



Semantic Information and the Complexity of Deduction

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Abstract

In the chapter “Information and Content” of their *Impossible Worlds*, Berto and Jago provide us with a semantic account of information in deductive reasoning such that we have an explanation for why some, but not all, logical deductions are informative. The framework Berto and Jago choose to make sense of the above-mentioned idea is a semantic interpretation of Sequent Calculus rules of inference for classical logic. I shall argue that although Berto and Jago’s idea and framework are hopeful, their definitions do not capture what is intended. This is so because the definitions are solely based on the logical complexity of an argument and they fail to capture the richness of the non-logical content of that argument. Then I will suggest some amendments to address this problem. Finally, I will extend the application of the definitions to first-order logic. It shall be observed that in some classical deductions, applying contraction may lead to hiding some epistemic impossibilities. This happens when formulae which contribute different propositions to the proof get contracted at the quantified level.

1 Introduction

In the chapter “Information and Content” of their *Impossible Worlds*,¹ Berto and Jago suggest a theory of semantic information (BJTSI) in deductive reasoning such that we have an explanation for why some, but not all, logical deductions,² are informative. This offers a solution to the problem of informativity of deduction, usually referred to as The Scandal of Deduction (SD). They also argue that their account of deductive information content provides an answer to the problem of Logical Omniscience (LO). In what follows, it will be argued that although their approach is promising, it needs some amendments in order to provide a satisfactory solution to the SD problem and that those amendments give us a different solution for the LO problem.

¹ Berto and Jago (2019).

² In this article, whenever we talk about logic without modification, we mean classical logic.

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First, we review the literature on semantic information and explain how BJTSI approaches it. We emphasize the connection between hyperintensionality and granularity of BJTSI on the one hand, and their reliance on the syntax of formulae on the other. In the Sect. 3, we review BJTSI and highlight its key features. Section 4 examines the technical definitions Berto and Jago use to capture their philosophical points about epistemic possibility. Considering the connection between the framework of BJTSI and the syntax of formulae, it will be argued that the semantic and the information content differ. Then it will be discussed that to fully capture Berto and Jago's philosophical intentions, the BJTSI definitions should keep track of information content rather than semantic content. In order to address this issue, some amendments will be suggested.

We extend BJTSI to First-Order Logic (FOL) in Sect. 5. This section begins by arguing for the significance of such an extension. Hidden Impossibilities (HI) shall then be introduced in SC as a result of the application of the structural rule of contraction. In addition, some BJTSI definitions will be updated based on HI. In the final section, we address the implications of modified BJTSI regarding the answer this theory provides to the Logical Omniscience problem.

2 Semantic Information

The first problem BJTSI seeks to address is how some deductions, but not all, are informative. This contrasts with a literature that holds that logical theorems are uninformative. Bar-Hillel and Carnap³ were pioneers in the development of the concept of information. Using a combination of inferentialism and Carnap's intensional understanding of declarative sentences, they developed a theory of information known as Semantic Information. According to SI, a statement's content is the set of possible worlds in which it is false. Alternatively, the content of statement S is the set of scenarios or possibilities that its truth excludes. The reason for this is that one can infer those contents based on a false assumption, namely S .

In this setting, nothing can be inferred from a statement that is always true while everything can be inferred from a statement that is always false. The first group of statements consists of logical truths. Since logical truths hold true in every possible scenario, they do not exclude any possibility. Therefore, they do not contain any information. Conversely, a logically false statement is false in every possible circumstance and everything can be inferred from it, thus conveying too much information. These are both counter-intuitive results that are considered to be disadvantages of SI. The first problem is typically referred to as the Scandal of Deduction (SD) and the second problem as the Bar-Hillel and Carnap Paradox (BCP).⁴

A variety of approaches have been taken in the literature to address SD and BCP. Hintikka argues, for instance, that since polyadic FOL is undecidable, theorems in this section of FOL are informative.⁵ Floridi and D'Agostino counter that undecidability

³ Carnap (1953).

⁴ Sequoiah-Grayson and Floridi (2022).

⁵ Hintikka (1973).

is not a suitable measure, and that the efficiency of the decision process is more appropriate.⁶ In their view, since the Tautology Problem is computationally difficult (NP-hard), that is there is no efficient method for determining theoremhood of any given expression, any system that consists of theorems cannot be applied to measuring information. As a system suitable for measuring deductive information, they propose a signed natural deduction system without theorems that has an efficient decision making process.

With regard to the BCP, Floridi argues that Bar-Hillel and Carnap's theory is insufficient to understand the notion of information, since truth supervenes information in SI. In his view, information should be defined as truthful data.⁷ A false statement, including logically false statements, does not fall within the definition of an informative statement if we limit information to true data.

In Floridi's responses to SD and BCP, a common theme is to begin with a notion of information. This is to define semantic information as a kind or subcategory of that notion. Semantic information can, however, be understood in other ways. One can, for example, begin with semantics and then develop a notion of information within it. The work of Bar-Hillel and Carnap may be understood in this manner. On the basis of Carnap's intensional semantics, they developed their theory of information.⁸ Thus, Floridi's responses to SD and BCP represent a methodological departure from Bar-Hillel and Carnap's SI.

Considering the distinction mentioned above, BJTSI attempts to address SD without a methodological shift. To be able to discuss informative logical truths, where information has semantic content, they adopt a hyperintensional approach to semantic content.⁹ Berto and Jago construct their hyperintensional semantics by adding impossible worlds to possible ones. They attempt to capture epistemically possible, but logically impossible scenarios that a competent, but not omniscient, rational agent might entertain in which premises are true, but conclusions are false. This is before the proof of the argument is presented. As it eliminates some epistemic possibilities, the proof of such an argument is an informative deduction. As Berto and Jago's work addresses the SD problem, BCP falls outside the scope of our discussion. Nevertheless, their approach opens up some possibilities for addressing the BCP issue as well.

The possible and impossible worlds in BJTSI are linguistic ersatzes. As Jago asserts, linguistic ersatzes are as fine-grained as the syntax of the language in question.¹⁰ Both he and Berto provide multiple explanations for this level of granularity.¹¹ We will discuss soon whether this level of granularity is desirable for information content. With this approach, we would like to emphasize that the measures that are used to measure the informativity of deduction should take into account its syntactic complexity. However, BJTSI definitions do not fully meet this requirement.

⁶ D'Agostino and Floridi (2009).

⁷ Floridi (2004).

⁸ This is also suggested in Sequoiah-Grayson (2008).

⁹ This approach has been explained and advocated in Jago's previous works like Jago (2015).

¹⁰ Idid, p.592.

¹¹ Ibid and Berto and Jago (2019).

This hyperintensional account of semantic content faces the challenge of modeling epistemic possibility with appropriate properties. By definition, logically impossible worlds behave illogically. It should, however, be rational, but not completely logical, i.e., it should not be fully closed under any logical consequence. If epistemic possibilities are closed under any logical consequence, then agents must be aware of all epistemic possibilities that logically follow from their existing knowledge. In their work, Berto and Jago refer to this as the Logical Omniscience (LO) problem. In Sect. 6, we will discuss LO in more detail.

3 Berto and Jago's Theory of Semantic Information

Using propositional classical logic, BJTSI proposes an approach to the SD problem. This approach is based on an idea, a framework that captures that idea, and a definition of what constitutes an informative argument within that framework. The idea is that a deduction—shown as $\Gamma \vdash \Delta$ —is informative if the set of epistemically possible but logically impossible worlds in which all premises are true and all conclusions are false—shown as $|\Gamma|^+ \cap |\Delta|^-$ and called ‘world-proof’ for reasons we shall see shortly—is not empty. An argument with an empty epistemic space is trivial. The triviality of an argument can be defined as follows:

Triv1: Argument $\Gamma \vdash \Delta$ is trivial if and only if the impossible world-proof $|\Gamma|^+ \cap |\Delta|^-$ has no content.

According to BJTSI, the information content of an argument is represented by world semantics in which the logically possible and impossible worlds are linguistic ersatz.¹² If an argument is valid, these worlds are epistemically possible, but logically impossible. In such a world, almost anything logical can go wrong. For example, a conjunction may be true while either or both of its conjuncts are false. Most of these logically impossible scenarios are still epistemically entertainable according to BJTSI; we may hold logically inconsistent beliefs. According to this view, the only thing that makes such logically impossible worlds epistemically impossible is that the same formula holds both true and false.

Berto and Jago employ a semantic interpretation of Sequent Calculus (SC) rules of inference for classical logic in order to explain the above-mentioned idea. In this interpretation, in a valid argument, inference rules link logically impossible worlds. A proof tree in SC represents a world in which the formulae on the antecedent side are true and the formulae on the succedent side are false. Thus, we have a semantic interpretation of each step of a proof in SC. From the root to the top of a proof tree, the set of true formulae in the upper step is the union of the set of true formulae in the upper step with the set of true formulae in the lower steps. The same applies to the set of false formulae.

An example will illustrate how the idea is applied to the framework. Consider a simple modus ponens instance such as $p, p \supset q \vdash q$. In this argument, the semantic content is the set of worlds in which p and $p \supset q$ are true and q is false. To be more

¹² Jago (2015), p.591.

specific, the worlds with $\{p^+, (p \supset q)^+, q^-\}$ as their subset. This is the proof tree for this argument:

$$1. \frac{p \vdash p \quad q \vdash q}{p, p \supset q \vdash q} L \supset$$

For the same argument, here are the corresponding worlds:

$$2. \frac{w_1 = \{p^+, p^-\} \cup \{p^+, (p \supset q)^+, q^-\} \quad w_2 = \{q^+, q^-\} \cup \{p^+, (p \supset q)^+, q^-\}}{w = \{p^+, (p \supset q)^+, q^-\}}$$

The argument for which we are interested in defining the semantic content is connected to two worlds at the top of the tree, w_1 and w_2 . It is the union of the set of sentences in w and the set of sentences in w_1 that constitutes the set of sentences in w_1 . Based on the values stipulated in the framework, we obtain $\{p^+, (p \supset q)^+, q^-, p^-\}$. For w_2 , the output is $\{p^+, (p \supset q)^+, q^+, q^-\}$. The semantic content of the argument $p, p \supset q \vdash q$, that is, the set of worlds in which premises are true and conclusions are false, includes two worlds w_1 and w_2 . Since both of these worlds contain a formula that is true and false at the same time, they are epistemically impossible. Therefore there is no epistemically possible world or scenario in the semantic content of the argument and as a result it does not convey any information. Here is a definition of empty content:

Empt1: The impossible world-proof $|\Gamma|^+ \cap |\Delta|^-$ has no content if and only if all impossible world-proofs immediately above it contain an identical formula A assigned as both true and false.

As can be seen, the leaves of the proof tree represent blunt epistemically impossible scenarios or worlds. In BJTSI, an informative deduction is defined according to how close its root is to these obvious epistemic impossibilities. A cut-free proof tree of $\Gamma \vdash \Delta$ in SC exists if $\Gamma \vdash \Delta$ can be classically proved. A world in which all members of Γ are true and all members of Δ are false is epistemically possible and logically impossible only if it is far enough from the obvious impossible worlds represented by leaves of the proof tree in which a proposition is taken to be both true and false. To put it another way, if the proof tree is not too small.¹³

The epistemic possibility is then modeled using non-trivial deductions. A model of this type is defined as a tuple of worlds, normal worlds, and a valuation function. The length of a tree is defined as the *rank* of its highest leaf. From here, A is regarded as a trivial consequence of Γ based on the rank of the $\Gamma \vdash A$ proof tree.¹⁴ There are two properties of triviality: monotonicity and nontransitivity. They are defined in the following manner:¹⁵

Mono: If $triv_n(\Gamma, A)$ and $\Gamma \subseteq \Delta$, then $(triv_n(\Delta, A))$.

N-tran: For $n \geq 1$ it is not the case that if $triv_n(\Gamma, A)$ and $triv_n(\Gamma \cup A, B)$, then $triv_n(\Gamma, B)$.

So is the epistemic possibility that is defined based on this concept of triviality. A consequence-like structure is provided by these definitions, which does not preserve truth, but prevents falsity.¹⁶

¹³ In this article propositions are used to talk about the semantic of formulae which are syntactic entities.

¹⁴ Berto and Jago (2019), p.204.

¹⁵ Ibid., p.205.

¹⁶ Ibid., p.204.

Berto and Jago argue that their way of modeling epistemic possibility has the following desirable features: There is little assumed about the behavior of epistemically possible but logically impossible worlds; the only assumption made is that impossible worlds in which one and the same formula is both true and false are obviously epistemically impossible. There is a non-trivial way of defining informative deductions; not all deductions are informative, only those with some level of logical complexity are informative. Additionally, it captures the vague nature of epistemic possibility by using the vagueness of deductive information. And finally, it explains the LO problem by explaining why it is not possible for us to know all of the consequences of what we know. This is due to the fact that epistemic possibility, as defined by triviality, cannot be transitive.

There are a few points that may require further explanation. The first claim that there is not much assumed about the behavior of impossible worlds is due to the properties of SC systems. Except for Cut, all SC rules have the subformula property. This means that in all of these rules, whether structural¹⁷ or operational,¹⁸ the formulae above the inference line are subformulae of those below. In particular, when it comes to operational rules, the focused formula below the line is more complex than the one(s) above. Consequently, in the BJTSI setting, as one moves from bottom to top, immediate subformulas of a given formula receive their own truth value. This is regardless of the value of the complex formula below.

For instance, in the case of Modus Ponens, as we saw in (1), $p \supset q$ is assigned positive, while p is positive and q is negative in either of the world-proofs above the line. As a result, it is clear at each stage of the reasoning what type of impossible world we are dealing with. Consequently, our impossible worlds are defined by directly assigning values to their elements (formulae) without assuming much more about them. And as mentioned before, almost everything logical could go wrong in these impossible worlds. Essentially, this semantics models epistemic possibility based on only one assumption: the logically impossible worlds in which a formula is both assigned positive and negative are epistemically impossible. Due to the properties of the chosen framework, Berto and Jago are able to overcome the challenge of defining what type of impossible worlds they are dealing with without making many assumptions about those impossible worlds.

The next point relates to the issue of LO; we will discuss it in greater detail in Sect. 6, but in the meantime some clarification is helpful. It refers to the question of whether knowledge is closed under any consequence relation. In epistemic logic, the modal operator K denotes the fact that an agent ‘knows that...’. A basic axiom of epistemic logic holds that if an agent knows that p (Kp) and $p \supset q$ ($K(p \supset q)$), then they know q (Kq).¹⁹ This is referred to as closure under implication, or more broadly closure

¹⁷ Structural rules are the rules with no logical operation. They express general rules of combining proofs. They are divided into right and left rules to define what inferential move is accepted in the right hand side and the left hand side of the turnstile respectively.

¹⁸ Operational rules define ways of treating formulae with particular logical constants as the focused formula. They come in form of right and left rules and define what can be inferred from the focused formula when it appears in the right hand side or the left hand side of the turnstile respectively.

¹⁹ Rendsvig and Symons (2021)

under the consequence relation. The level of idealization in epistemic logic has been criticized as not reflecting the reality of the human mind.

If one believes that closure under a consequence relation (consistent or paraconsistent) is required for our epistemic states to have a semantic structure, then they are facing a kind of dilemma or paradox: either our epistemic states have structure and are closed under a sort of consequence relation or not. If they are, we face the LO problem since we don't know all the repercussions of our knowledge. If this is not the case, i.e., if our epistemic states are not closed under any consequence relation, then there is no semantic structure.

As we discussed, epistemic possibility has some structure in BJTSI. As an example, it is monotonous, but not transitive. This very structure explains why we are not aware of all the consequences of what we know. We do not trivially know all the consequences of what we know due to the non-transitive nature of triviality or the non-informativeness of deduction. Sometimes, we must run through a lengthy deduction process before we can arrive at a conclusion. Therefore, if accepted, BJTSI would provide some structure to the epistemic possibility while also avoiding the dilemma between an unstructured epistemic space and the problem of LO.

From what have been said in the last two sections, the evaluation of the idea and framework of BJTSI is as follows: the idea that a logical deduction is informative if the possibility of the premises of the deduction being true while its conclusion is false is epistemically entertainable is promising. As a result of the process of deduction, these epistemic possibilities are eliminated. Intuition resonates well with the connection between being informative and ruling out some options. If we are looking for a car in a parking lot full of different colors, knowing that the car we are looking for is blue is helpful. This is because it prevents further consideration of the red car and the black car. It is difficult to apply this concept to logical deduction. Our intended car in the example was blue, but we could have entertained the possibility that it could also be red or black. The possibility that premises hold but the logical consequences do not is actually entertaining some impossibilities when examining classically valid deductions. Therefore, considering logically impossible scenarios or worlds is an effective way to extend the intuitive understanding of the informativity of expressions from contingent to logical truths. In particular, if we are seeking an objective measure of informativity, then worlds semantics may help us achieve this objective measure.

The SC presentation of classical logic is an effective framework for narrowing down the type of impossible worlds we are dealing with by allowing us to specify the class of impossible worlds based on the classical law that has been broken. Furthermore, it is compatible with the approach BJTSI has to impossible worlds, which is to view impossible worlds as linguistic ersatzes. This is also consistent with a known understanding of the semantics of SC, which states that asserting premises and denying conclusions is not logically correct.²⁰ Of course, what is logically incorrect is still epistemically possible and can be used to measure the competence of an agent. At least in the classical sense of logic, asserting and denying the same statement constitutes a paradigmatic case of logical incompetence. In SC, initial sequences represent this.

²⁰ This idea usually goes under the title 'bilateralism' and has been discussed as a way to justify classical consequence relation. Restall (2005) is a well-known example of this literature.

Therefore, the extent to which agents are able to realize these obvious impossibilities appears to be a good indicator of their logical competence as well as the informative value of deduction in terms of ruling out these impossibilities.

We have so far seen Berto and Jago's idea and framework. By utilizing a hyper-intensional account of semantic content, in which worlds are linguistic ersatzes, it explains the informativity of deduction by interpreting the inference steps in classical SC proofs. It is pertinent to note that this approach to the semantic content of a sentence is as fine-grained as the syntax of the language used in the deduction. Our next step will be to review BJTSI's definitions. It will be argued that although the idea and framework are useful, the definitions do not capture what is intended. It is because semantic and information content should not be treated as being identical within a BJTSI setting. It will be shown in the following section that this results in definitions that merely reflect the logical complexity of arguments and do not reflect the richness of their non-logical content. Furthermore, it will be argued that the richness of non-logical content is critical to capturing the information content of arguments. In order to address this issue, we will propose some amendments.

4 Logical Complexity and The Richness of Content

BJTSI proposes worlds as linguistic ersatzes as semantic content for SC deduction to capture deduction's informativity. In this way, the syntax and semantics of deduction are closely related. According to linguistic ersatzism, the content of a formula A is the set of all worlds in which A is true. As Jago has noted,²¹ this content is as fine-grained as the syntax of our deductive system. Thus, every single difference in syntax has a corresponding impact on semantics. For instance, the contents of p , $p \wedge p$, and $p \wedge \neg p$ are the sets of worlds in which p , $p \wedge p$ and $p \wedge \neg p$ are true respectively. The set of worlds in which $p \wedge \neg p$ is true is a subset of classically impossible worlds, and the worlds in which $p \wedge p$ is true differ from the worlds in which p is true.

Language ersatzism makes differences between formulae at the atomic level that are all understandable from an informational perspective. For instance, p and q convey different pieces of information, where p and q represent atomic propositions. In other words, differences in syntax correspond to differences in information content at the atomic level. When complex expressions are considered, problems arise. As an example, according to this semantics, p and $p \wedge p$ have different semantic contents. Our understanding of information is at odds with such a degree of granularity. The amount of information conveyed by p and $p \wedge p$ is the same.

The application of formal methods to philosophical problems often involves taking some philosophical notions for granted and justifying formal aspects of a theory on the basis of these assumed notions.²² Therefore, relying on some intuitive ideas about information can be considered acceptable, without much justification. The intuitions we are discussing here may, however, be rooted in an inferential understanding of

²¹ Jago (2015), p.592.

²² For example, Floridi takes certain ideas about information for-granted and then attempts to justify the formal side of his work based on those ideas about information.

information. In accordance with this understanding, two inter-derivable expressions convey the same amount of information. Therefore, for any given formula A , $A \wedge A$ and $A \vee A$ are equally informative, while $A \supset A$ is less informative. Negation changes the information content; $\neg A$ conveys a different piece of information from A . Regardless of whether or not these reasons apply, if we wish to follow the above-mentioned insights regarding information, semantic content is different from information content, since semantic content is more fine-grained than information content.

This difference is not taken into account by BJTSI. As a consequence, the formal definitions of BJTSI, which capture differences in the semantic content of inferential steps in SC, verify some inferences as informative even though they may not correspond with our intuitions about information. Those definitions include all the informative deductions, but do not exclude all the uninformative deductions. The BJTSI definition of *trivial consequent*, which is based on the rank of a world, is not an appropriate measure of triviality. The reason for this is that it does not include all instances of trivial deductions. Therefore, this definition overproduces when used to define non-trivial or informative deductions. Fortunately, since information content is less fine-grained than semantic content in BJTSI, and definitions are based on the latter, we can modify those definitions to capture semantic information.

Let us examine the relevant definition in more detail. Here is the definition of the rank of a world where model M is a tuple consisting of a set of worlds W , a set of normal worlds N which is a sub-set of worlds, that is $N \subset W$, and a valuation relation ρ :²³

Rank: Given a model $M = \langle\langle W, N, \rho \rangle\rangle$ and a world $w \in W$, we define w 's rank, $\#w$, as the size (number of nodes) in the smallest world-proof rooted at w , if there is one, and ω otherwise. The rank of model M is $\min\{\#w \mid w \in W\}$.²⁴

The logical complexity of a deduction can be accurately measured using this definition. Each node of a proof tree represents an inference rule, and when we are dealing with logical inference rules, as we are here, it represents the number of logical steps we must take to reach the initial sequents (in the form of $A \vdash A$). These sequents are the obvious inconsistent scenarios that are not even epistemically possible. This is all that is required to capture the epistemic space of a deduction if we use the semantic content to model epistemic possibility. However, if we wish to use information content to simulate epistemic possibility, then logical complexity is not the only factor to consider, as a rich non-logical²⁵ vocabulary is also important. Despite being logically more complicated than p , $p \wedge p$ conveys the same amount of information. Only at the atomic level do the gradients of information match the gradients of syntax. Thus, richness of the non-logical content is the aspect of syntax that captures information content.

²³ ρ relates (i) each atomic sentence to exactly one truth value in N , (ii) each sentence to zero or one or both in $W - N$ worlds. Berto and Jago (2019), p.204.

²⁴ A pointed model is a pair $\langle\langle W, \omega \rangle\rangle$ where ω is a world in M . Ibid, p.204.

²⁵ Non-logical content refers to the part of syntax that is not a logical connective or quantifier.

To some extent, this is also implicit in BJTSI. As an example of an informative argument, Berto and Jago use the sequent (3).²⁶ It is helpful to compare sequence (3) with sequence (4), where p_i s and q are atomic formulae:

3. $p_1, p_1 \supset p_2, p_2 \supset p_3, \dots, p_n \supset p_{n+1} \vdash p_{n+1}$
4. $p, (p \supset_1 (p \supset_2 (p \dots (p \supset_{n-1} p) \dots))) \supset_n q \vdash q$

Given that the meaning of the conditional, like the other logical connectives does not add to or remove any information from the context of deduction,²⁷ argument (3) creates a much wider epistemic space than argument (4). This is so even though they have the same rank due to having the same number of logical connectives. To prove P_{n+1} , one must first establish p_2 , up to p_n . Since none of p_2 , up to p_n are established information prior to the deduction, it is possible to resist them without making an obvious epistemological error. However, in order to prove q in argument (4) one must establish p a tedious number of times in order to establish q . Thus, holding p as true while refusing to hold q as true is not as epistemically plausible as holding p_1 true while refusing to hold P_{n+1} as true.

It should be noted that this observation presents no technical problem for BJTSI definitions, but it does demonstrate that they fail to capture the intended concept of epistemic possibility. In BJTSI, epistemic possibility is best explained by semantic information rather than semantic content. We have seen that these are two different things within the semantic framework of BJTSI. This argument can be extended to any definition that relies solely on the shape of a proof tree. It is not possible to capture the epistemic space intended by Berto and Jago with a definition such as this.²⁸ Thus, the notion of trivial sequent, which is based on the insight that proof trees with large rank create epistemic spaces, fails to model the epistemic space.²⁹

It is useful to dig deeper into the insight that argument (4) is not as informative as argument (3). Epistemic space is created when certain epistemically trivial impossibilities are not readily accessible, that is, they are not surfaced early in the proof process. An impossibility's rank is not an accurate indicator of its accessibility, however. At the root of the proof tree for argument (4) is the initial statement $p \vdash p$; however, from the second step of deductive inference onward, all sequents include only one atomic formula on each side.³⁰ In the second step of the proof tree, the obvious impossibility of the initial sequent $p \vdash p$ becomes evident. While in the case of argument (3), there are $n + 1$ initial sequents and only one surfaces in the second step of the proof, the rest surface in the deeper levels on the proof tree. As a result, argument (3) creates more epistemic space than argument (4).

²⁶ *ibid.*, p.203.

²⁷ Berto and Jago in *Ibid* p.199 emphasize that there is nothing special about the meaning of the conditional.

²⁸ One might think of the number of leaves in a proof tree as representative of content richness, but the proof tree of $p, (p \wedge_1 p \wedge_2 p \dots p \wedge_{n-1} p) \supset q \vdash q$ has as many leaves as the proof tree of argument (3) does while being as contently poor as argument (4). The shape of proofs does not necessarily reflect the richness of the non-logical content. And as we saw, this is that richness that is an appropriate measure of information content and this content is what better captures Berto and Jago's intended epistemic space.

²⁹ To check the definition see Berto and Jago (2019), p.204.

³⁰ Note that the conditional we are dealing with does not have a special semantic character similar to that of, say, relevance logic conditional.

The moral to be taken is that initial sequents (sequents in form of $A \vdash A$) are not the only instances of blunt epistemic impossibilities. It is true that they were so if we were to use semantic content to model epistemic impossibility. However, not if we were to rely on semantic information to model this type of impossibility. To use semantic information to model epistemic impossibility, we must change our formal definition of epistemically impossible proof-worlds such that they include not only the initial sequents, but also cases where the formulae on both sides of the turnstile convey the same amount of information. Our insights regarding information content³¹ suggest that our definition should include initial sequents, as well as sequents in which formulae contain either only A or only $\neg A$. Therefore, epistemic impossibilities (EI) can be defined as follows:

EI: The world-proof $|\Gamma|^+ \cap |\Delta|^-$ is epistemically impossible if and only if its corresponding sequent includes either an identical formula A on both sides or it includes formulae containing only A or only $\neg A$ on both sides.

After discussing modeling epistemic impossibilities, let us turn our attention to their role in the BJTSI explanation of informativity of deduction. According to BJTSI, the rank of these blunt impossibilities indicates the difficulty of reaching them. The further epistemic impossibilities are from the root of the proof tree, the greater the effort required to become aware of them and consider them. In technical terms, the rank of an impossibility determines whether the content of a deduction is empty.³² However, the relationship between the length of proof and the complexity of this mental journey requires further investigation. An example will help us better understand the problem. In our discussion of information and its inferential roots, we mentioned that $p \supset p$ is less informative than p . Due to the same reason, $p \supset (p \supset p)$ is less informative than $p \supset p$. Accordingly, the cut-free proof of sequent (6) is longer than that of (5) in SC.³³ The only difference between sequents (5) and (6) is that $p \supset p$ is replaced by $p \supset (p \supset p)$. That is, with the same premise p , it takes more effort to get the same conclusion, namely p , if we replace $p \supset p$ with $p \supset (p \supset p)$.

- 5. $p, p \supset p \vdash p$
- 6. $p, p \supset (p \supset p) \vdash p$

However, the mental effort required to prove arguments represented by sequents (5) and (6) is the same in the following sense: according to BJTSI, in both cases we entertain and rule out the same type of worlds, namely ones in which p is considered both true and false.³⁴ We might entertain such worlds for slightly longer in the proof of sequent (6), but nonetheless, the worlds are very similar. If we compare the cut-free proofs of sequences (5) and (6) with those of sequences (7) and (8), we realize that in the case of (7) and (8), we consider and rule out a wider range of worlds respectively. In (7), worlds with p both true and false in them, and worlds with q both true and false

³¹ For any given formula A , $A \wedge A$ and $A \vee A$ are as informative as and $A \supset A$ is less informative than A . And negation changes the information content; $\neg A$ carries a piece of information different from that of A .

³² As we reviewed in Sect. 3.

³³ The same goes for normal proofs with antecedent formulae as open assumptions and the consequent as conclusion in Natural Deduction.

³⁴ Recall, as mentioned in Sect. 3, in BJTSI we are working with cut-free proofs in SC.

in them. And in (8), worlds with p both true and false in them, worlds with q both true and false in them, and worlds with r both true and false in them.

- 7. $p, p \supset (p \supset q) \vdash q$
- 8. $p, p \supset q, q \supset r \vdash r$

If we extend Carnap and Bar-Hillel’s idea of informativeness, which has a direct relationship with ruling out more possibilities, to include impossibilities, we will then be able to bring the mental effort required for building deduction in line with its informativeness. In order to implement some of these insights in the BJTSI definitions, we expanded the definition of epistemic impossibilities. And to implement the rest of them, we are proposing to introduce the notion of *depth* which is stricter than rank. By excluding the sequents that proving them does not require consideration of a variety of impossible worlds, this notion intends to capture the discussed insights about mental effort and informativeness of deduction.

The rank of an epistemic impossibility does not capture the depth of it. For instance, in the proof tree (9), the same initial sequent appears in three different ranks. According to BJTSI’s interpretation of proof trees, each of these initial sequences represents a distinct impossible world. Nevertheless, when taking into account the depth of the obvious epistemic impossibility represented by the initial sequent $p \vdash p$, it is hardly convincing to claim that the one ranked 3 represents a deeper impossibility than the one ranked 1. The reason for this is that despite the fact that worlds $w_1 = \{p^+, (p \supset (p \supset (p \supset q)))^+, p^-, q^-\}$ and $w_5 = \{p^+, (p \supset (p \supset (p \supset q)))^+, (p \supset (p \supset q))^+, (p \supset q)^+, p^-, q^-\}$ contain different formulae, they are the same type of epistemically impossible worlds, namely the world represented by the initial sequent $p \vdash p$ where p is both true and false. And once such an impossibility came to our attention, its next appearance is no more informative.

$$\begin{array}{c}
 \frac{p \vdash p \quad q \vdash q}{p, p \supset q \vdash q} L \supset \\
 \frac{p \vdash p \quad \frac{p, p \supset q \vdash q}{p, p \supset (p \supset q) \vdash q} L \supset}{p, p \supset (p \supset (p \supset q)) \vdash q} L \supset \\
 9.
 \end{array}$$

The depth of an epistemically impossible world-proof should be stricter than the rank of it in order to exclude arguments represented by sequent (4) and the root sequent of proof tree (9) from the set of informative arguments, i.e. arguments with epistemically possible content. In order to achieve such a more restrictive definition, we may adopt the following definition:

Depth: The depth of an epistemically impossible world-proof $|A|^+ \cap |A|^-$ is equal to the smallest rank its corresponding world-proof appears in (from bottom to top).

This definition is justified by the fact that the smallest rank of an epistemically impossible world-proof encodes when we first became aware of the impossibility during the proof. In the case of inference rules that branch from bottom to top, this definition may lead to proof steps that involve world-proofs of varying depths. As an example, the topmost right branch of the proof tree (9) contains the sequents $p \vdash p$ with depth 1 and $q \vdash q$ with depth 3. The depth of a proof step may be defined in one of two ways: either by asking ourselves at what point we encountered either of these epistemic impossibilities, or alternatively by asking ourselves at which point we became

aware of both. Based on the branching nature of the proof step, the first question is appropriate. By adopting the first question, the depth of a proof step can be defined as follows:

PS-Depth: The depth of a proof step is the minimum of the depth of its proof-worlds.

Now with redefining the impossible world-proofs with no content,³⁵ we will have a definition that includes all the intended informative proofs and excludes all the trivial ones:

Empt2: The impossible world-proof $|\Gamma|^+ \cap |\Delta|^-$ has no content if and only if its shortest world-proof has no proof step with a depth bigger than 1.

In accordance with the above definition, a sequent has no information content if none of the proof steps in its proof have a depth greater than one. A proof step with depth 1 contains at least one sequent that represents an epistemically impossible world that is surfaced during the first step of reasoning. For instance, the sequent at the root of the proof (9) has no information content because it has a three-step proof and in each of these steps there is an initial sequent (epistemic impossibility) with depth 1. The initial sequent at rank 2 and one of initial sequents at rank 3, still has depth 1 due to being identical to the initial sequent that appears at the first step of the proof (from bottom). Sequents (5) and (6) have depth and rank 0 as they represent epistemic impossibilities.³⁶

By modifying the definitions in this way, sequents with long, but not very content-rich proofs will be excluded. The concept of depth is sensitive to non-logical content within a sequent, unlike the concept of rank. It is so because the identity of an initial sequent plays a crucial role in determining its depth. This approach captures the accessibility of epistemically impossible proof-worlds in a manner that is more in line with the intent of BJTSI. Moreover, the definition of triviality defined by this modified notion of empty content excludes unintended cases while maintaining the intended ones.

Moreover, this new definition of triviality retains all of the desirable properties of the original definition in BJTSI, such as monotonicity, reflexivity, and non-transitivity.³⁷ The original definition was based on rank, while the amended definition is based on PS-depth. It is important to note that rank and PS-depth are both quantitative measures; the only difference is that in the latter case, this quantity does not gain traction as easily as it does in the former. Therefore, the amended definition maintains the same desirable properties Berto and Jago wished for.

In Sect. 5, we will extend the definitions to first-order logic (FOL). It shall be observed that the application of contraction in some classical deductions may result in the concealment of some epistemic impossibilities. This happens when formulae which contribute different propositions to the proof get contracted at the quantified level.

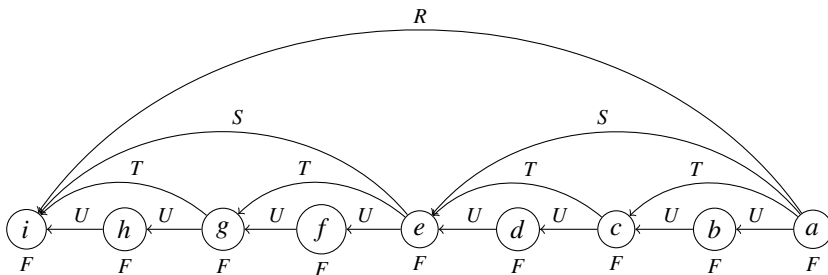
³⁵ Recall, $|\Gamma|^+ \cap |\Delta|^-$ is impossible when Δ classically follows from Γ .

³⁶ Sequents (5) and (6) are ruled out as uninformative by even by BJTSI definitions as their rank is 0.

³⁷ Berto and Jago (2019), p.205

5 Hidden Impossibilities

Berto and Jago consider mathematical theorems as paradigmatic cases of informative deductions.³⁸ The book section on informative inference, however, does not go beyond propositional logic (PL). It is both crucial and interesting to extend their analysis to FOL. Taking a look at FOL’s informativity is crucial if we want to know how mathematical deduction is informative. It is also interesting because FOL greatly expands our ability to conceive of and prove theorems. First-order languages allow us to express axioms. This means that we can encapsulate information about individuals, their properties, and their relationships. Consider the following structure, for instance:



If we express the facts about this structure by atomic propositions such as Rai (to be read as ‘ R relates a to i ’), Fb (to be read as ‘ b has the property F ’) and so on, it will take 24 atomic propositions to articulate all the facts. These facts can be summarized using only four expressions in first-order language:³⁹

10.

$$\begin{aligned} &\forall x\forall y\exists z(Rxy \supset (Sxz \wedge Szy)) \\ &\forall x\forall y\exists z(Sxy \supset (Txz \wedge Tzy)) \\ &\forall x\forall y\exists z(Txy \supset (Uxz \wedge Uzy)) \\ &\forall x\forall y((Uxy \wedge Fx) \supset Fy) \end{aligned}$$

From the above expressions, we can infer some other facts about the structure. For example, we can infer $\forall x\forall y((Rxy \wedge Fx) \supset Fy)$ or $\forall x\forall y((Txy \wedge Fx) \supset Fy)$ from them. It would be difficult to conceive the facts about this structure and argue about them without FOL and with PL.⁴⁰

By utilizing quantification, FOL allows us to express facts about individuals, their properties, and their relationship with one another in a compact manner. As a result of this difference in expressive power between FOL and PL, arguments in the former

³⁸ Berto and Jago (2019), p.192.

³⁹ This does not mean that these 4 expressions describe the above-mentioned structure uniquely. They can be true of many other structures.

⁴⁰ Especially if it hadn’t been graphically presented.

may be much more complex and informative than those in the latter. A PL argument comprehensible to us consists of a finite number of atomic propositions. And the set of atomic propositions in the initial sequents of any SC proof of that argument, if there is one, is a subset of the first set. In FOL, it is often not possible to attribute a specific set of atomic propositions to an expression. First-order expressions can be true of so many structures. Due to the nature of proofs as finite processes, if we find a SC proof for an argument in FOL, the set of atomic propositions appearing in the initial sequents of the SC proof will be finite. Therefore when we find a proof in PL, we have finalised a process which is decidable in principle, whereas when we find a proof in FOL, we have finalised a process which is not decidable in principle.

Considering all of the above, we should be convinced that expanding BJTSI to FOL is both important and interesting. Our focus in this section will be on the fact that, as we have seen, atomic propositions are the key to defining the information content of the modified BJTSI. In the modified BJTSI, quantified expressions are sources of semantic information as premises of arguments. In different arguments, first-order quantified expressions may contribute a different number of propositions (that is, different quantifier-free formulae), and therefore a different number of atomic propositions. Quantified well-formed formulae can be regarded as axioms. The number of times an axiom may be used in a proof varies depending on the theorem.

An example will help clarify this point. Considering the cut-free proof of argument (11) in SC, after applying quantification rules on the left and right, the expression $\forall x \forall y ((Uxy \wedge Fx) \supset Fy)$ contributes four propositions to the proof.⁴¹

11.

$$\begin{aligned} & \forall x \forall y \exists z (Sxy \supset (Txz \wedge Tzy)) \\ & \forall x \forall y \exists z (Txy \supset (Uxz \wedge Uzy)) \\ & \quad \forall x \forall y ((Uxy \wedge Fx) \supset Fy) \\ & \therefore \forall x \forall y ((Sxy \wedge Fx) \supset Fy) \end{aligned}$$

while if we consider cut-free proof of argument (12) in SC, after applying quantification rules on both sides, the expression $\forall x \forall y ((Uxy \wedge Fx) \supset Fy)$ contributes two different propositions to the proof of (12).⁴²

12.

$$\begin{aligned} & \forall x \forall y \exists z (Txy \supset (Uxz \wedge Uzy)) \\ & \quad \forall x \forall y ((Uxy \wedge Fx) \supset Fy) \\ & \therefore \forall x \forall y ((Txy \wedge Fx) \supset Fy) \end{aligned}$$

In a classical proof, which is the type of proof we are concerned with in this article, multiple copies of an axiom or premise can be contracted. Contraction is a structural

⁴¹ Check the appendix for proof of (11).

⁴² Check the appendix for proof of (12).

rule⁴³ that states that a valid deduction that utilizes multiple instances of a premise remains valid if it is used only once.⁴⁴ Here is the Contraction rule on the left:

$$13. \frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta} L\text{ Con}$$

Recall, SC rules except Cut have the subformula property. Consequently, as one moves up the proof tree, the complexity of the formulae reduces until the top leaves are all initial sequents with atomic formulae on both sides. However, we might arrive at an initial sequent before arriving at the topmost leaves. The topmost leaves are all initial sequents, but not all initial sequents are topmost leaves. From a bottom to top perspective, contraction of copies of a premise can be viewed as unfolding some possible ways to maintain the truth of premises and falsity of conclusions in SC proofs. When the argument is logically valid, all of these possibilities lead to initial sequents which are obvious impossibilities in BJTSI.

Such initial sequents are hidden impossibilities. They are so because there is only one instance of each of the premises at the root of the proof tree. One copy of a premise leads to a proposition and a number of initial sequents (epistemic impossibilities) following from that proposition. Moving up the deduction may require more copies of a premise, but the propositions that follow from those copies differ from each other and lead to different initial sequents. As this branching occurs in the upper steps of the proof tree, away from the root, it is reasonable to call them hidden, as they do not appear at the beginning of the deduction (the root). Hence, Hintikka's position that there are hidden contradictions, as mentioned by Berto and Jago, is supported by this evidence.⁴⁵

Thus, yet another criterion for distinguishing trivial from non-trivial arguments can be proposed in the context of FOL. The criterion states that arguments that can be proved only by ruling out some hidden impossibilities are non-trivial since they contain 'subtle inconsistencies'.⁴⁶ A more formal definition of hidden impossibilities (HI) can be found here:

HI: The impossible world-proof $|\Gamma|^+ \cap |\Delta|^-$ contains hidden impossibilities if and only if there is a formula A in $\Gamma \vdash \Delta$, at the root of the proof tree, such that two or more copies of A are required in that proof and each of those copies lead to a set S_i of initial sequents such that there are at least two sets S_i and S_j followed from two copies of A that are not the same, that is $S_i \neq S_j$.

Based on the above definition, trivial arguments in FOL can be distinguished based on whether or not they contain hidden impossibilities.

Triv2: Argument $\Gamma \vdash \Delta$ is trivial if and only if the impossible world-proof $|\Gamma|^+ \cap |\Delta|^-$ contains no hidden impossibilities.

⁴³ Structural rules in SC are rules with no logical connective in them. That is, their focus is on the structure of proofs regardless of the content of formulae in them. Usual structural rules are Weakening, Contraction, Exchange, and Cut.

⁴⁴ nLab (2022)

⁴⁵ Berto and Jago (2019), p.196.

⁴⁶ Ibid, p.196.

Hidden impossibilities provide an explanation of why a world in which premises are true but conclusions are false is a genuine epistemic possibility. Though it is logically impossible for all conclusions to be false while premises are true, this impossibility is only revealed to us after considering some impossible scenarios that follow from accepting both the falsity of a conclusion and the truth of the premises simultaneously. These plain impossibilities are not available or known to us at the root of the proof.

In regard to the informativity of deduction, we saw that Carnap-Bar Hillel measure for informativity is excluding scenarios. A proposition is more informative than another if its truth excludes a greater number of possibilities. We must consider both impossible and possible scenarios when applying this measure for the informativity of deduction. As a result, a deduction is more informative than another if it rules out a greater number of epistemic impossibilities. There is no obvious connection between this concept of informativity and hidden impossibilities. One can imagine a proof in which only one copy of each formulae in the root sequent is used, but the proof excludes more possibilities than another proof in which multiple copies of other formulae in the root sequent are used. It should be noted, however, that in the first proof, each formula leads to a single set of initial sequents. In the second proof, each formula leads to various sets of initial sequents, and not all of these sets are identical. It is evident that the latter eliminates more sets of impossibilities for each formula in the root sequence. Hence, if each premise is viewed as a source of semantic information in FOL, then the more copies of that premise are needed in the proof, the more information will be generated from that source, since the propositions that follow from multiple copies differ.

We examined some implications of extending the modified version of BJTSI to FOL in this section. The next section will discuss Berto and Jago's response to the LO problem and the implications of the expanded view on that problem. The fact that semantic information is closely connected to atomic propositions, and that first-order formulae are sources of introducing atomic propositions to deduction, has implications for the solution to the LO problem.

6 The Problem of Logical Omniscience (LO)

As discussed briefly in Sect. 3, the LO problem is that if knowledge is closed under any consequence relation, then one should know all the consequences of what one knows. Thus, if one knows all the formulae in Γ and $\Gamma \vdash A$, they also know A . This also means that they know all the theorems, that is all the special cases where Γ is empty.⁴⁷ This portrays too idealistic a picture of the rational capacity of humans, but at the same time it gives it some structure. The challenge is to give a more realistic, yet structured account of our rationality.

As one way of addressing this challenge, the epistemically possible logical impossibilities are taken into account. There needs to be subtlety in such impossibilities. By arguing for the vagueness of deductive information, Berto and Jago attempt to make

⁴⁷ Rendsvig and Symons (2021). To understand the level of idealization at play here, we should mention that theoremhood is NP-hard. That is there is no efficient method of defining if a given formula is a theorem even in PL. A method to solve a problem is efficient if it takes polynomial time to solve that problem.

sense of subtle impossibilities. As we saw in Sect. 3, this is manifested formally by stating that the notion of triviality (non-informativity) in BJTSI is not transitive. In other words, if two pieces of proof have a certain level of triviality, combining them changes the level of triviality. According to Berto and Jago, the non-transitivity of triviality allows us to address the LO problem as follows: The formal system provided by BJTSI enables us to define epistemically feasible spaces. This space has structure and some of structural properties of our epistemic space, namely non-transitivity of epistemic triviality, can explain why we are not logically omniscient. The consequences that require shorter proofs from Γ to them will be more trivial to us than the ones that require longer proofs to reach them from Γ . The less trivial a consequence, the less likely we are to know it.

The main reason Berto and Jago give for the vagueness of deductive information is that inference rules in PL are trivial.⁴⁸ Even if we accept their assessment of PL rules, quantifier rules are far from trivial inferences. This is a good reason to doubt that vagueness explains the informativity of FOL arguments. As we saw in the previous section, first-order formulae are sources of semantic information and contribute different numbers of propositions to different proofs. In Sect. 5, we argued that these propositions determine the triviality level of an argument. Based on our discussion of the ambiguity of propositional content in FOL, it seems that as we move from PL to FOL, the vagueness of deductive information is topped off with propositional ambiguity. This adds another layer to the notion of deductive information.

This additional layer of information provides another explanation for the LO problem. We do not know all the consequences of accepting an axiom, which is a quantified formula, because such an axiom contributes different propositional content to different arguments, and we cannot know that content before conceiving the arguments. It is crucial to note that this response depends on the complexity of the non-logical content of a deduction. The reason we are not logically omniscient is not just the complexity of the logical structure, but also the richness of the non-logical content of arguments.

It is possible to object to the suggested account of informativity of FOL deductions on the grounds that it renders arguments in any contraction-free logic uninformative, but all such arguments are unlikely to be trivial. There is no doubt that those arguments might be informative in some sense. An informative deduction can be defined in a number of ways, but semantic information is only one of them.

7 Conclusion

In summary, Berto and Jago's proposed framework based on an interpretation of SC derivations that includes logically impossible worlds as well as possible ones is a plausible way to model epistemic possibilities. A few difficulties were encountered with their definitions in terms of capturing the intended notion of epistemic impossibilities and the depth of such impossibilities. The difficulties resulted from overlooking the distinction between semantic and information content and the role of non-logical content in capturing this distinction in the epistemology of deduction. Then we proposed

⁴⁸ Ibid., p.196.

a way to overcome those difficulties without losing the desirable properties of Berto and Jago’s original thesis by taking into account the non-logical content of deduction.

Then we extended the modified BJTSI to FOL and observed that the non-logical content of formulae becomes ambiguous moving from PL to FOL. Furthermore, we observed that in classical first-order proofs, it is possible to contract two copies of a formula that are the root of two different sets of epistemic impossibilities. This was understood as a way of making sense of hidden impossibilities in a deduction in the spirit of what seems to be suggested by Hintikka.⁴⁹ Finally, we saw that ambiguity in non-logical content might suggest a different solution to the problem of LO.

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Appendix

Rules of inference used:

- $\frac{A, B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} L\wedge \quad \frac{\Gamma \vdash \Delta, A \quad \Gamma' \vdash \Delta, B}{\Gamma, \Gamma' \vdash \Delta, A \wedge B} R\wedge$
- $\frac{\Gamma \vdash \Delta, A \quad B, \Gamma' \vdash \Delta}{A \supset B, \Gamma, \Gamma' \vdash \Delta} L\supset \quad \frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \supset B} R\supset$
- $\frac{A(t/x), \forall x A, \Gamma \vdash \Delta}{\forall x A, \Gamma \vdash \Delta} L\forall \quad \frac{\Gamma \vdash \Delta, A(y/x)}{\Gamma \vdash \Delta, \forall x A} R\forall$
- $\frac{A(y/x), \Gamma \vdash \Delta}{\exists x A, \Gamma \vdash \Delta} L\exists \quad \frac{\Gamma \vdash \Delta, \exists A, A(t/x)}{\Gamma \vdash \Delta, \exists x A} R\exists$

The restriction on $R\forall$ is that y must not occur free in Γ, Δ , and $\forall x A$. The restriction on $L\exists$ is that y must not occur free in Γ, Δ , and $\exists x A$,⁵⁰

Proofs:

Proof of Argument (11):

To make the proofs fit the page, the following substitutions have been applied:

⁴⁹ This is based on Berto and Jago’s suggestion that refers to Hintikka (1975). Hintikka’s original idea of hidden impossibilities is based on surface and deep information which is far more complicated than what is discussed here. For an in depth more recent review of Hintikka’s idea see Sequoia-Grayson (2008).

⁵⁰ Negri and von Plato (2001) p.67.

Sequoiah-Grayson, S., & Floridi, L. (2022). Semantic conceptions of information. In E. N. Zalta (Eds.), *The Stanford encyclopedia of philosophy*. Metaphysics Research Lab, Stanford University (Spring 2022 ed.).

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