$

From this we can obtain first

$$a=a\wedge(((\neg(b')\to\neg(a'))\wedge((\neg b)'\to(\neg a)'))\supset b)\ \theta\ a\wedge((x\wedge y)\supset b)$$

and then

$$a\vee b\ \theta\ (a\wedge((x\wedge y)\supset b))\vee b=b.$$

Thus, we have $\langle a\vee b,b\rangle\in\theta$. Applying the same reasoning, by symmetry, we can obtain $\langle b\vee a,a\rangle\in\theta$, which implies $\langle a,b\rangle\in\theta$. ∎

Notice that, in fact, Lemma 3.4 holds not only for $\mathcal{T}$-lattices but for any algebra that has a $\mathcal{T}$-lattice reduct. This observation will turn out to be useful when we expand the language of $\mathcal{T}$-lattices with modal operators. Since the equality relation is a congruence, the above lemma immediately implies the following.

COROLLARY 3.5
For any $\mathcal{T}$-lattice $\mathbf{A}$ and all $a,b \in A$,

$$a = b \qquad \text{if and only if} \qquad a' = b' \quad \text{and} \quad (\neg a)' = (\neg b)'.$$

Although for the present section Corollary 3.5 would be enough, when we discuss the congruences of $\mathcal{T}$-lattices (Section 5) we will need the more general result stated in Lemma 3.4. Notice, however, that using the fact that the class of $\mathcal{T}$-lattices is a variety (hence, closed under quotients) it is not difficult to prove that Lemma 3.4 also follows from Corollary 3.5, so the two statements are in fact easily inter-derivable (we leave this as an exercise for the reader).

Let us also notice that Corollary 3.5 entails that every element $a \in A$ is uniquely determined by the pair a' and $(\neg a)'$. One might wonder whether a can actually be expressed in terms of these two elements, that is, if there is a binary term t in the language of $\mathcal{T}$-lattices such that $t(a', (\neg a)') = a$ for every $a \in A$. This is not the case, as shown by the following example. Consider the four-element full twist-structure $\mathbf{B_2}^{\bowtie}$, where $\mathbf{B_2}$ is the two-element Boolean algebra (which we view as a residuated lattice) with universe $B_2 = \{0,1\}$. By Proposition 3.3, we have that $\mathbf{B_2}^{\bowtie}$ is a $\mathcal{T}$-lattice. Now consider the element $\langle 0,0\rangle \in B_2^{\bowtie}$. We have that $\langle 0,0\rangle' = (\neg\langle 0,0\rangle)' = \langle 0,1\rangle$. It is easy to check that, if we close the set $\{\langle 0,1\rangle\}$ under under the $\mathcal{T}$-lattice algebraic operations, we obtain the set $\{\langle 0,1\rangle, \langle 1,0\rangle, \langle 1,1\rangle\}$. This implies that there is no term t in the language of $\mathcal{T}$-lattices such that $t(\langle 0,0\rangle', (\neg\langle 0,0\rangle)') = \langle 0,0\rangle$.

We are now able to prove the announced result.

THEOREM 3.6
Any $\mathcal{T}$-lattice $\mathbf{A}$ is isomorphic to a twist-structure over $\mathbf{A}'$ via the map $\iota\colon A \to A' \times A'$ defined, for all $a \in A$, as

$$\iota(a) := \langle a', (\neg a)'\rangle.$$

PROOF. Injectivity of ι follows from Corollary 3.5. It is obvious that $\pi_1(\iota(A)) = A'$. It remains to prove that ι is a homomorphism. By (E5) we have $\iota(e) = \langle e', (\neg e)'\rangle = \langle e', e'\rangle$. As to negation, by involutivity we have

$$\iota(\neg a) = \langle (\neg a)', (\neg\neg a)'\rangle = \langle (\neg a)', a'\rangle = \neg\iota(a).$$

Recall that, by definition, $\langle a,b\rangle \wedge \langle c,d\rangle = \langle a \wedge' c, b \vee' d\rangle$ for all $a,b,c,d \in A'$. Let us check the case of $\wedge$:

$$\begin{aligned}
\iota(a\wedge b) &= \langle (a\wedge b)', (\neg(a\wedge b))'\rangle \\
&= \langle (a\wedge b)', (\neg a \vee \neg b)'\rangle && \text{by De Morgan laws} \\
&= \langle (a'\wedge b')', ((\neg a)' \vee (\neg b)')'\rangle && \text{by Definition 3.1 (iii)} \\
&= \langle a' \wedge' b', (\neg a)' \vee' (\neg b)'\rangle \\
&= \langle a', (\neg a)'\rangle \wedge \langle b', (\neg b)'\rangle \\
&= \iota(a) \wedge \iota(b).
\end{aligned}$$

The case of $\vee$ follows from the previous two since the De Morgan law $a \vee b = \neg(\neg a \wedge \neg b)$ holds. The case of $\supset$:

$$
\begin{aligned}
\iota(a \supset b) &= \langle (a \supset b)', (\neg(a \supset b))' \rangle \\
&= \langle (a \supset b)', (\neg b * a)' \rangle && \text{by (E6)} \\
&= \langle (a' \supset b')', ((\neg b)' * a')' \rangle && \text{by Definition 3.1 (iii)} \\
&= \langle a' \supset' b', (\neg b)' *' a' \rangle \\
&= \langle a', (\neg a)' \rangle \supset \langle b', (\neg b)' \rangle \\
&= \iota(a) \supset \iota(b).
\end{aligned}
$$

And the case of $\subset$ concludes our proof:

$$
\begin{aligned}
\iota(a \subset b) &= \langle (a \subset b)',\ (\neg(a \subset b))' \rangle \\
&= \langle (a \subset b)',\ (\neg(\neg\neg a \subset \neg\neg b))' \rangle && \text{by involutivity of negation} \\
&= \langle (a \subset b)',\ (\neg\neg b * \neg a)' \rangle && \text{by (E6)} \\
&= \langle (a \subset b)',\ (b * \neg a)' \rangle && \text{by involutivity of negation} \\
&= \langle (a' \subset b')',\ (b' * (\neg a)')' \rangle && \text{by Definition 3.1 (iii)} \\
&= \langle a' \subset' b',\ b' *' (\neg a)' \rangle \\
&= \langle a',\ (\neg a)' \rangle \subset \langle b',\ (\neg b)' \rangle \\
&= \iota(a) \subset \iota(b).
\end{aligned}
$$

∎

4 Adding modal operators

In this section we show how to add modal operators to our twist-structures. As before, we start with a residuated lattice but we now assume that our lattice is endowed with one (or more) modal operators, defined as follows.

Given a lattice $\mathbf{L}$ with associated order $\sqsubseteq$, we will say that a unary function $f : L \to L$ is a *modal operator on* $\mathbf{L}$ if it is monotone, i.e. if $a \sqsubseteq b$ implies $f(a) \sqsubseteq f(b)$ for all $a, b \in L$.

This rather minimal definition in terms of properties of f corresponds to our intention of being as general as possible, so that we will be able to treat as a special case any other algebraic modal operator enjoying stronger properties (for instance, any necessity-style operator satisfying $f(x \sqcap y) = f(x) \sqcap f(y)$).

Definition 4.1
Let $\mathbf{L} = \langle L, \sqcap, \sqcup, \cdot, \backslash, /, f_1, f_2, g_1, g_2, 1 \rangle$ be a residuated lattice with modal operators f_1, f_2, g_1, g_2. The *full modal twist-structure* $\mathbf{L}^{\bowtie} = \langle L \times L, \wedge, \vee, \supset, \subset, \Box, \Diamond, \neg, \langle 1, 1 \rangle \rangle$ is defined as follows:

(i) the reduct $\langle L \times L, \wedge, \vee, \supset, \subset, \neg, \langle 1, 1 \rangle \rangle$ is the full twist-structure over the residuated lattice $\langle L, \sqcap, \sqcup, \cdot, \backslash, /, 1 \rangle$

(ii) the operations $\Box : L \times L \to L \times L$ and $\Diamond : L \times L \to L \times L$ are defined, for all $\langle a_1, a_2 \rangle \in L \times L$, as

$$
\begin{aligned}
\Box \langle a_1, a_2 \rangle &:= \langle f_1(a_1), f_2(a_2) \rangle \\
\Diamond \langle a_1, a_2 \rangle &:= \langle g_1(a_1), g_2(a_2) \rangle.
\end{aligned}
$$

A *modal twist-structure over* **L** is an arbitrary subalgebra **A** (w.r.t. to the language $\{\wedge, \vee, \supset, \subset, \Box, \Diamond, \neg, \langle 1, 1\rangle\}$, of $\mathbf{L}^{\bowtie}$ such that $\pi_1(A) = L$, where $\pi_1(A) = \{a_1 \in L : \langle a_1, a_2\rangle \in A\}$. We write $\mathbf{A} \leq \mathbf{L}^{\bowtie}$ to mean that **A** is a modal twist-structure over **L**.

Let us point out that in general one need not view $\Box$ as some kind of necessity operator, nor $\Diamond$ as a possibility operator. Such an interpretation, which may be suggested by our notation, will indeed make sense when we impose stronger properties on the modal operators of the associated residuated lattice **L**. The same applies to the following abbreviations that we will use: $\Diamond_1 := \neg\Box\neg$ and $\Box_1 := \neg\Diamond\neg$.

The construction of Definition 4.1 is obviously a generalization of (and was inspired by) the one introduced in [17, Definition 7] (see also [18]). Notice also that in our definition there is no requirement on the interaction among the four modal operators on **L**. So, any two of them may coincide. In particular, in the twist-structure constructions considered in [17] and [18] it is assumed that $f_1 = g_2$ and $g_1 = f_2$. In such a case it is easy to check that, as happens with classical modal Boolean algebras, the operator $\Diamond$ is definable as $\neg\Box\neg$. Thus, $\Diamond = \Diamond_1$ and $\Box = \Box_1$. Note also that our definitions imply that all operators $\Box, \Box_1, \Diamond, \Diamond_1$ are monotone w.r.t. the lattice order of $\mathbf{L}^{\bowtie}$.

Our next aim is to introduce an abstract equational presentation of a class of algebras that will turn out to correspond exactly to our modal twist-structures. Following the same argument used in Section 2, we begin by observing that, for any modal twist-structure $\mathbf{A} \leq \mathbf{L}^{\bowtie}$, we can define modal operators on the algebra $\langle A', \wedge', \vee', *', \supset', \subset', \langle 1, 1\rangle'\rangle$, which is defined as for non-modal twist-structures, as follows. For all $a \in A$,

$$\Box' a := (\Box a)' \qquad \Diamond'_1 a := (\Diamond_1 a)'$$
$$\Diamond' a := (\Diamond a)' \qquad \Box'_1 a := (\Box_1 a)'.$$

It is then easy to check that the algebra

$$\mathbf{A}' = \langle A', \wedge', \vee', *', \supset', \subset', \Box', \Diamond'_1, \Diamond', \Box'_1, \langle 1, 1\rangle'\rangle$$

is a residuated lattice with modal operators $\Box', \Diamond'_1, \Diamond', \Box'_1$. Moreover, $\mathbf{A}'$ is isomorphic to **L**. Let us also notice that the following properties hold, for all $a \in A$:

$$\Box' a = \Box'(a') \qquad \Box'_1 a = \Box'_1(a')$$
$$\Diamond' a = \Diamond'(a') \qquad \Diamond'_1 a := \Diamond'_1(a').$$

We are now able to introduce our equational presentation for modal twist-structures.

Definition 4.2
A *modal $\mathcal{T}$-lattice* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \supset, \subset, \neg, \Box, \Diamond, e\rangle$ such that the reduct $\langle A, \wedge, \vee, \supset, \subset, \neg, e\rangle$ is a $\mathcal{T}$-lattice and any operation $\circ \in \{\Box, \Diamond\}$ satisfies, for all $a, b \in A$,

1. $\circ(a \wedge b) \leq \circ b$
2. $(\circ a)' = (\circ(a'))'$
3. $(\circ_1 a)' = (\circ_1(a'))'$.

Consistently with our previous notation, we abbreviate $a' := \neg(a \supset e)$, $\Diamond_1 a := \neg\Box\neg a$ and $\Box_1 a := \neg\Diamond\neg a$ for any element $a \in A$.

It is obvious from the definition that modal $\mathcal{T}$-lattices form a variety. We have already observed that any modal twist-structure satisfies all items of Definition 4.2, therefore we immediately have:

PROPOSITION 4.3
Any modal twist-structure is a modal $\mathcal{T}$-lattice.

It remains therefore to prove that any modal $\mathcal{T}$-lattice can be represented as a modal twist-structure.
We adopt the same conventions used for non-modal $\mathcal{T}$-lattices, except that now we denote by $\mathbf{A}' = \langle A', \wedge', \vee', *', \supset', \subset', \Box', \Diamond'_1, \Diamond', \Box'_1, e' \rangle$ the residuated lattice *with modal operators* associated with the modal $\mathcal{T}$-lattice $\mathbf{A}$, where

$$\circ' a := (\circ a)'$$

for any operation $\circ \in \{\Box, \Box_1, \Diamond, \Diamond_1\}$ and for all $a \in A$.

THEOREM 4.4
Any modal $\mathcal{T}$-lattice $\mathbf{A}$ is isomorphic to a modal twist-structure over $\mathbf{A}'$ via the map $\iota: A \to A' \times A'$ defined, for all $a \in A$, as

$$\iota(a) := \langle a', (\neg a)' \rangle.$$

PROOF. We know from Theorem 3.6 that the map ι is injective and a homomorphism w.r.t. to the $\{\wedge, \vee, \supset, \subset, \neg, e\}$-reduct of $\mathbf{A}$. We only need to check that ι respects the operations $\Box$ and $\Diamond$. As to the former, we have that, for all $a \in A$:

$$\begin{aligned}
\iota(\Box a) &= \langle (\Box a)', (\neg \Box a)' \rangle \\
&= \langle (\Box a)', (\neg \Box \neg\neg a)' \rangle && \text{by double negation law} \\
&= \langle (\Box (a'))', (\Diamond_1 \neg a)' \rangle && \text{by Definition 4.2 (ii)} \\
&= \langle (\Box (a'))', (\Diamond_1 ((\neg a)'))' \rangle && \text{by Definition 4.2 (iii)} \\
&= \langle \Box'(a'), \Diamond'_1((\neg a)') \rangle && \text{by definition of } \Box' \text{ and } \Diamond'_1 \\
&= \Box \langle a', (\neg a)' \rangle \\
&= \Box(\iota(a)).
\end{aligned}$$

The case of $\Diamond$ is easily proved in the same way. ∎

5 Congruences

In this section we look at congruences of (modal) $\mathcal{T}$-lattices, which we now know to coincide, up to isomorphism, with (modal) twist-structures. Our aim will be to characterize the lattice of congruences of an arbitrary (modal) $\mathcal{T}$-lattice in terms of the lattice of congruences of its associated residuated lattice (with modal operators). In this way we will obtain a way of studying any universal algebraic property of any class of (modal) $\mathcal{T}$-lattices that depends on the structure of the congruence lattice (e.g., congruence distributivity, subdirect irreducibility, etc.) by looking at the congruences of the associated class of residuated lattices.

Let us first consider the case where $\mathbf{A}$ is a non-modal $\mathcal{T}$-lattice. Consider the map $H: \mathrm{Con}(\mathbf{A}) \to \mathrm{Con}(\mathbf{A}')$ defined, for all $\theta \in \mathrm{Con}(\mathbf{A})$, as

$$H(\theta) := \theta \cap (A' \times A'). \tag{5.1}$$

Given that all the algebraic operations of $\mathbf{A}'$ are defined by terms in the language of $\mathcal{T}$-lattices, it is easy to check that $\theta \cap (A' \times A')$ is in fact a congruence of the residuated lattice $\mathbf{A}'$.

We claim that its inverse $H^{-1}: \mathrm{Con}(\mathbf{A}') \to \mathrm{Con}(\mathbf{A})$ is given, for all $\eta \in \mathrm{Con}(\mathbf{A}')$, by

$$H^{-1}(\eta) := \{\langle a, b\rangle \in A \times A : \langle a', b'\rangle, \langle (\neg a)', (\neg b)'\rangle \in \eta\}. \tag{5.2}$$

Both H and H^{-1} are obviously monotone. Let us check that H^{-1} is well-defined.

Proposition 5.1
For any congruence η of $\mathbf{A}'$, the relation $H^{-1}(\eta)$ defined in (5.2) is a congruence of $\mathbf{A}$.

Proof. It is clear from the definition that $H^{-1}(\eta)$ is an equivalence relation. Also, compatibility with negation easily follows from double negation law. Now assume $\langle a, b\rangle, \langle c, d\rangle \in H^{-1}(\eta)$, i.e., $\langle a', b'\rangle, \langle (\neg a)', (\neg b)'\rangle, \langle c', d'\rangle, \langle (\neg c)', (\neg d)'\rangle \in \eta$. We will make frequent use of property (iii) of Definition 3.1. We have

$$(a \wedge c)' = (a' \wedge c')' = a' \wedge' c' \;\eta\; b' \wedge' d' = (b' \wedge d')' = (b \wedge d)'.$$

Using De Morgan laws, we also have

$$(\neg(a \wedge c))' = (\neg a \vee \neg c)' = (\neg a)' \vee' (\neg c)' \;\eta\; (\neg b)' \vee' (\neg d)' = (\neg b \vee \neg d)' = (\neg(b \wedge d))'.$$

Thus, $\langle a \wedge c, b \wedge d\rangle \in H^{-1}(\eta)$. Compatibility of $H^{-1}(\eta)$ with $\vee$ follows from the fact that $a \vee b = \neg(\neg a \wedge \neg b)$ for all $a, b \in A$. As to $\supset$, we have

$$(a \supset c)' = a' \supset' c' \;\eta\; b' \supset' d' = (b \supset d)'.$$

Using (E6), we also have

$$(\neg(a \supset c))' = (\neg c * a)' = (\neg c)' *' a' \;\eta\; (\neg d)' *' b' = (\neg d * b)' = (\neg(b \supset d))'.$$

Hence, $\langle a \supset c, b \supset d\rangle \in H^{-1}(\eta)$. A similar reasoning proves that $\langle a \subset c, b \subset d\rangle \in H^{-1}(\eta)$. ∎

Next we check that H and H^{-1} are actually mutually inverse.

Proposition 5.2
Let $\theta \in \mathrm{Con}(\mathbf{A})$ and $\eta \in \mathrm{Con}(\mathbf{A}')$. Then, $\theta = H^{-1}(H(\theta))$ and $\eta = H(H^{-1}(\eta))$.

Proof. Let $\theta \in \mathrm{Con}(\mathbf{A})$. By definition, $\langle a, b\rangle \in H^{-1}(H(\theta))$ means that

$$\langle a', b'\rangle, \langle (\neg a)'(\neg b)'\rangle \in H(\theta)$$

which is equivalent to $\langle a', b'\rangle \in \theta$ and $\langle (\neg a)', (\neg b)'\rangle \in \theta$. By Lemma 3.4, this is equivalent to $\langle a, b\rangle \in \theta$. Hence, $\theta = H^{-1}(H(\theta)$.
Now let $\eta \in \mathrm{Con}(\mathbf{A}')$. The assumption $\langle a, b\rangle \in H(H^{-1}(\eta))$ means that $\langle a, b\rangle \in H^{-1}(\eta) \cap (A' \times A')$, i.e., $a, b \in A'$ and $\langle a, b\rangle \in H^{-1}(\eta)$. This last assumption is equivalent to $\langle a', b'\rangle, \langle (\neg a)', (\neg b)'\rangle \in \eta$. Then, given that $a = a'$ and $b = b'$, we immediately see that $H(H^{-1}(\eta)) \subseteq \eta$. Conversely, if $\langle a, b\rangle \in \eta$, then $a = a' \,\eta\, b' = b$. To prove that $\langle (\neg a)', (\neg b)'\rangle \in \eta$, notice that $x \supset e \approx \neg\neg(x \supset e) \approx \neg(x')$ is obviously valid in every $\mathcal{T}$-lattice. Under the assumption that $a, b \in A'$, we have then $a \supset e = \neg(a') = \neg a$ and likewise $b \supset e = \neg b$. Using compatibility of η with the operation $\supset'$, we obtain then

$$(\neg a)' = (a \supset e)' = a' \supset' e = a \supset' e \;\eta\; b \supset' e = b' \supset' e = (b \supset e)' = (\neg b)'.$$

Hence, $\eta \subseteq H(H^{-1}(\eta))$ and so $\eta = H(H^{-1}(\eta))$. ∎

Joining the two above propositions, we obtain the result we have been aiming at (a similar one has been proved in [11, Proposition 6.2]).

Theorem 5.3
For any $\mathcal{T}$-lattice $\mathbf{A}$, the lattice $\mathrm{Con}(\mathbf{A})$ of congruences of $\mathbf{A}$ is isomorphic to the lattice $\mathrm{Con}(\mathbf{A}')$ of congruences of $\mathbf{A}'$ through the maps H, H^{-1} defined in (5.1) and (5.2). Consequently, for any twist-structure $\mathbf{A} \leq \mathbf{L}^{\bowtie}$, we have $\mathrm{Con}(\mathbf{A}) \cong \mathrm{Con}(\mathbf{L})$.

Notice that the second statement of the above theorem follows immediately from Theorem 3.6 (the same applies to the second statement of Theorem 5.4, which follows from Theorem 4.4).

Theorem 5.3 tells us, in particular, that subdirectly irreducible $\mathcal{T}$-lattices are exactly those isomorphic to twist-structures $\mathbf{A} \leq \mathbf{L}^{\bowtie}$ where $\mathbf{L}$ is a subdirectly irreducible residuated lattice. We are now going to see that this correspondence is still true when we add modal operators.

Theorem 5.4
For any modal $\mathcal{T}$-lattice $\mathbf{A}$, the lattice $\mathrm{Con}(\mathbf{A})$ of congruences of $\mathbf{A}$ is isomorphic to the lattice $\mathrm{Con}(\mathbf{A}')$ of congruences of the residuated lattice with modal operators $\mathbf{A}'$ through the maps H, H^{-1} defined in (5.1) and (5.2). Consequently, for any modal twist-structure $\mathbf{A} \leq \mathbf{L}^{\bowtie}$, we have $\mathrm{Con}(\mathbf{A}) \cong \mathrm{Con}(\mathbf{L})$.

Proof. We already know that the map H is an isomorphism between the congruences of the non-modal reducts of $\mathbf{A}$ and $\mathbf{A}'$. So we only need to check that $H(\theta) \in \mathrm{Con}(\mathbf{A}')$ for all $\theta \in \mathrm{Con}(\mathbf{A})$ and that $H^{-1}(\eta) \in \mathrm{Con}(\mathbf{A})$ for all $\eta \in \mathrm{Con}(\mathbf{A}')$. As in the non-modal case, the first claim is easy because any modal operator of $\mathbf{A}'$ is defined by a term in the language of modal $\mathcal{T}$-lattices. To prove the second, assume $\langle a, b\rangle \in H^{-1}(\eta)$, i.e. $\langle a', b'\rangle, \langle (\neg a)', (\neg b)'\rangle \in \eta$. Then we have

$$\Box'(a') = (\Box a')' = (\Box a)' \ \eta \ (\Box b)' = (\Box b')' = \Box'(b').$$

Using (ii) of Definition 4.2 together with double negation law, we obtain

$$\Diamond'_1((\neg a)') = (\Diamond_1((\neg a)'))' = (\Diamond_1(\neg a))' = (\neg\Box\neg\neg a)' = (\neg\Box a)'$$

and, similarly, $\Diamond'_1((\neg b)') = (\neg\Box b)'$. By assumption η is compatible with $\Diamond'_1$, therefore we have $\langle \Diamond'_1((\neg a)'), \Diamond'_1((\neg b)')\rangle \in \eta$ and so $\langle (\neg\Box a)', (\neg\Box b)'\rangle \in \eta$. Hence, $\langle \Box a, \Box b\rangle \in H^{-1}(\eta)$. A similar reasoning shows that $H^{-1}(\eta)$ is compatible with $\Diamond$, i.e., that it is a congruence of the modal $\mathcal{T}$-lattice $\mathbf{A}$. ∎

6 Logics of $\mathcal{T}$-lattices

Given that the main motivation for the study of twist-structures comes from non-classical logics, there is a particular interest in associating logical systems to the algebraic structures we have been considering in the previous section. One way of doing this is to use the general theory of algebraization of logics developed in [4]. This strategy, that we will pursue in the present section, has the advantage of allowing us to obtain logics whose associated consequence relation is essentially equivalent to the equational consequence relation of the classes of algebras we are interested in.

Let us denote by *Fm* the set of propositional formulas defined in the standard way over the the language of $\mathcal{T}$-lattices $\{\wedge, \vee, \supset, \subset, \neg, e\}$, which is going to be the non-modal fragment of the propositional language of the logics we consider. We can translate a formula $\varphi \in Fm$ into an equation (i.e., a pair of formulas) in the same language by letting

$$\tau(\varphi) := \varphi \approx \varphi \vee (\varphi \supset \varphi).$$

We extend the translation τ to sets of formulas in the following way. If Γ is a set of formulas, let $\tau(\Gamma) := \{\tau(\varphi) : \varphi \in \Gamma\}$. We may then define a consequence relation $\models_{\mathcal{T}}$ as follows: for all formulas Γ, φ,

$$\Gamma \models_{\mathcal{T}} \varphi \quad \text{iff} \quad \tau(\Gamma) \models_{\mathcal{T}\text{-Lat}} \tau(\varphi)$$

where $\models_{\mathcal{T}\text{-Lat}}$ denotes the equational consequence associated with the variety of $\mathcal{T}$-lattices. This obviously defines a sentential logic in the language $\{\wedge, \vee, \supset, \subset, \neg, e\}$. We are going to see that this logic is algebraizable, in the sense of [4], w.r.t. to the variety of $\mathcal{T}$-lattices. In order to prove this result, we have to define an inverse translation $\rho : Fm \times Fm \rightarrow Fm$ from equations into formulas, as follows: for any equation $\varphi \approx \psi$,

$$\rho(\varphi \approx \psi) := \{(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)\}.$$

We extend ρ to sets of equations in the same way as τ. Let us also abbreviate

$$\varphi \leftrightarrow \psi := (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi).$$

We may then state the announced result.

Theorem 6.1
The logic $\models_{\mathcal{T}}$ is algebraizable w.r.t. the variety of $\mathcal{T}$-lattices with defining equation $\varphi \approx \varphi \vee (\varphi \supset \varphi)$ and equivalence formula $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$.

Proof. By [4, Definition 2.8], we only need to check that the following condition is satisfied:

$$\varphi \approx \psi \mathrel{=\!\!|\!\models_{\mathcal{T}\text{-Lat}}} \tau(\rho(\varphi \approx \psi)).$$

Applying our translations, we have that $\tau(\rho(\varphi \approx \psi))$ is

$$\varphi \leftrightarrow \psi \approx (\varphi \leftrightarrow \psi) \vee ((\varphi \leftrightarrow \psi) \supset (\varphi \leftrightarrow \psi)) \tag{6.1}$$

It is easy to check that (6.1) is equivalent, in twist-structures, to $\varphi \approx \psi$. Therefore, by Theorem 3.6, we conclude that the same holds in $\mathcal{T}$-lattices. ∎

We employ the same translation τ defined above to define a modal logic $\models_{\mathcal{MT}}$ in the language $\{\wedge, \vee, \supset, \subset, \neg, \Box, \Diamond, e\}$. We immediately obtain the following result as a corollary of Theorem 6.1.

Theorem 6.2
The logic $\models_{\mathcal{MT}}$ is algebraizable w.r.t. the variety of modal $\mathcal{T}$-lattices with defining equation $\varphi \approx \varphi \vee (\varphi \supset \varphi)$ and equivalence formula $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$.

Thanks to the algebraizability results stated above, it would be possible to axiomatize our logics by taking as axioms the ρ-translations of the equations that axiomatize the variety of (modal) $\mathcal{T}$-lattices. For instance, the equation

$$(a \wedge b)' = (a' \wedge b')'$$

of Definition 3.1 (iii) would be translated into the following axiom:

$$(p \wedge q)' \leftrightarrow (p' \wedge q')'$$

which is a shorthand for

$$\neg((p \wedge q) \supset e) \leftrightarrow \neg(((\neg(p \supset e) \wedge \neg(q \supset e))) \supset e).$$

It is easy to show that by adding standard logical rules (e.g. modus ponens, adjunction) to this set of axioms, we may obtain a complete axiomatization for our logics, both the modal and the non-modal one. However, it is clear from the above example that such axiomatization is neither compact nor elegant as it involves some rather long axioms whose logical meaning is not immediately clear. It is certainly possible to introduce better axiomatizations for our logics, but we will not pursue this here as our main concern here has been rather to make sure that our logics are finitely axiomatizable (hence, finitary) rather than to look for a specific axiomatic presentation. We leave this topic to future research, together with others that we mention in the next section.

7 Conclusion and future work

The treatment of modal twist-structures and their associated logics presented in the previous section is clearly just a first approximation to the topic. From a logical point of view, we believe that the most interesting line for future research is the study of alternative (e.g. possible worlds) semantics for our logics. This may lead to a definition of global and local consequence relation, as is usual within modal logic, the global being the one that is likely to coincide with the algebraic consequence defined in the previous section.

Another related direction of research involves the study of many-valued Kripke frames where not only the valuation function, but also the accessibility relation, take value into some fixed (maybe finite) twist-structure (see [5], where such a study is developed for modal logics over residuated lattices).

From an algebraic point of view, a topic which has in our opinion an obvious interest is the purely structural investigation of our twist-structures from a universal algebraic and lattice-theoretical point of view, along the same lines as [16, 19].

Let us conclude these final remarks with an example of a specific open question in this respect. It is easy to see that the presence of the constant e in the language of $\mathcal{T}$-lattices (interpreted as the element $\langle 1, 1\rangle$ in twist-structures) is essential to the construction that allowed us to prove Theorem 3.6, which states that any $\mathcal{T}$-lattice can be represented as a twist-structure. In fact, the role of the constant e in our construction is very similar to the one played by the constant denoted by the same letter in [6].

From a logical point of view, it seems difficult to find a natural justification for the presence of this constant in the propositional language, especially if we take into account the fact that both e and $\neg e$ are theorems of our logic, which seems to be forcing (rather than just allowing) a paraconsistent behaviour. One can therefore wonder if it may be possible to define an alternative construction that avoids using the constant. In algebraic terms, this would amount to characterizing the e-free subreducts of $\mathcal{T}$-lattices. One obvious option would be to introduce the prime operation ($'$) as primitive in the language of $\mathcal{T}$-lattices and use it to reproduce the construction leading to Theorem 3.6. From a technical point of view this strategy seems to be equivalent to the one we have adopted in the previous sections. However, from a logical point of view the presence of the prime operation in the basic propositional language seems to be even harder to justify. For the time being we do not know of any other strategy that may lead to a satisfactory solution of the problem, but we believe that what we do know about twist-structures as presented in this paper is interesting enough to stimulate further research on the topic.

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