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Why do numbers exist? A psychologist constructivist account

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ABSTRACT
In this paper, I study the kind of questions we can ask about the existence of numbers. In addition to asking whether numbers exist, and how, I argue that there is also a third relevant question: why numbers exist. In platonist and nominalist accounts this question may not make sense, but in the psychologist account I develop, it is as well-placed as the other two questions. In fact, there are two such why-questions: the causal why-question asks what causes numbers to exist and the teleological why-question asks for what purpose numbers exist. I argue that in a psychologist constructivist account, in which numbers are understood to exist as referents of a particular type of culturally shared concepts, both why-questions can get plausible answers.

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1. Introduction

Why do numbers exist? I take it that this question divides philosophers of mathematics more in whether it makes sense at all than in what the particular answer could be. Indeed, historically the question seems to have been considered so misplaced that for a long time it did not play a notable role in the philosophy of mathematics (see, e.g. Benacerraf and Putnam 1984). While there have been debates concerning whether numbers exist and if so, *how* they exist, asking *why* they exist has not been viewed as a similarly important question. In this paper, however, I want to challenge that view. I will argue that the *why-question* is both well-placed and it can be feasibly answered.

My approach is to distinguish between three different types of questions concerning the existence of numbers. First and most fundamental

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is the '*do they'-question*, which asks whether numbers exist. In Section 2, I show that this is the first great divide between philosophers: platonists, for example, accept the existence of numbers, while nominalists explicitly deny it. The second question asks that if numbers exist, how do they exist? This *how-question* gets different answers and as I will show in Section 2, historically much of the discussion in philosophy of mathematics has revolved around it. But traditionally, numbers have been thought to exist – if at all – in a mind-independent manner. Understandably, this has not raised the question why numbers exist. Recent developments in philosophy and the cognitive sciences, however, have led to increasing popularity of *psychologist* and *social constructivist* views in the study of foundations of mathematical knowledge.[1] These two views can be related but are not necessarily so, as we will see, and the primary focus of this paper in on psychologist views. In psychologist approaches, the epistemology of mathematics is studied based on empirical data on human cognitive capacities. In this paper, I propose that empirical data suggest that numbers as abstract objects exist, but they do so in a mind-dependent manner. Thus, in Section 3, I argue that in psychologist (and social constructivist) approaches, in addition to the 'do they' and 'how' questions, we also face a third type of question concerning the existence of numbers, i.e. the why-question. In fact, I contend that there are two different why-questions which should both be considered to be *bona fide* questions for philosophers of mathematics. The first of these is the *causal why-question*: what causes numbers to exist? The second is the *teleological why-question*: for what purpose do numbers exist?

As I will argue, the most feasible psychologist understanding of the existence of numbers is that they exist as referents of a particular type of shared number concepts. Thus the matter should be studied in close connection to psychologist approaches to number concepts. These can be divided roughly into two categories: *nativist* views, according to

[1]The term 'psychologism' has bad associations in the philosophy of mathematics, as it is often connected to the kind of 19th century psychological accounts of numbers (e.g. Schröder 1873) that Frege (1884) famously, and justifiably, criticised. For this reason, I would prefer to use the term 'cognitivism' for the kind of approaches I am focused on in this paper. However, this term already has two different interpretations, both of which need to be distinguished from the view I am concerned with here. In psychology, cognitivism was a response to behaviorist psychology starting from the 1950s. This kind of cognitivism is generally compatible with the approach here, but more problematic is the use of the term in the literature on philosophy of mind, in which cognitivism is often associated with internalism about cognition. As pointed out by Menary (2007, 10), cognitivism does not need to be an internalist doctrine, but to be safe, I will use term 'psychologism' here, with the hope of reclaiming its use from the kind of writings that Frege opposed to.

which number concepts are innate, and *constructivist* views, according to which number concepts are acquired during ontogeny. In Sections 4 and 6, I will show that these two psychologist approaches give different answers to the how-question, but also to both the causal and the teleological why-questions. Based on philosophical conceptual analysis and empirical evidence, in this paper, I argue that the view best supported by the current state-of-the-art understanding of numerical cognition is a psychologist constructivist position. In Section 5, I elaborate what the existence of numbers means in the context of that position. In sum, in Sections 4–6, I will argue that the psychologist constructivist approach can provide answers to all three types of questions concerning the existence of numbers. In the process, I will establish that – contrary to the tradition in philosophy of mathematics – the 'do they', 'how' and 'why' questions are all important for philosophy.

2. Three types of questions concerning the existence of numbers

In the introduction, I presented two different types of why-questions concerning the existence of numbers:[2] the causal why-question and the teleological why-question. I propose that the fact that neither of these questions has traditionally played a major role in the philosophy of mathematics is due to the long history of *platonist* epistemology and ontology.[3] Platonist epistemology is based on the following tenet: since mathematical objects themselves cannot be accessed through sense perception, knowledge about the objects must be gained through reason (*The Republic*, 527a-b). The ontological counterpart of this epistemological account is that mathematical objects exist outside the realm of the physical, i.e. they are *abstract* (non-physical, atemporal and causally inactive). In modern philosophy, mathematical platonism is often presented as the conjunction of three claims (e.g. Linnebo 2018a). According to the *existence* claim, mathematical objects exist. The *abstractness* claim specifies that mathematical objects are abstract. And finally, the *independence* claim states that they are independent of human languages, thoughts and practices.[4]

[2]I am primarily focusing in this paper on the existence of natural numbers (0, 1, 2, 3, …). However, in Sections 6 and 7 I will also consider other number systems.

[3]This proposal is supported by the fact that the best-known (but often implicit) treatments of the why-question have taken place in the recent decades, as part of non-platonist accounts (see, e.g. Burgess 2008; Burgess and Rosen 1997; Cole 2015; Steiner 1978).

In the philosophical literature, it is commonplace to speak of abstract objects understood in the above fashion as being *mind-independent* (see, e.g. Dummett 2006). If it is accepted that mathematical objects are abstract and mind-independent, it is straight-forward to see why neither of the two why-questions presented above has found an important place in the philosophy of mathematics. When presented about abstract, mind-independent numbers, asking the causal why-question is akin to asking why, say, gravity exists. Aside from possible theist answers, it does not appear fruitful to look for the cause of numbers existing. The teleological why-question would seem to be similarly misplaced. For what purpose could the mind-independent numbers exist, if their existence is atemporal and as such does not depend on any agents and their aims?

The above platonist view of mind-independent mathematical objects has been popular in the history of Western philosophy, but in recent decades it has received competition from many explicitly non-platonist theories (e.g. Cole 2015; Field 1980; Hellman 1989; Kitcher 1983; Lakoff and Núñez 2000). However, it would be mistaken to treat platonism as an antiquated view. It is held in different forms by many prominent contemporary philosophers of mathematics, including Shapiro (1997), Brown (2008) and Linnebo (2018b). However, while presumably agreeing on the intelligibility of the why-questions, platonist philosophers offer essentially different answers to the *how-question* concerning in what manner numbers exist.[5] For Shapiro (1997), numbers exist as places in mind-independent ('ante rem') *structures* of numbers. Linnebo (2018b), on the other hand, argues that numbers are 'thin objects' that exist as referents of singular terms formed by *abstraction principles*, such as *Hume's principle* (HP), which states that the number of *F*s is equal to the number of *G*s just in case there is a one-to-one correspondence between the *Fs* and the *Gs* (Boolos 1998, 181; Frege 1884). Shapiro's structuralist account and Linnebo's abstractionist account are thus clearly two different answers to the how-question of characterising mathematical existence.

[4]In this paper, I follow the custom that Platonism with capital 'P' refers specifically to Plato's philosophy whereas platonism with a lower case 'p' refers to a more general realist metaphysical position on mathematics that fulfils the three claims stated above. Tait (2001) has suggested that instead of platonism, it would be clearer to talk about 'realism' than platonism. The only reason I use the term 'platonism' here is to follow the terminology of the relevant literature more closely.

[5]The importance of how-questions in metaphysics has been stressed by Schaffer (2009), who argues that the answer to the 'do they'-question is trivially 'yes' (since existence can also mean existing in fictions) and only the how-question is interesting.

However, there are also philosophers who argue that the how-question is misplaced. According to mathematical *nominalists*, abstract mathematical objects like numbers do *not* exist (Bueno 2020; Burgess and Rosen 1997). The how-question is therefore misplaced for nominalists simply because numbers are not thought to exist in the first place. This may sound unintuitive, given that mathematics is full of existence theorems that state that a certain object, like a number or a set of numbers, exists. However, as Shapiro (1997) has pointed out, this does not need to commit to anything more than a methodological 'working realism', according to which mathematicians work *as if* numbers existed.

For the nominalist, the questions regarding the existence of numbers end by answering the 'do they'-question negatively. Just as platonism comes in different forms, there are also different versions of nominalist philosophy of mathematics. One of the most important differences concerns whether mathematical statements can be *true*. According to *fictionalism*, mathematical objects are fictional entities and any existential claim about them can only be true as part of a particular fiction (Brock 2002; Field 1980). Other forms of nominalism are committed to the view that mathematical objects or structures do not exist in a mind-independent, abstract manner, but still contend that mathematical statements can be true. The nominalist approach can then be to either to reformulate mathematical theories in a way that does not imply any ontological commitment, such as in the case of the modal structuralism of Putnam (1967) and Hellman (1989), or to argue that mathematical theories do not involve any ontological commitment to mind-independent structures or objects (e.g. Azzouni 2000).

While nominalism is clearly an anti-platonist view, not all forms of anti-platonism are nominalist. According to one such view, mathematical objects or structures can be thought to exist, but they are not fundamentally abstract (see, e.g. Kitcher 1983). This view may sound *prima facie* impossible since certainly mathematical objects, such as sets, numbers and lines, cannot be concrete, physical objects. For one thing, many mathematical objects demand infinity, and it is not clear that even the physical universe as a whole can be considered to be infinite. Yet the anti-platonist position need not be committed to the view that mathematical objects or structures themselves are concrete. It could be that they are abstract, but not *independent* of concrete objects and structures. In this kind of anti-platonist account, mathematical objects and structures are thought to be abstract, but they are in some way *grounded* in physical objects and structures.[6]

A related view is social constructivism, according to which abstract mathematical objects like numbers exist as particular types of social constructs (Cole 2009; 2013; 2015; Feferman 2009). In this paper, I will explore a similar approach. In my approach, mathematical objects are not thought to be grounded on physical objects, but their character as social constructs is nevertheless strongly constrained by our experience of the physical world. Consequently, I will develop an account according to which mathematical objects like natural numbers are abstract constructions that exist as referents of a particular type of culturally shared concepts, namely, a type that is in an important way shaped by our evolutionarily developed cognitive architecture.

Before we continue, we should take stock of the kind of questions we can ask about the existence of mathematical objects or structures. Above I have outlined a trajectory of three types of questions concerning the existence of mathematical objects. First is the '*do they*'-question, to which platonists answer in the positive and nominalists in the negative. Second is the *how*-question, to which different platonist philosophers have different answers, while for nominalists the question is misplaced. Third is the *why*-question, which takes both the causal and teleological forms, and which, as we have seen, is not well-placed for either platonists or nominalists.[7] To make sense of the why-question, we need to follow a third path, according to which numbers exist but their existence is dependent on human subjects.

3. Psychologism and the how-question

I take it as uncontroversial that we can have mathematical knowledge and, whatever the status of mathematical objects may be deemed to be, the development and characteristics of this knowledge can be, at least in principle, analysed in terms of psychological processes. This is the case if we think of reason providing the epistemic access to a Platonic world of objects, but also if we think of mathematics ultimately being merely a shared fiction. Therefore, merely including psychological

[6]Pettigrew (2008) has called this type of account *aristotelianism* in mathematics and it has been supported by, among others, (Resnik 1997) and (Franklin 2014).

[7]It should be added that there is one type of why-question that both platonists and nominalists accept, namely the *constitutive* why-question. An example would be a set-theoretic account of natural numbers, in which the why-question concerning the existence of numbers would be answered by them being constituted of sets. In the present framework, however, the constitutive why-question is better understood as being included in the how-question. I thank Bahram Assadian for a helpful remark in this regard.

processes in the epistemology of mathematics is not necessarily connected to any particular epistemological or ontological position. However, using psychological methodology for studying philosophical problems has not been commonplace in philosophy of mathematics. This kind of approach has been called *psychologism* in the philosophical literature, following the use of the term by Ellis (1991).

The main tenet of the kind of psychologism concerning mathematical knowledge that I focus on in this paper is that studying the psychological processes involved in mathematical cognition is important for the epistemology of mathematics. It could be that the study of mathematical cognition ultimately points us towards either platonism or nominalism, or other epistemological and ontological views presented in the philosophical literature. However, as an approach, psychologism should be distinguished from any of those accounts. While platonists and nominalists, for example, make explicit ontological and epistemological claims, in psychologism the ontological and epistemological questions are approached in a bottom-up manner: we can speak of ontological and epistemological matters only when they are informed by cognitive considerations. In this way, psychologism as a general position is potentially compatible ontologically and epistemologically with various views, including platonism and nominalism. However, for present purposes, the most interesting type of psychologism differs from both platonism and nominalism in ontology. In this type of psychologism, numbers are thought to exist, but they are thought to exist in a *mind-dependent* manner. In the rest of this paper, when writing about psychologism, I am referring to this non-platonist and non-nominalist variation.

The type of psychologism that is committed to the mind-dependent existence of numbers can be divided into two approaches. Common to these approaches is that numbers are thought to exist in some way through *number concepts*. In the first approach, number concepts are due to some innate properties of the mind (Butterworth 1999; Gallistel 2017; Gelman and Gallistel 1978; 2004). This can mean several things, ranging from actual innate number concepts (Gallistel and Gelman 1992) to an innate ability to do arithmetic (Wynn 1992) or an innate approximate mental number line (Dehaene 2011). The shared key idea between these accounts is that there is an innate capacity for counting and/or arithmetic, which is ultimately responsible for the existence of number concepts. In this way, number concepts are thought to be a product of biological evolution (De Cruz and De Smedt 2010). Let us call this family of views *nativism* over number concepts.[8]

Not everybody agrees that this type of nativism over number concepts makes number concepts mind-dependent. According to those who disagree, mental states (and the mind itself) are mind-dependent only in a trivial sense, since they are not *constructed* by minds (see, e.g. Brock and Mares 2007, 40). According to this kind of view, mind-dependent entities are not simply mental entities, but something that *depend* on mental entities for their existence. I find this notion of mind-dependence appealing, but it is not my main reason for not focusing on the nativist view in this paper. The main reason for that is that, as we will see in the next section, nativism about number concepts in all the above forms is unsupported by empirical evidence.

Therefore my main focus will be on the second psychologist view that takes numbers to exist in a mind-dependent manner. This view takes number concepts to be mind-dependent but in a *culturally developed*, rather than nativist, manner. In such accounts, it is accepted that there can be innate quantitative, *proto-arithmetical* (or 'quantical'), abilities, but these should not be confused with arithmetical abilities concerning number concepts (De Cruz, Neth, and Schlimm 2010; Núñez 2017; Pantsar 2014). Number concepts are seen as the product of specific culturally shared practices, emerging as the product of *cumulative cultural evolution* (Boyd and Richerson 2005; Henrich 2015; Heyes 2018) and acquired in ontogeny through processes of *enculturation* (Fabry 2020; Menary 2015). According to this view, number concepts are something that humans *construct* by applying their cognitive capacities in interaction with their environment (see, e.g. Lakoff and Núñez 2000). As with nativism, there are many different forms of such *constructivism*, ranging from externalist views, according to which material symbols are central to the cognitive process of number concept construction (Overmann 2018; Zahidi 2021) to views in which genetically determined and cultural factors in tandem allow the construction (and acquisition) of number concepts (Beck 2017; Carey 2009). But, again, there is a distinct shared key idea: number concepts are human constructs emerging from cognitive practices, at least some of which are culturally developed.[9]

[8]It is important to note that 'innate' can mean several different things and should not be confused with 'present at birth'. It could be, for example, that number concepts are innate in the sense that the cognitive capacities that give rise to the emergence of number concepts are genetically determined. See Griffiths (2001) for more on the different interpretations of 'innate'.

[9]For an approach focusing on the history of mathematics from a constructivist background, see Muntersbjorn (1999; 2003). For more on the development of mathematics through cognitive practices, see Ferreirós (2015; Wagner 2017).

This psychologist's view is clearly related to social constructivism about mathematical objects, such as those presented by Cole (2013; 2015) and Feferman (2009). However, unlike those social constructivist accounts, the psychologist view I want to pursue here is (partly) based on empirical data on the development of numerical cognition and a specific framework of cultural learning, namely the enculturation account (Menary 2015). I believe that these differences are important and distinguish the present account from those of Cole and Feferman. Most importantly, as becomes clear in the next section, in the present account number concepts as social constructs are partly determined by our evolutionarily developed cognitive capacities. Therefore, in my account, there is a good explanation why number concepts are not completely *conventional*, which makes an important difference to the objectivity of number concepts.[10]

The fact that the present account is not conventionalist is particularly important because the question of objectivity is in the present account central to understanding how number concepts and *numbers* are connected. Above we have been discussing number concepts, but the topic of this paper is the existence of numbers. In the psychologist constructivist account I will develop in this paper, numbers exist as the referents of a particular type of culturally shared number concepts. The number four, for example, exists as an abstract object ultimately because within and across cultures, people share the number concept FOUR. Therefore, by employing the concept FOUR they refer to the same thing in their arithmetical expressions. This thing is a culturally developed construct, the abstract object number four.

How can we establish that that 'thing', the culturally developed construct, exists? A direct consequence of the present account is that the culturally shared number concepts refer to abstract objects. Since numbers as objects are only thought to be referents of culturally shared concepts, it is enough for their existence that the same number concepts are generally shared by members of societies. I will show in the next section that we can establish this to be the case because the number concepts are the

[10]In a nutshell, the weakness I see in particular in the account of Cole (2013; 2015) is that although mathematics is thought to play a representational function on reality, the account provides no convincing reason why mathematics – consisting of social constructs – would do so. Thus, although Cole's account as such is not conventionalist, there is no apparent reason why mathematics could not consist of deeply entrenched conventions in the sense of Warren (2020). In my account, this conventionalist threat is avoided because mathematics is thought to be based on our evolutionarily developed cognitive architecture, which is clearly non-conventional. More on this argument, which is pursued further in Pantsar (2023), in Section 5.

result of an enculturated development based on universally shared (at least by neurotypical humans) proto-arithmetical abilities. Thus the existence of numbers follows from the existence of culturally shared number concepts.[11] According to the platonist interpretation of psychologism, numbers are more than that, i.e. they have an existence independent of the number concepts shared by cultures. However, in the present account – as I will show – there is no motivation to make that assumption.

It should be noted that psychologist constructivism is open to many different answers to the how-question about the existence of numbers presented above, i.e. whether numbers exist, for example, as individual objects or places in a number structure. If numbers exist as cultural constructs shared by the members of social groups, they can exist equally feasibly as individual objects or places in structures. However, aside from these how-questions, constructivism provides a clear answer to another, more general how-question: do numbers exist mind-independently or mind-dependently? In this, psychologist constructivism differs essentially from all platonist views. In the present context, constructivism is a particularly interesting view since, in addition to the 'do they' and 'how' questions, it also proposes answers to the two kinds of why-questions: the causal why-question and the teleological why-question. In the next three sections, I will present those answers and argue that the psychologist constructivist account provides a feasible explanation to all three types of questions about the existence of numbers.

4. The causal why-question

Since psychologist nativism and constructivism take mathematical objects such as numbers to exist in a mind-dependent manner, it makes sense to ask *why* they are created by the mind (or minds, to be more precise). In Section 6, I will focus on the teleological why-question. But for now, let us concentrate on the causal why-question. This is a particularly important question when it comes to the present account, because, as we will see in Section 5, it also implies why it is sufficient for the existence of numbers that people possess shared number

[11] I have been asked whether this also implies that the culturally shared concept of, say, a dragon implies that dragons exist. There are many reasons why number concepts and dragon concepts are not comparable, but the most important one is that numbers as referents of shared number concepts are abstract. Clearly only abstract – if any – objects can be brought to existence via shared concepts. The more interesting question is how number concepts are different from the other kind of shared concepts that do not have physical referents, but don't have abstract objects as their referents, either. This problem is tackled in Section 5.

concepts, when clearly this is not the case for all things referred to by shared concepts.

The psychologist causal why-question is divided into two questions, one concerning nativism and one concerning constructivism. For the nativist account, the answer is simple. If numbers can exist as referents of shared number concepts, and number concepts are either innate or determined by innate cognitive capacities, the reason for the existence of numbers must be found in processes of natural selection that (mainly) drive biological evolution. Many researchers have found support for this position in the empirical data. It has been claimed that 'infants possess true numerical concepts' and 'humans are innately endowed with arithmetical abilities' (Wynn 1992, abstract). Similarly, there have been claims that there are genuine arithmetical abilities in many non-human animals, including monkeys, parrots and newborn chicks (Agrillo 2015; Hauser, Carey, and Hauser 2000; Pepperberg 2012; Rugani et al. 2009). These findings suggest that not only are numerical concepts and arithmetical ability evolutionary adaptations, but they are either evolutionary early ones or have taken place multiple times in the course of the evolution of different species. In any case, since infants and untrained non-human animals cannot be feasibly thought to possess culturally learned abilities, their capacity for numbers and arithmetic must be an evolutionary adaptation.

The problem with this view is that it rests on a questionable reading of the empirical evidence. A careful analysis of infant and non-human animal abilities suggests that they indeed process observations in terms of numerosities, but these capacities are far from being arithmetical and including proper number concepts. Standardly, the evolutionarily developed quantitative capacities are divided into two abilities (e.g. Agrillo 2015; Feigenson, Dehaene, and Spelke 2004; Hyde 2011). For quantities from one to four, there is a *subitising* ability that allows determining the cardinality of objects without counting. The subitising ability has been confirmed in infants and many non-human animals (Dehaene 2011; Spelke 2000; Starkey and Cooper 1980). Subitising is closely related to the ability to do multiple object tracking and it is thus now standardly thought to be due to the *object tracking system* (OTS) that allows for the parallel individuation of objects (Carey 2009; Trick and Pylyshyn 1994). The OTS is not numerosity-specific, but there is also strong evidence of another innate cognitive system for numerosities, called the *approximate number system* (ANS) in the literature (Dehaene 2011; Spelke 2000). The ANS makes it possible to estimate larger numerosities, although this comes with the price that the estimations lose accuracy as the numerosities become larger, following the so-called Weber-Fechner Law (Knops 2020).

It is commonly accepted among empirical researchers that humans share both the subitising and estimation ability with other animals (see, e.g. Cantlon and Brannon 2006). It is also commonly accepted that the evolutionarily developed abilities form a cognitive basis for our arithmetical ability. Some researchers believe that the OTS is the primary innate cognitive system in developing arithmetic (Beck 2017; Carey 2009; Izard et al. 2008) while others see the ANS as the primary system (Dehaene 2011; Halberda and Feigenson 2008). There are also researchers who believe that both systems play an important role in that development (e.g. Spelke 2011a ; van Marle et al. 2018). In addition, some researchers argue that OTS and ANS are in fact part of a unified cognitive system (see, e.g. Cheyette and Piantadosi 2020).

It is not possible here to go into the details of all those accounts (see (Pantsar 2021a) for a more thorough exposition), but the important thing here is to realise that none of them require that number concepts or arithmetic are in fact innate evolutionary adaptations. The subitising ability is limited to very small quantities while the estimation ability is approximate. Neither of them thus carries all the most important characteristics of natural number concepts: that they are discrete and form a linear progression that can be continued indefinitely. It is not yet clear how the OTS and the ANS represent numerosities (or indeed, *if* they represent anything; see (Zahidi 2021)) but it is crucial to note that those representations (if they are representations) are not yet number concepts. Infants and non-human animals show sensitivity to numerosities, but so far all the observed behaviour can be explained as being due to the OTS or the ANS. That is why it is important to make a distinction between *proto-arithmetical* and *arithmetical* abilities. Proto-arithmetical abilities (subitising and estimation) deal with numerosities, but only arithmetical abilities deal with number concepts (for details, see Pantsar 2014; 2021a; 2024).[12]

Some of the strongest evidence against the nativist position comes from the fact that there are human cultures in which individuals do not possess number concepts. The Amazonian peoples of Pirahã and Munduruku, for example, show similar proto-arithmetical abilities to Western

[12]In a similar manner, Núñez (2017) has proposed the term ‘quantical’ to distinguish proto-arithmetical abilities from arithmetical abilities.

people, but they cannot complete numerical tasks that would require proper number concepts, starting from simple addition tasks (Everett and Madora 2012; Frank et al. 2008; Gordon 2004; Pica et al. 2004). The reason for this cannot be due to a different evolutionary trajectory, since the Amazonian peoples have diverged relatively recently in the human genealogy. Instead, the reason can most likely be found in the fact that neither the Pirahã nor the Munduruku languages include a system of *numeral words* for exact quantities. Therefore, there is a double case against psychologist nativism concerning number concepts. First, all the observed infant and non-human behaviour can be explained without postulating number concepts: all that is needed are proto-arithmetical abilities, whose presence in infants and non-human animals is firmly established. Second, adult humans in cultures without numeral words do not appear to possess number concepts. All this evidence strongly points towards the position that number concepts are in fact a *cultural* development, not an evolutionary one.

However, it would be mistaken to conclude that number concepts are a purely cultural phenomenon, in the sense that their content is reducible to culturally shaped conventions. As I have argued in detail elsewhere (Pantsar 2019; 2021a), the acquisition of number concepts in the course of individual ontogeny is best understood in the framework of *enculturation*. Enculturation refers to the transformative effect of culturally shaped practices on cognitive capacities (Fabry 2020; Menary 2015). The neural plasticity of the brain makes structural and functional variations possible, which enables the acquisition of new cognitive capacities in the ontogeny of the individual. Novel abilities, such as reading and writing, are thus made possible by redeploying old, evolutionarily developed neural circuits for new, culturally specific functions (Dehaene 2009; Menary 2014).[13] Menary (2015) calls this feature of the brain *learning driven plasticity*.

In explaining number concept acquisition, learning driven plasticity plays a key role. Many researchers hypothesise that evolutionarily developed proto-arithmetical abilities for processing numerosities in the intraparietal sulci are recycled (or reused) for new, arithmetical functions. Together with linguistic abilities for numeral words and symbols, the OTS and the ANS are re-deployed, resulting in two different (but partially overlapping) systems for processing numerosities in the brain (Dehaene

[13]This principle is called *neuronal recycling*. Anderson (2010; 2015) has argued for a different, more general principle of *neural reuse*. In the context of arithmetical cognition, it has been argued that neural reuse provides a better explanation (Fabry 2020; Jones 2020).

2011; Dehaene and Cohen 2007; Menary 2015; Nieder and Dehaene 2009). It is not possible to go into the details of the empirical theories, but the exposition here is sufficient for moving into the philosophical question concerning the existence of numbers. If number concepts are a culturally developed phenomenon based on our proto-arithmetical abilities, what does that imply for the three questions concerning the existence of numbers? The first thing to note is that the enculturation framework is a bad fit with strict conventionalist and fictionalist accounts of number concepts. Since proto-arithmetical abilities are thought to partly determine the content of number concepts, we cannot think of number concepts as being purely conventional fictions. According to the best modern understanding of number concept acquisition, number concepts are (partly) based on either the OTS or the ANS, or both (Beck 2017; Carey 2009; Dehaene 2011; Izard et al. 2008). Since those systems are universally shared by humans, number concepts cannot be purely conventional. This is in line with the observation that while there are cultures without numbers concepts, different cultures (e.g. the Greek, the Chinese and the Mayans) have independently developed natural number concepts and arithmetical operations in highly converging ways (Everett 2017; Ifrah 1998). We will return to this topic in Section 6 when the focus is on cultural development, but for now, it is important to realise that if cultures develop number concepts, they tend to have similar characteristics.[14]

However, even though strict forms of conventionalism are a bad fit with the enculturation account of number concepts, it is of course possible to take a nominalist approach to enculturated *numbers*. Whatever number concepts may be cognitively, it does not follow that there exist numbers. In a reverse manner, the enculturation account is also compatible with platonism: whatever number concepts may be cognitively, the platonist would argue, it does not have an effect on what numbers are. While both views may be impossible to refute, I find them less than satisfactory. One central tenet of the psychologist approach I pursue in this paper is that the more we know about the cognitive development of number concepts, the more we know about the epistemology and ontology of the numbers themselves. Indeed, as mentioned before, I contend that the existence of numbers is best explained through us having culturally shared number concepts. The existence of numbers

[14]More precisely, the number concepts tend to have similar characteristics when it comes to finite positive integers and their arithmetical operations. There are important differences in arithmetical practices, such as proofs, as well as concerning zero and infinity (see, e.g. Ifrah (1998) for more).

could concern something more than that, as the platonist would argue (for example, numbers being atemporal entities), but in the present approach there is no reason to make that assumption. Indeed, through Occam-like reasoning, I see no reason to assume that numbers as abstract objects are anything more than referents of shared number concepts.[15] The question of existence of numbers is then best framed in philosophy through the question of the existence and characteristics of number concepts.

5. From number concepts to the existence of numbers

Based on the above analysis, I propose the following answer to the how-question concerning the existence of numbers under the present account: numbers as objects exist as referents of *culturally shared concepts* based on proto-arithmetical abilities. Numbers exist because new generations are enculturated in mathematics, making them acquire (mostly) the same number concepts as the earlier generations.[16] This could be understood in a nominalist manner, implying that numbers do not really exist at all, but that kind of argument only works against a platonist, mind-independent, view of the existence of numbers.[17]

Under the present account, the existence of numbers is mind-dependent, which raises two questions. First, how can we establish that mind-dependent abstract entities exist? Second, what is special about *numbers* in that they are brought into existence by shared number concepts, whereas not all shared concepts bring abstract objects into existence? Let us focus initially on this second question, which I take to be a *bona fide* potential problem for the present account. What makes numbers different from something like trade agreements, the concepts

[15]This is compatible with the account developed by Hodes (1984), who identifies numbers with numerical quantifiers. For Hodes, numbers are then fictions that are used because they encode facts about numerical quantifiers conveniently. The important difference between the present account and that of Hodes is that in my view numbers should not be considered to be fictions, since their existence is tied to the existence of culturally shared number concepts, which are not purely conventional.

[16]This is the case generally: we acquire our number concepts from the previous generation and pass them on to the next generation. However, at times the domain of number concepts changes. For example, in the case of natural number concepts, zero was included as a number concept relatively late. In addition, entirely new number concepts, like those of irrational numbers and complex numbers have been introduced.

[17]As Pettigrew (2008) has pointed out, even among anti-platonists there is a tendency to accept the platonist interpretation of mathematics as a default position to be argued against. However, I see no reason why a platonist reading of mathematical existence should be somehow *prima facie* privileged. Related to this, I have been asked whether this understanding of numbers existing as culturally shared concepts fails to distinguish between sense and reference, as brought to philosophy by Frege (1892). However, this question only makes sense if numbers are also thought to exist as something other than referents of culturally shared concepts, which is an assumption that I see no reason to make.

of which are widely shared, but only some of which exist? The Australia-EU trade agreement, for example, is widely discussed and we can assume that people possess similar concepts of it. However, unlike the Australia-US trade agreement, for example, the Australia-EU agreement does not exist.[18] Is my account of the existence of numbers doomed to not being able to make such distinctions, given that shared number concepts are thought to be enough to bring their referents (natural numbers) into existence?

To avoid the above problem of non-existence, I previously stated that numbers exist as 'referents of a *particular type* of culturally shared concepts'. Now is the time to explicate what that 'particular type' means. In the social constructivist account of Cole (2013; 2015), mathematical entities are thought to exist as *institutional* objects (Searle 1997; 2010), which means that there are constitutive rules for their existence. Abstract objects like money, laws, rules of games, etc. are brought into existence in a process of ratification (or *declaration* Cole (2015)) where these rules are agreed upon, but the constitutive rules can also be implicitly accepted. In that account, mathematical objects are no different. They exist as abstract objects because there are constitutive rules for their existence (e.g. axiomatic systems).

One clear strength of Cole's account is that it can easily explain the difference between the Australia-US and the Australia-EU trade agreements. Only the former is ratified by constitutive rules, hence only it exists. However, while this may be correct for some abstract objects, I don't think that is how mathematical objects like natural numbers are brought into existence. Historically, there must have been many stages in the development and use of numbers before anything like a ratification of arithmetic existed. One could argue that numbers only came into existence when such ratifications arrived, but I find this unsatisfactory. It is impossible to trace the origins of the first systematic use of numbers, but it is extremely unlikely that their introduction was similar to ratifying laws or rules. Most likely, it was a much more gradual development in which numeral words, number concepts and their applications developed in parallel (more on this in Section 6). Now the question is: at what point of this development were numbers as abstract objects brought into existence? It would be spurious to propose a precise answer, but if the notion of abstract objects existing based on human practices is thought to make sense, certainly the answer must be that this happened before any

[18]I am thankful to an anonymous referee for providing this example.

institutional ratification of arithmetic. I contend that it originally happened at the point when enough members of the culture shared numeral words that they could use to reliably communicate their shared number concepts, and communicate with them. [19]

How can we know that those people shared the number concepts? Here is where the proto-arithmetical origins are crucial to acknowledge. Contrasted with rules of games, laws and other institutional objects, number concepts are non-conventional in a stronger manner. As mentioned in footnote 10, I consider this to be the great strength of the present account, and in that respect it differs from Cole's account of institutional objects. Since in his account institutional objects can be brought into existence also by implicitly accepted constitutive rules, my above account of how numbers came to exist is compatible with Cole's. The difference is that Cole provides a very different type of account of why those particular constitutive rules regarding numbers came to be accepted. In Cole's account, numbers must have the features they do in order to serve their representational function. However, this does not provide a strong argument against the conventionalist threat, given that it provides no non-conventionalist reason why that representational function arose in human practice. Indeed, it is instructive that both Cole (2013; 2015 and Feferman (2009) draw parallels between chess and mathematics.

In contrast, my account explains why the practices that brought numbers into existence are different from many other types of practices. Clearly we cannot change rules concerning numbers like we can change the rules of games, for example, so the process of institutionalisation is different. Indeed, I contend that when it comes to natural numbers, their content is determined (partly) already in a pre-institutional fashion due to the shared number concepts based on our proto-arithmetical abilities. This is what I mean by saying that numbers are 'referents of a particular type of culturally shared concepts'. Number concepts are not like many other concepts, because their cognitive basis is shared (partly) already pre-institutionally. This, I argue, makes numbers as abstract objects different from objects being brought into existence entirely by their constitutive rules. That is also the reason why I believe the present

[19]One may ask whether this kind of explanation of the existence of numbers is causal, given that numbers as social constructs are thought to be abstract, i.e. causally inactive. However, under the present approach, if an object is created by humans, there can exist a causal explanation of its existence. In practice it is of course impossible to trace a causal chain of events to the ratification/declaration that first brought numbers to existence, but in the present account – as presumably in Cole's social constructivist account – such a chain must exist.

account to provide a stronger answer to the conventionalist threat than those of Cole and Feferman.

The above considerations bring us to the first question: how can we establish that numbers as mind-dependent abstract entities exist, i.e. that the referents of number concepts are abstract objects? Because numbers are not thought to exist in some abstract realm that exists mind-independently, we must explain how the existence of abstract entities can depend on concrete things, like human brains (or minds)? I cannot tackle the general question of the existence of abstract objects here, but my approach is generally compatible with approaches like (Thomasson 2008), which accept that human creations can exist as abstract artefacts.[20] Such artefacts are sometimes called 'intentional objects' in the philosophical literature and their ontological status has been contested (see, e.g. Crane 2001). Here I cannot take part in that debate, but my approach in this paper is different. I accept that numbers are objects, because that is how they are treated on all levels in mathematics. This concerns both grammatical practices (e.g. 'count to four') and the content of theorems ('there exists *n* such that … '). We could go through a nominalisation process and ground mathematical objects in, for example, processes, thus getting an object-free ontology of mathematics.[21] But in the present context, it is difficult to see the motivation for that, given that the existence of numbers as abstract objects is considered to be mind-dependent, based only on there being referents of culturally shared number concepts. Therefore, what is at stake is ultimately nothing more than whether we can accept that such culturally shared concepts can be thought to have referents, which seems quite unproblematic.

Of course, some philosophers are likely to disagree, based on this being a very thin account of mathematical objects. However, I hope to have shown that the above constructivist psychologist answer to the causal why-question is enough to explain how natural numbers as abstract objects are brought into being. Whether they are something more than that is a question I cannot discuss here. In fact, I believe the burden of proof should lie on the side that believes abstract objects to be something beyond the referents of shared number concepts. Indeed, the present way of treating the existence of numbers is both epistemologically and

[20]Although it is important to note that Thomasson does not see abstract objects as simply referents of shared concepts, as in the present account.

[21]Indeed, this is what I have previously done in arguing that mathematical objects can be understood as the metaphorical counterparts of mathematical processes (Pantsar 2015).

ontologically unproblematic, without making the unnecessary additional assumption that numbers coincide with an eternal abstract realm of mathematical objects.

In the previous two sections, we have answered the *how*-question for the psychologist constructivist account. In the enculturation account, we also got a partial answer to the causal *why*-question. The existence of numbers is possible because number concept acquisition is the product of enculturation. In numerical cultures, individuals have shared number concepts because they apply their proto-arithmetical abilities during ontogeny to acquire culturally developed number concepts. This is made possible by cultural practices that determine number concepts to be an important part of education. When a teacher (or parent, or someone else) instructs a child based on those practices, they directly influence the ontogenetic development of a child, enabling them to acquire and possess number concepts.

A direct consequence of this line of thinking is that if there were no numerical cultures, number concepts would cease to exist. Of course, if we accept the platonist interpretation, numbers would continue to exist. But if we reject the platonist interpretation and follow a constructivist line of thinking, there is nothing that the existence of numbers could mean outside the context of culturally shared concepts. Thus numbers would cease to exist along with number concepts. This final remark may sound absurd to those who think of numbers as universal and/or eternal (and this does not require that one is a committed platonist; see, e.g. Burgess 2008; Burgess and Rosen 1997; Linnebo 2018b). Burgess and Rosen (1997), for example, believe that since numbers lack temporal properties, even if there were no numerical cultures and thus no number concepts, there would still be numbers. Burgess (2008) writes that: 'one, two, three, and the other numbers, if they exist at all, do not have the same sort of spatial or temporal features as human ideas' (28). The key phrase here is 'if they exist at all', in which existence is already assumed to mean something different from the way human ideas exist as shared concepts. For example, Burgess (2008, 29) says that asking temporal questions about numbers is to commit a 'grammatical solecism'. But why would this be the case, unless it has already been decided that numbers must exist, if at all, in a completely atemporal manner?

In that respect my account differs from that of Cole (2013; 2015). He writes that 'As institutional facets of reality, surrogates [which numbers are thought to be – *Author*] are the products of declarations.

Consequently, their central features are as we represent them to be' (Cole 2015, 1111). Since a central feature of the way we represent numbers is that they are atemporal, the argument goes, we construct them as atemporal existents. I am not satisfied with this. While we clearly get to declare what the features of social constructs are, this liberty cannot reach such matters as temporality. For discursive purposes we can declare numbers to be atemporal, but philosophically more interesting is the question whether numbers as social constructs are somehow constrained by temporality. I contend that they are, simply because under the social constructivist accounts there was a time when numbers didn't exist.[22]

Even though it may seem in this way unintuitive that numbers are somehow temporally constrained, we should not be overly troubled by that. This is because the assumption that numbers are universal or eternal (or, to put it another way, their existence is completely outside temporal considerations) is by itself highly problematic. To see why, we need to move on to the question how number concepts have come around in the first place. After all, the enculturation framework can help explain how humans acquire number concepts during individual ontogeny, but it can only explain how that happens when the individuals are enculturated in cultures that already have number concepts.[23]

[22]Indeed, I see this as a big problem in Cole's idea that numbers are atemporal because we declare them as such. As I understand this idea, it implies that we cannot say that there was a time when numbers didn't exist, which goes against the very notion of them being constructed by humans. One may ask (as an anonymous reviewer did), whether this temporality of numbers commits my account to not being able to accept truths like 'Shortly after the big bang, the number of fundamental forces was greater than two.' But this is not the case. We simply need to re-structure the sentence as a counter-factual: 'Had there been numbers shortly after the big bang, the number of fundamental forces was greater than two.' Alternatively, we can avoid counter-factual statements and say that there were more than two forces after the big bang, but we could only express this fact after numbers were constructed (I thank another anonymous reviewer for this suggestion). If these explanations seem clumsy, I feel that it is a small price to pay for avoiding the position that numbers existed before they were constructed. For more on this matter, see (Pantsar 2021b).

[23]My position that the existence of numbers is not a completely atemporal matter should be explained a bit further. After all, I have accepted that numbers are abstract objects, which has traditionally been understood to imply that they are non-spatial and atemporal. However, as pointed out by Rosen (2020, sec. 3.1), not all abstract objects stand in a similar relation to space and time. The game of chess, considered as an abstract object, is clearly connected to spatial and temporal events (i.e. the location and time of its creation and development) in a meaningful way. According to the present account, numbers have a similar connection to temporal events. Them being abstract objects, it does not make sense to ask spatiotemporal questions like 'where did numbers exist last year?' However, it does make sense to ask questions like 'when did numbers come to existence?' It is important to distinguish between these two ways of dealing with the temporality of mind-dependent abstract objects.

6. The teleological why-question

Pelland, among others, has asked that if the enculturation account is correct for ontogeny, how did number concepts emerge in cultures when there previously were none (Pelland 2018; 2020)? In many cases, this has most likely been the result of cultural transmission: numeral words and number concepts have entered into new cultures through interactions with other cultures (see, e.g. Everett 2017). But clearly this could not have happened in all cases. Unless we accept a nativist account (as Pelland (2020) himself ultimately appears to do) we are faced with the challenge of explaining how number concepts could have emerged originally in a context where there were only proto-arithmetical cognitive capacities. In other words, as well as an account of the ontogeny of number concepts, we need an account of their *phylogeny* and cultural history.

That number concepts have originally emerged as a cultural development based on the proto-arithmetical abilities is accepted by many researchers (see, e.g. Carey 2009; Dehaene 2001; Spelke 2011b). Aside from nativist accounts, in the empirical literature on numerical cognition, it is commonplace to accept that arithmetical abilities, starting from those involving number concepts, are culturally developed in a process that applies proto-arithmetical abilities in one way or another. This supports the idea that number concepts are the result of cumulative cultural evolution (Pantsar 2021a). Cumulative cultural evolution refers to the way knowledge and skills are developed and transmitted across generations (Boyd and Richerson 1985; Henrich 2015; Heyes 2018; Tomasello 1999). In this process, practices and tools are improved upon in small (trans-)generational increments, which accounts for the way number concepts and arithmetic arise as the product of a long line of cultural development in which languages, artefacts and other factors have played a crucial role (Everett 2017; Fabry 2017; Ifrah 1998).

In the theoretical framework for explaining number concept acquisition, cumulative cultural evolution complements the enculturation account. Whereas enculturation can help explain the acquisition of number concepts in ontogeny, cumulative cultural evolution can help explain how number concepts have developed on a phylogenetic-historic timescale. However, this framework by itself does not provide an answer to Pelland's question above: how did number concepts initially emerge? In the process of cumulative cultural evolution, there must have been a stage in which no humans possessed number concepts, followed by a stage in which some humans did. What happened at this crucial stage? In addition to the ontogenetic question, we need also to answer this phylogenetic-historical question in order to have a full constructivist response to the causal why-question. As we will see, this question concerning the origins of number concepts is tightly connected to the teleological why-question.

What kind of answer does the teleological why-question get in the psychologist constructivist account? A fundamental tenet of the cumulative cultural evolution account is that, like adaptations resulting from biological evolution, the improved tools and practices are in some way beneficial for communities. As with biological evolution, what is beneficial depends largely on the environment in which the communities live. These observations are important when we try to locate the initial benefits involved in developing exact number concepts. As noted earlier, cultures like the Pirahã and the Munduruku have not developed extensive systems of numeral words and do not possess exact, consistent number concepts. Since they possess similar proto-arithmetical capacities to people in numerical cultures, the explanation for their lack of numerical knowledge must lie in different trajectories of cultural evolution. In the case of the Pirahã, for example, the cultural differences are enormous. As hunter-gatherers, they do not practice agriculture or store food. While they do practice trade, no record is kept (Everett 2017). I contend that it is from these types of observations that we can start approaching the teleological why-question. Something in cultures that developed number concepts made them beneficial, up to the point that in modern industrialised cultures numbers are ubiquitous (Núñez 2017). Whether implicitly or explicitly, number concepts – and hence numbers, in our non-platonist account – served some *purpose*.[24]

It is easy to see what purpose numbers serve in modern societies. Our financial system, for example, is thoroughly numerical. All areas of science require some kind of use of mathematical tools that involve number systems (Pantsar 2018). But it would be spurious to claim such modern developments as the purpose of developing number concepts. The first known symbolic numerical systems are from around 5000 years ago (Schmandt-Besserat 1996) and nobody knows how much verbal numerical systems predate the symbolic systems. In any case, it should be clear that the contemporary advantages of the earliest number concepts may

[24]Given the relatively late separation of those cultures from other South American cultures, it is possible that the ancestors of Pirahã and Munduruku possessed numeral words and number concepts, but later generations lost them (see, e.g. Everett 2017; Pantsar 2019).

have been highly different from modern ones. For this reason, when looking for answers to the teleological why-question, we should focus on the early history of number concepts.

In recent years, researchers have developed accounts of the emergence of number concepts based on Malafouris' *material engagement theory* (Malafouris 2013; Overmann 2018; Zahidi 2021). According to these accounts, the properties of number concepts, numeral words and numeral symbols, as well as norms for their use, have been determined by a socio-culturally evolved set of material practices. In the development of number concepts, the emergence of counting practices is seen as a crucial stage. For instance, in one of the most discussed accounts of number concept acquisition in ontogeny, the bootstrapping account of Carey (2009), the acquisition of a counting list of numeral words precedes the acquisition of number concepts (Beck 2017; Carey 2009). But how do counting practices emerge? As Flegg (2002) points out, counting is already quite an advanced process. Thus, the challenge becomes explaining how counting practices could have developed.

Zahidi (2021, 540) has recently argued that from putting items into one-to-one correspondence to discriminating collections according to quantity, the capacities that counting requires are based on proto-arithmetical abilities. Furthermore, with such capacities practices of material engagement can emerge, such as tallying and finger counting. In the practice of marking one animal with one notch, for example, no number concepts need to be present, yet this practice already provides information about the cardinality of collections. By comparing the amount of notches, one can establish which group of animals is more numerous. These kinds of practices can then lead to developing properties of counting sequences. Finger counting, for example, can help establish the linear order of counting (Bender and Beller 2012; Overmann 2018). Words for body parts can evolve into numeral words, and tallying marks can develop into numeral symbols (Ifrah 1998).

Research on cognitive practices of counting can provide an answer to Pelland's question of origins. As argued by Wiese (2007), it is likely that number concepts and numeral words co-evolved as a result of practices like body part counting (see also Dos Santos (2021)). With the emergence of numeral words, number concepts could be associated with further material practices. Grouping objects like pebbles, for example, can lead to norms about the 'plus one' operation and further about addition in general (Overmann 2016; 2018). These practices could then be applied in, for example, trade and other financial interactions. The

Mesopotamians are known to have used clay tokens for accounting for millennia, especially in the Neolithic era (8300–4500 BC) (Overmann 2018; Schmandt-Besserat and Hallo 1992). This slowly evolved across generations into practices where the tokens were grouped into units of different sizes. These groupings could then be associated with different numerical values through, for example, ten small clay cones being equivalent to one small sphere. With the emergence of cuneiform writing systems, these relations between different physical arrangements of clay tokens could then be transferred into symbols (Nissen et al. 1994; Overmann 2018).

This kind of account of material engagement can help explain the emergence of numeral words and numeral symbols in co-evolution with number concepts. It is not possible here to go into further details, but it should be clear how information on how number concepts have developed in this manner helps us answer Pelland's question. We cannot point to a single moment in history when shared number concepts, and hence numbers, came into being. Instead, this happened through a long development in which numerical words and symbols, as well as practices and norms for their use, co-evolved with number concepts.

This finally allows me to propose an answer to the teleological why-question, although the answer is complex. The purpose that culturally evolved numbers serve depends on the culture and its environment. It could be that for the Pirahã, there was no such purpose. For the Mesopotamians, we know that at least accounting was an early purpose of numbers and arithmetic. For the Mayans, who to the best of our knowledge developed arithmetic independently of other cultures, astronomy was an important application for numbers (Ifrah 1998). Whatever the purposes may have been, they must have somehow proven to be beneficial for social practices as well as intra- and inter-cultural interactions. This, I contend, is the answer to the teleological why-question concerning the existence of numbers. Numbers exist because human cultures have developed them through a process of cumulative cultural evolution. Whether recognised at the time or only later, numbers have proven to be useful for purposes deemed to be sufficiently important in the particular cultures. It is this cultural success of numbers that provides an answer to the teleological why-question. In some cultures, like the Mesopotamians and the Mayans, numbers were elevated into a cultural status in which manipulating them developed a value of its own. This is how arithmetic emerged as a 'pure' practice on its own, not necessarily tied to any specific applications (Boyer 1991; Ifrah 1998).

This emergence of pure mathematics can help us explain the emergence of number concepts beyond those of natural numbers. The history of new innovations in mathematics, after the early origins, is largely a history of pure mathematics. This history led to the emergence of the concepts of rational numbers, irrational numbers, real numbers, complex numbers and transfinite numbers. With many of these, the interest was originally purely mathematical (see, e.g. Boyer 1991). The teleological why-question for the existence of complex numbers, for example, can only be answered by understanding the value seen in mathematics in Europe. Although complex numbers later found important applications in science (e.g. quantum mechanics), the reason for their development – and hence, also their existence in the present account – is that mathematics existed as an independent intellectual pursuit. It was a better understanding of mathematics itself that motivated the introduction of complex numbers. The platonist would most likely agree with this statement, but they would insist that a better understanding of mathematics means a better understanding of the mathematical universe. However, I see no motivation to make this latter amendment.

The nominalist, on the other hand, could insist that even though enculturation and cumulative cultural evolution are central to the development and acquisition of number concepts, this does not mean that numbers exist as abstract objects. However, once numbers as abstract objects are understood as referents of a particular type of culturally shared number concepts as described in Section 5, and nothing more, I see no motivation to deny the existence of numbers, either. They are human constructions, but this does not make them non-existent. As I have argued in this paper, this kind of existence is mind-dependent. However, I see no reason to deny that it is existence nevertheless.[25]

[25]One question regarding the psychologist constructivist account of numbers is how they can account for the applicability of numbers in both science and everyday life. If numbers are social constructs, how can they help explain the world? Here it's important to note that according to the present account, numbers are a particular type of cultural constructs whose content is (partly) determined by evolutionarily developed cognitive architecture. This implies that numbers are closely connected to the way we observe the world around us. As such, it is no wonder that numbers find use in applications of explaining that world. Indeed, I see this as a great strength of the present account, especially compared to conventionalist and platonist theories (see Pantsar (2021b) for a detailed treatment of this topic). In that paper, I also pursue the present approach with regard to one key topic in the philosophy of mathematics, namely, what is the status of mathematical *truth* in the psychologist constructivist account?

7. Conclusion

One of the most famous quotes regarding the existence of numbers is attributed to the nineteenth century mathematician Leopold Kronecker. In the translation of Gray (2008, 153), the quote is that 'God made the integers, all else is the work of man'. Theist language aside, the quote suggests a different ontological status for integers compared to other numbers. In this paper, I have defended an account of natural numbers (i.e. non-negative integers) that rejects any such ontological differences. Natural numbers, just like all other numbers, are the work of human beings. However, this does not mean that there are no epistemological differences between natural numbers and other numbers. Anthropological studies reveal that relatively extensive numeral word systems, while not universal, are nevertheless highly common in human languages (Everett 2017). Numeral words for other kinds of numbers are far less common (Everett 2017). Anthropologically, there appears to be something special about natural numbers. I believe that this can be explained through the cognitive development and epistemology of natural numbers. For other number systems, the connection to our basic cognitive capacities is less direct.

The consequence of this is that the why-questions, whether causal or teleological, do not get the same answer regardless of which number system is discussed. As far as the causal why-question is concerned, the enculturation framework can accommodate the learning of all number systems, but the enculturation process differs in each case. Natural number concepts are something that almost all (neurotypical) children acquire in arithmetical cultures with relative ease. On the other hand, complex number concepts, for example, are comparatively rarely acquired by humans and almost exclusively as the product of a long and specified trajectory of education.

Similarly, we should expect differences when it comes to the teleological why-question. Whereas natural numbers were likely developed (at least in Mesopotamia) initially for very practical purposes, such as accounting, at least the initial purpose of more recent number systems has been motivated by mathematical research itself. To be able to answer the teleological why-question, we thus need to be aware of the cultural surroundings of the development of number systems. Above all, the great richness of modern mathematical number systems has required that pure mathematics is considered a worthy pursuit that should be supported by societies. Ultimately it is this development, carried out by different cultures in different times, that holds the answers to the question – both causal and teleological – why numbers exist. For the philosophy of mathematics, this is extremely important. As I have hopefully established in this paper, for non-Platonist and non-nominalist accounts, the why-questions concerning the existence of numbers are just as well-placed as the 'how' and 'do they' questions.

In addition to numbers, the present account can also be expanded for other types of mathematical objects. In addition to arithmetic, the kind of psychologist approach taken in this paper has been developed also for geometric cognition (Hohol 2019; Pantsar 2022). In addition to proto-arithmetical abilities, empirical research suggests that there are evolutionarily developed proto-geometrical abilities (see, e.g. Spelke 2011a). Through a similar analysis to the one conducted in this paper, geometrical objects could also be established as culturally shared concepts based on the proto-geometrical abilities. Indeed, there is no reason to think that this approach is limited to arithmetic and geometry. An important future development of the present work would be to generalise it to concern all mathematical objects.

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References

Agrillo, C. 2015. "Numerical and Arithmetic Abilities in non-Primate Species." In *The Oxford Handbook of Numerical Cognition*, edited by R. C. Cohen Kadosh, and A. Dowker, 214–236. New York: Oxford University Press.

Anderson, M. L. 2010. "Neural Reuse: A Fundamental Organizational Principle of the Brain." *Behavioral and Brain Sciences* 33 (4): 245–266. https://doi.org/10.1017/S0140525X10000853

Anderson, M. 2015. *After Phrenology: Neural Reuse and the Interactive Brain*. Cambridge, MA: MIT Press.

Azzouni, J. 2000. "Stipulation, Logic, and Ontological Independence." *Philosophia Mathematica* 8 (3): 225–243. https://doi.org/10.1093/philmat/8.3.225

Beck, J. 2017. "Can Bootstrapping Explain Concept Learning?" *Cognition* 158: 110–121. https://doi.org/10.1016/j.cognition.2016.10.017

Benacerraf, P., and H. Putnam, eds. 1984. *Philosophy of Mathematics: Selected Readings*. New York: Cambridge University Press.

Bender, A., and S. Beller. 2012. "Nature and Culture of Finger Counting: Diversity and Representational Effects of an Embodied Cognitive Tool." *Cognition* 124 (2): 156–182. https://doi.org/10.1016/j.cognition.2012.05.005.

Boolos, G. 1998. *Logic, Logic and Logic*. Cambridge, MA: Harvard University Press.

Boyd, R., and P. J. Richerson. 1985. *Culture and the Evolutionary Process*. Chicago: University of Chicago Press.

Boyd, R., and P. J. Richerson. 2005. *Not by Genes Alone*. Chicago: University of Chicago Press.

Boyer, C. 1991. *A History of Mathematics*. 2nd ed. Hoboken, NJ: John Wiley and Sons.

Brock, S. 2002. "Fictionalism About Fictional Characters." *Noûs* 36 (1): 1–21. https://doi.org/10.1111/1468-0068.00358

Brock, S., and E. Mares. 2007. *Realism and Anti-Realism*. Illustrated edition. Montreal: McGill-Queen's University Press.

Brown, J. R. 2008. *Philosophy of Mathematics: A Contemporary Introduction to the World of Proofs and Pictures*. 2nd ed. New York: Routledge.

Bueno, O. 2020. "Nominalism in the Philosophy of Mathematics." In *The Stanford Encyclopedia of Philosophy (Fall 2020)*, edited by E. N. Zalta. Metaphysics Research Lab, Stanford University. https://plato.stanford.edu/archives/fall2020/entries/nominalism-mathematics/

Burgess, J. P. 2008. *Mathematics, Models, and Modality: Selected Philosophical Essays*. Illustrated edition. New York: Cambridge University Press.

Burgess, J. P., and G. Rosen. 1997. *A Subject with No Object: Strategies for Nominalistic Interpretation of Mathematics*. 1st ed. New York: Clarendon Press.

Butterworth, B. 1999. *What Counts: How Every Brain is Hardwired for Math*. New York: The Free Press.

Cantlon, J., and E. Brannon. 2006. "Shared System for Ordering Small and Large Numbers in Monkeys and Humans." *Psychological Science* 17 (5): 402–407. https://doi.org/10.1111/j.1467-9280.2006.01719.x

Carey, S. 2009. *The Origin of Concepts*. Oxford: Oxford University Press.

Cheyette, S. J., and S. T. Piantadosi. 2020. "A Unified Account of Numerosity Perception." *Nature Human Behaviour* 4 (12): 1265–1272. https://doi.org/10.1038/s41562-020-00946-0.

Cole, J. C. 2009. "Creativity, Freedom, and Authority: A New Perspective on the Metaphysics of Mathematics." *Australasian Journal of Philosophy* 87 (4): 589–608. https://doi.org/10.1080/00048400802598629.

Cole, J. C. 2013. "Towards an Institutional Account of the Objectivity, Necessity, and Atemporality of Mathematics." *Philosophia Mathematica* 21 (1): 9–36. https://doi.org/10.1093/philmat/nks019.

Cole, J. C. 2015. "Social Construction, Mathematics, and the Collective Imposition of Function onto Reality." *Erkenntnis* 80 (6): 1101–1124. https://doi.org/10.1007/s10670-014-9708-8.

Crane, T. 2001. "Intentional Objects." *Ratio* 14 (4): 298–317. https://doi.org/10.1111/1467-9329.00168.

De Cruz, H., and J. De Smedt. 2010. "The Innateness Hypothesis and Mathematical Concepts." *Topoi* 29 (1): 3–13. https://doi.org/10.1007/s11245-009-9061-8

De Cruz, H., H. Neth, and D. Schlimm. 2010. "The Cognitive Basis of Arithmetic." In *PhiMSAMP. Philosophy of Mathematics: Sociological Aspects and Mathematical Practice*, edited by B. Löwe, and T. Müller, 59–106. College Publications.

Dehaene, S. 2001. "Précis of the Number Sense." *Mind & Language* 16 (1): 16–36. https://doi.org/10.1111/1468-0017.00154

Dehaene, S. 2009. *Reading in the Brain: The New Science of How We Read*. London: Penguin.

Dehaene, S. 2011. *The Number Sense: How the Mind Creates Mathematics*. Revised, Updated edition. Oxford: Oxford University Press.

Dehaene, S., and L. Cohen. 2007. "Cultural Recycling of Cortical Maps." *Neuron* 56 (2): 384–398. https://doi.org/10.1016/j.neuron.2007.10.004

dos Santos, C. F. 2021. "Enculturation and the Historical Origins of Number Words and Concepts." *Synthese* 199 (3-4): 9257–9287. https://doi.org/10.1007/s11229-021-03202-8.

Dummett, M. 2006. *Thought and Reality*. Oxford: Oxford University Press.

Ellis, B. D. 1991. *Truth and Objectivity*. Oxford: Blackwell.

Everett, C. 2017. *Numbers and the Making of us: Counting and the Course of Human Cultures*. Cambridge, MA: Harvard University Press.

Everett, C., and K. Madora. 2012. "Quantity Recognition Among Speakers of an Anumeric Language." *Cognitive Science* 36 (1): 130–141. https://doi.org/10.1111/j.1551-6709.2011.01209.x.

Fabry, R. E. 2017. "Cognitive Innovation, Cumulative Cultural Evolution, and Enculturation." *Journal of Cognition and Culture* 17 (5): 375–395. https://doi.org/10.1163/15685373-12340014

Fabry, R. E. 2020. "The Cerebral, Extra-Cerebral Bodily, and Socio-Cultural Dimensions of Enculturated Arithmetical Cognition." *Synthese* 197 (9): 3685–3720. https://doi.org/10.1007/s11229-019-02238-1

Feferman, S. 2009. "Conceptions of the Continuum." *Intellectica* 51 (1): 169–189.

Feigenson, L., S. Dehaene, and E. Spelke. 2004. "Core Systems of Number." *Trends in Cognitive Sciences* 8 (7): 307–314. https://doi.org/10.1016/j.tics.2004.05.002

Ferreirós, J. 2015. *Mathematical Knowledge and the Interplay of Practices*. Princeton: Princeton University Press.

Field, H. 1980. *Science Without Numbers*. Princeton: Princeton University Press.

Flegg, G. 2002. *Numbers. Their History and Meaning*. New York: Dover Publications.

Frank, M. C., D. L. Everett, E. Fedorenko, and E. Gibson. 2008. "Number as a Cognitive Technology: Evidence from Pirahã Language and Cognition." *Cognition* 108 (3): 819–824. https://doi.org/10.1016/j.cognition.2008.04.007.

Franklin, J. 2014. *An Aristotelian Realist Philosophy of Mathematics: Mathematics as the Science of Quantity and Structure*. 1st ed. 2014 edition. Basingstoke: Palgrave Macmillan.

Frege, G. 1884. *Foundations of Arithmetic*. Translated by J. L. Austin. Oxford: Blackwell.

Frege, G. 1892. "Über Sinn und Bedeutung." *Zeitschrift Für Philosophie Und Philosophische Kritik* 100 (1): 25–50.

Gallistel, C. R. 2017. "Numbers and Brains." *Learning & Behaviour* 45 (4): 327–328. https://doi.org/10.3758/s13420-017-0272-3

Gallistel, C. R., and R. Gelman. 1992. "Preverbal and Verbal Counting and Computation." *Cognition* 44 (1-2): 43–74. https://doi.org/10.1016/0010-0277(92)90050-r.

Gelman, R., and C. Gallistel. 1978. *The Child's Understanding of Number*. Cambridge, MA: Harvard University Press.

Gelman, R., and C. R. Gallistel. 2004. "Language and the Origin of Numerical Concepts." *Science* 306 (5695): 441–443. https://doi.org/10.1126/science.1105144

Gordon, P. 2004. "Numerical Cognition Without Words: Evidence from Amazonia." *Science* 306 (5695): 496–499. https://doi.org/10.1126/science.1094492

Gray, J. 2008. *Plato's Ghost: The Modernist Transformation of Mathematics*. 1st ed. Princeton: Princeton University Press.

Griffiths, P. E. 2001. "What is Innateness?" *The Monist* 85 (1): 70–85. https://doi.org/10.5840/monist20028518.

Halberda, J., and L. Feigenson. 2008. "Developmental Change in the Acuity of the" Number Sense": The Approximate Number System in 3-, 4-, 5-, and 6-Year-Olds and Adults." *Developmental Psychology* 44 (5): 1457–1465. https://doi.org/10.1037/a0012682

Hauser, M. D., S. Carey, and L. B. Hauser. 2000. "Spontaneous Number Representation in Semi–Free–Ranging Rhesus Monkeys." *Proceedings of the Royal Society of London B: Biological Sciences* 267 (1445): 829–833. https://doi.org/10.1098/rspb.2000.1078

Hellman, G. 1989. *Mathematics Without Numbers*. Oxford: Oxford University Press.

Henrich, J. 2015. *The Secret of our Success: How Culture is Driving Human Evolution, Domesticating our Species, and Making us Smarter*. Princeton: Princeton University Press.

Heyes, C. 2018. *Cognitive Gadgets: The Cultural Evolution of Thinking*. Cambridge, MA: Harvard University Press.

Hodes, H. T. 1984. "Logicism and the Ontological Commitments of Arithmetic." *Journal of Philosophy* 81 (3): 123–149. https://doi.org/10.2307/2026440.

Hohol, M. 2019. *Foundations of Geometric Cognition*. New York: Routledge.

Hyde, D. C. 2011. "Two Systems of non-Symbolic Numerical Cognition." *Frontiers in Human Neuroscience* 5: 150.

Ifrah, G. 1998. *The Universal History of Numbers: From Prehistory to the Invention of the Computer*. Translated by D. Bellos, E. F. Harding, S. Wood and I. Monk. London: Harville Press.

Izard, V., P. Pica, E. S. Spelke, and S. Dehaene. 2008. "Exact Equality and Successor Function: Two key Concepts on the Path Towards Understanding Exact Numbers." *Philosophical Psychology* 21 (4): 491–505. https://doi.org/10.1080/09515080802285354

Jones, M. 2020. "Numerals and Neural Reuse." *Synthese* 197 (9): 3657–3681. https://doi.org/10.1007/s11229-018-01922-y

Kitcher, P. 1983. *The Nature of Mathematical Knowledge*. Oxford: Oxford University Press.

Knops, A. 2020. *Numerical Cognition. The Basics*. New York: Routledge.

Lakoff, G., and R. Núñez. 2000. *Where Mathematics Comes From*. New York: Basic Books.

Linnebo, Ø. 2018a. "Platonism in the Philosophy of Mathematics." In *The Stanford Encyclopedia of Philosophy*, edited by E. Zalta. https://plato.stanford.edu/archives/spr2018/entries/platonism-mathematics.

Linnebo, Ø. 2018b. *Thin Objects*. Oxford: Oxford University Press.

Malafouris, L. 2013. *How Things Shape the Mind: A Theory of Material Engagement*. Cambridge, MA: MIT Press.

Menary, R. 2007. *Cognitive Integration: Mind and Cognition Unbounded*. London: Palgrave Macmillan.

Menary, R. 2014. "Neuronal Recycling, Neural Plasticity and Niche Construction." *Mind and Language* 29 (3): 286–303. https://doi.org/10.1111/mila.12051

Menary, R. 2015. *Mathematical Cognition: A Case of Enculturation*. Open MIND, MIND Group.

Muntersbjorn, M. 1999. "Naturalism, Notation, and the Metaphysics of Mathematics." *Philosophia Mathematica* 7 (2): 178–199. https://doi.org/10.1093/philmat/7.2.178.

Muntersbjorn, M. 2003. "Representational Innovation and Mathematical Ontology." *Synthese* 134 (1): 159–180. https://doi.org/10.1023/A:1022139715092.

Nieder, A., and S. Dehaene. 2009. "Representation of Number in the Brain." *Annual Review of Neuroscience* 32 (1): 185–208. https://doi.org/10.1146/annurev.neuro.051508.135550

Nissen, H. J., P. Damerow, and R. K. Englund. 1994. *Archaic Bookkeeping: Early Writing and Techniques of Economic Administration in the Ancient Near East*. Translated by P. Larsen. 1st ed. Chicago: University of Chicago Press.

Núñez, R. E. 2017. "Is There Really an Evolved Capacity for Number?" *Trends in Cognitive Science* 21 (6): 409–424. https://doi.org/10.1016/j.tics.2017.03.005

Overmann, K. A. 2016. "The Role of Materiality in Numerical Cognition." *Quaternary International* 405: 42–51. https://doi.org/10.1016/j.quaint.2015.05.026

Overmann, K. A. 2018. "Constructing a Concept of Number." *Journal of Numerical Cognition* 4 (2): 464–493. https://doi.org/10.5964/jnc.v4i2.161.

Pantsar, M. 2014. "An Empirically Feasible Approach to the Epistemology of Arithmetic." *Synthese* 191 (17): 4201–4229. https://doi.org/10.1007/s11229-014-0526-y.

Pantsar, M. 2015. "In Search of Aleph-Null: How Infinity Can be Created." *Synthese* 192 (8): 2489–2511. https://doi.org/10.1007/s11229-015-0775-4

Pantsar, M. 2018. "Mathematical Explanations and Mathematical Applications." In *Handbook of The Mathematics of The Arts and Sciences*, edited by B. Sriraman, 2587–2602. Cham: Springer.

Pantsar, M. 2019. "The Enculturated Move from Proto-Arithmetic to Arithmetic." *Frontiers in Psychology* 10: 1454. https://doi.org/10.3389/fpsyg.2019.01454
Pantsar, M. 2021a. "Bootstrapping of Integer Concepts: The Stronger Deviant-Interpretation Challenge." *Synthese* 199: 5791–5814. https://doi.org/10.1007/s11229-021-03046-2.
Pantsar, M. 2021b. "Objectivity in Mathematics, Without Mathematical Objects†." *Philosophia Mathematica* 29 (3): 318–352. https://doi.org/10.1093/philmat/nkab010.
Pantsar, M. 2022. "On the Development of Geometric Cognition: Beyond Nature vs. Nurture." *Philosophical Psychology* 35 (4): 595–616. https://doi.org/10.1080/09515089.2021.2014441.
Pantsar, M. 2023. "From Maximal Intersubjectivity to Objectivity: An Argument from the Development of Arithmetical Cognition." *Topoi* 42 (1): 271–281. https://doi.org/10.1007/s11245-022-09842-w.
Pantsar, M. 2024. *Numerical Cognition and The Epistemology of Arithmetic*. Cambridge: Cambridge University Press.
Pelland, J.-C. 2018. "Which Came First, the Number or the Numeral?" In *Naturalizing Logico-Mathematical Knowledge: Approaches from Philosophy, Psychology and Cognitive Science*, edited by S. Bangu, 179–194. New York: Routledge.
Pelland, J.-C. 2020. "What's New: Innovation and Enculturation of Arithmetical Practices." *Synthese* 197 (9): 3797–3822. https://doi.org/10.1007/s11229-018-02060-1.
Pepperberg, I. M. 2012. "Further Evidence for Addition and Numerical Competence by a Grey Parrot (Psittacus Erithacus." *Animal Cognition* 15 (4): 711–717. https://doi.org/10.1007/s10071-012-0470-5
Pettigrew, R. 2008. "Platonism and Aristotelianism in Mathematics." *Philosophia Mathematica* 16 (3): 310–332. https://doi.org/10.1093/philmat/nkm035
Pica, P., C. Lemer, V. Izard, and S. Dehaene. 2004. "Exact and Approximate Arithmetic in an Amazonian Indigene Group." *Science* 306 (5695): 499–503. https://doi.org/10.1126/science.1102085
Putnam, H. 1967. "Mathematics Without Foundations." *Journal of Philosophy* 64 (1): 5–22. https://doi.org/10.2307/2024603.
Resnik, M. 1997. *Mathematics as a Science of Patterns*. Oxford: Oxford University Press.
Rosen, G. 2020. "Abstract Objects." In *The Stanford Encyclopedia of Philosophy* (Spring 2020), edited by E. N. Zalta. Metaphysics Research Lab, Stanford University. https://plato.stanford.edu/archives/spr2020/entries/abstract-objects/
Rugani, R., L. Fontanari, E. Simoni, L. Regolin, and G. Vallortigara. 2009. "Arithmetic in Newborn Chicks." *Proceedings of the Royal Society B: Biological Sciences* 276 (1666): 2451–2460. https://doi.org/10.1098/rspb.2009.0044
Schaffer, J. 2009. "On What Grounds What." In *Metametaphysics*, edited by D. Chalmers, D. Manley, and R. Wasserman, 347–383. Oxford: Oxford University Press.
Schmandt-Besserat, D. 1996. *How Writing Came About*. Austin: University of Texas Press.
Schmandt-Besserat, D., and W. W. Hallo. 1992. *Before Writing: Volume 1: From Counting to Cuneiform*. 1st ed. Austin: University of Texas Press.
Schröder, E. 1873. *Lehrbuch der Arithmetik und Algebra für Lehrer und Studirende I*. Leipzig: Teubner.

Searle, J. R. 1997. *The Construction of Social Reality*. Illustrated Edition. New York: Free Press.

Searle, J. R. 2010. *Making the Social World: The Structure of Human Civilization*. Oxford: Oxford University Press.

Shapiro, S. 1997. *Philosophy of Mathematics: Structure and Ontology*. Oxford: Oxford University Press.

Spelke, E. S. 2000. "Core Knowledge." *American Psychologist* 55 (11): 1233–1243. https://doi.org/10.1037/0003-066X.55.11.1233.

Spelke, E. S. 2011a. "Natural Number and Natural Geometry." In *Space, Time and Number in the Brain*, edited by S. Dehaene, and E. Brannon, 287–317. Academic Press.

Spelke, E. S. 2011b. "Quinean Bootstrapping or Fodorian Combination? Core and Constructed Knowledge of Number." *Behavioral and Brain Sciences* 34: 149–150. https://doi.org/10.1017/S0140525X10002220

Starkey, P., and R. G. Cooper. 1980. "Perception of Numbers by Human Infants." *Science* 210 (4473): 1033–1035. https://doi.org/10.1126/science.7434014

Steiner, M. 1978. "Mathematics, Explanation, and Scientific Knowledge." *Noûs* 12 (1): 17–28. https://doi.org/10.2307/2214652.

Tait, W. W. 2001. "Beyond the Axioms: The Question of Objectivity in Mathematics." *Philosophia Mathematica* 9 (1): 21–36. https://doi.org/10.1093/philmat/9.1.21

Thomasson, A. 2008. "Existence Questions." *Philosophical Studies* 141 (1): 63–78.

Tomasello, M. 1999. *The Cultural Origins of Human Cognition*. Cambridge, MA: Harvard University Press.

Trick, L., and Z. W. Pylyshyn. 1994. "Why Are Small and Large Numbers Enumerated Differently? A Limited Capacity Preattentive Stage in Vision." *Psychological Review* 101: 80–102. https://doi.org/10.1037/0033-295X.101.1.80

van Marle, K., F. W. Chu, Y. Mou, J. H. Seok, J. Rouder, and D. C. Geary. 2018. "Attaching Meaning to the Number Words: Contributions of the Object Tracking and Approximate Number Systems." *Developmental Science* 21 (1): 12495. https://doi.org/10.1111/desc.12495

Wagner, R. 2017. *Making and Breaking Mathematical Sense: Histories and Philosophies of Mathematical Practice*. Princeton: Princeton University Press.

Warren, J. 2020. *Shadows of Syntax: Revitalizing Logical and Mathematical Conventionalism*. New York: Oxford University Press.

Wiese, H. 2007. "The Co-Evolution of Number Concepts and Counting Words." *Lingua. International Review of General Linguistics. Revue Internationale De Linguistique Generale* 117 (5): 758–772. https://doi.org/10.1016/j.lingua.2006.03.001

Wynn, K. 1992. "Addition and Subtraction by Human Infants." *Nature* 358 (6389): 749–751. https://doi.org/10.1038/358749a0

Zahidi, K. 2021. "Radicalizing Numerical Cognition." *Synthese* 198 (S1): 529–545. https://doi.org/10.1007/s11229-020-02956-x