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Where Does Cardinality Come From?

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Abstract

How do we acquire the notions of cardinality and cardinal number? In the (neo-) Fregean approach, they are derived from the notion of equinumerosity. According to some alternative approaches, defended and developed by Husserl and Parsons among others, the order of explanation is reversed: equinumerosity is explained in terms of cardinality, which, in turn, is explained in terms of our ordinary practices of counting. In their paper, 'Cardinality, Counting, and Equinumerosity', Richard Kimberly Heck proposes that instead of equinumerosity or counting, cardinality is derived from a cognitively earlier notion of *just as many*. In this paper, we assess Heck's proposal in terms of contemporary theories of number concept acquisition. Focusing on bootstrapping theories, we argue that there is no evidence that the notion of *just as many* is cognitively primary. Furthermore, since the acquisition of cardinality is an enculturated process, the cognitive primariness of these notions, possibly including *just as many*, depends on various external cultural factors. Therefore, being possibly a cultural construction, *just as many* could be one among several notions used in the acquisition of cardinality and cardinal number concepts. This paper thus challenges those accounts which seek for a fundamental concept underlying all aspects of numerical cognition.

1 Introduction

How do we grasp the notion of cardinality? This has been a central question in the philosophy of mathematics ever since the work of Frege, but traditionally it was studied from metaphysical and epistemological perspectives, without focus on the

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cognitive processes involved. According to the Fregean tradition, the concept of cardinal number is grasped in terms of equinumerosity. This relation is captured by Frege's (1884) implicit definition of *cardinal number* in terms of what became known as Hume's Principle:

> (Hume's Principle) The number of *F*s = the number of *G*s if and only if there is a one-to-one correspondence between the *F*s and the *G*s.

Frege and his neo-Fregean followers such as Hale and Wright (2001) and Linnebo (2018) hold that the right-hand side of Hume's Principle is epistemologically and explanatorily prior over its left-hand side. According to an alternative conception – defended and developed by Husserl (1891) and Parsons (1994), among others – the explanation runs in the opposite direction: one-to-one correspondence should be explained in terms of sameness of number, and sameness of number should, in turn, be explained in terms of our ordinary practice of counting. Against both of these two accounts, Richard Kimberly Heck (2000) points to stages of cognitive development in children that are in conflict with the proposed explanations. Heck argues that instead of counting and equinumerosity, children have an earlier, conceptually and cognitively less demanding, notion of *just as many* that makes acquiring the concepts of cardinal numbers possible.

The question we focus on in this paper is how, from the perspective of cognitive development, the notion of cardinal number is to be understood.[1] We explore the cognitive construal of the relations between cardinality, counting, and our pre-theoretical concept of cardinal number. To make our approach more precise, we use, drawing on Linnebo and Pettigrew (2011, pp. 241–2), the following notions of *primariness*. Suppose N_1 and N_2 are two notions, generally understood to include concepts, conceptions, theories, propositions, etc. Let us say that:

- N_1 has *logical* primariness with respect to N_2 if it is possible to formulate N_1 without appealing to N_2.
- N_1 has *cognitive* primariness with respect to N_2 if it is possible to cognitively grasp N_1 without first understanding N_2.
- N_1 has *justificatory* primariness with respect to N_2 if it is possible to motivate and justify the claims concerning N_1 without appealing to N_2.

For example, if the notion of cardinal number can be formulated without using the notion of one-to-one correspondence, then the former enjoys logical primariness with respect to the latter. But that, as such, does not mean that the notion of cardinal number has cognitive primariness with respect to one-to-one correspondence. Here is a possibility: in order to formulate and state claims about the notion of cardinal number, one may need not appeal to one-to-one correspondence. Yet, whenever we come to explain these claims to those unfamiliar with them, we inevitably appeal to

[1] So far, the cognitive approach, in this dialectical situation, has been rarely taken in the philosophical literature. For exceptions, see Decock (2008), Assadian and Buijsman (2019), Pantsar (2023).

one-to-one correspondence. In this sense, cardinal number does not enjoy cognitive primariness with respect to one-to-one correspondence. The same is true of justificatory primariness. The question of this paper focuses on cognitive primariness; and if a notion is cognitively primary to all other related notions, we call it *cognitively fundamental*.

In this paper, our discussion will mainly focus on the *de facto* thinking processes involved in the acquisition of numerical concepts as opposed to the *de jure* warrants for the correctness of thoughts. During individual ontogeny, how does a child acquire competence with cardinal numbers? Is one-to-one correspondence the cognitively primary notion, mirroring the (neo-)Fregean approach, or is counting the primary notion, mirroring the Husserl-Parsons account of cardinality? Is Heck's notion of *just as many* the relevant cognitively primary notion?

We argue that it is not necessary to see any of these notions – counting, equinumerosity, and *just as many* – as cognitively fundamental. As we show, when studied in connection with modern empirical understanding of the development of numerical cognition, the notion of cardinal number can be cognitively dependent on a plurality of factors, neither of which has a fundamental cognitive priority over others. These can feasibly include notions of equinumerosity, *just as many*, and practices of counting. This paper thus challenges those accounts – Fregean or otherwise – which seek for one fundamental concept underlying all aspects of numerical cognition.

To explore this problem, we shall examine the following two questions: (1) Is the notion of cardinal number cognitively dependent on the notion of equinumerosity? (2) Is the notion of cardinal number cognitively dependent on our practice of counting? In Section 2, we analyze the basic notions of counting, cardinality, equinumerosity, and *just as many*, mainly in light of Heck's work. In 2.1, we set the stage by introducing some of the key concepts. In 2.2, we focus on Heck's account of *just as many*, and analyze their arguments concerning the relations of cognitive primariness between the basic notions introduced in 2.1. The purpose of Section 3 is to test Heck's account in terms of modern empirical findings.

The focus on Heck's arguments has three main motivations. First, Heck explores the relation between the notions of cardinal number and equinumerosity in terms of empirical and cognitive evidence, which is the focus of our attention. Second, in Heck's view, Hume's Principle is not the fundamental source of our knowledge about cardinality and numerical identity; something weaker is needed, which is captured not in terms of one-to-one correspondence, but in terms of *just as many*. Third, since Heck's notion of *just as many* is cognitively less demanding than that of one-to-one correspondence, it is of crucial importance for our purposes to show that even such an impoverished notion is still subject to cognitive and cultural variations.

In 3.1, we review recent empirical work concerning the cognitive primariness of notions related to cardinality. Then in 3.2, we present theories of number concept acquisition based on *proto-arithmetical*, evolutionarily developed abilities. In 3.3, we motivate and focus on currently one of the most discussed accounts of number concept acquisition, the bootstrapping theory originally presented by Susan Carey (2009) and further developed by Jacob Beck (2017). In particular, we focus on the *enculturated* version of that account as presented in (Pantsar 2021a). Finally, in 3.4, we bring together the work on enculturated bootstrapping with Heck's argumentation, and develop it further. Instead of being a cognitively fundamental principle, we argue that *just as many* may itself be a culturally developed, complex notion, just like one-to-one correspondence. Furthermore, we argue that when it comes to culturally developed notions, we should not expect to find a single fundamental cognitive notion such as *just as many*. Instead, we argue that number concept acquisition is more likely to include a plurality of integral cognitive influences, based on their cultural development and application.

2 Cardinality, Counting, and Equinumerosity

2.1 Some Basic Notions

To begin our analysis, we need to clarify some of the distinctions between the key notions involved. Frege (1884) and his neo-Fregean followers such as Hale and Wright (2001) and Linnebo (2018) hold that in laying down Hume's Principle, the concept of cardinal number is introduced in terms of one-to-one correspondence. A one-to-one correspondence between the items of two collections establishes *sameness of cardinality*. This is the case, for example, when we establish that there are as many forks as there are knives on a table, without counting them to come up with a *particular number* that counts both the forks and the knives (see, e.g., Dos Santos 2021). So, Hume's Principle primarily gives us sameness of cardinality (often called *equinumerosity*) without sameness of number.

It is only after the concept of cardinal number is introduced in terms of one-to-one correspondence that grasping counting processes will be possible. Therefore, taking → to stand for cognitive primariness, the (neo-)Fregean order of primariness goes as follows:

one-to-one correspondence → sameness of cardinality → sameness of number → counting

According to a rival tradition – defended and developed by Husserl (1891) and Parsons (1994) – the order of explanation is reversed: one-to-one correspondence should be explained in terms of sameness of number, and sameness of number should, in turn, be explained in terms of our ordinary practice of counting. Thus, what is cognitively fundamental is our ordinary practice of counting.

One aspect of Hume's Principle that has not received sufficient attention is indeed how it can be understood in terms of cognitive primariness. During individual ontogeny, how does a child acquire competence with cardinal numbers? Is one-to-one correspondence the cognitively primary notion or is counting the primary notion?

By appealing to data from psychological research (Gelman and Gallistel 1978), Heck (2000, pp. 165–6) suggests that there is a sense in which Husserl and Parsons were right: children can have the ability to count and answer 'How many?'-questions – such as 'How many grapes are on the plate?' – while showing no, or only a minimal, understanding of the notion of one-to-one correspondence. However, this observation, by itself, does not undermine the Fregean point that counting already involves one-to-one correspondence: after all, to count the *F*s is to effect a one-to-one correspondence between the *F*s and the numerals from 1 to '*n*', for some (finite) *n*. But this does not mean that someone who is able to count has an understanding of the notion of one-to-one correspondence. In the terminology of the present paper, the point of divergence between the view of Husserl and Parsons, on the one hand, and Frege's, on the other, concerns the cognitive primariness of counting over one-to-one correspondence, and this is not established by the above data. In our view, what the data show is that there is a stage of development in which children do not exhibit an understanding of one-to-one correspondence, though they do show an ability to count. However, it is still possible that an *implicit* use of one-to-one correspondence is behind their grasp of counting processes. (We address the relevant notion of implicitness in Section 3.)

Before we continue, another distinction needs to be made. Benacerraf (1965, pp. 49–51) distinguishes between two types of counting. In *intransitive* counting, one simply recites the numeral word list in the right order, like when starting a game of hide and seek. In *transitive* counting, by contrast, the items on the numeral word list are used to enumerate items in collections, like when counting the grapes on a plate. As will be seen in Section 3, this distinction is crucial, since intransitive counting precedes transitive counting in cognitive development by a significant period of time (Carey 2009). In addition, it is a distinct possibility that while transitive counting could be cognitively dependent on the notion of one-to-one correspondence, intransitive counting would be cognitively prior to it.

2.2 Cognitive Primariness Among the Basic Notions

The first question to tackle is whether transitive counting is cognitively primary with respect to the notion of one-to-one correspondence. As Heck observes, one-to-one correspondence is cognitively demanding, and thus unlikely to be cognitively primary. By presenting the following experiment on their then three-year-old daughter, they seek to show how one can grasp counting and number ascriptions without an antecedent grasp of one-to-one correspondence:

> We had some Barbies and some hats and put them on the table. "How many Barbies are there?" I asked her. "One, two, three, four. Four Barbies!" she said proudly. And then we spent some time with the hats. We saw that we could put a hat on each Barbie – just one – there not being any left once each Barbie had a hat. "Just enough hats for the Barbies!" she said. So now the question: How many hats are there? No amount of prompting would elicit the inference: Four Barbies; one hat for each; so four hats. (Heck 2000, p. 165)

As noted by Heck (2000, pp. 165–6), this phenomenon was already known by psychologists like Gelman and Gallistel (1978). Children who are able to do transitive counting do not necessarily grasp the notion of one-to-one correspondence (it is of course, based on this kind of observation, possible that children like Isobel do grasp one-to-one correspondence, but they do not connect it to the transitive counting procedure).[2] The empirical data Heck appeals to seem to show that one can understand the following basic facts about sameness of number (Heck 2000, p. 166) merely in terms of transitive counting, without any prior grasp of one-to-one correspondence:

1. There are no *F*s iff the number of *F*s is 0.
2. If the number of *F*s is the same as the number of *G*s, and if an object is added to the *F*s and an object to the *G*s, then the number of *F*s and the number of *G*s remains the same.
3. If the number of *F*s is not the same as the number of *G*s, and if an object is added to the *F*s and an object to the *G*s, then the number of *F*s and the number of *G*s remains different.

There is a further, related, issue. Heck argues that mastery of counting and thereby successfully answering 'How many?'-questions is "compatible with one's having no concept of cardinality at all" (Heck 2000, p. 169). For instance, even if a child correctly answers 'How many barbies are on the table?', that does not, as such, mean that the child knows "what the answers to such questions mean – or indeed, what the questions mean" (Heck 2000, p. 168). As one sort of evidence, Heck cites Carey (1995), in which it is observed that children around three years old do not always count with conventional numeral words. Children of this age may understand the question "How many horses do carry the King's carriage?" as (implicitly) meaning: "Count the horses that carry the King's carriage". It seems then that they have no understanding of numerals as standing for cardinal numbers as specific objects. This prompts the question: what exactly is involved in understanding 'How many?'-questions?

In light of our discussion, it should be clear that the notion of one-to-one correspondence cannot be the right answer here. Indeed, Heck argues that children can grasp a more primary notion of *just as many*. As Heck observes, while the concepts of one-to-one correspondence and *just as many* have the same extension, they are intensionally different concepts. There is a one-to-one correspondence between the *F*s and the *G*s if and only if there are as many *F*s as there are *G*s. However, that does not mean that whenever one grasps *just as many*, they thereby grasp one-to-one correspondence. Therefore, testing for children's ability to grasp *just as many* and predict the outcome of a one-to-one correspondence would require different cognitive tasks.

But what is the concept of *just as many*? Heck (2000, pp. 170–1) proposes the following three principles to implicitly define the concept:

1*. If there are no *F*s, then (there are just as many *F*s as there are *G*s iff there are no *G*s).

[2] It should also be noted that some literature shows that children have difficulties in grasping zero (see, for example, Nieder 2016). Therefore, the reference to '0' in (1) – and (1*) below – may need to be replaced by a reference to singly-instantiated concepts. That would make Heck's principles more faithful to descriptions of children's early knowledge of cardinal numbers.

2*. If there are as just as many *F*s as there are *G*s, and if an object is added to the *F*s and an object to the *G*s, then there will be just as many *F*s as there are *G*s.
3*. If there are not just as many *F*s as there are *G*s, and if an object is added to the *F*s and an object to the *G*s, then there are not just as many *F*s as there are *G*s.

These principles, Heck suggests, show that one can grasp *just as many* without having any antecedent grasp of the notions of cardinal number and sameness of number. In Heck's (2000, p. 171) view, the starred principles are the fundamental truths about our pre-theoretical notion of cardinality.[3]

Before moving to the next section, let us spell out *just as many* and its relation with one-to-one correspondence a bit further. There are two general ways for further sharpening the gap between *just as many* and one-to-one correspondence; i.e., in terms of what we have called in Section 1 *cognitive primariness* and *logical primariness*. The idea behind cognitive primariness is that a child may be in a position to cognitively grasp *just as many* without first grasping one–one correspondence, even though the notions are extensionally the same, in the sense that there are just as many *F*s as there are *G*s if and only if there is a one-to-one correspondence between the *F*s and the *G*s. So, a child may be able to understand that – given than there are as many *F*s as there are *G*s – if an object is added to the *F*s and an object to the *G*s, then there will still be just as many *F*s as there are *G*s. But that does not ensure that the child has thereby grasped the more complicated concept of one–one correspondence, which stands for a bijection function between the collection of *F*s and the collection of *G*s, meaning that every element of the second collection is mapped to from exactly one element of the first collection.

The above point, moreover, has consequences concerning the logical primariness of *just as many* over one-to-one correspondence. In order to *formulate* claims about the notion of *just as many*, we do not need to appeal to one-to-one correspondence. For *just as many* is stated in terms of Heck's starred principles (1*)–(3*), which do not involve one–one correspondence.

In the present dialectical situation, our assumption is that children do possess the notion of *just as many*, and also have a reasonable mastery of it. This assumption is uncontentious in this context, since the aim is to explore whether *just as many* is fundamentally bound up with the notions of cardinality, one-to-one correspondence, and other cognate notions. Thus, determining how children possess and form

[3] As an anonymous referee of this journal pointed out to us, there are some limitations about these three principles: a child who understands them would only be able to recognize *some* situations where the relation *just as many* continues to hold or not to hold. To be able to do so generally, the child should *additionally* possess the ability to apply (2*) or (3*) recursively. Furthermore, to grant the child with the ability to recognize that the two collections instantiate *not just as many*, an additional principle would be needed – stating that whenever the items of two collections stand in the *just as many* relation and one adds items to *only* one of the collections, then the items do not stand in that relation any longer. Important as these observations are, it should be noted that similar additional principles would also be needed for generally recognizing one-to-one correspondence. Indeed, it is important to note that the principles (1*)–(3*) are primarily intended to give an *implicit definition* of *just as many*. As such, they are not intended to give the *ability* to recognize all situations in which *just as many* continues to hold or not to hold.

the notion of *just as many*, as well as how they acquire an ability to perform tasks involving that notion, requires appropriate empirical tests. Heck's starred principles merely formulate and expose *just as many*; they do not seek to characterize the notion to meet the experientialist demand; i.e., to provide us with enough information about the notion to enable testing whether children possess it or not. Heck's observations should, therefore, be regarded as an analysis of the notion of equinumerosity primarily in terms of *just as many*.

To sum up, in Heck's view, the notions of sameness of number and (transitive) counting have cognitive primariness with respect to one-to-one correspondence. Furthermore, *just as many* has cognitive primariness with respect to cardinal number. Section 3 examines to what extent, if at all, *just as many* is cognitively primary to counting. Before that, though, let us in the remainder of this section address the question of whether the notion of cardinal number has cognitive primariness with respect to counting.

It seems that the notion of cardinal number, and hence our specific ascriptions of number, are tightly connected with counting. To count the four dolls on the table is to establish that there are just as many dolls as there are numerals in the sequence from '1' to 4'. The following thesis states what specific assignments of number mean:

(TC) 'There are *n* *F*s' means that there are just as many *F*s as there are numerals in the sequence from 'one' to '*n*'.

If TC (short for "transitive counting") is accepted, there is a clear sense in which our grasp of cardinal number is not independent of counting. Attractive and plausible as TC may appear, Heck (2000, pp. 173–4) casts some doubts on it. Their argument is mainly based on the observation that children (and adults) can ascribe cardinal numbers to collections without explicitly recognizing or connecting the cardinality of the collection to how many numerals there are in the sequence. Their knowledge of cardinal numbers seems to be about *numbers*, not *numerals*. This can be understood in terms of two different sorts of knowledge that are generated by knowing that there are four dolls (*de re* knowledge) and knowing that there are just as many dolls as numerals between 'one' and 'four' (*de dicto* knowledge).[4] Based on this, Heck (2000, p. 175) asks two questions: (1) What does the statement that, for example, 'there are four dolls' mean, if not, as TC states, that there are as many dolls as numerals in the sequence from 'one' to 'four'? (2) How does counting establish how many dolls there are, if not by establishing that there are just as many dolls as numerals in the sequence from 'one' to 'four'?

[4] In this context, a *de re* knowledge belongs to a number as a particular object, whereas this kind of object-directedness is missing in *de dicto* knowledge. That is, when we know that there are four dolls, we know the particular number that is associated with the dolls. But knowing that there are as many dolls as numerals between 'one' and 'four' does not pick up – at least directly – a particular number. As Heck notes, this difference is due to Kripke (1992).

As for the first question, Heck (2000, p. 175) shows sympathy to Frege (1884, §55), where he advances a proposal about how we can contextually define numerically definite quantifiers using first-order logic with identity. On this proposal, 'The number of Fs = 0' means that $\forall x \neg Fx$. Likewise, 'The number of Fs = 1' means that $\exists x(Fx \wedge \forall y(Fy \rightarrow x = y))$. In Heck's account, one grasps the concept ZERO just in case one knows how many Fs there are when there are no Fs; one grasps the concept ONE just in case one knows how many Fs there are when there is one F; and so on. Thus, grasping the concept FOUR is not a matter of knowing that there are four Fs if and only if there are just as many Fs as numerals in the sequence from '1' and '4'.

Question (2) is directed at the practice of counting, but Heck does not tell us what counting, as they regard it, exactly is. What is clear, though, is that in their view, counting is not merely tagging objects with certain numerals or symbols. Heck's argument for this last thesis is essentially as follows: if counting were tagging, the meaning of number-ascription statements "wouldn't matter whether we counted with numerals or with days of the week or letters of the alphabet" (Heck 2000, p. 176).

All the same, what matters for our purposes is Heck's claim to the effect that the connection between our grasp of the notion of cardinal number and counting is not as tight as TC suggests. There can be specific ascriptions of cardinality without connecting the cardinality to how many numerals there are in the sequence. The notion of cardinal number enjoys some degree of cognitive primariness with respect to transitive counting.

As mentioned above, according to Heck, the fundamental truths about cardinality are codified by the starred principles which purport to implicitly define the notion of *just as many*. But is Heck correct that *just as many* is cognitively primary to counting? We will approach this question in light of empirical research on early numerical cognition.

3 Is *Just as Many* Cognitively Primary to Counting?

3.1 Cognitive Primariness

Heck's way of using cognitive considerations was not particularly solid empirically, since aside from one reference to the work of Gelman and Gallistel (1978), the evidence Heck presents consists of anecdotal observations, discussed above. However, Heck did not have nearly the kind of empirical resources available that we currently do. As it happens, though, Heck's anecdotal evidence has been corroborated by data on both accounts discussed above. Against the (neo-)Fregean primariness of the notion of one-to-one correspondence, Heck appeals to the phenomenon that was already known by psychologists like Gelman and Gallistel (1978): children who are able to do transitive counting do not necessarily grasp the notion of one-to-one correspondence.[5]

If one-to-one correspondence fails as the putative primary cognitive notion in grasping sameness of cardinality, one could assume that the Husserl-Parsons hypothesis fares better and (transitive) counting takes the role as the primary notion. But Heck's observations of their daughter, Isobel, suggests that being able to do transitive counting is not enough for grasping cardinality, either:

> What children of that age do when asked how many *F*s there are is to count, even if they've just finished counting those same objects. […] Such children also do not appear to understand what Frege called "ascriptions of number". If I had asked Isobel to give me three hats, her response would have been to grab some hats and hand them over. Whether she gave me three hats would have been a matter of chance. (Heck 2000, p. 169)

Again, Heck's evidence has been confirmed by empirical findings. Following the work of Wynn (1990), a standard method of ascertaining the level of child's development in numerical cognition is the so-called "give-*n*" test. In this type of experiment, a child is presented with a collection of objects and asked to give *n* of them. If the child consistently gives *exactly n* objects, they are thought to show understanding of the number concept *n*. It is now well known that there is a stage in children's development when they can count (transitively) to *n* but do not pass the give-*n* task, just like Heck's daughter giving a random amount of objects (see, e.g., Wynn 1990; Sarnecka and Carey 2008). When starting to pass the give-*n* test, children learn the first numbers in ascending order. At roughly two years of age, they pass the give-1 task, becoming *one-knowers*. Then in stages taking typically four to five months, children become *two-knowers, three-knowers* and *four-knowers* (Knops 2020). After that, children grasp something general about cardinal numbers and when they become *five-knowers*, they typically at the same time become *six-knowers*, *seven-knowers*, etc. (Lee and Sarnecka 2010). Therefore, when children are able to more generally match the last numeral uttered in the counting sequence with the cardinality of a group of objects, they are said to have become *cardinality-principle* knowers (Sarnecka and Carey 2008; Lee and Sarnecka 2011).

Importantly, there is an extensive period (roughly two years) during which children can engage in intransitive and transitive counting up to some number *n*, but they cannot pass the give-*n* task (Knops 2020). For Heck (2000, p. 179), this kind of evidence showed not only that (transitive) counting is not enough to grasp number concepts, but that children may not connect counting to cardinality at all. We interpret this to mean that, according to Heck, transitive counting cannot therefore be the cognitively primary notion in number concept acquisition, because the primary notion needs to be connected to cardinality. As we have seen, their solution was to look for primariness in the notion of *just as many*. Instead of one-to-one correspondence or

[5] See also (Sarnecka and Gelman 2004; Sarnecka and Wright 2013) and Buijsman (2019). For a review on empirical research on how children learn to use cardinal numbers to establish whether two sets are equinumerous, see (Muldoon et al. 2009).

counting being primary, the argument goes, children have a notion of *just as many* that predates both in the course of ontogenetic cognitive development.

But does this account get support from empirical data? Or, to be more precise, can the present empirical theories of number concept acquisition accommodate Heck's notion of *just as many*? That will be the topic of the rest of this paper.

3.2 Empirical Accounts of Number Concept Acquisition

To the best of our knowledge, there have not been experiments that have specifically tested the hypothesis that *just as many* is cognitively primary in number concept acquisition. However, this does not imply that the present accounts in the literature on numerical cognition cannot be assessed in terms of their fit with this hypothesis.

The empirical and empirically-informed philosophical literature on numerical cognition can be divided into distinct categories in more than one way. One characteristic difference is captured by a rough *internalist* versus *externalist* division. 'Internalism' here refers to forms of *nativism*, according which our numerical abilities are fundamentally based on evolutionarily developed capacities (e.g., Gallistel 2017; Gelman and Gallistel 1978, 2004; Butterworth 1999). In this view, number concepts and the capacity for arithmetic are, in one sense or another, considered to be *innate*.[6] The internalist view is contrasted with externalist accounts that take material engagement with the environment to be central to the development of number concepts and arithmetic (e.g., Overmann 2018; Zahidi 2021). In between the internalist and externalist views are accounts of *enculturation* that take evolutionary developed *proto-arithmetical* capacities as a basis for the acquisition of number concepts and arithmetic, but emphasize the transformative effect of culturally shaped practices in the ontogenetic development of numerical cognition (e.g., Menary 2015; Pantsar 2019; Fabry 2020; Jones 2020).

We believe that only accounts compatible with the enculturation framework can fully capture the intricate influences of different types of factors in the development of numerical cognition. The reason for this is two-fold. First, the evolutionarily developed proto-arithmetical abilities are not proper arithmetical abilities, and they do not involve proper number concepts. It is widely accepted that there are two proto-arithmetical abilities, due to two different cognitive systems. First is *subitizing*: the ability to determine the exact cardinality of objects without counting, up to three or four objects. The subitizing ability has been confirmed in human infants, as well as many species of non-human animals (Dehaene 2011; Spelke 2000; Starkey and Cooper 1980). It is standardly thought to be due to the *object tracking system* (OTS), which makes it possible to track several objects in a parallel fashion (Carey 2009; Trick and Pylyshyn 1994). The OTS and hence the subitizing ability, however, are limited to at most four objects. For larger numerosities, there is another cognitive system, called the *approximate number system* (ANS) (Dehaene 2011; Spelke 2000;

6 See Griffiths (2001) for an overview of several ways in which 'innate' can be understood.

Knops 2020). The ANS allows for the estimation of larger numerosities, but this estimation ability loses accuracy as the numerosities become larger.

It is crucial to distinguish between proto-arithmetical and arithmetical abilities, as well as the numerosity representations involved in the former and the number concepts involved in the latter (Pantsar 2014, 2015, 2018). The subitizing ability is limited to small cardinalities, while the estimation ability is imprecise. The abilities that infants and non-human animals possess are therefore not proper arithmetical abilities, and there is no reason to believe that they possess exact number concepts. Indeed, without suitable processes of enculturation, also adult humans remain anumeric, as evidenced by the Amazonian cultures of Pirahã and Munduruku (Gordon 2004; Pica et al. 2004). Importantly, the Pirahã and the Munduruku people possess similar proto-arithmetical abilities to people in arithmetical cultures. What they lack are the *cultural* practices involved in learning arithmetic. Therefore, the proto-arithmetical abilities alone cannot account for the acquisition of arithmetic and exact cardinal number concepts. (See Pantsar 2019, 2021a for a more detailed argument.)[7]

The second reason why the enculturation framework shows most promise is that accounts that take number concepts and arithmetic to be *purely* cultural constructs run into problems explaining the character of arithmetical knowledge. Such strict *conventionalist* accounts take mathematics to be ultimately about arbitrary rules of symbol manipulation that cannot be connected to anything more substantial (see, e.g., Wittgenstein 1976; Field 1980; Balaguer 2009). As argued in (Pantsar 2021b), ranging from the apparent objectivity of mathematical knowledge to the wide range of mathematical applications in science, there are many ways in which strict conventionalism about mathematics seems mistaken. In the present context, one reason is particularly important. While there are anumeric cultures like the Pirahã and the Munduruku, arithmetic has been developed independently in different cultures (e.g., Greek, Chinese and Mayan) in ways that converge when it comes to counting and basic operations like addition and multiplication (Ifrah 1998). What could explain such convergence if arithmetic was purely conventional, entirely a cultural construction? Even in cultures that do not possess developed arithmetical skills, extensive numeral systems are common (see, e.g., Everett 2017). Counting – whether with numeral words, body parts, systems of tallying, or by some other means – is not a universal human ability. But it is certainly too common and too similar across cultures to be written off merely as a coincidence.

Instead of strict conventionalism, the alternative explanation is much more appealing: processes of enculturation shape number concept acquisition by recruiting the proto-arithmetical abilities for new culturally specific functions (see Pantsar 2019, 2024 for a detailed argument). If this alternative explanation is accepted, we must develop an account of the acquisition of number concepts and arithmetic that

[7] According to the nativist view, number concepts are innate so instead of being anumeric, cultures like the Pirahã and the Munduruku must simply lack the tools (number words and symbols) to express exact number concepts. However, there is no evidence to support the nativist view and there are promising alternative accounts. According to one of them, number concepts and number words have *co-evolved* culturally, which provides a much more plausible explanation for the existence of anumeric cultures (for detailed arguments, see (Wiese 2007; dos Santos 2021; Pantsar 2024).

is sensitive both to evolutionarily developed and cultural factors. In the next section, we will focus on the most promising such account, which we will then analyze in terms of its suitability to Heck's idea that the notion of *just as many* is cognitively primary in number concept acquisition.

3.3 The Enculturated Bootstrapping Account

At present, there exists a wide and diverse literature on numerical cognition, with radically different views on how number concepts develop and are acquired. As we saw in the previous section, some researchers believe number concepts to be innate. Others argue that they are acquired primarily by applying the approximate number system (Dehaene 2011). Yet others see the object tracking system as the primary cognitive system. In the literature, these latter views are often associated with the *bootstrapping* account of number concept acquisition, the most influential version of which was presented by Carey (2009).

Here we focus on modern developments of the bootstrapping account on number concept acquisition, namely, how Carey's account has been explicated successfully by Jacob Beck (2017) and developed further in the framework of enculturation (Pantsar 2021a). We are mostly concerned with this latter account, which emphasizes the role of cultural influences for the bootstrapping process and sees a role for both the OTS and the ANS in it. One important reason for this choice is that there is increasing evidence that the ANS may also play a role in early number concept acquisition (see, e.g., Wagner and Johnson 2011; vanMarle et al. 2018). Unlike the OTS-based versions of Carey and Beck, the OTS-ANS bootstrapping account presented in (Pantsar 2021a) is compatible with this evidence. The second important reason is that this latter bootstrapping account provides a specific framework for the cultural influence on number concept acquisition.[8]

As originally formulated by Carey, the bootstrapping process consists of three stages. First is the acquisition of a numeral word list, which at that stage functions as a list of placeholders without semantic content. This corresponds to the stage described earlier in this paper, during which children can recite part of a counting list (i.e., engage in intransitive counting) without grasping its connection to cardinalities. They may also be able to, as in the case of Heck's daughter, tag objects when counting to connect one numeral with a particular collection of items (i.e., engage in transitive counting). But what the children at this stage cannot do is grasp that the last numeral uttered in the counting sequence refers to the cardinality of the collection, i.e., they do not pass the give-*n* task (see, e.g., Wynn 1990). The most likely explanation for this is that children at this first stage do not yet possess number concepts.

[8] By focusing on the enculturated bootstrapping account, we do not mean to suggest that it is representative of the wider present literature on number concept acquisition. However, we contend that that account provides a fruitful platform for evaluating Heck's account in terms of modern empirical findings. In what follows, our treatment is thus focused on, but not limited to, the enculturated bootstrapping account.

To explain the second stage of the bootstrapping process, we need to go into the details of how the object tracking system (OTS) functions. The idea supported by Carey (2009) is that the OTS works by each observed object occupying a mental *object file*. If one object is observed, one object file is occupied. If two objects are observed, two object files are occupied, and so on, until the limit of the OTS (four objects). Importantly, in observing, say, three cats, the mental model is not thought to include threeness. Instead, each cat occupies their own object file. In the second stage of the bootstrapping process, then, children are thought to associate the first members of the counting list with those mental models:

> The meaning of the word ‘‘one” could be subserved by a mental model of a set of a single individual {i}, along with a procedure that determines that the word ‘‘one” can be applied to any set that can be put in 1–1 correspondence with this model. Similarly, two is mapped onto a longterm memory model of a set of two individuals {j, k}, along with a procedure that determines that the word ‘‘two” can be applied to any set that can be put in 1-1 correspondence with this model. And so on for ‘‘three” and ‘‘four”. (Carey 2009, p. 477)

However, as explicated by Beck (2017), establishing this one-to-one correspondence is not done explicitly. Rather, it is due to “computational constraints” in how the mind processes representations in mental models, i.e., “procedures that govern how those representations can be manipulated” (Beck 2017, p. 116). The one-to-one correspondence, rather than being a notion that the child grasps explicitly, is thus based on constraints of how the mind manipulates the representations in the object files.

How do these representations in the object files turn into number concepts? Beck argues that children acquire number concepts through “counting games” in which they learn that the final numeral word in a (transitive) counting sequence is associated with the cardinality of the collection of items. At its simplest, a counting game consists of pointing to each member of a collection while rehearsing the ordered counting list (Beck 2017, p. 119). Thus, the counting list turns from a meaningless list of words into numerical representations, by consistently associating the last word on the list with the cardinality of the collection, “endowing the words in the count list with new conceptual roles” (ibid.). This, as we have seen, happens gradually as children become one-knowers, two-knowers, etc. Finally, at the third stage of the bootstrapping process, through inductive and analogical inference, children are able to generalize on this principle beyond the range of the OTS, becoming cardinality-principle knowers.

We can see how processes of enculturation are crucial for the bootstrapping account. The counting list of numeral words is culturally developed and not all cultures have counting lists (most famously, the Pirahã and the Munduruku). In addition, the counting games are cultural practices. However, the bootstrapping account includes also crucial factors that are products of biological, rather than cultural, evolution. The functioning of the OTS plays a central role in bootstrapping and it is thought to be an evolutionary adaptation (see, e.g., Knops 2020). Thus, the bootstrapping account seems to be a clear case of explaining cognitive phenomena by including both biologically and culturally evolved factors, which is in line with the enculturation account.[9]

This is important for the present topic, since the bootstrapping account appears to make clear postulations concerning the developmental primariness of different notions involved in number concept acquisition. In the first stage, children acquire (part of) the counting list of numeral words. This is only possible when the necessary cultural practices are in place. As seen in the case of the Pirahã and the Munduruku, without having a counting list, children do not acquire number concepts (Gordon 2004; Pica et al. 2004). In the bootstrapping account, this is easy to explain, since children lack the necessary placeholder list to enter the second stage of the bootstrapping process. In this way, intransitive counting might appear to be the primary cognitive notion in number concept acquisition. However, the matter is not so straightforward. Although acquiring the counting list is the first stage of the bootstrapping process, it does not lead to acquiring number concepts unless it is connected to the relevant mental models due to the OTS, which is done through counting games. In the first stage, children only recite a counting list: they count intransitively with numeral words, but they don't count with number *concepts*, i.e., they don't associate the numeral words with cardinalities of collections.

In contrast, in becoming one- through four-knowers, children learn to associate numeral words with cardinalities of collections. In the bootstrapping account, this is thought to take place by establishing some kind of matching between mental models and collections, and then associating the correct numeral word with this process. Does this mean that grasping one-to-one correspondence is primary to acquiring number concepts? While this conclusion may be tempting to draw, it would be mistaken. After all, as seen above, the idea is that in the second stage of the bootstrapping process, one-to-one correspondence is established only *implicitly*, i.e., children can go through the second stage without grasping that they are establishing a one-to-one correspondence. Whatever the matching process in question is, it is pre-conceptual and cannot be identified with grasping the notion of one-to-one correspondence. This is in line with Heck's remark that "[t]he notion of a one–one correspondence is very sophisticated; it is far from clear that five-year-olds, who do seem to grasp the concept just as many, have any general grasp of one–one correspondence" (Heck 2000, p. 170). In addition, the one-to-one matching involved in the OTS only functions for collections up to four items, which is an important difference to general principles of one-to-one correspondence.

[9] In the accounts of Carey (2009) and Beck (2017), the biological focus of the bootstrapping process is firmly on the object tracking system. However, empirical findings imply that already before becoming cardinality-principle knowers, children associate number words with approximate quantities (see, e.g., Wagner and Johnson 2011). This suggests that the ANS could also play a role in number concept acquisition. While this may clash with the accounts of Carey and Beck, in the bootstrapping account developed in (Pantsar 2021a), both the object tracking system and the approximate number system are thought to play a role in number concept acquisition. This could also explain the findings reported by Krajcsi and Fintor (2023) according to which children may possess number concepts larger than four already before they are cardinality-principle knowers.

We can thus see that in the bootstrapping account, both counting and one-to-one matching are, in one sense, primary to acquiring number concepts. But the notion of counting in this context is merely the ability to recite part of the counting list, and the notion of one-to-one matching refers to an implicit ability to connect mental models with cardinalities of collections. Neither of them is the kind of explicit notion that is relevant to Heck's pursuit of finding the primary cognitive principle in establishing the sameness of cardinal numbers. In addition, both counting and establishing a one-to-one correspondence seem to be culturally specific abilities. As reported by many researchers, the Pirahã cannot do one-to-one matching for collections of items larger than three (Everett and Madora 2012; Frank et al. 2008; Gordon 2004). This is evidence that not only explicitly grasping one-to-one correspondence, but also the ability to do one-to-one matching, even for relatively small collections, could be a cultural development that requires having a system of numeral words in place. The mechanism through which the OTS tracks quantities is clearly something different.

In this regard, some evidence found in the literature seems to clash with the Carey-Beck bootstrapping account. As reported by Izard and colleagues (2014), children (around the age of three) are in some contexts able to (non-verbally) establish exact quantities of 5 and 6 objects even though they do not seem to yet possess those number concepts.[10] Sarnecka and Gelman (2004), on the other hand, report that children seem to have some understanding of unfamiliar number words referring to specific numerosities even before they know exactly what those numerosities are. These are interesting findings that suggest that there is more in play with number concept acquisition than what is posited in the bootstrapping account. Where does the ability to deal with numerosities larger than 4 come from, if children do not possess those number concepts?

Here the enculturated bootstrapping account presented in (Pantsar 2021a) fares better than the Carey-Beck account. It includes a role also for the ANS, in particular in connecting the OTS-based abilities to numerosities beyond the OTS range (e.g., that numerosities form a progression also beyond the subitizing range). Hence that account is compatible with there being culturally developed numerical capacities beyond the OTS range also before children become cardinality-principle knowers. However, this does not imply that the OTS is not integral to acquiring the first number concepts, as the Carey-Beck bootstrapping account argues. The above evidence does not suggest that the OTS-based bootstrapping account is fundamentally wrong, but it does suggest that we need to reconsider the role of ANS and other (in particular cultural) factors in the bootstrapping process, which is what (Pantsar 2021a) aims to do.

[10] Interestingly, Izard and colleagues (2014) report that they can use one-to-one correspondence cues to establish set identity of groups of 5 and 6 objects. This could be understood as evidence that one-to-one correspondence can precede number concept acquisition. However, children in that stage of development typically do possess the first number concepts which could, in accordance with the bootstrapping theory, be acquired without a grasp of one-to-one correspondence.

The enculturated bootstrapping account also has another, perhaps more important benefit. Given the focus it gives to cultural factors in number concept acquisition, it can also provide a better explanation of how numerical content enters our concepts. As disclosed above, the concept of counting that precedes the bootstrapping process is only that of intransitive counting, i.e., repeating a counting list in order. The concept of one-to-one correspondence is one of implicitly matching object files with objects. Neither of these have *numerical* content, yet somehow the bootstrapping process is thought to give rise to number concepts. In the enculturated account, this is explained through the cultural input. The counting games only enable number concept acquisition because they are culturally established customs that instill numerical content to the counting list and its applications in the games (for more details, see Pantsar 2021a, 2024).

3.4 Bootstrapping, *Just as Many*, and Heck's Account

Finally, we are in a position to evaluate Heck's proposal that the notion of *just as many*, as encapsulated by (1*)–(3*) is cognitively primary to both counting and one-to-one correspondence. Where would *just as many* fit in the bootstrapping process? In the second stage, children are able to complete the give-*n* task successfully for *n* in the OTS range (one to four), but they are not able to make the kind of transitive inferences that when there are *n* Barbie dolls and as many dolls as there are hats, then there are *n* hats. By the time children are able to do this, they are in the third stage of the bootstrapping process, i.e., they are cardinality-principle knowers, and are thought to possess number concepts (Carey 2009; Sarnecka and Carey 2008). Is it feasible that at the second stage of the bootstrapping process, children have grasped the notion of *just as many*? There is nothing to suggest that, in the way formulated in Heck's principles. Children's abilities with exact numerosities are limited to the OTS range, and there is no evidence that they can grasp general principles such as (1*)–(3*).

As established in the previous section, what children at that stage have is, at best, an implicit ability to connect numeral words to occupied object files. This brings us to perhaps the most important open question in the bootstrapping account: how exactly do children use counting games to grasp that the mental model for *n* objects should be associated with the *n*th numeral word in the counting list? Directly related is the question, returning to Heck's topic, as to what the primary notion in this cognitive process is. Further empirical work is needed in order to establish more conclusive answers, but if the enculturated bootstrapping account is along the right lines, we should not expect there to be a single primary notion. If counting games are crucial for number concept acquisition, it is clear that the ability to do transitive counting precedes number concepts in cognitive development. However, as we have seen, an implicit application of the principle of one-to-one matching also plays a central role in the bootstrapping account.

What the enculturated bootstrapping theory suggests is that looking for a clear and specific trajectory of primariness of notions could be inherently mistaken when we focus on the early development of numerical cognition. In the bootstrapping account, different notions seem to be present (in early forms and implicitly) at the key stage of acquiring the first number concepts. However, it is crucial here to remember that bootstrapping is an *enculturated* process. As such, it should not be considered to be an account of universal development of numerical cognition, with uniform characteristics across cultures.[11] It could be that the counting games used to teach numbers, for example, have different foci in different cultures, resulting in different emphases in notions referring to specific cognitive phenomena involved in bootstrapping.[12]

Could *just as many* be one such notion, perhaps primary to the notions of counting and one-to-one correspondence? While this possibility should not be ruled out, there is no evidence to support that view, either. Just like in the case of counting and one-to-one correspondence, there is no evidence that *just as many* is universally shared among humans. Instead, it could be a cultural construct just like all other arithmetical notions. Recall that Heck pointed out that one-to-one correspondence cannot be a primary cognitive notion because it is a *complex* notion. But there is no reason to think that *just as many* is not a complex notion, either. It is possible that some similar notion is present in an implicit form in the early stages of number concept acquisition, e.g., in the second stage of the bootstrapping process. However, this should not be conflated with grasping the general notion of *just as many* anymore than it should with grasping one-to-one correspondence. Unlike the implicit notions involved in the proto-arithmetical abilities, *just as many* and one-to-one correspondence may be cultural developments, and if so, they should be considered to be complex and not 'unsophisticated' as suggested by Heck (2000, p. 198). If *just as many* plays a role in the development of numerical cognition, it could be one notion in a plurality of factors involved. But there is no evidence to suggest that it is a cognitively primary one.

Recall the two questions Heck asked about counting, as presented in Section 2: (1) What does the statement that there are four dolls mean, if not that there are as many dolls as numerals in the sequence from 'one' to 'four'? (2) How does counting establish how many dolls there are, if not by establishing that there are just as many dolls as numerals in the sequence from 'one' to 'four'? Now we are ready to answer these questions based on the enculturated bootstrapping account. The answer to (1) is straightforward: in an enculturated system of numeral words and counting practices, we associate number concepts with cardinalities of objects.

[11] This is not to say that these characteristics cannot have cross-cultural commonalities. Piantadosi and colleagues (2014), for example, report that children in the Tsimané farming-foraging culture in Bolivia learn numbers following a similar incremental trajectory to that in other cultures, only much later. Such findings are in line with the enculturated bootstrapping account: it is to be expected that some aspects of number concept acquisition are constrained by our cognitive architecture (e.g., the object tracking system) while others are shaped culturally.

[12] Indeed, this emphasis can also vary within a culture. Jara-Ettinger and colleagues (2017) report a study on the Tsimané in which children learn to count at a relatively late age and with high variance. The study reports that some children master counting but don't understand (exact) equality, whereas some children understand exact equality but don't master counting. This suggests that counting and establishing equality are separate notions cognitively and neither can be considered to be primary within the culture. This would be in line with the present approach.

When becoming CP-knowers, we possess number concepts and the statement that there are four dolls means that we associate the number concept FOUR with the cardinality of the collection of dolls. This is in line with Heck's response to the question: grasping the number concept FOUR is (generally) not a matter of knowing that there are four numerals in the sequence from 'one' to 'four'.

The answer to (2) is equally clear. As detailed in Section 2.2, Heck argues that tagging can be done with any linguistic item, but (transitive) counting demands the notion of *just as many* in addition to tagging. However, as discussed above, to the best of our knowledge, there is no evidence of such notion of *just as many* being a universal cognitively primary principle to counting or one-to-one correspondence. *Just as many* as a notion may be a cultural construct, like counting and establishing one-to-correspondence are. Therefore, the way counting establishes that there are four dolls can differ based on the way an individual is enculturated in counting processes. This process can involve a multitude of notions, one of which may be *just as many*.

The present account is, in one sense, consistent with Heck's argument. According to the bootstrapping theory, what makes transitive counting and judgments of cardinality possible is the acquisition of number concepts. Thus, knowing that there are four dolls typically concerns our knowledge of a particular number, just as Heck argued. Cardinality statements like 'There are four dolls on the table' do not typically concern numerals, even if the acquisition of numeral words is a key stage in acquiring number concepts. The primary subject of cardinality statements is *number concepts*. Therefore, we can be confident that our analysis in this section is consistent with Heck's purpose of explaining what cardinality is about. But while consistent with their approach, our work develops Heck's account significantly. For example, Heck did not detail what they mean by counting, which we have done. Furthermore, by analyzing Heck's framework in terms of the enculturated bootstrapping theory of number concept acquisition, this paper places it in a modern, empirically informed context. This is an important factor, given the great development in the field of numerical cognition in the past decades.

Perhaps most importantly, our account manages to do what Heck's account does not: provide an empirically valid characterization of what *just as many* could be like as a notion. Heck (2000, p. 198) points out that we need an 'unsophisticated' answer to what *just as many* means, i.e., an answer that does not involve more complex notions. With the enculturated bootstrapping account, we can suggest such an answer. *Just as many* means a particular enculturated notion of expressing equinumerosity. It is possible that equinumerosity can be grasped without possessing number concepts, as in the example of forks being equinumerous with knives. However, data point to one-to-one matching being an ability that is connected to grasping number concepts, as reported in the case of Pirahã (Everett and Madora 2012; Frank et al. 2008; Gordon 2004). It is possible that *just as many* is similarly connected to having a notion of cardinality that arises from acquiring number concepts. If this is true, *just as many* is also one culturally developed way of expressing that two collections have the same cardinality, just like equinumerosity is – but unlike the culture-independent proto-arithmetical abilities.

If the enculturated bootstrapping account is correct, then grasping *just as many* may be cognitively dependent on counting processes and the implicit matching of object files with observed objects. As Heck argued, *just as many* may be a more primary notion than one-to-one correspondence. However, if the considerations in this section are correct, even in this case, it is not necessarily the kind of fundamental notion from which the notion of cardinality generally comes.

4 Conclusion

In this paper, we have shown how in light of the bootstrapping account of Carey (2009) and Beck (2017), developed further in (Pantsar 2021a, 2024), it is plausible that instead of being the kind of cognitively fundamental notion as proposed by Heck, the notion of *just as many* is one of several enculturated factors on which the acquisition of number concepts depends. While there is no evidence, to the best of our knowledge, that this notion is cognitively fundamental, it is nevertheless possible that it has played an important role in the cultural development of numerical cognition and arithmetic. However, *just as many* may be a culturally developed notion, and it is possible that humans (as well as nonhuman animals) can make (at least, implicitly) cardinality judgments also without it. Nevertheless, in acquiring the culturally developed concept of exact cardinality in terms of natural number concepts, *just as many* could be an integral notion. Its role could also vary between cultures, which advances an interesting question for empirical research: are there cultures (and languages) in which *just as many* plays a more important role than in others? To the best of our knowledge, no such studies have been conducted.

It should be noted that the present analysis is not limited to the bootstrapping account in its current forms. It could be that the details of the bootstrapping theory turn out to be mistaken, for example, when it comes to the way numerosities are represented in mind. All the same, in the face of the best empirical data on number concept acquisition, there are good reasons to think that the enculturated bootstrapping theory is along the right tracks. Our argumentation in this paper should be generally compatible with such an account, and the details need to be adjusted as we gain a better empirical understanding of the development of numerical cognition and number concept acquisition.

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