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Strange and Wonderful: numbers through a new (material) lens

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Abstract: I respond to P. McLaughlin and O. Schlaudt's critique of my analysis of the cross-cultural origins of numbers, noting that my work draws extensively upon number systems as ethnographically attested around the globe, and thus is based only in part on the important Mesopotamian case study. I place the work of Peter Damerow in its historical context, noting its genesis in Piaget's genetic epistemology and the problems associated with applying Piaget's developmental theory to societies. While Piaget assumed numeracy involves invariant mental transformations, ongoing research in numerical cognition has been largely unsuccessful in identifying specific brain-bound mechanisms for numerical structure. Accordingly, I suggest the extended mind paradigm from the philosophy of mind may be a more fruitful approach, and detail such an approach using Material Engagement Theory.

Keywords: abstraction, representation, origin and development of numbers, cuneiform, historical epistemology, cognitive archaeology, material engagement

§1. Introduction

§1.1. Responding to McLaughlin and Schlaudt (2023) means placing Damerow's work on numbers in historical context. In the history of science, opinion has shifted from seeing psychology as having no role in numbers to seeing numbers as produced by the brain, whose form and function are the purview of the cognitive sciences. The psychological view also saw the brain as responsible for the numerical formatting that Piaget called logico-mathematical structure. In adopting the psychological view, Damerow specifically drew on Piaget's then-influential developmental work. However, the ground underneath the psychological view has since shifted. The cognitive sciences, despite closely investigating brain form and functions over the past several decades, have largely failed to discover how and why the brain might create numerical content, organization, and structure. Nor have they been able to explain satisfactorily why some non-Western societies have few numbers or organize their numbers differently when the ones in the Western tradition are extensive, highly elaborated, and decimal. These failures have inspired new paradigms that see cognition as the interaction between brain, body, and world, and material devices as having a role in conceptualizing numbers (Malafouris, 2013). On this account, the material devices used to represent and manipulate numbers are the source of their interesting properties, and differences between societies become simply a matter of whether devices are used, which ones are used, and how they are used (Overmann, 2023).

§2. The Historical Context of Damerow's Work

§2.1. As concepts go, numbers are strange and wonderful, if not more than just a little bit weird: They work the same way for everyone, and whatever we discover about them and regardless of whenever or wherever we discover it, we all discover the same things (e.g., prime numbers) and we all agree with an unusual conviction that they are the same things (i.e., the essence of mathematical proof). Around 400 BCE or so, this universality led Plato and many philosophers and mathematicians ever since to consider numbers and other logico-mathematical concepts as mind-independent objects (Linnebo, 2018; Maddy, 1990). When numbers are considered objective truths tractable to empirical discovery, they become independent of the human mind, and when mind and brain are considered synonymous, it follows that the sciences that study the brain have little to offer. The mathematician Gottlob Frege put this into blunt terms in the late nineteenth century: "die Psychologie bilde sich nicht ein, zur Begründung der Arithmetik irgendetwas beitragen zu können" ["psychology should not imagine it can contribute anything to the foundation of arithmetic"] (Frege, 1884, p. VI).

§2.2. Of course, not everyone agrees with Plato that numbers are somehow real and we just discover and then understand them better over time, or with Frege that psychology and the brain have no place in discussions of what numbers are and how we get them. Indeed, if numbers are not external to us, then the brain is the next most likely place to look for them, including explanations of their universality and origin (Brouwer, 1981). A key question, then, is what it is about the brain that structures numbers so strongly that they are functionally identical and identifiable over vast differences of circumstance, time, and language, apart from surface variability in things like the organizing base (e.g., decimal vs. sexagesimal).

§2.3. Assuming the brain structures numbers, Swiss psychologist Jean Piaget formed a hypothesis about how the ability to think develops ontogenetically, including thinking about and reasoning with numbers (Inhelder & Piaget, 1958; Piaget, 1936, 1952). Implicit to this hypothesis was a *genetic epistemology*, the idea that invariant logico-mathematical structures develop through experience (for Piaget, *genetic* was "a synonym for developmental"; Hopkins, 2011, p. 1). Genetic epistemology explained that children developed numerical concepts and abilities by interacting with objects, and they developed the same numerical concepts and abilities despite any differences in their experiences. The specific mechanism was *reflective abstraction*, the idea that an individual derives knowledge by contemplating his or her experience.

§2.4. Given that no two children have identical experiences, reflective abstraction could produce invariant logico-mathematical structures only if experiential differences had no effect. In Piaget's model, experience interacts with cognitive tendencies for structure that are biologically determined and which structure numbers only in certain ways. Numbers end up with the same final form, regardless of the content of the experiences that prompt them, and this means that the objects encountered and any actions performed with them are epiphenomenal to the result (Nicolopoulou, 1997). This is plausible at first glance because quantity is quantity, regardless of the objects that instantiate it. For example, two objects are *two* regardless of what the objects actually are (e.g., fingers, fish, or Ferraris). Similarly, rearranging four objects into two groups of two works the same, regardless of the objects involved.

§3. Damerow's Piagetian Model of Sociohistorical Change in Number Concepts

§3.1. Some decades later, Peter Damerow, a mathematician and historian of science, leveraged Piaget's ideas of genetic epistemology and reflective abstraction as his basis for modeling sociohistorical change in mathematical concepts, including numbers (Damerow, 1994, 2007, 2010). In Damerow's four-stage model, as was true in Piaget's developmental model, experience is subjected to invariant mental transformations: "die abstrakten, logisch-mathematischen Begriffe, zu denen insbesondere auch der Zahlbegriff gehört, ... intern repräsentierte Invarianten von Transformationen begreifen, denen die Objekte im handelnden Umgang mit ihnen unterworfen werden [sind]" ["abstract, logico-mathematical concepts, which include the concept of number in particular, ... [are] internally represented invariants of transformations to which the objects [of experience] are subjected in dealing with them"] (Damerow, 1994, p. 256). The "coherence of the developed structure is the reason for the status of logical necessity of mathematical knowledge, since the latter is determined by the [cognitive] system and not by the real objects [of experience] that are assimilated—that is, interpreted in terms of this system" (Damerow, 2010, p. 307).

§3.2. Damerow based his first two stages of sociohistorical development on the ancient Near East, the Mesopotamian numbers that are the world's earliest unambiguous numbers; his latter two stages were based on the history of mathematics and logic as specifically developed in the

West by Greek and later European thinkers. In doing so, Damerow became part of a longer intellectual tradition that sought to explain not just conceptual change within the Western tradition over time but also differences in thinking between societies, particularly between Western and non-Western societies (e.g., Lévy-Bruhl, 1910, 1922, 1927).

§3.3. While multiple points of disagreement between history and ontogenesis have since been recognized (e.g., Bjorklund, 1997; Franco & Colinvaux-de-Dominguez, 1992; Siegel, 1982), "one of the basic assumptions" of genetic epistemology was the idea that historical change in the mathematical thinking of societies "parallels" the ontogenetic development of mathematical thinking in individuals (Sfard, 2008, p. 17). That is, when Piaget's stages of ontogenetic maturation in numbers are applied to societies, the adults in societies with few numbers are assumed to conceive of numbers in the same way children do. This has had the unfortunate result of classifying some societies as childish and others as adult, and indeed, Damerow's model positions Mesopotamia as the early childhood of an adult Western mathematical tradition, rather than one of its roots or intellectual influences.

§3.4. When they are compared side by side (Table 1), Damerow's four stages of historical development track directly with Piaget's four stages of ontogenetic maturation. While there is a tendency to downplay the parallels in the way McLaughlin and Schlaudt (2023) have done, Damerow seems to have been aware of a general squeamishness regarding their implications, remarking that "the responsibility for the conclusions drawn and especially for any misinterpretations of our results in light of questions pertaining to developmental psychology rests solely with me" (Damerow, 2010, p. 304). Notably, while McLaughlin and Schlaudt are technically correct that a word search of Damerow's (2010) anthology fails to bring up the term "concrete number," it is not too much of a stretch to see the number concepts associated with Piaget's concrete stages—and thus with Damerow's corresponding stages, given their parallels with Piaget's model as shown in Table 1—as definitionally concrete. The idea that some numbers are "concrete" (or attached to whatever it is they enumerate) and others are "abstract" (or not so attached) is generally attributed to the ancient Greeks. Aristotle, for example, distinguished countable and counting numbers; the former were connected to the things they counted, while the latter were not (Katz, 2023; Klein, 1992).

Table 1: Comparison of Piaget's ontogenetic stages with Damerow's historical stages

Ontogenetic development (Piaget, 1952; also see Nunes & Bryant, 2009)	**Historical development of numbers in the ancient Near East [ANE] and beyond (Damerow, 2007, pp. 47–48)[1]**
Stage 1, Sensorimotor stage (from birth to year 2): In this stage, children experience the world through sensation and movement. Infants are able to appreciate quantity and object permanence.	*Stage 0, Pre-arithmetical quantification* (ANE before 10,000 BCE): "No arithmetic activities. All judgments about quantities are based on direct comparisons of amounts and sizes. Communication and transmission only by transmittable techniques of comparison and by comparative expressions of language."
Stage 2, Preoperational stage (years 2–7): In this stage, children begin to use words and pictures to represent objects (emergence of symbolic thought). Thinking is very concrete and lacks logic. Initial understanding of *one-to-one correspondence*.	*Stage 1, Proto-arithmetic* (ANE 10,000–3000 BCE): "Quantities are precisely identified by one-to-one correspondences. Communication and transmission with the aid of conventional counting sequences and tallying systems." [The 1994 version says quantities are precisely identified by "eineindeutige Zuordnungen" [unique assignments].

Table 1: Comparison of Piaget's ontogenetic stages with Damerow's historical stages

Ontogenetic development (Piaget, 1952; also see Nunes & Bryant, 2009)	**Historical development of numbers in the ancient Near East [ANE] and beyond (Damerow, 2007, pp. 47–48)**[1]
Stage 3, Concrete operational stage (years 7–11): In this stage, children become able to think logically but only concretely so, in being restricted to matters involving the physical manipulation of objects. They begin using *inductive logic*, reasoning that proceeds from specific information to a general principle. [Note: Piaget's Stage 3 may reflect cultural developments particular to the West, rather than an ontogenetic stage applicable to human societies generally (Buck-Morss, 1975; Chen, 1986; Maynard, 2008; Molitor & Hsu, 2019).]	*Stage 2a, Symbol-based arithmetic with context-dependent symbol systems* (ANE, "until the invention of the sexagesimal number system" around 2000 BCE): "Quantities are structured by metrological systems. Communication and transmission of these systems and of the corresponding mental constructs through complex symbol systems and developed techniques for the transformation of symbol configurations."
	Stage 2b, Symbol-based arithmetic with context-independent symbol systems (up to "the beginning of Classical Antiquity" around 500 BCE): "Quantities are structured by abstract numerical systems with object-independent arithmetic operations. Communication and transmission of these systems by unified, context-independent, but culture-specific symbol systems for the representation of arbitrary quantities, including abstract 'rules of calculation'. Emergence of first forms of 'pre-Greek mathematics' that are abstract but dependent on culture-specific symbol systems."
Stage 4, Formal operational stage (years 12 and older): In this stage, children become able to think abstractly (scientific reasoning), solve complex problems, and be aware of and understand their own thoughts. They begin to reason about hypothetical problems. They also begin to use *deductive logic*, reasoning that proceeds from general principles to specific conclusions. [Note: Piaget's Stage 4 is widely viewed as a development particular to Western culture, rather than a universal ontogenetic stage (Buck-Morss, 1975; Chen, 1986; Maynard, 2008; Molitor & Hsu, 2019).]	*Stage 3a, Concept-based arithmetic with deductions in natural language* (Classical/Late Antiquity, Middle Ages, Early Modern Era; "until the emergence of analytical mathematics" in the 18th century CE): "Abstract number concept with 'a priori' provable properties. Communication and transmission with the aid of a written representation of 'propositions' about abstract numbers and their mathematical properties. Propositions are logically ordered and systematically arranged by deductive theories according to the model of Euclid's *Elements*."
	Stage 3b, Concept-based arithmetic with formal deductions (the "modern mathematical tradition until the present"): "Formal understanding of arithmetic structures and expansion of the number concept by constructing new arithmetical structures. Communication and transmission with the aid of formal language systems."

[1] The 2007 version differs in minor respects from the 1994 German version used by McLaughlin and Schlaudt (2023); the former is used here because of its prior translation into English.

§3.5. Rather than considering the experience of objects and any actions with them to be as thoroughly epiphenomenal as Piaget had, Damerow focused on several of the technologies used to represent numbers—the tokens, numerical impressions, proto-cuneiform notations, and cuneiform numbers used between the fourth and third millennia BCE. As explained in McLaughlin and Schlaudt's exegesis (2023), Damerow considered numerals written on tablets to constitute "second-order representations" in representing tokens, which in turn were "first-order representations" of the concrete objects they enumerated (e.g., sheep). Second-order representations were thought to enable, for the first time, numbers to be conceived as objects in themselves (i.e., "abstract numbers"), rather than as numbers attached to the objects they enumerated (i.e., "concrete numbers"). The mechanism transforming numbers into objects is presumably the same one providing their invariant structure, as operating on representations now twice removed from the original objects of counting.

§3.6. Writing undeniably had important effects on how numbers were conceptualized, though explanations different from Damerow's are certainly possible. For example (see Overmann, 2016b, 2019b, 2023), relative to the technologies preceding it, writing was a highly concise way to represent numbers—perhaps not so much with the early impressions or proto-cuneiform notations that first appeared around the middle of the fourth millennium BCE, but certainly with the cuneiform numerals that had emerged by the beginning of the third millennium BCE (Figure 1). Conciseness enabled not just the collection of numerical relations into tables but also the presentation of these data for simultaneous viewing. Scribes recreated the tables as part of their training and thus learned numerical relations to a greater extent than had been possible with tokens. Tables also enabled the appreciation of whole-part relations in ways that were not possible with tokens, where wholes disappear once they are rearranged into parts. In viewing volumes of simultaneous relational data, scribes had opportunities to notice patterns and to think of numbers in terms of their relations to a greater extent. This too would have informed the reconceptualization of numbers, not just as entities but as entities defined by their relations with each other. Relatedly, tables and memorized relations gave scribes new possibilities in calculating: They could still manipulate tokens (or whatever form of abacus may have developed; see Woods, 2017), or they could look up relations in tables or recall them from memory, shifting calculation from the manipulation of physical tokens to the manipulation of numerical relations.

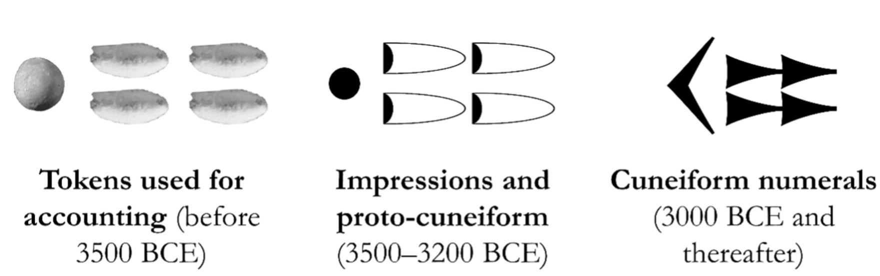

Figure 1: *Comparison of representations across technologies.* The number 14 is represented by a sign for 10 and four signs for 1 in (left) tokens, (middle) numerical impressions and proto-cuneiform notations, and (right) cuneiform numerals. These technologies share form (shape and size) and function (repetition and bundling). Damerow's model does not explain why written numerals are necessarily "second-order" representations when they are, for all intents and purposes, identical in form and function to their predecessors (e.g., tokens and impressions) and were likely thought to refer to the objects being counted in the same way their predecessors did.

§4. Issues with Damerow's Version of Piaget's Model

§4.1. A few issues suggest that Damerow's account is unlikely to be the final word. First, it is possible to agree with Damerow that writing changed the way numbers were conceptualized, and yet disagree that his model explains how and why this occurred. For example, handwriting involves specific neurological reorganizations (e.g., Overmann, 2016a, 2021c, 2022) that are distinct from reflective abstraction, genetic epistemology, and ontogenetic maturation. When we write characters by hand, we train the fusiform gyrus, a part of the temporal lobe with an evolved function for recognizing physical objects, to recognize written characters as if they were physical objects (Dehaene, 2009; Dehaene & Cohen, 2007; McCandliss et al., 2003). Being written as an integral character, rather than represented with collections of loose clay tokens, thus had the potential to influence numbers toward being conceived of as entities or objects in their own right, rather than as collections of objects. A similar change would have occurred earlier in the sequence, when numbers changed from equivalences ("as many as the fingers on both hands") to collections. In other words, the changes associated with writing by Damerow and his apologists are predictable outcomes of using a systematized sequence of material forms for numbers.

§4.2. Second, it is unclear just when the reconceptualization associated with writing would have occurred. It was unlikely to have happened immediately or all at once. Long before they were written, numbers may have been thought of as objects, as influenced by language (e.g., in a phrase like "there are seven," the number is nominal, not adjectival). Long after writing was available, numbers were still thought of as connected to the things they counted, as the sexagesimal place value system (SPVS) did not emerge until the end of the third millennium BCE (Robson, 2008), more than a thousand years after the advent of writing. The SPVS was a "calculating device" that took metrological quantities (i.e., the kind of numbers considered "concrete" in being attached to objects like containers of grain) and reconfigured them as "independent entities that could be manipulated without regard to absolute value or metrological system" (i.e., the kind of numbers considered "abstract" in not being attached to any objects) (Robson, 2008, pp. 77–78). This suggests the effect Damerow construed for numbers would have occurred long after writing—and through the SPVS, rather than writing per se or a mental transformation associated with it. It should be noted that the SPVS, rather than adding another referential layer interposing between the numerals and the objects they count, intensifies the relations between numbers.

§4.3. A third issue is that the representing technologies in question—tokens, impressions, proto-cuneiform notations, and cuneiform numerals (Figure 1)—share form (shape and size) and function (repetition and bundling).[2] They look, mean, and behave in exactly the same way; they are just produced differently, so Damerow's model is unclear as to why they would necessarily be understood differently or constitute different orders of representation. If it is the use of the representing technology away from the objects being enumerated, then perhaps tokens or precursors like tallies or fingers would qualify. If instead it is the representation of tokens or the use of tablets, then impressions on tablets should qualify, though McLaughlin and Schlaudt explicitly rule this out. If instead each new material form disconnects a representation from its antecedent, then perhaps each new impression or proto-cuneiform notation—even those made on the same tablet—could trigger the disconnection effect by being temporally and spatially distinct from those preceding it. If none of these are considered to have an effect, then the model should explain why not. If it is writing and only writing, then the model should specify whether the transition from first- to second-order occurs with the first proto-cuneiform or cuneiform notation, or only sometime later with reflective abstraction, and it should also say what happens if that reflective abstraction does not occur. On the other hand, the model, at least as articulated by Piaget, considers different experiences to produce invariant transformations, so it is also unclear why any material forms or actions with them should have an effect on conceptual structure. The cause of the purported transition between first- and second-order representations thus remains indeterminant (admittedly, this indeterminacy is problematic only if conceptual change is seen as involving multiple orders of representation; it disappears once conceptual change is seen as involving change in a single order).

§4.4. A fourth issue is Damerow's equation of second-order numerical representations with written numerals. This assumes a sharp, sudden distinction in numerical conceptualization based

[2] Minor differences related to substance and manufacture (e.g., tokens are convex, impressions made with tokens concave; notational forms were made with a stylus) are not considered significant for the purpose of this analysis.

on whether or not societies have writing, effectively replacing the abstract-concrete distinction with the similar one drawn between oral and written cultures (e.g., Goody, 1977; Goody & Watt, 1963; see Chamberlin, 2004 for criticisms). A distinction that is sharp and sudden is not necessarily warranted, as societies without writing are certainly able to explicate relations between numbers in a way that facilitates their calculating with them (Overmann, 2021a). For example, Inkan, West African, and Polynesian numbers were added and subtracted through their explicit relations, even though these numerical traditions did not incorporate writing prior to European contact (Florio, 2009; Overmann, 2020; Verran, 2000).[3] Granted, fewer explicit numerical relations would have been available to these reckoners, compared to those of the ancient Near East, where writing eventually made volumes of explicit relations possible. But the absence of writing does not entail the absence of all explicit relations or the ability to manipulate numbers using them, nor does its presence mean an immediate and inexplicable transition to a full set of numerical relations with knowledge-based calculation.

§4.5. Fifth, the technologies that would have preceded tokens—fingers and tallies—are not addressed in the 1994 version of Damerow's model, though he would later acknowledge that tallies had a role in one-to-one correspondence (Damerow, 2007, p. 35). This omission represents a general trend in the literature, which often treats tokens as if they were the first technology used for numbers in the ancient Near East. Yet cross-culturally, the fingers have this role, as terms for *five* and *ten* "are usually based on 'hand'" (Epps et al., 2012, p. 67). Indeed, the earliest number systems of Mesopotamia suggest the fingers were used in counting: the Sumerian words for *six*, *seven*, and *nine* were compounds of *five* plus the appropriate smaller number (Blažek, 1999; Edzard, 1980, 2005). Akkadian numbers were decimal, as is common for Semitic languages; Elamite numbers included a unique decimal system for tokens and early notations (Englund, 2004; Friberg, 2019; Kitchen et al., 2009; Lipínski, 2001). For tallies, archaeological evidence suggests they may have been used in the late Paleolithic (e.g., Reese, 2002), and textual evidence shows tallies were used throughout the Bronze Age (e.g., Henkelman & Folmer, 2016; "Translation of 'The Debate between Grain and Sheep,'" 2005). Despite their omission from Damerow's model, these technologies demonstrably influenced the structure of Mesopotamian numbers. For example, the sub-base of 10 in the sexagesimal number system most likely reflected counting to *ten* with the fingers.

§4.6. Sixth, the Piagetian model lacks explanatory power. The problem is not that it hypothesizes multiple orders of representations, as multilevel constructs are not uncommon in the cognitive sciences. One is Theory of Mind (ToM), the ability to understand that the mental states of others can differ from one's own. Like Piaget's model for numbers, the ToM construct struggles to explain cross-cultural differences. Variability in how many ToM orders are represented is considered an outcome of "what children [in different cultures] are taught about minds rather than providing any objective measure, or explanation, of the degree to which a particular type of ToM representation is more difficult [or complex] than another" (Conway & Bird, 2018, p. 1408). Nonetheless, the issue is not merely how well a multilevel construct deals with cross-cultural variability, but also whether it accurately describes the cognitive phenomena of interest. Here the Piagetian model falls short. In ToM, "I think that you think that she thinks X" is arguably more complex than "I think X"; on the other hand, it does not make sense to see cuneiform numbers as meaning proto-cuneiform notations that mean impressions that mean tokens that mean tally notches that mean fingers that mean the things being counted. Rather, cuneiform notations, just like all their predecessors, mean the things being counted because these, not the predecessor counting technologies, are the valuable items of interest. Further, because the Piagetian model sees conceptual change in numbers as mental and invari-

[3] The Inka recorded their numbers with knotted strings known as *khipu*; the knots were analogous to the tokens used in Mesopotamia before the advent of writing (Overmann, 2023). The non-numerical component of khipu remains untranslated and likely reflected a system of mnemonic prompts. In Polynesia, graphic signs have been found on only the easternmost island, Rapa Nui. Known as the Rongorongo script, it remains untranslated. It has not been shown to contain numerals, which is significant because numerals are generally identifiable in otherwise untranslated scripts (e.g., Elamite [Dahl, 2018; Englund, 2004]; Linear A [Corazza et al., 2021]). Attempts to interpret certain Rongorongo signs as phonetic numbers have been inconclusive to date (e.g., Davletshin, 2012).

ant, it does not merely glorify the brains thought to form the higher numerical orders. Rather, if conceptual change is purely mental, then the brain-bound capacities responsible for the logico-mathematical formatting predicted by the Piagetian model should be discoverable, but to date, they have not been identified. Similarly, if conceptual change is invariant—especially when the material forms involved are deemed epiphenomenal—the Piagetian model cannot explain why numerical concepts take different forms cross-culturally.

§4.7. And finally, a developmental model—any developmental model, not just Piaget's—is not an ideal framework for understanding sociohistorical conceptual change. A developmental model explains how and why children, as they mature into adults, become better able to understand things, think rationally, and make decisions. In other words, children become increasingly capable of mastering the knowledge made available to them by their society and environment, an important mechanism in the transmission of that knowledge. It is important to note that what children increasingly master is *existing* knowledge, and when that knowledge does not already exist, a developmental model lacks a ready mechanism for conceptual change involving *new* knowledge. The problem is compounded in a model like Piaget's, where invariant mental transformations occasioned by epiphenomenal experiences seemingly exclude the possibility of creating new knowledge altogether (this is not a problem if numbers are considered things that exist independently of the mind, a la Plato; in this case, conceptual change merely signals a better understanding of those independently existing things). If genetic epistemology and reflective abstraction are to be the mechanisms for creating new knowledge, their activity must also be able to explain cross-cultural variability—why some societies have few numbers and others highly elaborated traditions—without implying that some societies simply think better than others.

§4.8. In following Piaget in seeing numbers as mentally produced, contemporary researchers assume that brain-bound mechanisms underlie, influence, and/or create numerical and mathematical structure (Nieder, 2017a, 2017b). Nonetheless, the process whereby symbolic numbers are realized remains a mystery in their eyes (Núñez, 2017a, 2017b). Certainly, they have not successfully managed to define or isolate the kind of innate logico-mathematical predispositions hypothesized by Piaget, with two possible exceptions. One is the ability to perceive quantity that humans share with other species, often misleadingly called the *number sense* (Dehaene, 2011). The other is the mental number line, linear structuring that may be a cultural effect that emerges from, for example, exposure to a system of writing, rather than an innate tendency for structure (Aiello et al., 2012; Núñez, 2011; Pitt et al., 2018; Stoianov et al., 2008). Researchers have also largely failed to identify how and why human cognitive predispositions for numbers differ from those of other animals, as they plausibly must, given the dramatic differences between human and non-human species in their material and linguistic expressions of numbers. In fact, comparative researchers often take the opposite tack, seeing the mental structures producing numbers as shared with humans to the extent that honeybees understand zero (Howard et al., 2018; Nieder, 2016) and chimpanzees have a concept of rational numbers (Clarke & Beck, 2021). Beyond the difficulty of verifying that animals understand such concepts in a way that can be compared meaningfully to those of humans is the need to explain why such concepts would emerge in animals spontaneously and without the involvement of material culture and language. Certainly, for humans, concepts like *zero* and rational numbers represent conceptual efforts and refinements spanning centuries to millennia, and they do not just include but absolutely depend on material and linguistic support.

§5. The Shifting Ground: beyond the brain

§5.1. If psychology has a role in understanding numerical concepts but cannot explain numerical structure or elaboration through the brain alone, theories that are not brain-centric are the next logical place to turn. In this category is the embodied model of Lakoff and Núñez (2000), which combines mental conceptualization (the "conceptual blending" of Fauconnier & Turner, 1998, 2002) with the experience of objects to propose—as Piaget once did—that number concepts are mental outcomes of experiences with quantity as it occurs in collections of physical objects in the world. And like Piaget, the embodied model does not see the material forms used to represent and manipulate numbers as having any role in numerical conceptualization. The representational devices remain epiphenomenal in the same way that

Piaget considered them to be; they are considered merely the passive recipients of the mentally formed content that is deliberately externalized to them at some subsequent point in time. [See Overmann, 2023 for a longer critique of the embodied model.]

§5.2. Another model that is not brain-centric is the Material Engagement Theory of Lambros Malafouris (2010, 2013). This model has three major commitments: first, *cognition is extended* (that is, the mind includes not just the brain and body but also material objects); second, *materiality has agency* (what things are informs and influences what we can do with them); and third, *signification is enactive* (things acquire meaning when and because we interact with them). As applied to numbers, the foundational insight is the idea that material forms make perceptible quantity tangible and tractable to manipulation and sharing (Malafouris, 2010, 2013; also see discussion in Coolidge & Overmann, 2012). As developed by Overmann (2019b, 2023), the extended model has expanded to include the material forms used cross-culturally to represent and manipulate numbers—exemplars of small quantities, the fingers, technologies like tallies that accumulate, technologies like tokens that accumulate and group, and written numerals. These material forms are seen as interacting with brains, bodies, and behaviors to produce numerical content, organization, and structure. This perspective is unique, for it is the only model of numerical origins to consider the material devices used to represent and manipulate numbers as contributing substantially to their conceptualization, rather than being epiphenomenal or passive receptacles for externalized mental content.

§5.3. To examine concept formation and change (Figure 2), the extended model starts with the same model of conceptualization (Fauconnier & Turner, 1998, 2002) used in the embodied model (Lakoff & Núñez, 2000). The mental inputs similarly include knowledge, habits, expectations, perceptions, etc. However, conceptualization is then anchored and stabilized by adding a material domain as an input (Hutchins, 2005; Malafouris, 2013). The added material input provides a mechanism for the properties of the devices used for numbers to contribute to their conceptualization (Overmann, 2016b). Mental abilities and material qualities connect and merge within an enactive space, the locus created by the interaction of the brain, body, and material forms (as distinct from processing that occurs strictly inside the brain, as in the original model).

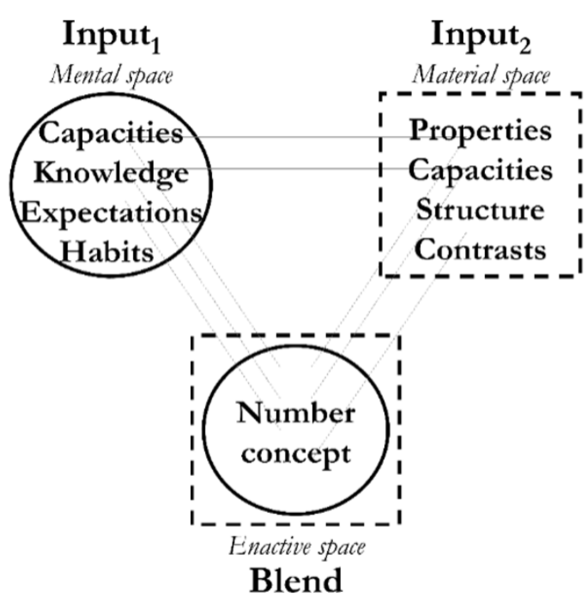

Figure 2: *Materially anchored conceptualization as used in the extended model.* Where the embodied model includes only mental inputs (Lakoff & Núñez, 2000), the extended model includes a material anchor (Hutchins, 2005), shown here as Input$_2$. Image adapted and redrawn from Malafouris (2013, Fig. 5.2, p. 101).

§5.4. Interaction between the inputs includes processes such as noticing, attending, recognizing, and selecting, as governed by attention and as interpreted through previous knowledge; outputs inherit some of their structure from the inputs, and novel structure can emerge from the interaction of the inputs (Malafouris, 2013). In essence, when we interact with a material form, our psychological, physiological, and behavioral capacities influence what we can do with it and how we respond to it, just as its substance permits or inhibits particular behaviors and elicits specific psychological responses; this interaction then informs the resultant concepts. As originally conceived, the materially anchored model (Hutchins, 2005) is agnostic regarding what happens inside the brain; it instead focuses on the interaction between brain, body, and world and how the different capabilities and limitations they represent influence conceptual outcomes. In comparison, the extended model draws upon standard psychological constructs (e.g., perception, categorizing, abstracting, sequencing, object and pattern recognition, working memory, learning, training effects, biases) to explain how and why the brain interacts with the body and material forms in the way it does in numbers cross-culturally.

§5.5. Material anchoring is important for more reasons than just shifting the focus from brain-bound processing to the engagement of objects (the arena also highlighted by Damerow and Piaget). Material anchoring also provides a mechanism whereby material forms can influence conceptual outcomes, whether they are counted with numbers or are used to represent them. An example of the former is found in Polynesia, where an innovative method of counting-by-sorting involved singles, pairs, or groups of four or eight as the unit of counting, as determined by object size and weight, creating multiple object-specified counting sequences; counting differentiated by object size and weight may have once similarly influenced the Mesopotamian counting sequences toward object-specified counting (Overmann, 2020). An example of the latter is the hand, the first coherent structure used cross-culturally in counting; this explains why decimal organization is the most prevalent structure found in number systems across the globe. Ten-ness, however, is not foreordained; rather, it is merely a consequence of having five fingers per hand (the norm for the human species) and then choosing them as the elements of counting. If the segments of the fingers or the spaces between them are chosen instead, numbers like *fourteen* and *twelve* (depending on whether or not the thumb is included) or *four* and *eight* (depending on the use of one or both hands) result. What the hand is and how it is used determine the structural outcome as quinary/decimal (5/10), quaternary/octal (4/8), or duodecimal/quadrodecimal (12/14) (Overmann, 2021b). This structuring also means the numerical outcomes cannot accurately be described as invariant mental transformations involving epiphenomenal material forms.

§6. The Extended Model

§6.1. Neither the cognitive sciences—nor philosophy, for that matter—understand what a "concept" really is. In fact, how physical phenomena like neurons, electrical impulses, neurotransmitters, and synaptic responses yield conceptual meaning remains a central mystery in studying the brain. While the physical level of interaction can be described (e.g., how cells communicate with each other through action potentials and neurotransmitters; which parts of the brain are involved and how they are connected), the conceptual level of interest cannot be. Because of this, researchers craft functional models that describe how conceptualization might work, suppositions grounded in the available observational and experimental data. The model offered by Damerow and Piaget is one such model; the materially anchored conceptualization developed by Hutchins and incorporated by Malafouris is another. The availability of competing models is not necessarily undesirable: models differ according to their focus and what they include (and exclude) to achieve that focus. When models focus on the same thing, the criteria for deciding between them should include how well they fit the available data and the assumptions they must make to do so. The first criterion governs the reliability and validity of the model with respect to the construct it is intended to elucidate; the second is the idea that fewer assumptions are better (the principle of parsimony, sometimes called Occam's razor).

§6.2. Damerow's model envisions a process in which experiences with material objects stimulate invariant mental transformations that create logico-mathematical structure. The transformation associated with writing is seen as causing a higher-order mental representation to develop, comprising a mental advance, if not a mental advantage, for the societies that have writing. Implicit to the model is the nineteenth-century notion that societies mature in their thinking in a way comparable to how children mature in their thinking as they grow up. As the conceptual transformations are deemed mental and invariant, the material devices used for numbers are effectively rendered epiphenomenal (or mostly epiphenomenal). The model also struggles to explain how new knowledge is created, or why numerical structure and elaboration vary cross-culturally. These issues were discussed at length in previous sections.

§6.3. In comparison, the extended model sees material objects as a critical input to conceptualization, and numerical concepts as varying and changing according to their mental and material inputs (Figure 2). Numerical realization and elaboration are outcomes of changes in the content of the material input (i.e., the devices used for numbers) and the mental responses to them (e.g., noticing, pattern recognition). Noticing something or recognizing a pattern is vastly different from the kind of invariant mental transformation proposed in Damerow's Piagetian model. These processes are anchored cross-culturally by a common starting point (i.e., the ability to perceive small quantities, the ready availability of five-fingered hands, etc.), and they are systematized cross-culturally by the predictable capabilities and limitations of the devices used. New knowledge is created through the incorporation and use of new devices, as these add new properties to the interaction and occasion new behavioral and psychological responses in the users. The addition of any new device does not immediately change the way numbers are conceived; differences may be behavioral at first, with conceptual explication following at a later date (perhaps much later, and possibly not at all, as the Egyptians and Romans apparently never explicated the zero implicit to the abacus design). As new material forms are added (and new properties gained thereby), concepts become distributed over multiple material forms, with the result that numerical concepts become seemingly independent of any particular form of representation.

Table 2: Types and chronology of material devices used in numbers

Numerical device	Descriptions, limitations, solutions, and persistent structure
Distributed exemplars • *Two:* eyes; arms; deer footprint • *Three:* tripod; bird claws; bird footprint; rubber seed; pronged fishing arrow • *Four:* spotted animal skin; brother (to *three*)	• **Description:** Commonly encountered natural and cultural objects whose quantity is appreciable, reliable, and expressible iconically (e.g., by means of a display of the fingers or a phrase describing an exemplar) or indexically (with a gesture toward an exemplar or a phrase drawing attention to it). • **Limitation:** Expressible quantities are typically limited to the subitizing range, and the methods of expressing them are ephemeral. Exemplars do not comprise a contiguous (single) device, so they do not significantly influence numbers toward organization or structure. • **Solution:** A material device (e.g., fingers) that can make quantity percepts tangible and manipulable, transcend the subitizing range, and influence numbers toward organization and structure. • **Persisting structure:** Forms and features of numerical language that conform to the limits of quantity perception or have etymological roots in material objects with subitizable quantity.

Table 2: Types and chronology of material devices used in numbers

Numerical device	Descriptions, limitations, solutions, and persistent structure
The hand • Fingers (bent, flexed, tapped, etc.) • Finger segments or joints • Spaces between fingers	• **Description:** The hand is the first contiguous device used for numerical representation because of the neurological interconnection between the parts of the brain that appreciate quantity and "know" the fingers and because the hands are readily available, visually salient, and easily used for expression. The fingers are used either directly to instantiate and display quantity or indirectly to gesture at an exemplar of quantity. • **Limitation:** As a material device, the fingers provide little persistence and have limited capacity. • **Solution:** Devices capable of doing what fingers do (e.g., accumulate with linearity and order) but which also address their lack of persistence and capacity. • **Persisting structure:** Discreteness, linearity, stable order, *ten-ness.*
Devices that accumulate • Notched tallies • Knotted strings • Torn leaves • Marks on surfaces • Pebbles or corn • The human body • Collaborative finger-counting	• **Description:** One-dimensional devices accumulate to amounts that exceed the fingers' capacity. They also persist longer than the fingers, with duration governed by durability of the substance used: Bone persists longer than wood, fiber longer than leaves, etc. • **Limitation:** Quantities higher than *about three or four* are increasingly indistinguishable (a limit inherent in the perceptual system for quantity), necessitating that items be matched to known exemplars or recounted. • **Solution:** Devices capable of doing what one-dimensional devices do (e.g., accumulate with capacity and persistence) but which also address the problem of visual indistinguishability through grouping. • **Persisting structure:** Accumulation, capacity, persistence; public in a way that bodies tend not to be; harnesses the power of material objects to accumulate and distribute cognitive effort.

Table 2: Types and chronology of material devices used in numbers

Numerical device	Descriptions, limitations, solutions, and persistent structure
Devices that accumulate and group • Counting boards and calculi (jettons) • Mesopotamian tokens • Abacus • Inka khipu • Collaborative strategies • Sorting strategies	• **Description:** Two-dimensional devices accumulate like one-dimensional devices do, but they also bundle the accumulated elements, either as appreciable quantities (*one* to *three* or *four*) or as amounts conforming to a well-exemplified quantity (e.g., often *five* or *ten*, the number of fingers on the hands). • **Limitation:** Loose elements lack integrity of form and are indistinguishable in higher (non-subitizable) quantities. The first may prompt enclosure, which contains but removes access to the elements; the second may inspire replacement by conventional forms understood as bundled values, which reduces the number of elements but requires the user to learn the bundling conventions. Khipus lack manipulability, so they cannot be used as a technology for calculation. • **Solution:** Devices capable of what two-dimensional devices do (e.g., group) but which add integrity of form. For manipulable forms, a fixed technology for recording is added (e.g., in Mesopotamia, bullae and then notations were used with tokens). For fixed forms, a manipulable form for calculation is added (e.g., in the Inka system, counting boards and yupana were used with khipus). • **Persisting structure:** Grouping (exponential structure), relations, manipulability.
Notations • Handwritten notations	• **Description:** Notations accumulate and group like two-dimensional devices but add integrity of form. They are concise, increasing the density of simultaneously viewable elements and enabling relational data to be recorded at volumes far exceeding those possible with earlier technologies. If they are handwritten with sufficient repetition, learned relational data and the neurological reorganizations associated with literacy will influence numbers toward being reconceptualized as relational entities. • **Limitation:** Notations are fixed, so calculating must be performed manually (e.g., with an abacus) until it can be supplemented with algorithms involving mental knowledge and judgements (e.g., long division performed using paper and pencil). • **Persisting structure:** Conciseness, entitivity, conceptualization of numbers as a system of relational entities.

Note: Versions of this table were previously published in Overmann (2018, 2019a, 2019b, 2023).

§6.4. The extended model (Table 2) starts with the ability to perceive quantity, as well as material forms whose quantity is perceptible and manipulable. The ability to perceive quantity is demonstrated by human infants at very young ages (Izard et al., 2009; Xu et al., 2005), it works the same whether societies have few numbers or highly elaborated ones (Henrich et al., 2010), and it is evolutionarily ancient, phylogenetically distributed in mammals, birds, fish, reptiles, amphibians, and perhaps even insects (as reviewed in Coolidge & Overmann, 2012). These qualities position it as a likely starting point for explaining the properties and evolutionary origin of numbers. Numbers emerge as the recognition that perceptible quantities are shared between sets of objects,[4] as typically expressed with the fingers or by means of a quantity exemplar (e.g., the eyes or arms for *two*). Numbers become elaborated when societies incorporate new devices to represent and manipulate them, as new devices add new properties. Societies select new devices on the basis of the capacities they share with previous forms and their ability to resolve the limitations of previous forms in some way.

§6.5. Beyond avoiding the thorny cross-cultural issues that perplex the Piagetian approach, the extended model also provides a reason why the cognitive sciences are not finding the source of numerical structuring inside the brain: simply, they are looking in the wrong place. The structure accumulates from the set of devices used to represent and manipulate numbers, rather than being generated mentally. While the brain continues to be a critical node in the cognitive system for numbers, it is relieved of the responsibility for generating all of the structuring that numbers acquire as they elaborate. Its role has been reduced to the things the brain is really good at doing, like recognizing patterns and forgetting details.

§6.6. The central idea of the extended model can be summed up as follows: All humans and all human societies are essentially identical in their neurological, morphological, and behavioral makeup, as we all have the same human brain, body, and behavioral capacities. In numbers, humans share the same ability to perceive quantity, the same quantity of fingers, the same neurological wiring in the brain[5] and the same perceptual salience for the hands[6] that predispose us to use the fingers in counting, and the same behavioral tendencies. These factors provide a basis for initial structure and a starting point for the relative cross-cultural uniformity that characterizes number concepts. Variability between cultural systems then becomes a matter of whether material devices are used to represent and manipulate numbers, which ones are used, and how they are used.[7] The recruitment of new material forms is motivated by a society's need for numbers, as driven by increases in complexity that emerges from larger population size, more frequent intergroup contact, or both, and as systematized by the capabilities and limitations of the devices used. The claim is not that the concept of number is isomorphic to any particular material form, but rather, that material forms and their precursors directly influence how numbers are conceived, and how numbers are conceived, in turn, influences how the material forms used to represent and manipulate them are understood and used. This role is generally recognized with respect to written numerals (Schlimm, 2018); the extended model merely expands it to include their precursors.

§6.7. Simply, numbers start with the perceptual experience of quantity. Recognitions of shared quantities are represented gesturally with the fingers and verbally with quantity exemplars. Since quantity exemplars are distributed, the fingers are perishable, and both have finite capacity, a society may be motivated at some point to incorporate a device like a tally that does what these forms do but adds persistence and the ability to accumulate to higher numbers. Since the quantity we can perceive visually is limited to *about three or four*, at some point the indistinguishability of an accumulation on a tally is a limitation that is addressed by adding a manipulable technology like tokens (or pebbles, kernels, beans, etc.). Hand-

[4] The definition comes from the work of Bertrand Russell (1920): the quantity of a set is a property of that set, not a number; quantity shared by sets is a number.

[5] The perceptual system for quantity is neurologically interconnected with the ability to "know" the fingers ("finger gnosia") to an extent that the latter predicts mathematical ability (Marinthe et al., 2001; Penner-Wilger et al., 2007; Reeve & Humberstone, 2011).

[6] Interestingly, congenitally blind people do not count on their fingers (Crollen et al., 2011; Marlair et al., 2024), which suggests that the visual experience of one's own hand, rather than some internal mental capacity, underlies its recruitment and use as a material form for numbers (Overmann, 2023).

[7] Most numerical researchers, to the extent they treat the material forms used to represent and manipulate numbers, segregate the different technologies, treating separately the fingers as embodied, written numerals as symbolic, and physical devices like tallies and tokens as material. Damerow, to the extent he mentions material devices, follows this same general pattern. In comparison, the extended model includes all these devices as united by their material substance and as individuated by their differences.

writing is associated with specific neurological and behavioral reorganizations that help influence numbers toward being conceived as objects in themselves, rather than collections of objects (Overmann, 2022). Handwriting also adds substantial conciseness; this quality enables numerical data to be collected at unprecedented volumes and presented for simultaneous viewing (e.g., tables of multiplication) and influences numbers toward being conceived in terms of relations and calculation toward the manipulation of relational knowledge.

§6.8. The ability to perceive quantity remains an influence throughout the entire elaborational trajectory of any cultural system of numbers. Its influence starts with the initial expression of numbers, as gestures recreating quantity and terms describing exemplars are limited to subitizable quantities, or the small quantities that fall within the subitizing range of *one* to *about three or four*. This perceptual limitation motivates the use of the hand to realize the first quantities beyond the subitizing range, which, as a result, are typically *five* and *ten*. The influence of the perceptual system for quantity remains detectable even when numbers are highly elaborated. It is why, for example, quantities higher than *about three or four* notches as accumulated on a tally are indistinguishable, a circumstance that motivates the recruitment of devices like tokens that can be grouped and rearranged (Overmann, 2016b, 2018a). It is why written numerical elements are rearranged into appreciable (subitizable) groups of elements, as seen in cuneiform and Egyptian numerals; it is why written forms involving a single group of subitizable elements (e.g., the Roman numerals I, II, III; Chinese numerals 一, 二, 三; and Western cursive forms 1, 2, 3) tend to be conserved (Overmann, 2021c).

§6.9. If a timeline for the Western numerical tradition and its Mesopotamian and Egyptian roots were to be formulated using the extended model, the result would generally resemble the timeline offered by Damerow (Table 1, rightmost column). On the other hand, if the material devices used for numbers are the mechanism of their realization and elaboration, we are unlikely to see their true beginnings, since distributed exemplars, finger-counting, and tallies made of perishable materials tend not to leave any archaeological trace of themselves behind. At the other end of elaboration, writing emerged in Mesopotamia in the mid-fourth millennium BCE and had been elaborated as systems of literacy and mathematics by 2000 BCE (Overmann, 2022; Robson, 2008). This means that beyond writing, conceptual advances in mathematics cannot be described as outcomes of adding new material devices with new properties, or behavioral and neurological adaptations to their use. Rather, later conceptual advances are matters of refining and extending the concepts realized through the sequence of material devices, as aided by new material expressions (e.g., concise, semasiographic signs for arithmetic: + − × ÷ = [Schulte, 2015]; equations and variables).

§6.10. Speaking of the conceptual advances that have occurred in the several thousand years since numbers supposedly became abstract in the ancient Near East, we need to keep in mind the vastly different circumstances we face today in learning Western numbers and mathematical operations. Western numbers have become highly elaborated, not just with signs, equations, and variables, but with numbers that are transfinite and even imaginary. They are not just notationally mediated, but their forms are ciphered, so the quantity they mean is no longer expressed with the number of their elements (e.g., 4 instead of 𒐉).[8] How we acquire these complex concepts is plausibly quite different from the way their antecedents would have been acquired by Bronze Age scribes, Neolithic farmers, or Paleolithic hunter-gatherers. For the most part, ancient peoples merely learned to use the tools available to them for recording and calculating with numbers, within social contexts that influenced the use of numbers as a technology for managing complexity (less for hunter-gatherers, more for urban scribes). This suggests that today's experience of the kind of mental insights that occur in learning long division or calculus may not be the

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[8] Ciphered notations like 4 never developed in the cuneiform writing system, but they did next door in Egypt as the demotic notations that emerged in the last centuries of the cuneiform lifespan (Chrisomalis, 2010; Geller, 1997). Non-ciphered notations like 𒐉, which express quantity through the number of their elements, visually aid the performance of arithmetic; a modern example is the Kaktovik numerals developed to represent the vigesimal Inuit numbers (Tillinghast-Raby, 2023). Ciphered numerals remove this visual support, so arithmetic must depend instead on memorized numerical relations or the use of a calculating device.

[9] Introspection is the method of studying psychological processes through the systematic self-observation and

best model for the mental component of ancient numeracy. In fact, it is rather introspectionistic[9] of modern researchers to think it would be.

§7. The Way Ahead

§7.1. For decades, the cognitive sciences have recognized that Piaget's theories are limited, both by the methodology he used (observing his own upper-class, educated European children and basing his conclusions about all children on them) and the generalizability of his results (not well, particularly for children in non-European cultures). Today, Piaget's theories inspire the mandate for instructional and testing methods to be developmentally and culturally appropriate. Their application cross-culturally, let alone to entire societies, has become highly sensitized, if not anathema (Bloch, 2012). It is thus somewhat curious to find that they persist, and not just in the history of science. Arguably, the cognitive sciences have not yet moved on from them sufficiently, given the continued assumption that logico-mathematical formatting is an entirely brain-bound process.

§7.2. Where do we go from here? For Assyriologists interested in sociohistorical change in numerical cognition and in leveraging the interesting Mesopotamian number system to understand it, there is now a choice of models. Damerow's Piagetian approach can be retained in the way McLaughlin and Schlaudt (2023) have recommended; the alternative is one of the newer models that address its shortfalls and draw on recent research in the cognitive sciences and the philosophy of mind. Synthesizing the two approaches, while a laudable goal, is more easily accomplished in theory than in practice, for the differences between the two models may well be irreconcilable. Models that look beyond the brain may differ too fundamentally to incorporate (or be incorporated into) brain-centric models. Ultimately, time and analytical output will determine which models survive and in which form(s), recognizing that no single one is likely to provide the final word on every subject.

§7.3. Mesopotamia has perhaps the most important number system in history. Its numbers are the earliest that are unambiguous to modern eyes, since neither notches on Palaeolithic bones nor clay objects unaffiliated with bullae can be conclusively identified as having represented numbers. Its extensive archaeological and textual evidence also demonstrate a clear chronology in the technologies used for numbers, a critical element for understanding how numbers become elaborated. Nonetheless, using the Mesopotamian numbers as a case study for numerical cognition means it is time to relinquish some of the now-outdated ideas they have accumulated over the years: seeing these numbers as inherently different from those of other cultural traditions because of their multiple counting sequences, thinking of tokens as representing particularly rudimentary numbers (e.g., Schmandt-Besserat, 1974, 1977, 1978, 1981, 1982, 1992a, 1992b), and seeing writing as enabling a sudden, second-order leap in their intellectual development (e.g., Damerow, 1988, 1994, 1996a, 1996b, 2007, 2010, 2012; Malafouris, 2010). Our familiarity with the models of the past, like our rightful veneration of their influential authors, should not blind us to the very real need for change.

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self-report of thoughts, perceptions, and feelings; its profound subjectivity makes it inherently unreliable and significantly ungeneralizable.

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