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S.I.: INFINITY

In search of $\aleph_0$: how infinity can be created

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Abstract In this paper I develop a philosophical account of actual mathematical infinity that does not demand ontologically or epistemologically problematic assumptions. The account is based on a simple metaphor in which we think of indefinitely continuing processes as defining objects. It is shown that such a metaphor is valid in terms of mathematical practice, as well as in line with empirical data on arithmetical cognition.

Keywords Infinity · Aleph-null · Arithmetical cognition · Metaphor · Process · Object

1 A Short history of the philosophy of $\aleph_0$ and an epistemological thesis

The history of infinity goes much further, but when it comes to the philosophy of mathematics, one standard place to start from is Aristotle. In *Physics* (Book 3, chapter 6) he writes:

> For generally the infinite has this mode of existence: one thing is always being taken after another, and each thing that is taken is always finite, but always different.

With this understanding of infinity, it becomes evident that the set of natural numbers—defined with the usual Dedekind-Peano axiom (or equivalent) stating that for each natural number n, its successor $n + 1$ is also a number—must be infinite. But as can be seen from Aristotle's formulation, he is not describing a *set*. Rather, he is describing

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a *process*. In this paper, I will argue that this distinction between objects (like sets) and processes is at the very heart of how we acquire the concept of infinite—among many other concepts—in mathematics. But let us first see how that distinction came to be widely ignored.

Aristotle's focus on infinite processes is not a coincidence, as he makes a clear distinction between *apeiron dunamei* (the *potential* infinity) and *apeiron hos aphorismenon* (the *actual* infinity).[1] Aristotle only believed in the former and his work was followed by more than two thousand years of tradition in which infinity in mathematics was generally considered to be potential. As is well known, all this was changed by Cantor (1892), who treated infinite sets as actual and showed that there are infinities strictly larger than others. Such was the break in tradition that he had to invent a new term for the numbers defined by infinite sets. Hence we still talk about *transfinite* ordinals and cardinals. The lowest transfinite cardinal is of course called $\aleph_0$ and it is the cardinality of the natural numbers.

While intuitionists questioned the use of actual infinities in mathematics, the result of the foundational debate was that by and large mathematicians chose to follow Cantor. However, the widespread acceptance of actual infinity among mathematicians has come with a price, albeit a philosophical one, rather than mathematical. The solution most mathematicians seem to be content with is to simply ignore the philosophical concerns involved with actual infinities. But it must be remembered that Cantor made a radical step. It is easy to agree with the Aristotelian tradition that there is such a thing as potential infinity, as it fundamentally means nothing more than accepting that some mathematical processes are unending. It is quite another thing to claim that infinite sets actually exist.

Mathematicians in their work mostly ignore this question. They often talk of *completed* or *definite* infinity, perhaps partly in order to circumvent the ontological question. But in the philosophy of mathematics the matter is not that simple. For (Cantor 1932, pp. 395–396), actual infinity was not just some metaphor he used for hypothetical pursuits involving transfinite cardinals and ordinals. He thought that his transfinite numbers were "forms or modifications of the actual infinite".

Ultimately, Cantor seems to have believed that his analysis of the infinite leads in one way or another to God, which also likely led to his theological pursuits later in life. Although we may feel that this connection is feeble, it must be noted that Cantor tackled an important issue. If we use actual infinities in our mathematics, surely we must in philosophy strive to explain their ontological status. Cantor gravitated toward God, but what is the fundamental difference to various platonist explanations for the actual infinite? Where do the actual infinite sets of numbers exist?

In addition to that ontological question, the problem of epistemic access to infinity looms large. If actual infinities exist, how can we get knowledge of them? From that background, it makes sense to pursue the option without those ontological or epistemological problems: that there exist no such sets. The concept of mathematical infinity is simply something we have created ourselves. But that is also an ontological claim and it seems needlessly limiting. In addition, the epistemological problems

[1] *Physics* 208a6.

often seem to be unrelated to any metaphysical questions about mathematical objects. Trying to refute platonism on epistemological grounds, for example, seems as futile a pursuit as showing that Cantor's God does not exist. Hence, to avoid commitment to ontological positions, I start the argumentation here from an epistemological thesis: if there are actual infinities in the world, we cannot have epistemic access to them.

In what follows I will try to show the plausibility of this claim, but ultimately it will remain to be a thesis. There is currently no way of showing that there is, for example, no special epistemic faculty that mathematicians apply when they study transfinite cardinals and ordinals. But what I hope to show here is that no such epistemic faculty or heavy ontological assumptions about mathematical objects are needed for a perfectly satisfactory account of mathematical infinity. Once we understand the talk about actual infinities in a proper metaphorical sense, we can build a coherent and epistemologically plausible account of transfinite numbers based on the quite reasonable premise that all mathematical infinities are potential.

Let me briefly defend this thesis before we begin. It in no way purports to deny the *use* of actual infinities in mathematics. Although finitist mathematics is still an interesting endeavor, in this paper I am not out to revise mathematics. The theory of transfinite cardinals and ordinals is coherent, mathematically fruitful, and illuminates our understanding of the concept of infinity. But at the same time, there is absolutely nothing in it to suggest a literal philosophical reading of *actuality*. We can take, for example, modal or constructivist approaches to mathematics and give corresponding treatments of transfinite numbers.

Platonism may be losing popularity and as such the thesis here is hardly revolutionary any longer. But of course there is one plausible case in which infinity could be actual rather than potential: if our universe is infinite. Would we not have a case of actual infinity right there? Naturally this is a possibility, which is why I must stress that the starting thesis here is an epistemological one. Even if the universe were infinite, the thesis states, that is not where the concept of infinity in mathematics comes from.

At this point it should be quite reasonable to assume that we do not have epistemic access to the universe as a totality. Thus, even if there were infinite sets, all our mathematical considerations about infinity must be based on reasoning about finite collections. It is by considering finite collections of natural numbers that we establish that there are potentially infinitely many of them. In short, whether there are infinite collections somewhere in the universe or not, we do not discover them. We may succeed in *characterizing* them through our mathematical work of determining the properties of abstract structures, but in essence, we have created infinity—starting from $\aleph_0$. That is the main premise of this paper and it will be defended as the argument goes along. Let us now see how this creation can take place.

2 How the human mind can create numbers

If we claim that all mathematical infinities are man-made, surely the system of natural numbers—being infinite—can in some relevant way be created by us. While the most common position in the literature has historically been that the human mind does *not* create natural numbers, in recent times there has been considerable popularity for approaches that challenge this position. Perhaps the best-known contemporary account is Kitcher (1983), which takes mathematics to be ultimately empirical.[2] At the root of his theory is the conviction that mathematics is "an idealized science of operations which we can perform on objects in our environment" (p. 12). More recently, Lakoff and Nunez (2000) have made an effort to explain how this science has developed. Their work is controversial, but at the same time it has gained status as a standard work for an empiricist explanation of the development of mathematics.

The above quotation of Kitcher is so easy to accept as to appear trivial—as indeed are most of the explanations of Lakoff and Núñez. But the key word in the quote is "can". Without doubt, this is true in the case of arithmetic—and it shows an insight into learning arithmetic. Actions on pebbles, apples or something similar are a natural way to familiarize children with arithmetic. Even the most avowed platonist will accept that arithmetic can be learned with empirical methods. But he will—as Frege (1884) already noted in his *Grundlagen*—insist that that is not what arithmetic is *about*. We can learn mathematics by idealizing on the operations in our environment, but the real question is *why* we can do that? What is the connection between the abstract numbers and our everyday operations with physical objects? That is a key question also with the present approach when we try to explain the nature of mathematical infinity. To know what $\aleph_0$ is about, we need to first know what natural numbers are about.

Such questions used to belong almost exclusively to the domain of philosophy, but nowadays we also have a solid body of relevant empirical data. While data from cognitive science, psychology and neurobiology cannot—at least yet—give us a philosophically satisfactory answer to what natural numbers are, there are many ways in which such empirical data can already have philosophical importance. If we get empirical insight into how number concepts are formed, we can gain important knowledge about the epistemic access to abstract mathematical objects. Particularly relevant would be empirical clarification of the move from playing around with pebbles into understanding that the sequence of natural numbers can be continued indefinitely. If we could understand this process, it seems that we would be close to explaining how we can create infinity.

Let us see what the best empirical data currently says about such questions. Perhaps the most important insight we have gained from the data is that processing observations in terms of quantities is not an ability exclusive to human beings with developed linguistic abilities. There is a great deal of evidence of animals processing observations based on the quantity of objects. This has been detected in primates like chimpanzees, but also in rats and small fish (Rumbaugh et al. 1987; Mechner 1958; Mechner and Guevrekian 1962; Church and Meck 1984). It has been established, for example, that mosquitofish can learn to choose the right hole to go through based only on the number of objects drawn above the hole. Remarkably, the fish were able to make the distinction even when the combined surface area and illumination of the objects were kept constant (Agrillo et al. 2009).

Such abilities have also been detected in human infants. The famous experiment of Wynn (1992) showed that infants reacted to the unnatural arithmetic of $1 + 1 = 1$ in

[2] Historically, the most famous empiricist account of mathematics is Mill (1843) *System of Logic*.

experimental settings. When they saw two dolls being placed behind a screen—one by one —, they expected to see two dolls and were puzzled when there was only one—the other having been removed without the child seeing. This tendency has later been established to occur in many variations of the experiment: including ones where the size, shape and location of the dolls are changed. The infants still found the changing quantity most surprising (Simon et al. 1995; Dehaene 2011, pp. 40–48).

With such results and many others, I believe it has been established that competence with quantities is not only the language-dependent developed human ability it was long thought to be.[3] However, it is important to remember that such basic abilities to estimate and process quantities should not be confused with the arithmetic of natural numbers as we understand it. The title of Wynn's paper, for example, was "Addition and subtraction by human infants." But that seems quite misleading as it already presupposes that the infants are dealing with arithmetical operations. There is a perfectly valid alternative explanation that the infants were keeping track of the quantity of the objects they expect to see, thus holding only one numerosity in their minds. Postulating the ability to do arithmetical operations assumes a much more developed ability in the infants than necessary. This is an important danger to acknowledge. Primitive abilities to deal with numerosities should be considered relevant when we study the origins of mathematical knowledge, but we must not confuse that with developed arithmetical thinking. That is why the infant and animal abilities are better described as *proto-arithmetical* (Pantsar 2014).

If not arithmetical operations, what are the infant and animal abilities to deal with numerosities? The ability used by the infants in Wynn's experiment is called *subitizing* (or *object tracking*) and it enables determining the amount of objects in one's field of vision without counting. This ability works only for small quantities, usually three or four, but at most five (Dehaene 2011). But the ability with numerosities does not stop there. For larger quantities, there is an estimation system in place. This system is in cognitive science often called the *analogue magnitude system* or the *approximate number system* (ANS) (Brannon and Merritt 2011; Nieder and Dehaene 2009). Unlike subitizing, the ANS can be used for quantities beyond four or five.[4] When dealing with more than three or four objects, however, the ANS quickly starts to lose accuracy (Xu et al. 2005; Dehaene 2011, pp. 17–20). It is important to recognize just how different the proto-arithmetical system of ANS is from developed arithmetical thinking, which seems to require a certain level of verbal competence (Spelke 2011). But at the same time, there are good reasons why among the empirical researchers subitizing and the ANS are generally thought to be the precursors to actual arithmetical ability.

To see this, we must start from the very beginning. What happens in the brain when we observe our environment? The full story is long and complicated, but the philosophically important main idea is that the brain operates through many different

[3] For more extensive information about the state of empirical study of numerical cognition, Nieder (2011), Dehaene (2011) and Dehaene and Brannon (2011) are good places to start. For a more detailed philosophical interpretation of the empirical results, see Pantsar (2014).

[4] It has been suggested that subitizing and the ANS could be part of the same ability, since ANS allows accurate estimation of small quantities. Expirements show, however, that subitizing and ANS have different characteristics, thus supporting the idea that they are separate systems (Revkin et al. 2008).

types of “filters” in the neural activity. When we see something, an enormous amount of activity goes on in the brain in order to gather the relevant information from our observation. This is why it has been so difficult to develop visual recognition in robots. In order to separate the relevant parts of the visual field from the irrelevant, the robot has to be programmed in excruciating detail. Our brain does such things automatically because it is accommodated to recognize the aspects that are important. Crucially to the matter at hand, part of this activity has to do with quantities (Nieder 2011).

We now know quite a lot about those “numerosity filters” (Nieder 2012). At the proto-arithmetical level, our ability with numerosities is properly discrete only for small quantities, after which it becomes increasingly approximate. But subitizing and the ANS can be crucial in explaining the origin of our ability to deal with numerosities. Evidence shows, for example, that there are not only distinct areas of the brain where quantities are processed, but within those areas there are specific sets of neurons which represent certain quantities (Nieder 2011, 2012). When a monkey is presented with two objects, a specific set of neurons activate. When the number of objects is three, a partly different set is activated. The experiments have been controlled for other variables, and the scientists have been able to tease out the effect of a particular quantity in the monkey brain. The brain, however, is a complex organ, and while there are specific neurons for each small numerosity, those neurons do not activate completely discretely. When the neurons for the numerosity “two” are activated, so is a small part of the neurons for “one” and “three”. And just like the behavior of monkeys predicts, as the numerosities become larger, the bigger the “noise” is between the different groups of neurons. Distinguishing between four and five is much more difficult than between one and two because in the former case more of the same neurons activate. Our natural capacity to deal with numerosities is one of approximate estimations that loses accuracy as the quantities become larger. What happens in the brain mirrors this.

Such data strongly suggests that subitizing and the ANS are hard-wired into the brain structure of monkeys. Fortunately, unlike many other animals that show capacity to deal with quantities, monkeys also have the ability—after extensive training—to understand symbols assigned to concepts, including numerosities. What Diester and Nieder (2007) established is that, to a large part, the same neurons in the prefrontal cortex were activated regardless of whether the monkey saw two objects or the symbol 2. So when the monkey learned to use number symbols, it assimilated the new knowledge with the primitive ability it already had about quantities. A similar thing happens with counting. Counting is no doubt fundamental to our developed capacity to deal with numerosities and enables us to formulate natural numbers exactly. In Nieder et al. (2006), monkeys were presented objects one by one, to simulate a non-verbal account of counting. As expected, there were differences in the parts of the brain that activated compared to the task of seeing a group of objects at once. However, the study found that at the end of the enumeration, a large part of the activated neurons were the same as with subitizing.

What do these results suggest our ability with numerosities to be? A plausible hypothesis seems to be that our primitive ability to deal with small numerosities is based on subitizing and the approximate number system. But as we develop the linguistic ability to count, we no longer lean only on the primitive ability in processing numerosities. We never lose subitizing and the ANS, but the primitive systems are increasingly supported by a language-based ability with numerosities. This obviously gives us a lot of added expressive power, in particular in grasping the successor operation, which makes arithmetic as we know it possible (Carey and Sarnecka 2006; Spelke 2011).[5]

These subjects need a lot of further study, but there is starting to be way too strong evidence for a connection between the ANS and our arithmetical ability to be explained away simply as a coincidence. The data clearly points to the direction that our verbal ability to deal with numbers was built to accommodate the primitive non-verbal system. That is of course nothing extraordinary. Although there are different areas of the brain involved in subitizing and the ANS from the ones dealing with counting and recognizing symbols for numerosities, it would be surprising if there were no connections between them at all. The brain in general has evolved to facilitate learning and the existing information is used to assimilate new data. If we have one mode for dealing with numerosities, it would seem unlikely that the brain starts to build another one completely from scratch instead of utilizing the existing connections.[6]

In the case that subitizing and the ANS are indeed the foundation for arithmetical thinking, how does arithmetical knowledge develop? In order to get a conclusive answer we would need to have a much better empirical understanding of the development of mathematical thinking than is currently available. However, there are some empirical results which suggest an answer. The hypothesis that ANS and subitizing are the foundation for arithmetical thinking includes several predictions which have received corroboration from experiments. Studies have shown that we do not lose ANS when we develop symbolic means to deal with quantities (Butterworth 2010; Spelke 2011). In both older children and adults there is a strong correlation between improved performance in symbolic mathematics and the non-symbolic processing of quantities. Furthermore, it is known that the same brain areas activate when dealing with symbolic and non-symbolic representations of quantities and that these areas activate in the brains of non-human primates (Piazza 2010).

Much of the current data suggests a fundamentally simple and coherent picture. Starting very soon after our birth, we have a non-symbolic ability to deal with small quantities. This ability is then developed into arithmetical ability in a process where the development of language is likely to play an important role (Spelke 2011). Exactly how this happens is still largely unknown, but there is evidence that grasping the idea of successor is central in this development. Based on subitizing we have the ability to distinguish between small quantities, usually from one to four. This means that we have different neural representations in our brain for those small numbers. One hypothesis is that children grasp the idea that these numbers form a progression that can be continued (Butterworth 2010). When one is added to three, even an infant can tell that the numerosities are different. But this ability comes from the proto-arithmetical system and it is only later that the child learns that adding one to four is similar; that

5 This account is developed in more detail in Pantsar (2014).

6 Of course one important question to ask is whether the results concerning monkeys can be applied to the human brain. There is a lot of evidence of this. The same areas in the human brain activate as in monkeys (Piazza et al. 2007) and college students have shown similar patterns as monkeys in number-ordering and quantity estimation tasks (Cantlon and Brannon 2006).

there also is a distinct number for the end product of that process (Feigenson 2011). From that there is a short way to addition of two numbers, which in turn enables the process of multiplication and so on.

The above results give us a strong hypothesis of how the human mind can create natural numbers, and importantly, it does not need to involve any special capacity for mathematical knowledge. Only a proper development of our proto-mathematical ability to deal with quantities is required to reach our familiar natural number system. The exact nature of this process is something that empirical research will slowly give details about. It can hardly be expected that in a foreseeable future we will have a full psychological account of how ANS turns into arithmetical thinking, yet already at this point we should in philosophy be interested in possible ways of expanding the picture. If the ANS-based theory is correct, how do we acquire more sophisticated mathematical concepts? In particular, how can the concept of infinity in all its mathematical richness develop from such primitive origins?

3 How the human mind can create infinity

There is of course nothing new in the idea that infinity is created rather than discovered. Indeed, it would seem that every non-platonist account of mathematics that employs actual infinities has to embrace that idea. Still, very few explicit hypotheses about the creation of infinity have been presented. If there is currently something resembling a baseline hypothesis for mind-created infinity, it has to be the one in *Where Mathematics Comes From* by (Lakoff and Nunez 2000, pp. 155–180). They start by rejecting the suggestion that we could explain the notion of infinity in a negative fashion as "not finite":

> ...this does not give us any of the richness of our conceptions of infinity. Most important, it does not characterize infinite *things*: infinite sets, infinite unions, points at infinity, transfinite numbers. To do this, we need not just a negative notion ("not finite") but a positive notion—a notion of infinity as an entity-in-itself. (p. 155, Italics in the original)

With the focus on infinite things, it seems clear that Lakoff and Núñez want to explain the mathematical notion of actual infinity. Their hypothesis is that all cases of actual infinity are applications of one conceptual metaphor, what they call the basic metaphor of infinity (BMI). The general idea of BMI is that processes that continue indefinitely are thought to have an ultimate result. Just like we speak of the final resultant state of a completed iterative process (such as counting to ten), with the help of BMI we can speak of the "final resultant state" of iterative processes that go on and on. For example, the definition that the successor of each natural number is a natural number gives us an indefinite iterative process. Once we realize that the process goes on and on, we no longer expect a final resultant state. Rather, we evoke a metaphorical "final resultant state", which is the concept of actual infinity we use in mathematics. And just like the end process of a completed iterative process, the "final resultant state" is unique and follows every non-final state of the indefinite iterative process. (pp. 158–159).[7]

Many valid criticisms of the account of infinity by Lakoff and Núñez have been presented. Henderson (2002), for example, presents a metaphor for infinity based on projective geometry that does not seem to be an application of BMI. Gold (2001) shows that the way Lakoff and Núñez derive the first infinitesimal number with the help of BMI is inconsistent with the way mathematicians define infinitesimals. The reply of Lakoff and Nunez (2001) makes for an interesting discussion, for they defend that the "granular" numbers that BMI produces—which include the property of having the least infinitesimal number—should be treated separately from formal mathematics. This approach seems to be either mathematically misguided or makes BMI crucially lacking as a metaphor. As Voorhees (2004) suggests, there is a big problem in considering the granulars as mathematical relevant at all if they can contradict with our established mathematical definition of infinitesimals. But even if we grant Lakoff and Núñez their position that the granulars should be considered as mathematical ideas only in a non-formal sense, there remains the problem how these BMI-generated numbers are turned into the mathematical objects we know and use. It seems that either BMI does not do all that it claims to, or does it erroneously.

Even with the above problems, however, the account of Lakoff and Núñez is not without its appeal. It can be easily forgotten just how slippery the concept of actual infinity can be. Lagendoen (2002), for example, in his review of Lakoff and Núñez argues that we do not need a metaphor in order to understand the concept actual infinity:

> However, the result of an unending process can be understood without the use of metaphor. Understanding of ordinary universal quantification is sufficient. One who understands a simple English sentence such as *every number is interesting*, and also understands that there is no end to the number sequence 1, 2,..., thereby understands the concept of actual (denumerable) infinity without the use of metaphor. So the use of metaphor is not necessary for the understanding of such mathematical concepts as absolute infinity. (Italics in the original)

But that argument seems misguided. What is the sequence 1, 2,...in Langendoen's argument? Certainly it cannot be an actually infinite set to start with, since understanding the sequence as actual infinity is the thing he is trying to show. So he must first be talking about a potentially infinite sequence. Since the characterization is that "there is no end" to the sequence, that seems to be a valid interpretation. In that case, however, we are not talking about a sequence as an object, but—as Langendoen mentions in the beginning of the quotation—an unending *process*. This implies that quantifying universally over the members simply means that "every number given

[7] In Nunez (2005) this account is amended and BMI is characterized as a "double-scope conceptual blend", after Fauconnier and Turner (2002), This means that BMI has two input spaces, one coming from Completed Iterative Processes and the other from Endless Iterative Processes. The idea is that that both inputs are used in BMI to reach endless processes with final resultant states. Instead of Basic Metaphor of Infinity, in this updated account BMI stands for the Basic *Mapping* of Infinity. While the new account is conceptually more coherent, I do not see the difference as great enough to require independent treatment. Hence, in this paper I focus on the account of BMI given in Lakoff and Nunez (2000).

by the process is interesting"—clearly understandable as *potential* infinite and thus missing its target.

However, it is not clear why there should be anything wrong in starting from potential infinity. Recall the empirical results of the last section. They suggested that our arithmetic of natural numbers is based on a proto-arithmetical ability on small numerosities, which is then generalized to include larger quantities as we develop sufficient linguistic ability. At the root of this generalization, just like at the root of axiomatic representations of natural numbers, is understanding the concept of successor and its general applicability over natural numbers. In short, that is how we learn to count. We count from one to ten by using the successor operation nine times. It does not take children long, however, to realize that this process can be extended indefinitely. Usually this happens around the age of 8. At that stage, children start understanding infinity. And they understand it by understanding the nature of a *process,* i.e., counting (Falk 2010).

Process clearly appears to be the key concept here. While I do believe that Lakoff and Núñez are on the right track with a metaphorical account of infinity, it seems that their account deals with the wrong metaphor. Where Langendoen's argument above fails is that it suddenly moves from an unending process to an infinite sequence. But this is exactly the step that must be *explained* by an account such as Lakoff and Núñez's. In this step, it seems that the difference between *applying* a process and *understanding* a process is crucial to the matter at hand. A child learns to count at a young age, but it takes considerably longer to understand that there is no end to the counting process. At that latter stage, the child understands something essential about the nature of the process, instead of just applying it.

This step of treating processes as objects of study is of course crucial to practicing mathematics. Take (following Lakoff and Núñez) the Fibonacci sequence as an example. The sequence

$$0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

is defined recursively by the function

$$F(0) = 0, F(1) = 1 \text{ and for all } n : F(n) = F(n-1) + F(n-2).$$

Clearly we are talking about an infinite *process* here, yet in mathematics the Fibonacci sequence is treated as an *object*.

This, I propose, is the key to explaining how we can successfully use infinities in mathematics. We make use of objects defined by unending processes. This also seems to be the foundation of BMI for Lakoff and Núñez. We can talk about infinite processes mathematically, because we use the metaphor of the "final resultant state" for such processes. It seems unclear to me, however, why we need to invoke the problematic concept of "final resultant state." BMI seems to leave unexplained one crucial difference between the two types of iterative processes. When we are talking about a completed iterative process, the final state is actually final. We can reach the number ten by counting from one without any deeper understanding of the process. But we can only understand the "final resultant state" of an indefinite counting process by understanding the infinity of natural numbers. We are talking about two different ways of treating processes here. One where the process is actual counting, the other where we understand the general process of counting. As such, the difference between a final resultant state and "final resultant state" is too vast for BMI to be a satisfactory metaphor.

Fortunately, there is a much more natural explanation available—and one that fits perfectly within mathematical practice. Our mathematical concept of infinity seems to be based on an understanding of indefinitely continuing processes, and the best characterization of such processes would seem to be the one that Lakoff and Núñez initially reject: they are *not finite*.

As we remember, the problem Lakoff and Núñez had with that negative characterization was that it cannot characterize infinite *things*, such as infinite sets and sequences. Of course the whole premise of their—as well as the present—theory is that there are no such infinite things. What they are after is an explanation how we can in mathematics nevertheless *speak* of infinite things in a coherent manner. But for that purpose, I suggest that the negative definition is enough.

Lakoff and Núñez seem to make things too hard for themselves by insisting that we need to have a way of talking about infinite things, when all we need is a way of talking about endless *processes*. Take the Fibonacci sequence as an example. It is obvious that no finite part of the sequence will ever give us the infinite series, nor will it—based on the old "Kripkenstein" argument (Kripke 1982)—unassailably give us the rule the series follows. But of course the recursive definition we gave above *does* give us the Fibonacci sequence unequivocally. What it does not give us, however, is the series as a *thing*. It gives us the series as a process, which is clearly not finite.

What are we doing when we talk about actual infinities in mathematics? The simple explanation is that we are treating the things defined by such potentially endless processes as objects of mathematics. Let us call this the "Process → Object Metaphor" (or POM). Clearly POM is a metaphor: unending processes are not objects and we do not need to understand things defined by such processes as literally existing. Just as clearly, it is a widely used metaphor in mathematics. We constantly talk about objects defined by potentially infinite processes, like in the case of the Fibonacci sequence.

But what is POM based on, philosophically speaking? It fundamentally includes nothing more than Aristotle's idea of potential infinity. As we remember, Aristotle described potential infinity as a process, whereas Cantor's approach was to treat infinite sets as actual. But if it is established that no mathematics is lost by taking Cantor's infinities to be metaphorical, we should strive to return to Aristotle's ontologically and epistemologically less problematic characterization of the infinite.

In the Fibonacci case, we only used the understanding that the Fibonacci sequence, defined recursively as above, is endless, i.e., *not finite*. The recursive mode of presentation of course conforms to mathematical practice and under a reasonable interpretation it is metaphysically unproblematic. We see that the Fibonacci sequence as defined by the recursive process cannot be finite. But there is no problem in having unending processes. Understanding that the Fibonacci sequence is infinite does not involve a metaphor of an infinite "final resultant state", either. It only involves the realization that the process is *not finite*.

These are the two things we need in order to make sense of potential infinity. We have a recursive definition for a sequence (or a set) of natural numbers. The definition implies that the recursion is endless. As a conclusion, the sequence is not finite. In order to speak of this sequence as an object—that is, to move to actual infinity—we only need a third thing: the Process $\rightarrow$ Object Metaphor.

In a nutshell, the theory here is that three things are needed in order for us to reach $\aleph_0$, the cardinality of the lowest actual infinity:

1. A notion that some recursive processes can be continued indefinitely.
2. A way of defining such recursive processes in finite steps (e.g., a basic understanding of an axiomatic system or implicit definition).
3. Treating things defined by such potentially endless processes as objects of mathematical study.

With the first two points we reach potential infinity in the mathematical sense and with the third we move to the mathematical concept of actual infinity in an unproblematic way. This account is much simpler than the one of Lakoff and Núñez and seems to correspond more naturally to mathematical practice. To give an example, POM clearly captures how infinity is introduced to set theory. In standard set theory, the axiom of infinity gives us the existence of the set of natural numbers (the inductive set):

$$\exists S(\emptyset \in S \land \forall x \in S((x \cup \{x\}) \in S).$$

What the inner part of the formula describes is clearly a *process* of forming new sets from previous ones—and just as clearly the process is an unending one. The axiom simply tells us that the set defined by that process exists. This is move from a process to an object in exactly the kind of way that POM suggests.

POM also captures the way mathematical concepts are often taught. Functions, for example, are taught as *picking* one element from the image of the function for every element of the domain. Another way to teach functions is to see the elements of domain as the *input* and the elements of the image as the *output*. Both of these ways speak of functions as processes and it is only much later that the students learn about treating functions as sets of ordered pairs, i.e., as objects.

The same applies to many important mathematical concepts in different fields. In analysis, for example, convergence is a fundamental concept. Already in the terminology it becomes obvious that processes are the key to convergence. A sequence is said to *converge* and the numbers are said to *approach* the limit of the sequence. But if a sequence is a static object consisting of numbers, that kind of terminology cannot make sense—unless it is meant to be metaphorical. Numbers do not actually approach anything. The terminology not only makes sense to us, but it is also the one that mathematicians everywhere use. I believe the reason for this is easy to explain with the help of POM. We do not see sequences (or functions) only as static objects. Often we see them as dynamic processes. As such, it is natural to say that the numbers (or values of a function) approach a limit: it corresponds to an intuitive understanding of sequences (or functions). For example, when we use the (ϵ, δ)-definition for the limit and convergence of a function, we treat the function as an object. But while this method was an important advance mathematically, it is also notoriously hard for many students to comprehend. However, treating the function as a process—by drawing a graph, for example−helps students understand the nature of convergence (Dawkins 2014). When they gain knowledge of the process, they also understand the object better−just as POM predicts.

In these four examples (the Fibonacci sequence, the axiom of infinity, functions and convergence) we see the power of the simple metaphor that is POM. In none of the examples did we take a well-known mathematical concept and use an arcane way of using processes to define or teach them. Instead, we have conformed tightly to mathematical practice. Recursive definitions like the one for the Fibonacci sequence are used because they capture the process of formulating new members of the sequence. The axiom of infinity encapsulates the idea behind infinity: an indefinitely continuing process. When functions are taught as processes, we are immediately aware of their purpose in mathematics. Functions are not meant merely to be sets of ordered pairs; they are meant to connect the domain with the image in some mathematically interesting way. And when subjects like limits and convergence are taught in terms of processes, students find them easier to understand.

Understanding processes appears to be at the heart of mathematics at every stage, from learning to count to providing definitions of formal theories. But why is this the case? In the psychology of mathematics, there is a widely accepted principle, due to Piaget (1970), that the nature of a mathematical concept is closely connected to its development in the individual mind. In philosophy, such principles have traditionally found little support. Ever since Frege (1884), it has been commonplace to separate the genealogy of mathematical concepts (whether historical or individual) from their nature. That kind of thinking, however, presupposes that mathematical concepts can have meanings that are independent of the work of mathematicians. While I do not want to deny that possibility, it seems plausible that the meanings of mathematical concepts are generally tightly connected to their development, and their use in mathematical practice. When we see mathematics this way as essentially a human endeavor, we see why processes play such a major role in mathematics. Human beings are active agents that learn new things dynamically. From this background, it is not surprising that recursive definitions, for example, play such a major role in mathematics. They capture processes, and we associate mathematics with processes ever since we learn to count with our fingers.

4 Comparison to other accounts

POM seems to be a simple metaphor that corresponds generally to the way we often reason in mathematics and particularly to the way we first reason about infinite mathematical objects. But what kind of a metaphor is it conceptually speaking? The use of metaphors in language has been an active field of study in the recent decades. One of the most important theories in the area—usually called the *Conceptual Metaphor Theory*—was presented in Lakoff and Johnson (1980). In Lakoff (1993), this account of conceptual metaphor is developed further and its influence is crucial in the metaphorical treatment given to mathematical infinity in Lakoff and Nunez (2000).[8]

In conceptual metaphor theory, metaphors are thought to consist of three main components. First is the *source domain*, the conceptual domain of the metaphorical expressions. Second is the *target domain*, the conceptual domain we try to clarify with the metaphors. In one standard example of conceptual metaphor theory, "love is war", *war* is part of the source domain and *love* part of the target domain. The third part is a *mapping* between the two domains, a partial set of correspondences between elements from the two domains. This mapping should preserve the structure of the elements of the source domain in elements of the target domain. Obviously the mapping does not need to be a function: it does not need to map every element of the source domain to the target domain. The idea is that a metaphor properly used will enable us to use structure of the source domain to explain something about the concept of the target domain. In the standard example above, our knowledge about war is supposed to help us understand the nature of love. While there are many important questions concerning the technicalities of such a mapping, in this context we do not need to go further into them. The main idea is that being familiar with a metaphor means being familiar with the mappings between the source and target domains.

How should POM be characterized in terms of conceptual metaphor theory? The source domain obviously consists of mathematical concepts that refer to *objects* such as sets, sequences, etc. We use these expressions in mathematics without necessarily believing in their objective existence. In the case of the Fibonacci sequence, for example, we are used to speaking of an object, while what we are actually discussing is a process. Hence, the unending process of creating new members of the Fibonacci sequence is the concept belonging to the target domain. The metaphor of POM is successful if the inferential structure of sequences is preserved in the inferential structure of unending processes.

Is this the case? It is hard to think of a better example of a metaphor being successful in terms of the conceptual metaphor theory. We have seen how objects like infinite sets and sequences are defined via processes. This way, POM clearly corresponds to mathematical practice. The source and target domains of the metaphor are also clearly specified. We would not accept a mathematical process unless it can be explicitly stated, and the same applies to definitions of mathematical objects. Furthermore, while we may disagree about the existence of mathematical objects like sets and sequences, mathematicians do not disagree that sets and sequences can be characterized in terms of processes. And if in any discourse the mappings used in a metaphor are known to preserve the inferential structure, this happens in mathematics where the correspondences are explicitly stated as definitions.

Understanding the Fibonacci sequence, for example, means understanding the process of creating the next number in the sequence. And when we understand that process, we understand that it can be continued indefinitely. When this is done, we can introduce a recursive definition for the Fibonacci sequence. Importantly, that is essentially *all* there is to understand about the Fibonacci sequence as a process. Under-

[8] The Conceptual Metaphor Theory has received a lot of criticism, as well. See Kovecses (2008) for a review of some of the most important problems and a proposal for a solution.

standing the Fibonacci sequence as an object, in turn, only adds the use of the Process $\rightarrow$ Object Metaphor.

So far we have focused on metaphorical accounts of infinity, whether POM or the Basic Metaphor of Infinity of Lakoff and Nunez (2000). But not all cognitive accounts of infinity involve the use of metaphor, at least not explicitly. The BMI may be the best-known effort to explain how the embodied mind can create the concept of infinity, but it is by no means the only interesting one. Related to POM, in mathematics education the importance of processes in mathematical thinking is often seen to be central. Gray and Tall (e.g., Gray and Tall 1994; Tall 2008) have characterized this as *proceptual* thinking. Roughly put, a "procept" refers to the way we use mathematical symbols to refer to both processes and concepts. More specifically, an *elementary procept* is said to be an amalgam of the process which produces a mathematical object and a symbol which is used to represent either the process or the object. A procept is then defined as a collection of elementary procepts which have the same object. Under this characterization, natural numbers are elementary procepts. The symbol "3" refers both to the object (the number three) as well as the process of counting from one to three. But we can talk about 3 also as a procept, since it has the same object as, for example, as the procepts $2 + 1$, $3 \cdot 1$ and $6/2$—as well as counting to three. In this way:

> The symbol "3" inextricably links both procedural and conceptual understanding. But conceptual understanding implies that the relationships inherent in all of the different components that form 3 are also available (1 and 1 and 1; 2 and 1; 1 and 2; one less than 4 etc). The symbols $1+1+1$, $2+1$, $1+2$, $4-1$ all have output 3 and together form part of the procept 3. All these different proceptual structures allow the number 3 to be decomposed and recomposed in a variety of ways either as process or object. In this way the various different forms combine to give a rich conceptual structure in which the symbol 3 expresses all these links, the conceptual ones and the procedural ones, the processes and the product of those processes. This combination of conceptual and procedural thinking is what we term proceptual thinking. (Gray and Tall 1994, p. 123)

Gray and Tall argue that it is the flexibility of procepts that make them so useful to mathematical thinking. When children learn that mathematical symbols can be seen as objects that can be decomposed and recomposed, they become more competent mathematically.

Philosophically, the procept of course raises one crucial question: what is the nature of the mathematical objects that help form the amalgam of elementary procept? Gray and Tall want to remain somewhat ambivalent on the issue, which is understandable since their focus is not a philosophical one. Obviously there is a great deal of mathematical agreement on what we understand by, say, the object "three", even if our philosophical leanings may differ. But in a philosophical account we need to be more specific, because there is the problematic scenario that mathematical objects are *nothing* beside shorthand for mathematical processes.

This may not seem particularly problematic in the case of finite numbers, as we can easily agree that the strings of symbols $1 + 1 + 1$, $2 + 1$,$1 + 2$ and $4 - 1$ all refer to the same number. But what is that number, beyond using the successor function on the number 0 three times (or two times on the number 1)? If we can get a full understanding of numbers as processes, where do objects enter the picture? Indeed, can we understand procepts to be separate from mathematical objects (if any)?

It seems that the procept account in no way implies that mathematical objects play a role in mathematical thinking. Rather, it works equally well under the interpretation that all we have in mathematics are processes, and any talk of objects is purely metaphorical. $2 + 5$, counting to seven, $10 - 3$ and $(10 + 4)/2$ each form part of the procept 7. But assuming that the procept 7 has an existence outside of these (and the other) processes is not necessary. Understood this way, the procept sounds very much like our Process $\rightarrow$ Object Metaphor. At least a considerable part of mathematics seems to be about treating the end products *as if* they were independently existing objects, but it is hard to see why we should assume any other domain for mathematical objects. If that is indeed the case, the procept has the troubling potential to be an amalgam of a process and an object, in which the object is actually a postulate reached from the process by a metaphor. It seems that ultimately one part of the amalgam, the object, would be redundant.

Aside from such general considerations, we must ask how $\aleph_0$ specifically should be understood as a procept? If we see it in terms of the Process $\rightarrow$ Object Metaphor, we understand that in mathematics it is advantageous to speak metaphorically about the objects defined by unending processes. I claim that this is the *only* way the concept of procept makes sense when it comes to transfinite cardinals, without assuming a heavy platonist ontology. The talk of objects makes sense to us when it comes to something like the number three. We have an understanding of three objects that is tied to our primitive proto-arithmetical ability. I have argued that we develop arithmetic based on that ability. Although strictly speaking it may be ontologically false to say there are mathematical objects like natural numbers, there at least would seem to be something objective (or at least maximally intersubjective) that small natural numbers refer to.

However, what is our understanding of $\aleph_0$? Do we somehow acquire the concept of countable infinity without recourse to unending processes? If so, how is this done? Is there a mathematical object that the cardinal $\aleph_0$ refers to and to which we have some sort of epistemic access? Such questions seem ontologically quite problematic. Fortunately, with the Process $\rightarrow$ Object Metaphor there is no need to speak of objects like $\aleph_0$ outside of the unending processes that define them. With that metaphor, the concept of procept makes perfect sense. That does, however, come with a price: the talk of objects in defining a procept is now made redundant. What a procept is under this new interpretation is merely a combination of processes and understanding those processes metaphorically as objects. The procept may be an illuminating concept to use in many cases, but without something like the Process $\rightarrow$ Object Metaphor, it leaves important matters unexplained. With the Process $\rightarrow$ Object Metaphor, however, the need for the concept of procept is questionable, as we immediately recognize the importance of processes and metaphorical thinking in creating mathematical objects.

Treating processes as objects is by itself nothing new in the study of mathematical thinking, and the procept is not the only account that applies this duality. Dubinsky (1991), for example, includes something similar to POM in his Piaget-influenced account of mathematical thinking. For Dubinsky, as for Piaget (e.g., Beth and Piaget 1966), the key to mathematical thinking is reflective abstraction. As one part of this theory is a four-fold image of learning mathematical concepts, the so called APOS model (Dubinsky and McDonald 2002; Dubinsky et al. 2005a, b). The first level is by *action*, e.g., trying out single cases. Second level is understanding the general *process* by interiorizing the actions. The third level is encapsulating the process as a cognitive *object*. The fourth and final level consists of organizing the actions, processes and objects into a *schema*. For the present purpose, it is the second and third levels that interest us the most. The connection between processes and objects goes both ways: sometimes it can be beneficial to treat objects as processes. Dubinsky (1991) uses functions as an example of a mathematical concept that we initially grasp as processes but start treating as objects. Indeed, it is something we *must* do in order to include sets of functions and quantify over them.

In this way, it should be stressed that as such, there is nothing new in emphasizing the importance of the process-object duality in mathematical thinking. Of course that duality is ubiquitous in teaching mathematics, whether elementary of more advanced. In addition to functions, the axiom of choice, for example, is almost always presented informally in terms of making selections—of course in addition to a varying degree of formal set theory. Accounts of infinity like the one given in Dubinsky et al. (2005a, b) closely resembles the approach here. It rejects the BMI-based theory of Lakoff and Núñez and especially its problematic feature of a "final resultant state" which, among other things, can lead to the kind of misunderstandings of infinity we hope to avoid, such as seeing infinity as the largest number. I believe that Dubinsky et al. argue convincingly that infinity should be taught and is best understood by treating endless processes as totalities.

But what I have argued for in this paper is something much stronger. I do not believe that the Process → Object Metaphor is merely a useful tool in teaching and understanding mathematics. I believe it is at the heart of a plausible theory of what many mathematical concepts actually *are*.[9] There is a clear sense in which processes are primary to objects.

Specifically to the matter at hand, I believe that POM captures the reason why infinity in mathematics and the whole theory of transfinite numbers makes sense. In a loose way, the account here could be seen as a hybrid of the views of Lakoff and Núñez on the one hand, and Gray and Tall, Sfard and Dubinsky on the other. It seems to me that Lakoff and Núñez are on the right track in emphasizing the importance of metaphorical reasoning in the development of the concept of infinity. It also seems to me that Gray and Tall, Sfard and Dubinsky make a crucial point in emphasizing—to different degrees—the importance of the process-object duality for mathematical concepts. I hope that the current account catches the best features of both approaches.

Finally, if we are trying to find out the metaphors behind the development of mathematical knowledge, it seems unlikely that they are specific to developed concepts like infinity. If the BMI of Lakoff and Núñez plays a role in acquiring knowledge of transfinite cardinals, it is plausible to think it is based on a more general and basic metaphor. A candidate with much potential to be that metaphor can be found in POM. Researchers in mathematics education have emphasized the importance of project-

[9] In this regard, the closest relative to the approach here is probably the account of Sfard (1991), who also emphasizes the primary nature of processes in the process-objects duality.

object duality, without doubt for a good reason. But the real philosophical question is *why* such duality should play an important role? The answer implicated by POM, that mathematical concepts are in fact often constructed metaphorically as objects in terms of processes, seems to make a great deal of sense.

5 Developing the hypothesis

Although POM seems to be more intuitive and a better fit with mathematical practice than the BMI of Lakoff and Núñez, there is one way in which their account seems to be more fitting. That has to do with the most common *error* people commit with infinities. The notion that some recursive processes can be continued indefinitely is so elementary that it hardly needs an explanation. By understanding the process of counting, children early on start to realize that the set of natural numbers cannot be finite. Of course they do not talk about sets and their understanding of infinity is often flawed. In the classical image of the playground understanding of infinity, infinity is seen as smaller than "infinity plus one," let alone "infinity plus infinity". Similar confusions about infinity continue to exist in the minds of students throughout their schooling (Monaghan 2001; Singer and Voica 2008). One of the most common mistakes is to think of infinity as something like large finite numbers (Pehkonen et al. 2006). Ironically, this confusion seems to arise when students learn about the concept of infinity. First-year university students, for example, have been found to think of infinity quite commonly as the largest number (Tall and Schwarzenberger 1978). A child can understand that there is nothing special about counting to, say, hundred. The process will continue similarly regardless of how big the numbers are. Indeed, school-aged children commonly think of infinity in terms of processes (Tirosh 1999; Monaghan 2001). However, when they learn about the concept of infinity, they may revert to thinking about it as the largest number (Pehkonen et al. 2006; Falk 1994; Sierpinska 1987).

With their account of BMI, Lakoff and Núñez are perhaps better able to explain this latter mindset. Although they of course make it clear that infinity is not in fact a number, they (p. 166) use BMI to explain how people get the idea of largest "integer" ∞, which is understood as a number but which cannot be used for calculations. But it is hard to see the philosophical value of using BMI to explain such a confused concept of infinity. Of course as cognitive scientists their job is also to explain such errors, but even for that purpose, I believe a much simpler explanation exists: people think of infinity as a number mostly because they are *miseducated* about it. There is empirical evidence of this. In Tsamir (1999) it was found that it is common for prospective teachers to speak of infinite sets as having the properties of finite sets. That POM does not give us a way to think of infinity as a largest number should not count against it, especially since thinking of infinity as a process may be the natural way for children to understand it, as suggested by the above studies of Tirosh and Monaghan.

But while POM certainly seems plausible, is it the *actual* metaphor we use in mathematics? While POM seems to capture the essence of the way we use processes to define objects in mathematics, do we know that it is the actual principle used, or just one explanation? Do we even use metaphorical thinking, whatever it ultimately is? It should be clear that we are far from a level where we can study the actual neural processes involved in forming a concept of infinity. Hence all pursuits like the present one are for a large part hypothetical. Nevertheless, the account based on POM seems to carry many advantages. Lakoff and Núñez were not content with the negative "not-finite" characterizations of infinite, because those could not apply to infinite things. The approach here does not have that problem. As we have seen, the existence of the denumerably infinite set is postulated in set theory in a manner equivalent to POM. The same applies to any recursively definable infinite set with the cardinality of $\aleph_0$. Such a definition consists of rules formulating the next member of the set in terms of the previous ones, that is, it defines a process. With POM we make the jump to the set defined by that process and hence have the tools to talk about infinite things. But all the while the account is based only on the negative "not-finite" understanding of infinity.

With that understanding of infinity come perhaps the greatest strengths of the present account—aside from fitting well with mathematical practice—which are epistemological and ontological. No special faculty for mathematical knowledge is postulated and arithmetical knowledge can be based completely on our experience with finite collections. No infinite amount of things is assumed, let alone some kind of epistemic access to infinite collections. We do not need to assume the independent existence of any mathematical objects if we see them metaphorically as defined by processes. Yet absolutely nothing is taken away from mathematics. The account conforms to the way denumerably infinite sets are defined. The cognitive details remain to be found out empirically, but there is an inherent plausibility in an account that takes mathematical objects to be what mathematicians describe them to be. However we understand those objects philosophically—and it should be remembered that the current account is perfectly compatible also with platonist explanations—mathematics is largely about using finite definitions to describe infinite things. That is precisely at the heart of the POM hypothesis.

In order to be mathematically sufficient, however, POM should be able to explain cardinalities greater than $\aleph_0$. That is without doubt one of the most important questions in developing the present theory further. A detailed account of that will demand another paper, but there is nothing to suggest that it cannot be done. The key, most likely, is in multiple uses of the Process $\rightarrow$ Object Metaphor. Cantor's diagonal argument, for example, can be easily explained in terms of POM. To mirror Cantor's (1892) original argument, we first use POM to give us the set of all infinite sequences of binary numbers. This is a paradigmatic case of POM use. Cantor in his paper begins by showing a few examples of the process, which is used to characterize the object, that is, the set of all infinite binary sequences. Then he famously describes the process of formulating a number which is not part of this set—another use of POM. With indirect proof Cantor then concludes that the set of all binary sequences can not be countable, that is, the process of enumeration does not give every infinite sequence of binary numbers. The Process $\rightarrow$ Object Metaphor can be seen many times in Cantor's argument.

Of course the diagonal argument is just an elementary result when it comes to transfinite cardinals, and there are many uses of infinity in mathematics that require their own treatment. The question of infinite ordinals is also something that cannot be dealt with in this paper. But POM is clearly a metaphor that can be used multiple times, in a manner similar to Lakoff and Núñez (p. 160) using BMI multiple times when going beyond $\aleph_0$. This way, higher infinities may not need to be treated actual in a literal sense any more than the denumerable infinity is.

Admittedly, just how POM (or indeed any explanation of infinity) can be used to explain all the uses of higher infinities is an open question. I do not wish to claim that POM provides an account that rescues all the uses of actual infinity in mathematics. But there should be no controversy in trying to develop the account further. If we can have a satisfactory metaphorical reading of denumerable infinity, it would be strange to stop pursuing the metaphorical path for higher infinities. To echo the idea behind Kronecker's famous quip about God only creating the integers: if there is one infinity that mathematicians would be ready to claim to exist, surely it is the denumerable one. A satisfactory metaphorical reading of the denumerable infinity seems like a good starting point for approaching higher transfinite cardinals and ordinals. In any case, POM, unlike the BMI of Lakoff and Núñez, is not a metaphor exclusive to questions regarding infinity. As such, the evidence for it should be expected to come widely from mathematical practice. Above I have mentioned a few examples, but this work should be expanded on.

But can we conceive of mathematical practices which would count as evidence *against* POM? This is a very interesting question. Since POM, if the hypothesis is correct, is so thoroughly embedded in the way mathematics is practiced, it is hard to conceive of possible counter-examples to it. However, if it could be established, for example, that students understand mathematical concepts such as functions and sequences more easily when they are *not* presented in terms of processes, that would put the hypothesis in question. In the case of infinite sets, in particular, that would be strong evidence against POM. Based on the available evidence, however, that seems quite unlikely. Abstract concepts are notoriously hard for children to learn. For example, the "New Math" experiment during the 1960s introduced topics such as modular arithmetic, symbolic logic and abstract algebra to American grade schools. The result was that children had more trouble learning basic mathematical skills like arithmetic (Kline 1973). Similarly, it is hard to see how children could grasp the abstract concept of an infinite set without any prior understanding that some mathematical processes can be endless.

Finally, for developing the account, it is clear that POM should be considered in a wider mathematical context, not just infinite sequences and sets. Geometry, in particular, is a subject that deserves a lot more attention than can be given here. However, from Euclid to projective geometry, it is also a subject that has always appeared to have much use for the Process $\rightarrow$ Object Metaphor. Euclid's postulates are described by (depending on the translation) verbs like "draw", "produce" and "extend". In projective geometry, the key concepts of real projective line and extended real number line are usually described by "adding" points at infinity to the real number line.

But aside from the uses of POM in sophisticated mathematics, for it to be a valid metaphor in mathematical thought, it must also apply to more primitive cases of mathematical cognition. In this seems to lie one of the great strengths of POM. Let us consider perhaps the most primitive conceptual introduction to mathematics: counting.

Of the quantities that cannot be subitized, the first access a child has comes through counting. When she counts to, say, eight, she most probably uses fingers one by one to reach the end product. But the end product of the counting is not the eight extended (or unextended, depending on the culture) fingers. It is the *quantity* eight. The process of extending fingers gives us the object, which we recognize in arithmetic as the natural number eight. This is the simplest case of using POM—a metaphor that seems to be ubiquitous in mathematics.[10]

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