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# On Radical Enactivist Accounts of Arithmetical Cognition

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Hutto and Myin have proposed an account of radically enactive (or embodied) cognition (REC) as an explanation of cognitive phenomena, one that does not include mental representations or mental content in basic minds. Recently, Zahidi and Myin have presented an account of arithmetical cognition that is consistent with the REC view. In this paper, I first evaluate the feasibility of that account by focusing on the evolutionarily developed proto-arithmetical abilities and whether empirical data on them support the radical enactivist view. I argue that although more research is needed, it is at least possible to develop the REC position consistently with the state-of-the-art empirical research on the development of arithmetical cognition. After this, I move the focus to the question whether the radical enactivist account can explain the objectivity of arithmetical knowledge. Against the realist view suggested by Hutto, I argue that objectivity is best explained through analyzing the way universal proto-arithmetical abilities determine the development of arithmetical cognition.

**Keywords:** enactivism; arithmetical cognition; philosophy of mathematics; number concept acquisition; enculturation

## 1. Introduction

Traditionally, cognitive science has relied heavily on the view that the human mind works through, or is at least best explained by, including mental representations and computations (see, e.g., Chomsky 1965/2015; Fodor 1975; Marr 1982; Newell 1980). This commitment to representations can be understood in two ways. Ontologically, the question is whether mental states are realized by mental representations and computations. Epistemologically, the question is whether mental states are best *explained* by postulating mental representations

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and computations. More recently, both views have been challenged by various *enactivist* accounts, according to which at least some cognitive phenomena can be explained without invoking mental representations or computations, nor is there any empirical or theoretical reason to believe in the existence of representations and computations in the mind in relation with those phenomena. According to the enactivist views, cognition arises through embodied interactions between organisms and their environments (see, e.g., Varela et al. 1991/2017). In *radically enactive (or embodied) cognition* (REC), the most basic forms of cognition are assumed not to involve mental representations or mental content (Hutto & Myin 2013; 2017). Representations (i.e., contentful mental states) only arise through the development of linguistic truth-telling practices (Hutto & Myin 2013; Zahidi & Myin 2016; Zahidi 2021).

The central challenge that the radical enactivists face is explaining how contentful cognition and representations can be acquired if basic minds are assumed to be contentless and representation-free. Even if basic cognitive abilities could be explained by non-representational accounts, it is problematic how this "scales up" to include higher cognitive capacities like thought and imagination (Downey 2020). Such "representation-hungry" cognition is often seen as the main challenge for the enactivist accounts (e.g., Clark & Toribio 1994; Kiverstein & Rietveld 2018). One cognitive phenomenon identified as a particularly difficult challenge is the development of *numerical* cognition (see, e.g., L. Shapiro 2014; Zahidi & Myin 2016; Zahidi 2021). In this paper, I analyze the empirical and philosophical feasibility of the REC accounts as presented by Gallagher (2017; 2019), Zahidi and Myin (Zahidi & Myin 2016; 2018; Zahidi 2021) and Hutto (2019). There are differences between these accounts, but they all share the fundamental tenet of radical enactivism: numerical representations are only present in minds that have access to linguistic and sociocultural resources to engage in truth-telling practices.

I will assess the feasibility of the REC accounts of numerical and arithmetical cognition in light of recent work both in empirical research and philosophy of mathematics. To tackle the scaling-up problem, I determine how the REC position fits with the well-established existence of evolutionarily developed *proto-arithmetical* abilities, that is, subitizing and estimating (Pantsar 2014; 2018). Through an analysis of empirical research on quantitative cognition in human infants and non-human animals, I show that the REC account is compatible with philosophical theories of arithmetical knowledge as the product of *enculturation* (Menary 2015; Fabry 2020; Pantsar 2019; 2020). According to these theories, the development of numerical cognition and arithmetical abilities draws on the proto-arithmetical abilities but is made possible by structured learning in sociocultural contexts. While the enculturation framework does not imply radical enactivism, I will argue that the REC account can be understood in terms of the enculturation account, including the hypothesis that arithmetical cognition is (partly) based on proto-arithmetical abilities.

However, that line of argumentation appears to go against the radical enactivist view of mathematics proposed by Hutto (2019), who argues that we cannot account for the objectivity of mathematical truth if arithmetic is thought to be based on proto-arithmetical abilities. Hutto wants a radical enactivist account of mathematical cognition that focuses on its roots in evolutionarily developed abilities while also embracing a realist position as the source of objectivity of mathematical truth. However, his proposed realism, whether understood in a Platonist, physicalist or other way, is potentially a problematic fit with the REC account. Yet, as I will argue, the REC account of arithmetical (and other mathematical) cognition does not require commitment to mathematical realism. I will show that the enculturated development of arithmetic based on proto-arithmetical abilities can provide a satisfactory explanation of mathematical objectivity, one that does not rely on the kind of problematic assumptions that a realist account includes. This does not mean that radical enactivist explanations of arithmetical cognition are not without potential problems. I will show that the REC account receives important challenges from the empirical data on proto-arithmetical cognition. However, although I am not a proponent of radical enactivism, I will argue that the REC view of arithmetical cognition is at least possible to integrate with the best current empirical and philosophical understanding.

In Section 2, I present an overview of radical enactivism and the research of numerical cognition, pointing out a tension between them. According to the REC account, basic minds have no representations and content. However, many researchers believe that there are innate numerical cognitive abilities that are contentful. Unfortunately, the empirical literature on numerical cognition contains many terminological and theoretical confusions, which I clarify in Section 3 by distinguishing between proto-arithmetical and arithmetical abilities. In Section 4, I then present the radical enactivist accounts of the development of mathematical cognition, focusing for the most part on the work of Zahidi and Myin, which is currently the most fully developed account in the literature. In Section 5, I present challenges on the REC account based on proto-arithmetical cognitive abilities (subitizing and estimating). I argue that the feasibility of the REC account of arithmetical cognition depends on our understanding of what counts as representation and content: namely, whether proto-arithmetical abilities can be feasibly understood as being representation-free and contentless. In Section 6, I move the focus to the question of objectivity of mathematics, as brought to the literature by Hutto (2019). Against his account, I will argue that the radical enactivist account is more feasible when associated with enculturated development of arithmetical cognition based on proto-arithmetical abilities, rather than when associated with a realist stance on mathematical knowledge.

## 2. Tension between Radical Enactivism and Numerical Cognition

### *2.1. Representation-Free Basic Minds?*

Enactivism as an umbrella term refers to the position that action is central to cognition (for an overview, see, e.g., Stewart et al. 2010). However, as pointed out by Rupert (2016) and others, there is a great deal of variation in how this main enactivist tenet is understood. Rather than trying to make sense of the wide field of research, here I will focus on one proposed enactivist account, namely the radically enactive (or embodied) cognition account, or REC, presented by Hutto and Myin (Hutto & Myin 2013; 2017; Myin & Hutto 2015). REC is enactivist in virtue of its focus on the dynamic interaction of an organism with its environment in the development of cognition. What makes it radical is the way it abandons three seemingly central tenets of traditional cognitive science for basic minds, that is, minds not influenced by processes of enculturation. These are the notions that cognition involves computation, that cognition is representational, and that cognition is contentful (Hutto & Myin 2013).

It is important to note that the REC account does not imply that cognition *cannot* involve computation, representations or contents. Humans can develop linguistic (and perhaps other) cognitive practices which clearly are computational, representational or contentful. However, the REC position states that such cognitive processes are always the result of shared sociocultural practices. In radical enactivism, *basic cognition* is equated with "concrete spatio-temporally extended patterns of dynamic interaction between organisms and their environments" (Hutto & Myin 2013: 5). This position is defended by a combination of two lines of argumentation: the "don't need" and the "can't have" strategies. According to the former, reflecting the epistemological question concerning representations, postulating representations in theories of cognitive science adds no explanatory power compared to purely dynamical explanations (Chemero 2009). According to the latter, reflecting the ontological question of representations, there is no theoretical reason to include representations, content and computations in explanations of basic cognition (Hutto & Myin 2013). Hence, the REC thesis goes, there are no grounds for thinking that basic minds are contentful, or that they include representations, or that they perform computations.[1]

The REC account has received vehement opposition from philosophers. In his review of Hutto and Myin (2013), L. Shapiro (2014) asks:

1. From here on, I will drop computation from the discussion, since the topic at hand is whether there is contentful or representational numerical cognition.

> Imagine that you have a contentless mind. [. . .] What does it mean to say that the word 'cat' means cat, or that '2 + 3 = 5' means that 2 + 3 = 5? External symbols acquire their meaning from meaningful thoughts—how could it be otherwise? (L. Shapiro 2014: 489)

Here I will not, alas, be able to focus on cats, but the meaningfulness of arithmetical expressions provides an interesting question for the present topic. Certainly no radical enactivist is likely to think that, when reading the expression "2 + 3 = 5", one does not engage in contentful cognitive processes. Instead, their position is that engaging in contentful cognition is due to a socioculturally shaped ontogeny during which one has learned the content of arithmetical symbols. It is the *basic* mind that is contentless, and an arithmetically capable mind is not a basic one.

However, this explanation immediately prompts the question how basic minds can acquire content. If we accept the REC view, we are all born as basic minds, without mental content or representations. Through actions in our sociocultural environment, we engage in linguistic and other activities which include representations and content. External symbols and words do acquire meaning, but it is not from having pre-linguistic meaningful thoughts. The expression "2 + 3 = 5" makes sense to arithmetically enculturated people, but not because there are already thoughts in place that are then connected to the symbols "2", "3", "5", "+" and "=" (or their natural-language equivalents). Basic minds do not have such (or indeed any other) content. But what exactly happens when basic minds engage with content? This is perhaps the central question concerning the REC account and here I cannot give it a general treatment. Instead, I focus on the specific question of the emergence of contentful numerical cognition.

Far from being just one cognitive domain amongst others, however, I believe that numerical cognition poses a particularly important challenge for the REC account. It is, presumably, easier to explain how basic minds acquire content about things like cats. We see and hear cats and learn to associate a certain sound other people make in connection with those observations. This sound we then learn to associate with other cats. We learn to mimic the sound and much later establish that a certain physical configuration of symbols represents that sound. By this time, we have probably learned many things about cats from other people. There remains the question how language originally emerged and developed—and that is a huge question—but it is at least initially plausible that we can achieve contentful cognition and linguistic (and perhaps other, e.g., visual) representations about cats, even though we were born as basic minds.

Numbers, however, are potentially different and the ontogenetic acquisition of number concepts provides a more difficult challenge in the REC framework.[2] How do we develop contentful numerical cognition? Unlike cats, numbers are abstract. While we can see, hear and touch cats before we have mental representations of them, how can one adopt sociocultural practices involving numbers if there are no representations or mental content about numbers beforehand?

An equally important challenge is the phylogenetic development of numbers. We get enculturated in numerical cognition through our parents, teachers and other numerically skilled humans, but one consequence of the radical enactivist account is that there was a time when all human minds were basic when it comes to numerical cognition. How did contentful numerical cognition emerge from actions performed by humans with basic minds (Pelland 2018; 2020)?

To answer these questions, it needs to be clear what is meant by contentful cognition. In the REC framework representations and content are tightly connected. Zahidi, for example, defines representations as "mental states with content" (Zahidi 2021: 529). The main feature of the REC understanding of contentful cognition is that it should not be conflated with cognition that processes *information*. As presented by Hutto and Myin:

> [A]nything that deserves to be called content has special properties—e.g., truth, reference, implication—that make it logically distinct from, and not reducible to, mere covariance relations holding between states of affairs [. . .] states that happen to be inside agents and which reliably correspond with external states of affairs [. . .] don't "say" or "mean" anything just in virtue of instantiating covariance relations. (Hutto & Myin 2013: 67).

This analysis leads Hutto and Myin (2013: 67) to distinguish between *information-as-covariance* and *information-as-content* (see also Griffiths 2001). The radical enactivist position is thus that, unless truth-bearing representations are involved, information-processing cognition should be treated through the concept of information-as-covariance. According to the REC account, there is plenty of information-based cognition that guides behavior, but the vast majority is based on information as covariance. It is only when subjects possess sufficient, socio-

2. Here by "numbers" I refer to the ordered set of natural numbers (0, 1, 2, 3, . . .) and by "number concepts" the concepts (whatever they may be) referring to them, often written in the philosophical literature as ONE, TWO, THREE, etc. When discussing non-arithmetical perceiving of cardinalities, I use the term "numerosity". These distinctions are not generally made in the empirical literature, which leads to potential confusions (see Pantsar 2021a for more). Some authors claim that we can perceive numbers without possessing number concepts (see, e.g., Clarke & Beck 2021) but this is conceptually confusing (see, e.g., Núñez et al. 2021).

culturally developed, linguistic ability that contentful cognition emerges (Hutto & Myin 2013: 82). Since basic minds only include information-as-covariance, humans begin their cognitive lives with no treatment of information-as-content, but during ontogeny within a sociocultural environment they acquire linguistic capacities which make contentful cognition possible. When it comes to numerical cognition, the REC account therefore implies that any pre-linguistic quantitative cognition has to be contentless and thus not involve representations.

### 2.2. *Contentful Numerical Cognition in Basic Minds?*

Numerical cognition is a particularly important challenge for the radical enactivist view since most researchers believe there to be solid evidence of innate, non-linguistic, numerical abilities. Perhaps the most important experiment in this field was reported by Wynn (1992) in her paper "Addition and Subtraction by Human Infants." In the paper, Wynn argued that infants possess innate numerical concepts that they use for rudimentary arithmetical operations. In the experiment, infants demonstrated surprise through longer looking times in settings that violated expectations. Namely, when infants saw one and one dolls put behind a screen, they were surprised to see only one doll when the screen was lifted (the other having been secretly removed). This experiment has been replicated for different variables (e.g., Simon et al. 1995) and it is generally accepted that infants are indeed sensitive to numerosity, rather than another variable, such as visible surface area. Similar explicit claims of innate arithmetical ability have been made for non-human animals, such as newborn chicks (e.g., Rugani et al. 2009; Agrillo 2015).

In Section 3 we will return to the feasibility of such claims of arithmetical ability in human infants and non-human animals, but if the claims are correct, we face the prospect that the subjects have some kind of innate contentful cognition, which would go directly against the REC account. After all, it does not seem plausible that arithmetical cognition could be completely contentless. Dehaene (1997/2011) has argued that this is due to an innate *number sense*, which allows for the estimation of quantities. The estimation ability is often thought to be due to a numerosity-specific and evolutionarily developed *approximate number system* (ANS) (e.g., Dehaene 1997/2011; Hyde 2011). In the literature, different models have been proposed as the neural basis of ANS-based estimations (see, e.g., Meck & Church 1983 and Dehaene & Changeux 1993), but common to these models is that some kind of mental representations of quantities are included on a functional level. Dehaene (2003) has proposed that quantities are represented on a logarithmic *mental number line*. This has received support from data showing that when asked to place numerosities on a physical line, people in cultures with limited or no numeral systems (such as the Pirahã and the Mundurukú of the Amazon) do so in a way that is best modelled as logarithmic (D. Everett 2005; Dehaene et al. 2008; Pica et al. 2004; but see also Núñez 2011 and Stapel et al. 2015). Thus Dehaene (1997/2011) argues that arithmetical ability is based on the innate number sense, and the basic numerical representation is the logarithmic mental number line.

Clearly in Dehaene's account numerical information is seen to be processed as *information-as-content* rather than merely *information-as-covariance*, so it goes directly against the radical enactivist account of numerical cognition. If Dehaene is correct, basic minds are born with or quickly develop numerical representations and thus at least some content is in place before they acquire linguistic capacities. Numerical discrimination abilities have been reported already in 53-hour-old infants (Antell & Keating 1983). This kind of neonate numerical ability would be a bad fit with any account that takes contentful cognition to only develop through processes of enculturation and scaffolding, as the REC does. But if the infant ability is accepted as being numerical, and the numerosities are represented by a mental number line, how can the radical enactivist defend their position?

It is evident that Dehaene believes that the above type of reasoning goes against enactivism, even though he does not explicitly target the REC account. The following quotation (used in Zahidi 2021), for example, reads like a direct anti-thesis of the REC views:

> [Y]oung children have much to learn about arithmetic, and obviously their conceptual understanding of numbers deepens with age and education—but they are not devoid of genuine mental representations of numbers, even at birth! (Dehaene 1997/2011: 33)

In this passage, Dehaene both implicitly accepts that children have conceptual understanding of arithmetic that precedes education and explicitly points out that they have mental representations of numbers at birth. Two claims less fitting with radical enactivism could hardly be found.

## 3. Proto-Arithmetic and Arithmetic

Is there contentful innate numerical cognition that would make the REC account untenable? First, I want to question Wynn's (1992) interpretation of the infant behavior. Already before Wynn's experiment, it had been established that infants can distinguish between small numerosities (Starkey & Cooper 1980). This ability, called *subitizing*, allows determining the numerosity of objects in our field of vision without counting, but it stops working after three or four objects.[3] What Wynn argued was that, based on the subitizing ability, infants can carry out rudimentary addition and subtraction operations and the observed behavior was due to infants reacting with surprise to the "unnatural arithmetic" of 1 + 1 = 1. However, as I have argued before (Pantsar 2018), this conclusion is unwarranted. The behavior of the infants can be explained by ascribing a cognitive mechanism or procedure for keeping track of one small quantity at a time. When the quantity of the dolls did not match their expectations, the infants were surprised. Importantly, under this interpretation nothing like an arithmetical operation, however rudimentary, is thought to take place in the cognitive process. What is presupposed is simply an ability to track small quantities, which gets wide support from empirical data (Spelke 2011; Carey 2009).

This is an important distinction. While the infants' behavior can be *described* with the help of arithmetical language, we should not ascribe arithmetical abilities to them. I have proposed a distinction between *proto-arithmetical* and arithmetical abilities to prevent these types of problems (Pantsar 2014; 2015; 2018). If genetically determined proto-arithmetical abilities (i.e., subitizing and ANS-based estimation) are enough to explain some behavior, we should not ascribe arithmetical abilities to the subjects demonstrating that behavior. Similarly, I have proposed that the word "number" should be reserved for arithmetic, while it would be clearer to speak of "numerosities" in the context of the proto-arithmetical abilities (Pantsar 2018).[4] Following this distinction, I use the term "numerical cognition" to refer to cognition involving number concepts and "arithmetical cognition" to refer to cognition involving number concepts and operations (addition, multiplication, etc.) on them. When targeting cognition about numerosities that does not involve number concepts or arithmetical operations, that is, subitizing and ANS-based estimating, I use the term "proto-arithmetical cognition".[5]

It is crucial to note that the above distinctions are not mere terminological issues. The quotation of Dehaene in the previous section, for example, reads quite differently when re-interpreted with the new distinctions in place. Instead of young children having "much to learn about arithmetic", it is now clear that they have *everything* to learn about arithmetic, because they only previously possess proto-arithmetical abilities. Instead of having "genuine mental representations of numbers" at birth, we can now say that—in Dehaene's account

3. This is the case for human infants, but also for adult humans and many non-human animals (see, e.g., Dehaene 1997/2011; Knops 2020).

4. A similar distinction has been suggested by De Cruz et al. (2010), as well as Núñez (2017), who proposes the term "quantical" ability roughly for what I call proto-arithmetical.

5. In empirical literature, "numerical cognition" includes generally also what I call proto-arithmetical cognition and "numbers" can mean either numerosities, numbers or number concepts.

at least—they have representations of *numerosities*. This conceptual analysis is central in the present context since it changes the task of the radical enactivist in explaining the ontogenetic and phylogenetic emergence of number concepts and arithmetic. The move from proto-arithmetic to arithmetic now becomes the target phenomenon that the REC account must explain, without recourse to representations or content in basic minds.

I have argued in Pantsar (2019) that the move from proto-arithmetic to arithmetic is best explained in a framework of enculturation, which refers to the transformative processes in which interactions with the surrounding culture shape the way cognitive practices are acquired and developed (Menary 2015; Fabry 2020; Fabry & Pantsar 2021; Pantsar 2019; 2020; 2021a; 2021d). By combining contributions of proto-arithmetical capacities with culturally shaped learning, the enculturation framework can accommodate the empirical data about the acquisition of arithmetical abilities in ontogeny. The enculturation framework is in close connection to the phylogenetic account of *cumulative cultural evolution* (Boyd & Richerson 1985; 2005; Tomasello 1999; Henrich 2015; Heyes 2018). In cumulative cultural evolution, practices and tools are improved upon in small (trans-)generational increments, which can explain the way number concepts and arithmetic arise as the product of a long line of cultural development in which languages, artifacts and other factors have played a crucial role (Ifrah 1998; C. Everett 2017; Pantsar 2019).

If the framework of enculturation and cumulative cultural evolution is along the right lines, arithmetical ability is (partly) based on the proto-arithmetical abilities and number concepts are (partly) based on the proto-arithmetical abilities to engage with numerosities. This is commonly accepted among empirical researchers of both arithmetical and proto-arithmetical cognition, although researchers disagree on which proto-arithmetical abilities play a role in this development and how they contribute to the emergence of arithmetical abilities. Dehaene (1997/2011) and Halberda and Feigenson (2008), for example, argue that ANS-based estimations are the primary resource in the development of number concepts and therefore also in the development of arithmetical ability. Others take subitizing as the prevalent ability in that development (e.g., Carey 2009; Izard et al. 2008; Sarnecka & Carey 2008; Carey et al. 2017; Beck 2017; Cheung & Le Corre 2018). There are also researchers (Spelke 2011; Pantsar 2014; 2015; 2019; 2021a; vanMarle et al. 2018) who argue that both abilities play a central role in the process.

The important point in the present context, however, is that the REC challenge is altered in an important way. It is not enough to show that number concepts and arithmetical cognition are not present in basic minds. Instead, as I will show, the radical enactivist has to argue for at least one of two positions. Either they must argue against the view that number concepts and arithmetic are based at least partly on proto-arithmetical cognition (*no connection*), or they must argue that proto-arithmetical cognition is not contentful and does not involve representations (*no content*). The "no connection" view will be seen as highly implausible. The "no content" view, on the other hand, can be defended. However, as we will see in Section 5, it faces important challenges. But before we move on to that, let us first take a more detailed look in the literature on radical enactivist numerical cognition.

## 4. Radical Enactivist Numerical Cognition

### *4.1. No Content or No Connection?*

The most detailed account of radical enactivist numerical cognition in the literature is presented by Zahidi and Myin (Zahidi & Myin 2016; 2018; Zahidi 2021). In the latest paper, Zahidi (2021) has proposed radical enactivist answers to both the ontogenetic and the phylogenetic problems, that is, how number concepts and arithmetical ability can be acquired by children with no prior numerosity representations, and how cultures could develop number concepts and arithmetic when there previously were none. Gallagher (2017) and Hutto (2019) have also argued for radical enactivist views regarding the ontogenetic and phylogenetic development of arithmetical cognition. The REC accounts of arithmetical cognition show some differences and thus I will treat them as separate from each other. However, they also share many important characteristics. For starters, they do not question the *existence* of the proto-arithmetic abilities, or that they are basic cognitive capacities. Zahidi states this explicitly:

> Clearly, the subitizing capacity is a basic cognitive capacity as conceived of by radical enactivism, since it is exhibited by agents that do not (yet) partake in sociocultural practices. (Zahidi 2021: 533)

Furthermore, Zahidi and Myin explicitly accept the kind of distinction I make between proto-arithmetic and arithmetic, while also accepting that the latter arises at least partly out of the former:

> Of course, arithmetical abilities don't arise out of nothing. They do have precursors, but the abilities out of which arithmetic arises are not arithmetical abilities themselves. (Zahidi & Myin 2016: 62)

The account of Zahidi and Myin therefore clearly adopts the "no content" approach specified in the previous section. They accept that proto-arithmetical cognition is basic cognition and that arithmetical cognition is at least partly based on it. Because of that, they are committed to the view that proto-arithmetical cognition is neither contentful nor involves mental representations.

But at least in Hutto's (2019) account, also the "no connection" strategy plays a role. Hutto wants to question the view that arithmetical abilities are based on proto-arithmetic abilities:

> There is evidence that distinct neural systems are brought into play when we approximate quantities, on the one hand, and when we calculate with discrete numbers, on the other. Moreover, there is evidence of some degree of neural overlap between these systems, even though it is an open question just how much these systems overlap. Of course, on its own, any evidence of overlap, however great, ought not lead us to conclude that our mathematical competence in using discrete numbers is in any way importantly grounded or draws on our ancient capacity to detect approximate quantities. (Hutto 2019: 831–32)

It is important to note that Hutto does not deny the possibility that arithmetical ability (involving discrete numbers) draws on the proto-arithmetical ability (involving approximate quantities). Rather, he argues that this should not be accepted as given, saving therefore place for both the "no connection" and "no content" strategies.[6] Clearly radical enactivists cannot accept that proto-arithmetical abilities, being basic cognition, are contentful. However, the "no connection" possibility pointed out by Hutto opens up another argumentative possibility. Even in case it proved to be difficult to argue against proto-arithmetical abilities being contentful, the radical enactivist could use another strategy and argue against the view that arithmetical cognition draws on proto-arithmetical abilities. In Section 6 we will be in a better position to evaluate the "no connection" strategy. For now, let us focus on the "no content" strategy.

### 4.2. *Ontogeny of Number Concepts*

Presenting his REC account, Zahidi (2021) argues against representations in what he calls the "standard accounts of numerical cognition" in the literature (e.g., Dehaene 1997/2011; Butterworth 1999). By standard accounts, he refers to views that not only accept that advanced numerical ability is based on basic

6. Unfortunately, Hutto does not specify what evidence he is referring to. I assume that he accepts the most commonly cited works also used by Zahidi, e.g., Dehaene (1997/2011) and Butterworth (1999).

(here, proto-arithmetical) capacities, but also that the basic capacities of subitizing and estimation involve numerosity representations. In the empirical literature, this view has been presented in many different ways, ranging from Butterworth's (1999) "number module" to Dehaene's (1997/2011) number sense and mental number line. As Zahidi (2021: 534) points out, some researchers (e.g., Gallistel 2017) explicitly state that the existence of proto-arithmetical abilities implies that *numbers* exist in the brain. However, this is due to the incongruent terminology between disciplines. Rather than numbers as abstract objects, what Gallistel refers to is better described as either innate number concepts or innate numerosity representations.

Whether we understand the standard accounts as including innate number concepts, numerosity representations, a number module or a mental number line, clearly they go against the radical enactivist position that basic minds are contentless and do not involve representations. But troublingly for the REC position, there are empirical data that seem to support basic numerosity representations. Jordan et al. (2005) and Jordan et al. (2008), for example, have reported experiments in which monkeys successfully match the cardinalities of stimuli across sensory modalities. There is also neuro-physiological evidence in support of this. Not only are there distinct areas of the brain where quantities are processed, but within those areas there are specific sets of neurons connected to particular numerosities (Nieder 2011; 2012; 2016). When a monkey is presented with two objects, a specific set of neurons activate. When the number of objects is three, a (partly) different set is activated. The experiments have been controlled for other variables, and the scientists have been able to tease out the effect of sensory stimuli involving a particular numerosity in the monkey brain across modalities (Nieder 2012; 2016).

Are such so-called "number neurons" bad news for the radical enactivist?[7] After all, such modality-independent activation could be seen as evidence of numerosity being represented neuronally in those very neurons. Zahidi (2021) does not believe so. To see why, he distinguishes between two versions of the standard account. In the first, which Zahidi calls the "semantic version", there are innate numerosity representations, or even innate number concepts. In the second, the "deflated interpretation," numerosity representations are deflated to neuronal activity like the number neurons of Nieder. We will soon treat the semantic version, but let us first focus on the deflated interpretation. Zahidi argues that neuronal activity associated with the common cardinality of the stimuli should not be seen as evidence that the activation of the neurons somehow represents numerosity:

7. With the present terminology in place, "numerosity neurons" would be a better term.

> However, the fact that the state of variable X (e.g. spiking activity) causally covaries with the state of another variable Y (e.g., the numerosity of a stimulus) does not entail that X is a representation of Y. (Zahidi 2021: 536–37)

What Zahidi refers to here is the "receptor notion of representation" that has been criticized by, among others, Ramsey (2007). In the receptor notion, "because a given neural or computational structure is regularly and reliably activated by some distal condition, it should be regarded as having the role of representing [. . .] that condition" (Ramsey 2007: 119). But as Ramsey (2007: 124–25) argues, there is no *prima facie* reason to believe that such activation serves as a representation. To conduct a similar analysis in terms of the two different notions of content presented earlier, what the existence of number neurons suggests is the treatment of quantitative information as *covariance* rather than *content*. Zahidi (2021) sees this as the main problem in treating the activation of number neurons as representations of numerosities. All we know is that there is causal covariation, and there is no reason to believe that the cognitive process involves numerosity concepts or content. Thus the deflated interpretation, Zahidi argues, is perfectly compatible with radical enactivism.

What about the semantic version? Since it refers to the view that there are innate numerosity representations or number concepts, it cannot be deflated into covariances of neuronal activities and stimuli. The semantic version of the standard account thus postulates representations on top of the covariances, and similarly treatment of information-as-content on top of information-as-covariance. It is hard to see how empirical evidence could rule out the existence of numerosity representations, so the way radical enactivists argue for their position is by trying to show that the development of numerical cognition can be explained without invoking numerical representations or content in basic minds. To see how this argumentation goes, it is important to understand what they ultimately see as the main problem with the standard account.

The foundation of the argumentation for the REC account of numerical cognition is that the semantic interpretation of the standard account is mistaken in the kind of postulations that are made to account for the behavior shown by infants and non-human animals in various experiments. Zahidi and Myin quote, for example, the following passage of Dehaene:

> How can a 5-month-old baby know that 1 plus 1 equals 2? How is it possible for animals without language, such as chimpanzees, rats, and pigeons, to have some knowledge of elementary arithmetic? My hypothesis is that the answers to all these questions must be sought at a single source: the structure of our brain. (Dehaene 1997/2011: xvii)

In this passage, it is assumed that babies and non-human animals *know* arithmetical sums. Furthermore, the knowledge comes directly from the structure of their brains. This postulation of knowledge for infants and non-human animals is troubling for philosophers, but with a charitable reading we can understand knowledge here simply as some kind of contentful cognition. Even so, it is clearly against the REC position. For radical enactivists, after all, content only arises once there are truth-telling practices in place, even though arithmetical abilities have "precursors" in basic minds (Zahidi & Myin 2016: 62). Now the question is how genuine arithmetical abilities can arise from the contentless precursor abilities. This challenge is present on both the ontogenetic and the phylogenetic level.

When it comes to ontogeny, Zahidi and Myin (2018) want to question the line of thinking among many empirical researchers that the totality of findings from experiments like that reported by Wynn (1992) can only be explained by infants having innate conceptual knowledge about numbers. This argument has been developed in most detail by De Cruz and De Smedt (2010) and the crux of it is that no lower cognitive faculties are able to explain all the data. But as Zahidi and Myin (2018: 225–26) point out, even though having innate number concepts and rudimentary arithmetical knowledge could explain the infant behavior in the experiments, this does not mean that they in fact have number concepts and propositional attitudes toward them. While De Cruz and De Smedt (2010) are correct in pointing out that there are data which are not as easily explained by lower cognitive faculties as the Wynn (1992) experiment, I share the general concern of Zahidi and Myin about postulating advanced cognitive capacities to account for infant and animal behavior. In particular, I share it when it comes to number concepts and arithmetical ability. It is not possible here to go into the details, but it is important to note that attributing the infant behavior to having number concepts is at best an inference to the best explanation. There is no empirical data to directly support the theory that there are innate number concepts. Ultimately, we still know very little about proto-arithmetical cognition outside of what is based on behavioral data alone. Thus, the argumentation of Cruz and De Smedt (2010) is perhaps best understood in the present context as a challenge to provide a better explanation.

Indeed, that is what Zahidi and Myin (2018) aim to do. However, their account is only a very rough sketch of the kind of influences that govern the acquisition of number concepts and numerical knowledge. The main idea is that infants have non-epistemic abilities and behavioral dispositions which they enact in an "epistemically loaded sociocultural environment" (Zahidi & Myin 2018: 228). Drawing from Williams's (2010) account of language learning, they see the acquisition of number concepts as a "triadic relation between novice, master and the world" (Zahidi & Myin 2018: 229). Since there are shared ways of responding to numerical stimuli in the world (i.e., proto-arithmetical abilities), these can form the basis of training by the master, which is then applied through the socioculturally acquired knowledge of the master (Zahidi & Myin 2018: 229; see also Williams 2010: 217). Through this kind of learning, the child can acquire the number concepts shared by the other members of her sociocultural surroundings, as well as norms for their application. In this manner, Zahidi and Myin argue, conceptual knowledge can arise in ontogeny when there originally were only non-epistemic abilities.

## *4.3. Phylogeny of Number Concepts*

In the next section, I will raise questions about the ontogenetic account, but we should also ask how this account can be transferred to the *phylogeny* of number concepts and numerical knowledge. Pelland (2018; 2020) has asked an important question about this connection. Assuming that the above account is along the right lines, how did the master acquire their numerical knowledge? Presumably it came from another master, but this chain of passing on knowledge cannot be continued indefinitely, even if we allow that numerical knowledge (and perhaps number concepts themselves) has been subject to modifications during its history. At some point, Pelland points out, there must have been a situation in which number concepts emerged from a state in which there were no number concepts. The nativist account supported by De Cruz and De Smedt (2010) does not run into this problem, since number concepts are thought to be the result of biological rather than cultural evolution.[8] But how does the REC account of Zahidi and Myin explain the emergence of number concepts in phylogeny?

Zahidi (2021) uses the *material engagement theory* of Malafouris (2013) and the work of Overmann (2018) to argue that the conceptual properties of numerals and norms for their use have been determined by a socioculturally evolved set of material practices. Just like in the case of ontogeny, the foundation of Zahidi's (2021) phylogenetic account is that we share non-epistemic proto-arithmetical abilities that are the result of biological evolution. As pointed out by Flegg (2002), the emergence of counting practices is crucial to the development of number concepts, but counting in itself is an advanced process. Zahidi notes that counting presupposes several important capacities, from putting objects into one-to-one correspondence to discriminating collections according to quantity, which he believes to be based on the proto-arithmetical abilities (Zahidi 2021: 540). With such capacities in place, there can emerge practices like finger counting, tallying, and other forms of material engagement, which determine the concep-

8. That account does prompt the question how number concepts evolved through biological evolution in a setting where there previously were none. But this question generally applies to any evolutionarily developed trait and is not specific to the nativist account of number concepts.

tual properties of numerals and norms for their use. Through finger counting, for example, the linear order of counting can be established (Overmann 2018). Body part names can evolve into numerals and tallying marks can develop into number symbols, etc. (Ifrah 1998). Indeed, it is likely that number concepts and number words have evolved together as a result of body-part counting procedures (Wiese 2007). This has made it possible, in turn, to manipulate material objects which can further develop numerical ability in the manner suggested by Malafouris (2013). Grouping collections of pebbles can lead to norms about operations like "plus one" and furthermore about general addition, and so on (Overmann 2018).

The topic of the emergence of number concepts is a much-discussed topic and there is a wealth of empirical data relevant to it, but the exposition here should suffice to describe Zahidi's REC account of the development of number concepts (for more detailed accounts, see e.g. C. Everett 2017; Overmann 2018; Pantsar 2019; Dos Santos 2021). The key point is that in this development, only the presence of evolutionarily developed proto-arithmetical, non-epistemic, abilities is assumed. Thus in the creation of number concepts and norms about them, there is no "outward projection of inner meaning" involved (Zahidi 2021: 543). Number concepts and numerical knowledge have evolved socioculturally through practices of material engagement. When we acquire numerical knowledge in ontogeny, the argument goes, we do not apply prior numerosity representations or contentful numerical cognition, let alone number concepts. We only apply non-conceptual abilities with numerosities, with conceptual knowledge and representations emerging where there previously were none.

## 5. Challenges against REC: The Proto-Arithmetical Abilities

The REC argumentation as presented by Zahidi and Myin carries a great deal of power against nativist positions like the one presented in Gelman and Gallistel (1978) and Gallistel (2017). This power comes both from the "don't need" and "can't have" tenets of radical enactivism. The nativist position assumes that number concepts exist in the mind independent of cultural learning. But mirroring the "can't have" argumentation of radical enactivism, there is no theoretical or empirical reason to think that this is the case. While the empirical data conclusively shows that infants and non-human animals have abilities to engage with numerosities, there is no data to suggest that they possess mature number concepts. Closely related, mirroring the "don't need" part, there is also no need to postulate innate number concepts to explain the behavior of infants and non-human animals in the experiments. Hence if we accept that contentful and representational numerical cognition requires the postulation of innate number concepts, the REC position is strong. The radical enactivist criticism is also well placed against accounts like that of Dehaene (1997/2011) where the proto-arithmetical abilities are conflated with having numerical *knowledge*. Based on the empirical data available at present, there is no reason to assume that infants and non-human animals possess number concepts or numerical knowledge.

However, while possessing number concepts or numerical knowledge clearly is contentful and involves representations, there could be contentful cognition concerning numerosities that does not require ascribing number concepts or numerical knowledge to the cognizing subjects. This is highly relevant for the present topic since, as we have seen, different types of explanations have been proposed for proto-arithmetical abilities in a manner that does not postulate mature number concepts for subjects not enculturated in numerical cognition. The bulk of the argumentation of Zahidi and Myin (Zahidi & Myin 2016; 2018; Zahidi 2021) is against the nativist position, but while I believe that their argumentation mostly hits its target, I also think that the more interesting question is whether it also applies to weaker forms of quantity representation. It could be the case, for example, that quantities are represented in basic minds through *visual* representation, initially tied to observations of physical objects. An animal could, for example, remember a visual image of a group of animals that it uses to represent the size of a dangerous group of animals. The possibility of such numerosity representations seems feasible, but it is clearly different from the animals having innate number concepts. However, as detailed in Section 2, also such weaker representations contradict radical enactivism.[9] Indeed, any representationalist account of the proto-arithmetical abilities is in conflict with the REC account. This is particularly important for the present topic since, as we have seen, the standard terminology among empirical scientists when discussing the approximate number system is to evoke its representation in the mind. But, if we follow the REC argumentation, representations in that context must mean something different from what is discussed in the philosophical literature on mind and cognition. Certainly, the ANS representations *can* be interpreted in a semantical way, for example, in terms of a mental number line that represents numerosities (as done by Dehaene 1997/2011). But the argumentation of Zahidi and Myin aims to show that there is no reason to make that interpretation. In their account, the presence of an innate estimation ability due to the ANS is not put into question, but its reliance of numerosity representations is.

While the empirical data presently leaves many such matters open, in general the position of Zahidi and Myin is at least *prima facie* plausible. We don't know enough about the neural mechanisms involved in the application of the ANS to make any conclusive judgments, but the current data do not give suf-

9. See Kiverstein and Rietveld (2018) for an REC account of visual imagery.

ficient reason to postulate numerosity representations in the sense meant by Zahidi and Myin. The type of thing that *is* known is that the intraparietal sulcus, which responds to symbolic numerical stimuli, also activates in estimation tasks (e.g., Cantlon et al. 2006). This is evidence that cognitive abilities with symbolic numerosities draw in some way on the ANS, and since symbolic numerical cognition involves representations, it could be tempting to make the conclusion that numerical representations are in place already in subjects (such as infants and non-human animals) that only possess the proto-arithmetical estimation ability.

However, we can accept that genuine arithmetical ability builds on the ANS without assuming that they involve the same type of neural mechanism. This is the idea behind the neuronal recycling hypothesis as applied by Gallagher (2017) in his REC account of numerical cognition, which draws on the idea of mathematical cognition as the product of enculturation (Menary 2015). Roughly put, in the presence of arithmetical practices involving cognitive tools, number words and symbols in the sociocultural environment, neuronal circuits originally evolved for the ANS are recycled when acquiring number concepts and learning arithmetic (Menary 2015; Pantsar 2019). Instead of mere recycling of particular neural circuits, this process can include a more general reuse of neural resources. Anderson (2015) has presented his notion of *neural reuse* on this basis, and it has been argued that the development of arithmetical cognition is in fact better understood as a case of neural reuse than neuronal recycling (Fabry 2020; Jones 2020).[10]

It is not possible here to go into the details of that topic, but one important feature of the neuronal recycling and the neural reuse accounts in the present context is that neither of them requires that representations enter the explanations before there are sufficiently developed number words and symbols in place. While Dehaene (1997/2011) has used his account of neuronal recycling to argue that exact number representations are reached by employing and enhancing approximate numerical representations, this is by no means an inevitable consequence of the neuronal recycling account. Even if neural circuits associated with the ANS were recycled for new, arithmetical purposes, it is possible that numerosity representations are a feature that emerges only after the recycling process. Thus, the REC view appears to survive the ANS challenge when it comes to representations. It is plausible that infant and non-human animal cognition with estimating numerosities can be explained without postulating representations or contentful cognition. As we have seen, the existence of "number neurons", for example, can be explained as covariance rather than content. If the activation of such numerosity-specific neurons is behind the ANS, it seems

10. It should be noted that Hutto (2019) has argued against Gallagher (2017) that neural reuse is a better fit with the REC account of the development of numerical cognition.

possible that the estimating ability can be explained without evoking representations or information-as-content. More empirical data is required, but based on the current evidence we cannot rule out the possibility that the functioning of the ANS is based on quantitative stimuli activating groups of neurons, showing causal covariance without the need to postulate mental content.

While the REC view seems to be able to meet the challenge of ANS-based estimations, we must remember that there is also another proto-arithmetical ability, namely subitizing. Indeed, there are many researchers who downplay or even outright deny the role of the ANS in the development of arithmetical cognition (most famously Carey 2009). The subitizing ability is closely linked to the ability to individuate objects in a parallel fashion and according to a widely accepted hypothesis, it is made possible by the *object tracking system* (OTS), also sometimes called the "parallel individuation system" (Knops 2020). Dehaene originally (1997/2011) included the OTS as a part of the "number sense", but most researchers treat the OTS as a distinct cognitive system from the ANS (see, e.g., Hyde 2011). Subitizing and estimating have different behavioral signatures and they are also reported to have different neural correlates (Cutini et al. 2014).

The most discussed position in the literature that takes OTS to be fundamental for the acquisition of number concepts and the development of arithmetical cognition is the *bootstrapping* account presented by Carey (2009). In her account, subitizing is explained by the quantity being recognized by forming distinct mental representations for each of the observed objects. These mental representations work by employing separate *object files*. Three dots in the field of vision, for example, are represented in three distinct object files, not as a representation of the quantity three. The object files are not specific to numerosities, since they are thought to merely represent objects observed in a parallel fashion. However, they are thought to allow the detection of numerosity by detecting how many object files are being employed (Carey 2009; Beck 2017).

Such implicit representation of quantity through object files, Carey (2009) argues, allows grasping the first four number concepts (in ascending order) by associating a number word with the amount of object files being occupied (see also Beck 2017 and Pantsar 2021a). Since the OTS stops working after four objects, at that point the child needs to make a qualitative leap to become a *cardinality-principle* (CP) knower, that is, in order to have the ability to generally match the last numeral uttered in the counting sequence with the cardinality of a group of objects (Sarnecka & Carey 2008; Lee & Sarnecka 2011). While the first four number concepts are acquired in stages separated by 4–5 months, at this stage of development the child "bootstraps" a general meaning of numbers and in addition to "five" grasps the meaning of the numerals "six", "seven", and so on (Lee & Sarnecka 2010).

The bootstrapping account has been much discussed in the literature but this level of detail is sufficient for the present purposes (for more thorough expositions, see Carey 2009; Beck 2017; and Pantsar 2021a). But is the bootstrapping account compatible with the REC view? After all, as we saw above, the bootstrapping theory is based on object files *representing* quantities. Even though this representation is implicit, it is clearly understood as mental, non-linguistic, representation—in other words, of exactly the type that the REC account denies. As a challenge to radical enactivism, the one provided by the OTS thus appears stronger than the ANS-based challenge. There are two main reasons for this. First, the object files postulated in the OTS account are the kind of representations that basic minds could feasibly have. The idea of innate number concepts is unsupported by any evidence and there are reasons to be skeptical about the existence of an innate mental number line, as well. But the object files are simply thought to perform a cognitive function that is clearly performed by *some* capacity in the mind, given that infants and non-human animals are able to track multiple objects (up to four) simultaneously.

The second reason why the OTS challenge appears more serious than the ANS-based one is that the object files that are assumed to implicitly represent numerosity are not seen as mere causal covariance. The "number neurons" mentioned earlier are often discussed as representations, but all we know is that there is a causal covariance between them and the quantity of the observed objects. They may or may not involve representations, but the present empirical and theoretical understanding of the number neurons does not require postulating representations. The object files, on the other hand, are by the very fundamental assumption of the OTS account employed as representations of discrete, observable objects. If three object files are employed when observing, say, three birds, how can we deny that the three object files (implicitly) represent the numerosity of the observed birds?

I can envision two main responses by radical enactivists to the OTS challenge. First is to deny that the kind of implicit representations postulated as the functioning principle of the OTS are proper representations of the type rejected by the REC account. This kind of response would certainly solve the problem if radical enactivism were only against innate number concepts. In the bootstrapping account based on the OTS, only non-basic minds enculturated with number words possess number concepts. But how can one determine the numerosity of occupied object files without possessing number concepts? To move from implicit representation of quantity to explicit representation, that is, to establish that three occupied object files, for example, represent the numerosity three, one would need to possess number concepts. Thus, the radical enactivist account would be safe, since there is no reason to believe that basic minds possess number concepts.

However, the REC account is not only against innate number concepts, but against all representation in basic minds. Thus, the above type of reasoning only moves the question to the level of individual objects files. An argument could be made that the object files represent objects, and thus the OTS challenge against radical enactivism hits its target. But although there is clearly a relation between the object files and observed objects, it is hard to imagine that the radical enactivist would be particularly concerned about this challenge. The reason for this is that object files according to the OTS account are only employed by observations of objects; they are not thought to involve any representation of the objects as that *particular* object with physical characteristics. Thus, the form of representation is ultimately a very weak one and the radical enactivist would not be likely to see it as a representation at all. Instead of being contentful cognition, it would fall within the range of radical enactivists' deflationary treatment of mental representation as information-as-covariance.

The second radical enactivist response to the OTS challenge I can envision is based on the object files being at this point postulations of a certain theory of our cognitive capacities, and nothing more. While the OTS provides an explanation of the functioning of multiple-object tracking and subitizing, to the best of my knowledge there is no empirical evidence directly supporting the existence of object files. With this lack of direct evidence, it is possible that new explanations of object tracking can emerge that do not require the postulation of mental object files. The functioning of the OTS, and indeed whether there is an independent innate cognitive system for object tracking, are ultimately empirical questions. With the current empirical data, it appears that the OTS can be interpreted to involve representations, but as we have seen, there are also plausible counter-arguments against that position. Indeed, whether the object files represent objects, for example, seems to be the kind of question that radical enactivists and representationalists fundamentally disagree on. If one is ready to follow the REC account in general, I cannot envision either the OTS or the ANS providing a cause to abandon the view.

## 6. Challenges against REC: The Objectivity of Mathematics

Let us put aside the worries presented in the previous section and assume that the REC account of number concepts and arithmetic survives the OTS and ANS challenges. That is, let us assume that only subjects with sufficient linguistic ability and enculturated in a suitable sociocultural environment possess any kind of contentful cognition on numerosities. However, this provides the radical enactivist with a general challenge: if numerical cognition is culturally shaped, as implied by the REC account, are we in danger of rendering all numerical knowledge a matter of convention? And if numerical knowledge is a matter of convention, is all mathematical knowledge likewise entirely culture-dependent? This is a troubling prospect because if mathematical knowledge is merely a matter of convention, we cannot *prima facie* rule out the possibility that mathematical conventions fundamentally differ in different cultures. Therefore we would be in danger of losing *objectivity* from mathematics. Mere material engagement would not appear to give enough constraints to ensure that number systems and arithmetic develop in convergent ways. If the radical enactivist account is correct, what is there to prevent people in different sociocultural environments from developing mathematics, beginning from arithmetic, in radically different ways?

This kind of strict conventionalism would be a problematic position since we know that arithmetic has developed independently several times during the human history, with fundamentally similar content when it comes to finite natural numbers and operations on them (Ifrah 1998). Moreover, even in cultures that haven't developed arithmetic, but have somewhat extensive systems of number concepts, there is great similarity regarding, for example, the recursivity of numeral words (C. Everett 2017). While there also exist anumeric cultures, it appears that if a culture develops number concepts, they are likely to follow the same principles to a large degree for finite numbers and basic operations (addition, multiplication) on them (Pantsar 2019).[11]

Gallagher (2017) in his radical enactivist account of mathematical cognition aims to solve this problem by basing his view on, on the one hand, the constructivist image of mathematics presented by Lakoff and Núñez (2000) and, on the other hand, the neuronal recycling account developed by Dehaene (2009) and applied by Menary (2014; 2015). The fundamental tenet of the account of Lakoff and Núñez (2000) is that mathematical cognition develops through embodied actions. The neuronal recycling account, as shown earlier, is based on redeploying neural circuits that have evolved for other purposes to new culturally specific functions. In the case of arithmetic, the idea of Gallagher (2017) in combining the two accounts appears to be that embodied action leads to neuronal recycling of proto-arithmetical neural circuits in acquiring number concepts and developing arithmetical cognition.

Gallagher's fellow radical enactivist Hutto, however, sees an important problem with applying the account of Lakoff and Núñez (2000) because it grounds mathematical content on "unconscious, inference-preserving neural mechanisms", which Hutto sees at odds with "enactivism that offers itself as an antidote to such neural fetishism" (Hutto 2019: 830). That is why he also sees a problem in applying the neuronal recycling account:

11. However, there are clearly also differences when it comes to numeral systems, methods, tools, applications and many other aspects. See Pantsar (2019) for details.

> Insofar as these driving commitments of the neural recycling account are retained there is a residual privileging of the neural as an ultimate source of a key element of mathematical understanding. What is attractive for enactivists about the neural recycling proposal is that is gives center stage to the idea that our advanced mathematical abilities only come into being through enculturation in relevant social practices. What is problematic with the proposal is that it keeps faith with the idea that ancient neural systems contribute importantly to mathematical understanding. (Hutto 2019: 832)

It is this "residual privileging of the neural" that Hutto wants to liberate the REC account of mathematics from, and for this reason he sees the neural reuse account of Anderson (2015) as a better fit with radical enactivism. Based on these criticisms, Hutto presents his wishes for what a radical enactivist account of mathematics should be like:

> My recommended formula for creating a satisfactory enactivist account of mathematical cognition is to enact the following procedure: Subtract any residual commitment to mental representation, information-processing stories, and neuro-fetishism. Add, in place of these items, a more Andersonian account of neural reuse—one that focuses on the pluripotent, protean brains and which places the greater weight on the contributions of socio-cultural practices in establishing mathematical content and competencies (see Zahidi & Myin 2016). Subtract any residual constructivism, anti-realism, and idealistic elements from the account. Finally, subtract any lingering psychologism about mathematics and its content. (Hutto 2019: 835)

Hutto clearly moves the focus of explaining the development of numerical cognition toward the sociocultural aspects, a move that is in accordance with the enculturation account of Menary (2015). But by subtracting "any lingering psychologism", Hutto's account is much more radical than Menary's, and as such it faces problems explaining how different cultures have developed number concepts and arithmetic in converging ways. Therefore Hutto's account may initially appear to run the danger of succumbing to strict conventionalism. After removing the proto-arithmetical roots in the name of eradicating "neuro-fetishism", there would seem to be no reason left why arithmetic, or mathematics in general, is objective in any strong sense.[12]

12. By "any strong sense", I mean objective in stronger sense than human conventions are considered to be objective. The rules of chess, for example, are objective for individual humans,

However, Hutto doesn't agree. In fact, according to him it is the *psychologist* position of Lakoff and Núñez that runs into problems explaining the objectivity of mathematics. Here Hutto's criticism has a wider significance that goes beyond radical enactivist accounts. One of the main purposes of Lakoff and Núñez (2000) in developing their account was to avoid a realist interpretation of mathematics. For Lakoff and Núñez, mathematics is a human creation and nothing else: "mathematics as we know it arises from the nature of our brains and our embodied experience" (Lakoff & Núñez 2000: xvi). As Hutto (2019: 835) points out, this kind of constructivism initially seems like a good fit with radical enactivism. However, Hutto argues that this constructivist account undermines the objectivity of mathematical truths:

> Lakoff and Núñez (2000) take this consequence to be a virtue of their account; they are satisfied in rejecting the romantic ideal just so long as they can avoid endorsing the idea that the meaning of mathematics is generated by arbitrary social conventions. However, their account of the truth conditions of mathematics only secures that it is non-arbitrary with respect to our shared embodiment—hence it falls a long way short of being an account of the objectivity of mathematics. (Hutto 2019: 835)

What Hutto wants instead is to detach mathematical *truth* from the constructivist position while accepting *conceptual* constructivism. This way, he argues that we can embrace both mathematical realism about truth and conceptual constructivism of the radical enactivist type, and it is possible that the subject matter of mathematics is objective and mind-independent (Hutto 2019: 835).

If the subject matter of mathematics were indeed mind-independent, clearly objectivity would be saved. But what does "mind-independent" mean in this context? Elsewhere Hutto (with Satne) has argued that truth-telling, content-involving practices save their objectivity through being correct (or incorrect) regardless of conventions:

> In contrast with other intelligent dealings with the environment, [. . .] content-involving practices contain a special sense of going wrong: this is not just falling in line with what is acceptable for the community but being correct or incorrect according to how things are anyway. (Hutto & Satne 2015: 534).

but not in a strong sense since it is entirely up to the community of chess players whether we want to change the rules.

This is line with McDowell's (1998) Kantian idea of "intuitive notion of objectivity", which is meant to distinguish genuine objectivity from conventions. But in case of mathematical knowledge, what can this kind of non-conventional objectivity be based on? In particular, since Hutto rejects the constructivist account of mathematical truth, what is there to distinguish his account from Platonist philosophy that postulates an abstract world of mind-independent mathematical structures or objects?[13] Hutto's account here is not fully fleshed out, but he explicitly decrees that a satisfactory REC account needs to subtract any "anti-realist elements". The resulting realist account points either toward platonism, or perhaps a kind of physicalism, over mathematics.

While platonist views are still entertained in contemporary philosophy of mathematics (e.g., S. Shapiro 2007; Brown 2008), they are seen as increasingly difficult both ontologically and epistemologically (see, e.g., Benacerraf 1973; Linnebo 2018; Pantsar 2021b; 2021c). The main ontological problem involves the status of mathematical objects (or structures), which—being non-causal, non-spatial and non-temporal—would be unlike that of any other objects. Importantly for the present context, this would go against the general ethos of radical enactivism, where the existence of representations is denied partly because it is an ontological assumption made without sufficient reason. While the two questions are independent of each other, one must wonder how postulating abstract, mind-independent mathematical objects could be ontologically acceptable while postulating mental representations is not.

A physicalist interpretation of realism would not be any less problematic. In the early version of Maddy's realism, she suggested that mathematics is realist as part of the physical world and we can perceive mathematical reality through observations of sets of objects (Maddy 1990). It is not possible to discuss Maddy's account here in detail (for a more thorough presentation, see, e.g., Tieszen 2005) but in its most recent incarnation, her idea is that our cognitive architecture has evolved to detect the logical structure of the world (Maddy 2014: 234). De Cruz has similarly argued that the adaptive behavior central to the development of proto-arithmetical abilities can only be explained by it detecting the structure of the world (De Cruz 2016). However, I see no reason to make such a hypothesis. Given that the proto-arithmetical abilities are evolutionary adaptations, the only thing we need to assume is that they have been advantageous in processes of natural selection. This may or may not be due to them in some way detecting the structure of the world (Pantsar 2021b).

Given such potential problems, should we develop the REC account of arithmetical cognition along realist lines as suggested by Hutto? As I see it, the key

13. Here I follow the custom that Platonism with capital "P" refers specifically to Plato's philosophy while platonism with a lower case "p" refers generally to realist metaphysical positions on mathematics (Balaguer 2016).

question in this respect concerns objectivity. A realist account—whether platonist, physicalist, or other—can provide an explanation for the objectivity of arithmetical knowledge. The big epistemological problem of realism, however, is to establish how our mathematical knowledge corresponds to the mind-independent reality. This is a problem that Hutto's proposed account also faces. He accepts that "our existing ways of engaging with and thinking about mathematics are open to change and development" (Hutto 2019: 835). But how can we establish that these ways of engaging with mathematics approach knowledge about the mind-independent subject matter of mathematics? Hutto mentions as a merit of his proposal that an enactivist account combined with realism can endorse an extensionalist theory of truth like that of Tarski (1936/1983). While this is correct, at best this move can only save the objectivity of mathematical truth, while saying nothing about how mathematical practices can help approach truth, or indeed how we can recognize mathematical truth if we have reached it.

However, even though I see Hutto's suggested account as a less than convincing response to the objectivity challenge, I believe that he raises an important point. As mentioned earlier, losing objectivity of mathematical discourse is an important potential threat for radical enactivist accounts of mathematics. But is Hutto correct in stating that the kind of constructivist approach suggested by Lakoff and Núñez "falls a long way short of being an account of the objectivity of mathematics"?

The first thing that needs to be clarified is what is meant by objectivity of mathematics. Clearly objectivity of mathematical knowledge is a central tenet of platonist philosophy of mathematics (see, e.g., Panza & Sereni 2013). But in order to avoid circular descriptions, we should analyze mathematical objectivity without assuming the existence of mind-independent mathematical objects. Previously, I have argued that the target phenomena we should first be concerned with are the *apparent objectivity* of mathematical discourse and mathematical *applications* in science (Pantsar 2021b). By apparent objectivity, I referred to the widespread belief that mathematical truths are objective. Instead of treating this as an argument for the actual objectivity of mathematics, however, my purpose was to explain how we can explain the apparent objectivity without assuming that there are mind-independent mathematical objects. As I argue in detail in that paper, it is plausible that at least arithmetic appears objective to us because it is based on universally shared proto-arithmetical abilities. Against the kind of realist view that Hutto's position implies, I argue that cultures develop number concepts and arithmetic in convergent ways because some psychological processes and dispositions related to quantitative observations are shared by individuals across cultures. Because proto-arithmetical abilities are the product of biological evolution and universal to humans (as well as present in many non-human animals), these cognitive processes are as widely shared as any human cognitive processes. Hence, I have previously called such abilities *maximally intersubjective* (Pantsar 2014).

In Pantsar (2021b), I argue that discourse and knowledge based on maximally intersubjective psychological processes and dispositions are likely to appear as objective to us. Whenever cultures have developed arithmetic, they do so in ways that converge over finite number concepts and basic operations (Ifrah 1998; Pantsar 2019). The best explanation for this is that arithmetic is developed, both in ontogeny and phylogeny, on the basis of the universal proto-arithmetical abilities. Whether we consider the proto-arithmetical abilities to have evolved as adaptations to our environment or not, it is clear that they are a fundamental part of how we initially process quantitative information in our observations. It is therefore to be expected that arithmetic developed based on the proto-arithmetical abilities appears objective to us. It is also to be expected that scientific theories based on quantifying phenomena will end up applying the mathematical concepts based on proto-arithmetical and possibly other proto-mathematical abilities. (Pantsar 2021b).

We can hence explain both reasons for believing in the objectivity of arithmetic (as well as other areas of mathematics) without making any problematic realist assumptions. Now the question is whether this account of arithmetical objectivity is compatible with radical enactivism. Certainly my account seems to be against Hutto's conception of REC, as it draws from exactly the kind of sources that he dismisses as "neuro-fetishism." But my account of mathematical objectivity seems perfectly compatible with the REC accounts proposed by Zahidi and Myin and Gallagher. As I understand those views, they are in line with the position that arithmetical cognition draws in an important way on the proto-arithmetical abilities. What those accounts require is accepting two views. First, that the proto-arithmetical cognition does not involve representations or mental contents. Second, that non-representational and contentless cognitive abilities can influence the acquisition of abilities that are representational and contentful. This second view I take to be uncontroversial for the radical enactivist. The first one has been discussed in Section 5, and while the (empirical) jury remains out, the view is at least plausible.

But while unlikely based on the evidence, it is also possible that the proto-arithmetical abilities do not influence the development of arithmetical abilities. Now we can finally return to the matter set aside in Section 4. Recall that in addition to the "no content" strategy concerning proto-arithmetical abilities, the radical enactivist could also use the "no connection" strategy: denying that arithmetical cognition draws on proto-arithmetical abilities. This possibility does not play a role in the argumentation developed by Zahidi and Myin, but it is explicitly stated by Hutto (2019: 831–32) and seems to form an important part of his wish for a satisfactory REC account of mathematical cognition.

However, in this scenario the objectivity of mathematics relies entirely on a realist epistemology of mathematics. Indeed, mirroring the argumentation above, this scenario falls into two stages of problems. First, it is in danger of succumbing to the conventionalist threat because no common cognitive ground can be established between independently developed cultures in their development of numerical cognition. Second, if the conventionalist threat is solved by evoking realism, it faces all the problems of realist philosophy of mathematics mentioned earlier. Philosophically, the independence of arithmetical cognition from proto-arithmetical cognition is thus a highly problematic prospect. But equally worrying is the bad fit with the data on the development of numerical cognition. While culturally specific aspects are crucial in shaping the development of number concepts and arithmetical knowledge, there is a wealth of evidence that one or more of the proto-arithmetical abilities play a role at least in early number concept acquisition. These range from neuroscientific evidence of (partly) the same brain regions being associated with both proto-arithmetical and arithmetical cognition (most significantly the intraparietal cortex) to behavioral data showing that proto-arithmetical ability levels predict number concept learning and arithmetical ability levels (see, e.g., Piazza et al. 2007; Izard et al. 2008; Sarnecka & Carey 2008; Nieder & Dehaene 2009; Brannon & Merritt 2011; vanMarle at al. 2018; for more details, see Pantsar 2014; 2019; 2021a).

In face of all these problems, maintaining the position that arithmetical cognition is not in any significant way based on proto-arithmetical cognition becomes unfeasible. Fortunately, in accounts like the ones developed by Menary (2015) and Pantsar (2019; 2021a), there are better explanations of the development of arithmetical cognition available; ones that are both supported by empirical data and don't make epistemologically or ontologically problematic assumptions. Nevertheless, while I have proposed that the account I developed in Pantsar (2021b) is compatible with the REC account, Zahidi and Myin seem to be skeptical about this type of argumentation. What they (Zahidi & Myin 2016: 69–70) are worried about is that such "neuro-centric" accounts promote the brain as having the central locus of cognition, as done by Lakoff and Núñez:

> Ideas do not float abstractly in the world. Ideas can be created only by, and instantiated only, in brains. Particular ideas have to be generated by neural structures in brains, and in order for that to happen, exactly the right kind of neural processes must take place in the brain's neural circuitry. (Lakoff & Núñez 2000: 33)

Zahidi and Myin (2016: 69–70) explicitly argue against this kind of centralization of cognition in the brain. In particular, they criticize dismissing the importance of extra-cranial activity in the process. With the importance of sociocultural processes for the development of numerical cognition, this criticism seems well placed. What is needed for the REC account, in order to save the objectivity of arithmetic, is then a view that takes arithmetical cognition to draw from proto-arithmetical abilities without being tied to the kind of neuro-centrism of Lakoff and Núñez (2000).[14]

However, while I am not against eradicating excessive neuro-centrism, this prompts the question where exactly ideas are located. Clearly, in the radical enactivist view, ideas are not only in minds, but somehow in the wider extra-cranial (and extra-bodily) world. But where are the loci of mathematical ideas? As I have argued above, we cannot postulate a platonist ontology as the answer, since all that does is replace a problematic notion (representations in basic minds) with much more problematic notions (mind-independent abstract objects and epistemological access to them). So the radical enactivist faces an important challenge. Mathematical ideas are not located in the mind, but neither can they exist in a mind-independent manner. Where are they, and in what way do they exist?

I do not want to suggest a radical enactivist answer, but as I understand the matter, any feasible explanation needs to be compatible with the above argumentation I presented on the objectivity of mathematics. The components of development that I have identified, that is, proto-arithmetical abilities and socio-culturally shaped processes of enculturation, can both be present in a radical enactivist account of the development of arithmetical (and other mathematical) cognition. As a consequence, whatever the ontology of number concepts and arithmetical ideas may be in the REC account, if it is based on enculturation and proto-arithmetical cognition, the objectivity of arithmetical knowledge can be explained. For radical enactivist philosophy of mathematics, this would be an important move forward.[15]

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14. Indeed, the problem of neuro-centrism is present already on the level of the proto-arithmetical abilities. In addition to the neural characteristics, the abilities need a suitable environment of macro-level objects. This influence of the environment on our cognitive abilities is often spelled out in terms of *affordances* (Gibson 1979), which can refer to properties of the environment (e.g., Turvey 1992) or properties of the system consisting of the organism and the environment (e.g., Chemero 2009). However, given that the proto-arithmetical abilities appear to be universal among humans (see, e.g., Dehaene 1997/2011; Pica et al. 2004; C. Everett 2017), it seems that highly different environments provide the appropriate affordances for proto-arithmetical abilities. This is in line, *mutatis mutandis*, with the argumentation of Jones (2018), who claims that affordances relevant for the development of mathematical knowledge are present generally in human environments.

15. There is a potential realist counter-argument against my account, spelled out by Clarke-Doane (2012). He has argued that 1 + 1 = 0, for example, could realistically construed be a mathematical truth, while the logical truth equivalent to 1 + 1 = 2 would be the evolutionary advantageous one. Thus arithmetic based on proto-arithmetical abilities does not necessary coincide with arithmetical truth under a realist interpretation. While this possibility cannot be rejected, for the present account it does not cause problems. 1 + 1 = 0 goes against our proto-arithmetical abilities and *because* of that it cannot be a mathematical truth. We can certainly envision such deviant

## 7. Conclusion

Is a radical enactivist account of arithmetical cognition feasible? I have tried to argue that, in addition to future empirical data, this depends ultimately on what we understand by representations and contentful cognition. Echoing the "don't need" strategy of the REC account, Zahidi and Myin argue that REC is consistent with there being innate mechanisms for numerosities but it is not necessary to posit any representational structures to account for them (Zahidi & Myin 2016: 86). I agree that there is neither any reason nor need to assume that we are born with innate number concepts or innate arithmetical ability. I also agree that processes of enculturation and material engagement are crucial for the development of arithmetical cognition. Furthermore, I agree that discourse on numerical knowledge is best restricted to truth-telling practices, which in turn requires linguistic capacities in sociocultural settings. And, finally, I agree with the general ethos of the REC program that we should not make unnecessary ontological assumptions when explaining behavior.

Consequently, I share the radical enactivist view that infants do not possess number concepts, arithmetical ability, or indeed numerical knowledge of any kind. All these arise in sociocultural contexts through processes of enculturation, but they are not present in basic minds. Yet it is important to note that none of this precludes the possibility of there being numerosity representations in basic minds. Postulating innate number concepts and arithmetical abilities is flawed, but there could still be innate dispositions for representing numerosities. The resulting representations could be, for example, related to visual experiences of particular physical objects.

The focus of the REC literature so far has been on the approximate number system but, as argued in Section 5, the object tracking system potentially provides a more difficult challenge for the radical enactivist account, for at least two reasons. First, in what I see as the strongest current theory for number concepts acquisition in ontogeny, the bootstrapping account of Carey (2009) and Beck (2017), the OTS plays a more important role in the acquisition of number concepts than the ANS does.[16] Second, the best explanation of the functioning of the OTS includes a mechanism that can be plausibly understood as employing representations. If the occupancy of object files can be seen as a representation of quantity, then strictly speaking basic minds do have numerosity representations. These are a far way off from innate number concepts, but still problematic for the radical enactivist account. It is in this and other

arithmetical systems, but unless we are already committed to mathematical realism, it is difficult to see their relevance in the present context (see Pantsar 2014 for more).

16. However, as I have argued in Pantsar (2021a), even if we accept the bootstrapping account, it is likely that the ANS plays some kind of role in the process of number concept acquisition.

similar questions—concerning, for example, the functioning of the ANS—that the feasibility of the REC account for the development of numerical cognition is ultimately assessed.

My skeptical and critical points about the radical enactivist literature are not meant to suggest that there could not be a feasible explanation of both the OTS and the ANS in terms of non-representational, contentless mechanisms. Instead of arguing against the REC account, I have wanted to locate the real crux of the matter when it comes to numerical and arithmetical cognition. Clearly the criticism of Zahidi and Myin is to the point if the targets are putative innate mature number concepts, arithmetical abilities and arithmetical knowledge. But as I have tried to show, a representationalist account of arithmetical cognition does not need to include such unfeasible assumptions. The representations being discussed can be something much less problematic, as in the case of occupied object files.

As a positive proposal, I have submitted that the REC account is compatible with other accounts of mathematical knowledge that take it to draw on our proto-arithmetical and other proto-mathematical abilities (e.g., Pantsar 2014; 2015; 2016; 2018; 2021a; 2022). This has important consequences when we consider the question of objectivity in mathematics. I have argued that REC accounts face the danger of conventionalism since mere material engagement could feasibly lead to highly divergent trajectories of numerical abilities in different cultures. As I have argued, however, this is not supported by empirical evidence and the best explanation for this is that universal proto-arithmetical abilities partly determine the content of arithmetical practices and knowledge. This is compatible with the REC account if the proto-arithmetical abilities do not involve mental representations, thus giving an explanation of arithmetical objectivity based on radically enactivist numerical cognition. And unlike the view presented by Hutto (2019), this account does not face the prospect of problematic realist epistemology or ontology.

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