

“Every mathematical theorem is eventually proved trivial.”

- Gian Carlos Rota

The Eternal Unprovability Filter – Part I

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Abstract

I prove both the mathematical conjectures $P \neq NP$ and the Continuum Hypothesis are eternally unprovable using the same fundamental idea. Starting with the Saunders Maclane idea that a proof is eternal or it is not a proof, I use the indeterminacy of human biological capabilities in the eternal future to show that since both conjectures are independent of Axioms and have definitions connected with human biological capabilities, it would be impossible to prove them eternally without the creation and widespread acceptance of new axioms. I also show that the same fundamental concepts cannot be used to demonstrate the eternal unprovability of many other mathematical theorems and open conjectures. Finally I investigate² the idea's implications for the foundations of mathematics including its relation to Godel's Incompleteness Theorem and Tarsky's Undefinability Theorem.

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² A more mathematical interpretation and the implications are in Part - II

Background

If we really understand the problem, the answer will come out of it,
because the answer is not separate from the problem.

- J. Krishnamurti

Problems cannot be solved at the same level of thinking that created them

- Albert Einstein

Why does every volume of the Talmud begin with page 2 and not page 1?

To teach us that no matter how much we learn, we have not yet reached the first page!

- R. Levi Yitzhok of Berditchev

This essay has its origins somewhere between Cristopher Moore and Stephan Mertens's The Nature of Computation, Gian Carlos Rota's Indiscrete Thoughts and Scott Aaronson's call for a more philosophical probing of the P vs NP question. I started out looking to understand the philosophical implications of the P vs NP question and eventually stumbled onto a potential link with the Continuum Hypothesis. I have mostly stayed away from previous attempts to resolve these famous conjectures focussing rather on the foundations and philosophy of mathematics which is what my intuition lead me to. Most of the ideas here were developed while reading some³ of the works of Avi Widgerson, Lance Fortnow, Oded Goldreich, Scott Aaronson, W V Quine, Hilary Putnam, Reuben Hersh, Raymond Wilder, George Lakoff, Yuri Manin, George Chaitin and of course Gian Carlos Rota. If anything is considered new thinking here, you can safely assume that it is primarily a minor rephrasing of the ideas of these pioneers. All mistakes and ideas considered crankworthy are entirely my creation.

³ References will be in the final version of this essay (if required)

The Eternal Unprovability Filter

The lack of real contact between mathematics and biology is a tragedy,
a scandal, or a challenge; it is hard to decide which.

- Gian Carlos Rota

Our mathematical ideas in general match our world
for the same reason that our lungs match earth's atmosphere.

- Reuben Hersh

The point of philosophy is to start with something so simple as not to seem worth starting,
and to end with something so paradoxical that no one will believe it .

- Bertrand Russell

Saunders MacLane, one of the creators of Category Theory wrote a paper called "Proof is Eternal". Eternal Validity is indeed the critical difference between statements with a mathematical proof and Scientific theories. A Scientific theory is the best causality seeking "explanation" of a part of the real world. It remains eternally unprovable but could still have the greatest benefits for mankind and solve its most difficult problems like Einstein's Theory of Relativity or its intellectual equal Quantum Physics. A Scientific theory remains forever open to evidence in the future to be replaced by a better "explanation" to use a word emphasized by David Deutsch. But once a mathematical theorem is proved, it remains valid for eternity.

That said, Proof is eternal only in the Abstract Axiomatic Space where the Axioms are considered "in principle" true because they are rooted in our natural understanding of the world around us or as George Chaitin says they are considered as true, otherwise we face infinite regress. So though our axioms have their origin in human biology and the physics of the real world, they help create an abstract world which is independent of its creator and the real world even though it may not remain useful if our underlying biology and physics change in the eternal future. Hence a proof using

these axioms remains eternally true only in this independent abstract world. Proof itself is an abstraction whose causality is the human desire for certainty. Certainty may or may not exist in the real world but is something the human mind seems to strongly need.

There is [, thus,] a hidden circularity in formal mathematical exposition. The theorems are proved starting with definitions; but the definitions themselves are motivated by the theorems that we have previously decided ought to be correct.

- Gian Carlos Rota

To restate Saunder's "Proof is Eternal", if a statement or conjecture can never be proven to be eternally true or false then it is eternally unprovable. David Deutsch in his "The Beginning of Infinity" also talks about how proofs may not exactly be eternally valid as they could be dependent on the underlying theories of physics. But I believe this is not entirely true because mathematics aims to be in an independently abstracted world. A perfectly abstracted world implies that it remains independent of the underlying physics of the world. Even if the underlying theories of physics change or something even more dramatic, the world starts behaving completely differently in the future, the mathematical abstractions we have created remain independent of physics. They could become less important to us but remain true nonetheless. For example, Euclid's geometry is now proven to be not an accurate representation of the real world because of Einstein's theory of space and time. But Euclidean geometrical theorems retain their eternal validity because Euclid and others created an abstract space using axioms which remain valid even though they are not an exact representation of reality.

Now there could be some mathematical statements which we currently believe to be proven which are not independent of the underlying theories of physics and this would need further probing. But in principle, most of the mathematical theorems can

be fully imagined to be in an independent abstract axiomatic universe with no relation to the underlying physics. For example, gravity could completely disappear tomorrow and $1 + 1$ would be equal to 2 primarily because of the underlying axioms we have defined which are independent of physics. The role of the foundation of mathematics is to keep mathematics both useful to mankind and entirely abstract based on a few principles considered to be true by their very nature in relation to us.

But having said this, the intuition of this paper is that there are mathematical conjectures and statements which are currently not in an Abstract Axiomatic Universe independent of human evolution which in itself remains indeterminate. Imagine the biological capabilities of man about a million years in the future. Regardless of whether we accept that a million years ago, human biological capabilities were different than they are now, we can absolutely not be certain how they would be a million years from now. For example studies of the brain tell us that there seems to be some sort of hierarchical abstraction in its functioning. Is it not possible then, that in a few million years, the human brain adds more layers to these layers of abstraction and is able to see, understand things in a different way than today. It is imperative to emphasize that a proof that is not valid a billion years from now is not considered a proof even by today's standards.

With this intuition, let's conceptualize an Eternal Unprovability Filter hinged on the indeterminacy of human biological evolution in the eternal future. I use the term filter here to indicate that the statements or conjectures that do not pass through this filter are eternally unprovable and will be considered in principle to be so by mankind. In a way, this intrinsically implies that these statements and conjectures are not fully in an independent mathematical abstract space. If there are mathematical statements which are connected in any way to current human biological capabilities, its eternal

provability is not possible because future human biological capabilities remain indeterminate. And this is regardless of whether someone believes in evolution or not, as the key is that both sides accept that we cannot say for sure that human biological capabilities will stay the same for eternity.

So think of the Eternal Unprovability Filter, to put it very simplistically and not mathematically for now, as asking the question

Is this mathematical conjecture or proven statement completely independent of human evolution ?

If it is independent then its proof is eternal and hence stays as a valid proof. If it is dependent on human biological capabilities then there cannot be a proof as its eternal validity is not possible. Let's understand that at its core a proof has to be convincing about its eternal validity. If there is any doubt about its eternal validity then its not a proof in the truest sense of the term. Now, this could be construed to be a phenomenological debate and it is in some sense, but the goal of mathematics is to create a useful but abstract world where there exists the concepts of certainty and eternal validity unlike in the real world and hence using ideas originating in phenomenology is fair game if it leads us to that goal.

“The question for the ultimate foundations and the ultimate meaning of mathematics remains open; we do not know in which direction it will find its final solution nor even whether a final objective answer can be expected at all. “Mathematizing” may well be a creative activity of man, like language or music, of primary originality, whose historical decisions defy complete objective rationalization.”

- Hermann Weyl

The Question of P vs NP

Intuitively, the P vs. NP problem is formidable, because being a universal statement about mathematics, which says that discovery is hard, it can potentially preclude its own proof and be independent of the axioms of set theory.

- Ketan Mulmuley

Let's use the Eternal Unprovability Filter rooted in biological evolution on one of the most interesting and unproven questions of our time – The question of P vs NP. Assume someone is able to prove that $P \neq NP$. Let's apply our Eternal Unprovability Filter which is the following question.

Is this mathematical conjecture or proven statement completely independent of human evolution ?

A few million years from now, the human brain could very well evolve to be able to solve NP problems in polynomial time with its newly developed biological capabilities. The set of newly developed biological capabilities can be anything from a few more layers of hierarchical abstractions, new type of neurons in the brain to an entirely unimaginable mutation. All parts of the human body from the eye to the hand have all evolved from pretty much nothing. So on a scale of a few million or more years, nothing is in the range of impossibility. On a scale of billion years based on our current levels of understanding, I believe it would be surprising if human biological capabilities remain as they are now. Remember the complexity of the brain even today is a scientific fact and that this came about from nothing means what the eternal future holds for the development of the brain is not something in the current range of human imagination.

The acceptance of a new mathematical truth is a sociological process.

- Steven Krantz

Which means that $P \neq NP$ is eternally unprovable. Again, our filter does not tell us anything about whether the statement could be actually true or not, just that there is no way to prove it eternally. Very simply it tells us whether the statement or conjecture lies outside of the independent abstract world of Mathematics. Then where exactly would be the error in the mathematical proof? The error would basically be in the axioms selected for the proof. Axioms are accepted in principle because they seem true and justified based on our natural understanding of the world for the past 5000 odd years. But evolution is an idea with only a history of around 150 years and it has no representation in the axioms. In a Universe where mankind thrives for a billion years we cannot conclusively say that the axioms selected by us in the 20th century are the only ones that are in-principle true for the next billion years. As mankind evolves, the axioms to create a fully abstracted mathematical space will need to evolve. For proof itself to remain a useful entity, we need to remain open to the introduction of new axioms through time.

Functionally a definition is not a premise to theory, but a license for rewriting theory by putting definiens for definiendum or vice versa. By allowing such replacements a definition transmits truth: it allows true statements to be translated into new statements, which are true

by the same token.

- W V Quine

It is not difficult to imagine a point in time of mankind's existence where Mathematics transforms itself and becomes dramatically different compared to what it is today with a systematic procedure of bringing new axioms into the fold and entirely different axioms producing more useful and practical applications than what the mathematics of the present can provide. The broad outline being, the abstract mathematical world will still need to be independent but get closer to the underlying reality of the world. Not to mention the obvious unimaginable fields that could

replace Mathematics itself, as we currently understand it. To think that mathematics will look more or less like it does now in the 21st century in an eternal Universe seems rather underwhelming when seen in this context.

The truth of a mathematical theory is distinct from the correctness of any axiomatic method that may be chosen for the presentation of the theory.

- Gian Carlos Rota

Let's say that after being convinced by the eternal unprovability of $P \neq NP$, someone designs a new axiom that is able to create an Independent Abstract Axiomatic Universe which is additionally independent of human biology and physics (as already indicated by David Deutsch), and then goes on to prove $P \neq NP$. This would remain a fully valid proof because it does not imply that mankind cannot evolve to solve NP problems by the very definition of the Independent Abstract Axiomatic Universe created for the proof. Whether that is a more realistic way to a proof for $P \neq NP$ or otherwise is something only time can tell. But regardless, the definitions of P, NP would be then something completely delinked from the human condition and possibly lose any philosophical implications along with it.

To put it more simply, a proof about $P \neq NP$ can never be accepted by mankind unless it is in an independently abstract Universe with no connection to the human condition because it will never be convincing for the reasons mentioned here. What is also interesting to think about is whether the difficulty of finding a proof for $P \neq NP$ is in some way related to the currently accepted axioms being not enough.

So what about $P = NP$ then? Counterintuitive as it may sound, the eternal unprovability filter cannot be applied as simply as before and $P = NP$ is not as clearly eternally unprovable. But you could apply it now to the part about the human capabilities to generate difficult decision problems. Now that we have created new

knowledge and new answers to old questions, couldn't we use that knowledge to create more difficult NP+ questions in the future? That will not change that $P = NP$ is eternally proved by the given proof, but it does not directly imply that mankind cannot again generate newer decision problems which have no solution in deterministic polynomial time. Coming up with a solution in deterministic polynomial time for a previously NP decision problem crosses the previous limits of human creativity, imagination and intuition. But the same creativity, imagination and intuition can again be used to design newer decision problems that push those boundaries of human capabilities. So $P = NP$ does not directly imply that human faculties such as creativity can now be forever automated as is currently believed. At the very least it will need a separate proof. This conclusion however surprising and counterintuitive is not an isolated view as many people including Donald Knuth have the same opinion though different reasonings for the same.

Is $1 + 1 = 2$ still eternally true ?

The question now really is wouldn't using human biological evolution as a filter for verifying a proof's eternal validity become an impediment for all known proofs and mathematical statements. If it is so then it clearly has no value. But it becomes almost immediately clear that most mathematical statements are true in their abstract axiomatic space and their validity does not change even after a billion years. $1 + 1$ remains equal to 2 regardless of human biological evolution which includes new organs that mankind might develop and mutations of the brain. It is possible that mankind's use for number theory ends completely in a billion years but $1 + 1 = 2$ would still remain true based on the already well accepted axioms. We might have a newer notation for a newer number system but $1 + 1 = 2$ remains so in the older one.

The more interesting area which might need investigation is how is the measure of complexity related to the number system. This is not a trivial investigation because you cannot even prove $1 \neq 2$ without the foundational axioms of mathematics. So if we cannot prove $1 \neq 2$ without axioms why do we believe we can prove $P \neq NP$ without additional axioms?

A good definition is "justified" by the theorems that can be proved with it, just as the proof of the theorem is "justified" by appealing to a previously given definition.

- Gian Carlos Rota

In fact, it is entirely possible to show and axiomatize complexity itself as a measurable entity for which we have notations P , NP etc as being exactly similar to $1, 2, 3, \dots$ as notations for a counting of objects system axiomatized by mankind based on its natural observations of the world around it. Think of our current Peano Axioms for example which apply to the number system as a system designed by mankind coming to an in principle agreement based on the way we began to naturally "count" and "measure" objects around the 20th century. Similarly we can come up with Complexity Axioms as a system designed by mankind in the 21st century by coming to an "in principle" agreement based on the way we began to naturally "sense" differences in complexity of decision problems. In the number system, we have the concept of infinity which is beyond our biological capacity for counting. Couldn't complexity be something that also needs a similar measuring system with another imagined concept of infinity? Clearly we can imagine an infinite number of objects but never be able to count and so can we imagine an infinite level of complexity and never be able to solve. If one studies the history of mathematics, it becomes clear that an interesting idea when proved to be unprovable has a strong tendency towards morphing into an axiom eventually.

What about the Continuum Hypothesis ?

The Continuum Hypothesis very simply put, states that there is not another level of infinity between the infinite set of positive integers and the infinite set of real numbers which Cantor proved earlier to have different cardinalities. Let's apply our simple Eternal Unprovability Filter on again one of the major questions in mathematics – The Continuum Hypothesis.

Is this mathematical conjecture or proven statement completely independent of human evolution ?

Both the set of positive integers and the set of real numbers are creations of mankind. Though admittedly, once created they remain in an independent abstract axiomatic space and can be believed to have an existence of their own. Again, the simple Eternal Unprovability Filter tells us that there is no way to eternally prove that mankind cannot come up with another abstraction for an infinite set with a cardinality between the cardinalities of the set of positive integers and the set of real numbers in its eternal future where its biological capabilities can evolve to be something radically different. Again with the most basic agreed upon principle of the indeterminacy of future human biological capabilities currently non axiomated, the Eternal Unprovability Filter tells us that The Continuum Hypothesis is eternally unprovable.

There's no getting round a proof... even if it proves that something is unprovable!

- Apostolos Doxiadis and Christos H. Papadimitriou, Logicomix

Now, there are very interesting similarities between the P vs NP question and The Continuum Hypothesis

- i. Both have been shown to be independent of Axioms
- ii. Both are not independent of mankind's biological capabilities
- iii. Proofs for both have remained elusive for a long time

- iv. Both questions are trivially resolved as eternally unprovable (though not the question of whether $P = NP$) by the Eternal Unprovability Filter which basically tests for any relation with human biological capabilities
- v. One involves the non existence of an equality and the other involves the non existence of another level of infinity.

"Some people are always critical of vague statements. I tend rather to be critical of precise statements:

they are the only ones which can correctly be labeled 'wrong.'"

- **Raymond Smullyan**

The Questions that drive us

In mathematics the art of proposing a question must

be held of higher value than solving it.

- **George Cantor**

The question now is whether the non existence of something connected to human biological capabilities being eternally unprovable is a general reality of our world. Another interesting question is whether the reason the proofs for these have remained elusive has some kind of causality with this underlying reality and could we come up with a proof, taking cognizance of these intrinsic contradictions. The question is whether there could be some in principle agreement on some of the ideas here as something that could lay the foundations of a new kind of axiom.

Again it's not like every conjecture unproven till date is eternally unprovable by this logic. For example Reimann Hypothesis and all other major open problems firmly remain in their fully abstracted axiomatic space independent of human evolution and hence are not deemed eternally unprovable using this filter.

Epilogue

The folly of mistaking a paradox for a discovery, a metaphor for a proof, a torrent of verbiage for a spring of capital truths, and oneself for an oracle, is inborn in us.

-Paul Valery

Primarily because of their definitions being on the edges of mathematics, $P \neq NP$ and the Continuum Hypothesis can never be proved to be eternally true as the indeterminacy of human evolution in the eternal future will preclude any proofs that claim eternal truth and never be convincing about their eternity. A proof has to be eternal or it is not a proof and eternity is possible only in an independent axiomatic abstract mathematical space with no definitions involved in the conjecture with any direct relation to human biological capabilities.

In part II, I attempt to use the language of mathematics to communicate the concepts in this paper along with another philosophical attempt to look for a relation between the idea of the Eternal Unprovability Filter, Godel's Incompleteness Theorem and Tarsky's Undefinability Theorem. As you will probably acknowledge, if there is actually a need for part II then a lot of people should be writing it.