An extensionalist's guide to non-extensional mereology

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1 Introduction

Formal mereology should be of interest to metaphysicians because it allows us, in a rigorous way, to determine the consequence relations between metaphysical theses and theories, many of which are of metaphysical interest. Take, for example, the many and various puzzles of constitution, and the metaphysical theories that have been advanced to solve them. Ideally, it should be possible to analyse these puzzles and their solutions in a formal way — to describe, for example, the puzzle of Dion and Theon, or of the statue and the clay, as a set of logically inconsistent propositions in a formal mereological language. By doing that we could more easily explore the space of possible solutions, since every solution would consist in denying one of those propositions. Ideally, it should be possible to state some (but not all) of the content of metaphysical theories concerning the part-whole relation formally, and that would enable us to better evaluate objections to those theories, since it would give us procedures for determining what is and what isn't a consequence of such a theory. Formal mereology is a great tool, and it should be used.

Unfortunately, existing formal mereology is not well-geared to this. The best understood mereology — classical mereology — is way too strong. Its theorems contain principles that are themselves matters of metaphysical debate, and so it cannot be used in a neutral way to determine what the logical relationships are between participants in those debates. Much discussion in the literature sees some authors offering philosophical objections to classical mereology, and others resisting those objections.

What we would like would be a neutral, or *minimal mereology* whose theorems are only those truths that are analytic, or conceptual truths of the mereological concepts, or at least widely accepted among rival metaphysical positions. Note that I say “only” here, not “all and only” — there may well be conceptual truths of the mereological concepts that are not formalisable in the style of a formal mereology — conceptual truths linking mereological concepts to other, non-mereological concepts for example. Also note that I am not saying that a minimal mereology should be completely certain and immune to any philosophical controversy — that is impossible as philosophers will no doubt disagree about which are conceptual truths of mereology as well as about which are the metaphysical ones. But that should not stop us from trying. I at least would like to have a formal tool that allows me to represent not only my own metaphysical thinking about mereology but the thinking of those I disagree with as well, and I think I can do that.

In this paper I describe what seems to me to be a minimal mereology of this kind. It is neutral on a number of issues that classical mereology settles: on the circumstances under which some collection of things have a fusion, for example; and, importantly, on the question of mereological extensionality — on whether there can be two things, so to speak, “made of the same stuff”. Though there are mereologies in the literature that attempt this,[1] they all have drawbacks, the nature of which will become clear when compared with my view, I think. The mereology I describe is an *atemporal* one — it regards the part-whole relation as two-place, and has no explicit representation of times or change of parts over time. This may seem to be metaphysically partisan — in my view, however, the best way to represent metaphysical theories on which the part-whole relation holds only at a time, and not simpliciter, is by modifying a theory of the kind I describe here, so that finding an atemporal minimal mereology is the first step towards finding a way to be neutral between atemporal and temporal approaches.[2]

2 SPO and its extensions

It is best to begin in *medias res* and lay out the details of my preferred minimal non-extensional mereology, SPO. In my view, this theory captures all the the formal conceptual truths of the mereological relation — a weaker system would allow some models that are mereologically incoherent; a stronger one would ban some models that are not incoherent. SPO is the first order theory with the fol-

[1] Casati and Varzi’s “Minimal mereology”, for example. (Casati and Varzi 1999, p. 39) (Varzi 2009)

[2] I hope to say more about this in a later paper.

lowing axioms and definitions:

$$x < x \tag{Refl}$$

$$x < y \wedge y < z \rightarrow x < z \tag{Trans}$$

$$x \circ y \Leftrightarrow (\exists z)(z < x \wedge z < y) \tag{Def$\circ$}$$

$$\neg x < y \rightarrow (\exists z)(z < x \wedge \neg z \circ y) \tag{SSP}$$

(I use the symbol $\Leftrightarrow$ for "is equivalent by definition". Definitions containing $\Leftrightarrow$, such as Def$\circ$, are to be understood as abbreviated *rules* to the effect that any sentence may be re-writtten eliminating the symbol being defined. I want to be able to mention definitions in order to say that they are or are not admissible in such and such a theory. Alternative symbols for definition, such as $\equiv_{df}$, suggest that it is being stipulated that a definition is admissible, whereas I want to be able to refer to definitions without stipulating that they are admissible).

"SPO" stands for *supplementary pre-ordering*, which is my name for the type of relation $<$ that satisfies these axioms. "Pre-ordering" is a standard name for relations that are both transitive (satisfy Trans) and reflexive (satisfy Refl); a "supplementary" relation is one that satisfies the *strong supplementation principle*, SSP. The intended interpretion of SPO is that $<$ is to mean "is part of" and $\circ$ is to mean "mereologically overlaps" (or "has a part in common with").[3] So interpreted, SPO says that the part-whole relation is reflexive, transitive, and supplementary.

The idea that part-whole should be transitive is familiar from the literature. My toe is part of my foot, my foot is part of me, therefore my toe is part of me. I shall not defend or question it further here.[4] So to is the idea that part-whole should be reflexive. It is a terminological convention in the literature that each individual counts as its own part. There is also a perfectly good sense of "part" in which no individual is its own part — that sense is called "proper part", and is interdefinable with the reflexive relation we are focusing on.[5]

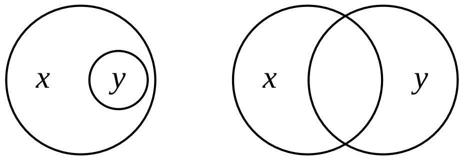
That leaves the idea that part-whole should be supplementary — i.e. that it should satisfy the SSP axiom. SSP encodes what is (I believe) an essential and distinctive feature of our mereological concepts: that if x is not part of y, then some part of x is disjoint from (i.e. does not overlap with) y. To make this seem

[3]The concept of overlap is needed to make the axiom SSP less cumbersome, and more recognisable (SSP is usually formulated in this way in the literature), and it will also come in handy when we describe some extensions of SPO.

[4]For further discussion of the question of whether the part-whole relation is transitive, see Simons (1987, pp. 107-108).

[5]Beware! I here sweep under the carpet some issues about the definition of "proper part" that will return to bite us in section 3.1.

plausible, consider the following argument by cases: first suppose that x and y are disjoint. Then x itself is the part of x that is disjoint from y, so SSP is satisfied. Now suppose that x and y overlap. If x overlaps, but is not part of y, then there are two possibilities: either y is part of x (as shown in the diagram on the left), or x and y "merely overlap" (as shown in the diagram on the right):



Either way, we would like to say, there is some of x not in y — some part of x, that is, not part of y. So, no matter how x and y are related, mereologically speaking, SSP is satisfied. So it is reasonable to suppose that the part-whole relation is supplementary.

SPO is a non-extensional mereology. It allows for two individuals to *coincide* — to be "made of exactly the same stuff". As we will see in section 3, it is a controversial matter exactly how this relation of coincidence should be explained in terms of part-whole. For present purposes, we will say that individuals coincide iff they are parts of each other. That is, if we write $x \sim y$ for "x and y coincide", then coincidence may be defined as shown in MPC (for *mutual parthood definition of coincidence*) below:

$$x \sim y \Leftrightarrow x < y \wedge y < x \qquad \text{(MPC)}$$

Note that each individual coincides with itself, because the part-whole relation is reflexive. This is so even in an extensional mereology. What makes SPO non-extensional is that it allows that *two* individuals coincide with each other; or to put it another way, it allows that there are cases of *proper coincidence*, where for x and y to *properly coincide* is for x and y to coincide while being numerically distinct.

MPC may seem an odd way to define coincidence, but in the setting of SPO it makes perfect sense. Say that x and y are *mereologically equivalent* iff they occupy, so to speak, the same position in the mereological heirarchy — iff they stand in all and only the same purely mereological relations to every other individual.[6] Since all mereological relations are definable in terms of the part-whole

[6]A *purely* mereological relation is one definable in terms of part-whole alone. Identity, for example, is not purely mereological, since if it was, no two things would count as mereologically equivalent. Similarly, the relation of non-identical parthood defined in section 3 is not purely mereological, as it can only be defined in terms of part-whole and identity, not in terms of part-whole alone. In contrast, strict parthood (as defined in section 3) as well as overlap and disjointness are all purely mereological relations.

relation, x and y are mereologically equivalent iff they stand in all and only the same part-whole relations to every other individual; i.e. iff they have all and only the same parts and are parts of all and only the same things. Trans says that if x is part of y, then whatever is part of x is part of y; therefore, if x and y are parts of each other, then they must have all and only the same parts. For similar reasons, if x and y are parts of each other, then they must be part of all and only the same things. Therefore, if x and y are mutual parts, then they are mereologically equivalent. Conversely, if x and y are mereologically equivalent, then they have all and only the same parts; since, by Refl, y is part of itself, y must be part of x, and by the same reasoning x part of y. Therefore, if x and y are mereologically equivalent, then they are mutual parts.

What we have just seen is that, on the assumption only of Trans and Refl, mutual parthood *is* mereological equivalence. If it is possible for two things to be mereologically equivalent (and, in SPO, it is), then it seems reasonable to define coincidence as mereological equivalence — and that is equivalent to defining coincidence as mutual parthood. *That* is the key insight behind the mutual parthood tradition in non-extensional mereology.

But is it possible for two things to coincide in SPO? Yes; to verify this, consider the following simple finite model: let there be two mereological atoms, and two composites, both of which have all four individuals as parts, as shown in the Hasse diagram[7] in figure 1 below. It can easily be verified that this model satisfies

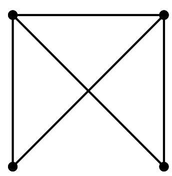


Figure 1: A model of SPO with coincidence

all the axioms of SPO. In this model, the two composite objects coincide.

The model shown in figure 2, however, which appears similar to figure 1, is

[7]A Hasse diagram is a convenient way of visualising a finite pre-ordered set. Members of the domain (for our purposes, individuals) are shown as dots. Lines on the diagram represent the pre-ordering relation (for our purposes, the relation of part to whole). x stands in the pre-ordering relation to y (i.e. x is part of y) iff there is any way of reaching x from y by travelling only downward or horizontally along the lines. Reachability is transitive, and each node is reachable from itself, so the diagrams assume that they are representing a pre-ordered set, but not that the set is supplementary. Thus they are useful for visualing both models of and non-models of SPO. (Strictly speaking these are "extended Hasse diagrams" — extended with a convention for depicting pre-ordered sets, rather than just posets).

not a model of SPO, as it fails to satisfy SSP. The two composite object on the left is not a part of the one on the right, but has no part which fails to overlap the composite object on the right. The impossibility of this model is a typical feature of mereological theories that contain SSP. (Simons 1987, pp. 28-29) Also, note that the model of figure 2 does not feature any coincidence in the sense in which coincidence was defined above; the two composite individuals are not mereologically equivalent, as each one has a part the other lacks, namely itself.

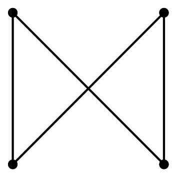


Figure 2: A non-model of SPO

It is also possible, in SPO, for non-composite individuals to coincide. Consider the model in which there are exactly two individuals, which are parts of each other, and nothing else, as shown in figure 3. It is again easy to verify that this satisfies all the axioms of SPO.

Figure 3: A model of SPO with coincidence between atoms

I expect that this type of model will raise interesting philosophical issues. First issue: is it plausible that two mereological atoms may coincide? I don't see why this is any worse than the usual philosophical examples of coincidence (persons and their bodies; statues and the clay they are made of). It's easy to imagine the usual philosophical examples taking place on the subatomic level. Ignore for a moment the fact that protons are composite individuals made up of quarks, and imagine that they are simple bodies, as Rutherford supposed. Now consider a hydrogen atom that becomes ionised by having its electron annihilated. Chemical atoms, it seems reasonable, *become* ions, they are not destroyed in the process of ionisation. But a hydrogen ion is constituted by a lone proton; before being ionised both the hydrogen atom and the proton existed, and were distinct; neither, it seems reasonable to say, have been destroyed; after the ionisation there is no mereological difference between them; therefore, after the ionisation two things coincide.[8] Since we are pretending that protons are mereological atoms,

[8] This is a subatomic version of Chrysippus's puzzle of Dion and Theon (Burke 1994); or van Inwagen's Descartes and Descartes-minus; (van Inwagen 1981); or Geach's Tibbles and Tib (Wiggins 1968).

two mereological atoms coincide.

This argument is hardly watertight; but it seems no worse than arguments for coincidence on the basis of persons or cats losing parts of their bodies, which it resembles. Arguments for coincidence at the subatomic level are no worse than analogous arguments at the macroscopic level. So there is no particular reason that I can see to reject coincidence between mereological atoms while allowing it between composites.

Second issue: by what right do I describe the model shown in figure 3 as a case of coincidence between atoms anyway? Isn't an atom something that has no parts other than itself? The elements of the model in question both have parts other than themselves; ergo they are not atoms.

What I meant by "mereological atom" when I said that SPO allows for coincidence between atoms was "something that has no parts that do not coincide with it". In that sense, the elements of the model in question *are* atoms. But what makes my definition of "atom" the right one, and the definition of the previous paragraph the wrong one? I will be able to answer this question later in this paper (see section 3.3). What *has* been established now, however, is that the model of figure 3 *is* a model of SPO, however we describe it in natural mereological language.

2.1 Extensional mereology and beyond

SPO is my attempt at a minimal mereology; a fortiori, it is a weak mereology, much weaker than classical mereology, and weaker than the minimal extensional mereologies found in the literature. Stronger mereologies than SPO, in my view, include among their theorems some material that is philosophically controversial. Nevertheless, it is interesting to see how to formulate some familiar and/or interesting stronger mereologies by adding axioms to SPO.

The first axiom we might add is ASym, below:

$$x < y \land y < x \to x = y \tag{ASym}$$

ASym says that $<$ is anti-symmetric. Relations that are anti-symmetric, reflexive, and transitive are called "partial orderings", and the theory that results from adding ASym to SPO may be called SO, or the theory of *supplementary orderings*. SO is an *extensional* mereology; in SO, it is impossible for two things to coincide (as follows obviously from ASym together with the definition of coincidence, MPC). It is equivalent to Varzi's (2009) EM.

SO is still weaker than classical mereology however. A famous feature of classical mereology is its so called “principle of unrestricted composition”, or *general sum principle* which says that any non-empty collection of individuals has a “mereological sum”, or “fusion”. To state this principle formally requires more than the normal resources of first-order logic, as the quantification over “collections” in the previous sentence would suggest. Some authors treat collections as sets, making classical mereology an extension of set theory; others change the underlying logic from first-order to first-order plural or second-order logic; I shall use the popular schematic approach (Simons 1987, p. 38) in which the principle is presented as a schema using a schematic variable in place of quantification over “collections”. The general sum principle (GSP) may thus be stated:[9]

$$(\exists z)(\Phi) \rightarrow (\exists x)(\forall y)(x \circ y \leftrightarrow (\exists z)(\Phi \wedge y \circ z)) \qquad \text{(GSP)}$$

The theory whose axioms are those of SO, plus ASym and GSP is *classical mereology*. Following the naming scheme I used for SPO and SO, however, it might called the theory of *sum-complete supplementary orderings* (SSO), where “sum-complete” refers to the formal property that a relation $<$ has when it satisfies GSP, such as “supplementary” refers to the formal property of satisfying SSP.

We could also add GSP directly to SPO, without adding ASym. This would produce the theory of *sum-complete supplementary pre-orderings* (SSPO), which is weaker than classical mereology (SSO), but stronger than SPO. SSPO closely resembles classical mereology, except that it allows for coincidence, and does not require that a non-empty collection of individuals have a *unique* sum. Any two sums of the same collection of individuals, however, must coincide.[10]

2.2 SPO-models as posets

There is a special relationship between the theories SPO and SO, which makes it easy to reason about SPO and its extensions if you are used to reasoning about SO and its extensions. While every model of SO is a model of SPO, there is also

[9]Paul Hovda has recently called attention to some different possible meanings of “fusion”, and correspondingly different formulations of GSP-like principles. My version of GSP says, in effect, that for any satisfiable sentence Φ, there is what Hovda (2009, p. 58) calls a “Type-1 fusion” of those things that satisfy Φ. The differences between different definitions of summation (i.e. between “types” of “fusion”) that Hovda calls attention to, however, need not trouble us because we have assumed the strong supplementation principle (SSP) from the start, and formulations of GSP using different types of fusion are are equivalent if SSP is assumed. (Hovda 2009, p. 70)

[10]The easiest way to prove these facts about SSPO is via the technique described in the following section.

a sense in which every model of SPO is also a model of SO. Taking advantage of this fact can make it easier to feel at home with non-extensional mereology if you are used to extensional mereologies.

Models of SPO are *pre-ordered sets*, pairs of a domain D (of individuals) and a reflexive transitive relation $<$. It turns out that every such structure can be represented without loss of information by a partially ordered set, or *poset*, which is a pair of a domain and a reflexive anti-symmetric transitive relation.[11]

Let $[x]$ be an abbreviation for $\{y \in D : y < x \text{ and } x < y\}$ — the set of all individuals coincident with x. It follows that $[x] = [y]$ iff ($x < y$ and $y < x$) — that x and y belong to the same such set iff they coincide.

Now let the *poset representation* of $\langle D, < \rangle$ be the pair $\langle D', <' \rangle$ where:

$$D' = \{[x] : x \in D\}$$
$$[x] <' [y] \text{ iff } x < y$$

Putting this in words: the elements of the poset representation of an SO-model are sets of coincident individuals, and the ordering relation is the relation that holds between two such sets iff members of one are parts of the members of the other.

It's easy to see that $<'$ is reflexive and transitive if $<$ is. Also, $<'$ is anti-symmetric *even if* $<$ *is not*, since if $[x] <' [y]$ and $[y] <' [x]$, then $x < y$ and $y < x$ (by the definition of $<'$) and then $[x] = [y]$ (by the definition of $[...]$). So $\langle D', <' \rangle$ is a poset.

If an identity-free sentence of our mereological language — a sentence whose only predicates are $<$ and $\circ$ — is true in the model $\langle D, < \rangle$, then it is true in its poset representation $\langle D', <' \rangle$, and vice versa.[12] SSP is an identity-free sentence; so it is true in a model iff it is true in that model's poset representation. So the poset representations of SPO models are SO models. For similar reasons, the poset representations of SSPO models are SSO models (i.e. models of classical mereology).

The upshot of all this is that if you want to check whether whether a structure is a model of SPO, it's easy to do this by checking whether its poset representation is a model of SO (which is a fairly well-understood extensional mereology).

[11]For a similar application of this feature of pre-orders, see Cotnoir (2010, p. 403-405).

[12]I won't go through the proof here, but it would proceed by induction over the structure of sentences, with the base clause of the induction relying on the fact that $[x] <' [y]$ iff $x < y$. The restriction to identity-free sentences is needed because the sets $[x]$ and $[y]$ may be identical when the individuals x and y are distinct.

Likewise, if you want to check whether something is a model of SSPO, you can do that by checking whether its poset representation is a model of SSO (which is the extremely well-understood classical mereology).[13]

I find it much easier to think in extensional terms than non-extensional. I do not think that I am unusual — often opposition to non-extensional mereologies is derived less from philosophical objections than from a feeling that the clear structure of classical mereology is slipping away into the fog. But we should not feel foggy about SPO (still less SSPO) for they are closely related to extensional mereologies, differing from them only in more than one individual is permitted to occupy each place in the mereological network.

2.3 Axiomatisation in terms of overlap

There is something unlovely about the axiomatic basis of SPO with which I introduced it. The axioms Refl and Trans are stated in terms of the primitive predicate $<$, but the SSP axiom contains both $<$ and the defined predicate $\circ$. It would be nice if we could state the whole theory without having to use any definitions. Of course this could be done by writing SSP in primitive notation — just expanding out the occurence of $\circ$ in it. That, however, just makes SSP verbose and hard(er) to understand, and hard to recognise as the principle named by Simons.

There is however, another elegant way to axiomatise SPO without any definitions, and it is a way that sheds some light on the significance of SSP. Let the theory O (for *theory of overlap*) be the first order theory with the following axioms:

$$x \circ x \tag{ORefl}$$

$$x \circ y \to y \circ x \tag{OSym}$$

$$x \circ y \to (\exists z)(\forall w)(w \circ z \to (w \circ x \wedge w \circ y)) \tag{OP}$$

The axioms of O are supposed to capture conceptual truths of the concept of mereological overlap that was defined in SPO. ORefl and OSym are particularly obvious (each thing is a part of itself, so each thing overlaps itself; if x and y have a common part, then y and x have a common part). OP, the *overlap principle*, is a bit more interesting. It says that if two things overlap, then there is something

[13] A corollary of this applies to Hasse diagrams. Take a Hasse diagram, and for each pair of dots linked by a horizontal line, move those two dots on top of each other, so that they look like one. Then you have the Hasse diagram of the poset representation of the model represented by the original Hasse diagram. If what you now have is a model of SO (/SSO), then what you started with was a model of SPO (/SSPO).

they, so to speak, *overlap on*. If x and y overlap, then there's something that if you overlap it, then you overlap x and y both. You can think of this as saying, using only the language of overlap, that if two things overlap, then they have a common part. ORefl, OSym, and OP are all theorems of SPO.[14]

This is all very well, but not much use unless we can define "is part of" in terms of "overlap". The following definition is frequently used in mereologies that take overlap as their primitive:

$$x < y \Leftrightarrow (\forall z)(z \circ x \rightarrow z \circ y) \tag{Def<}$$

Let us call the theory that results from adding Def< to O, O+. O+ is the same theory as SPO.[15] So SPO has an axiomatic basis in which $\circ$ is the primitive, and $<$ is defined in a standard way for such bases.

[14]The proofs of ORefl and OSym are trivial, and already alluded to in the text above. The proof of OP in SPO forms part of the proof given in the following footnote that O+ and SPO are the same theory.

[15]To show that this is the case, I first show that both of the definitions, Def< and Def $\circ$, are admissible in both theories, then that each of the axioms of O+ is a theorem of SPO, and then that each of the axioms of SPO is a theorem of O+.

Def< is admissible in SPO. Proof: (a) The left-to-right direction of Def< is a consequence of Trans and Def$\circ$. If $z \circ x$, then there is some common part of both, w. If $x < y$, then by Trans, w is part of y. So w is a part of both y and z, so $z \circ y$. (b) Now take the right-to-left direction of Def<. We will prove the contrapositive, $\neg x < y \rightarrow (\exists z)(z \circ x \wedge \neg z \circ y)$, by conditional proof. Start by supposing that $\neg x < y$. (c) By SSP, there is some z such that $z < x$ but $\neg z \circ y$. (d) Since $z < x$, $z \circ x$ (using Def$\circ$ and Refl). (e) Therefore $(\exists z)(z \circ x \wedge \neg z \circ y)$, and we have proven the contraposed form we set out to.

Def$\circ$ is admissible in O+. Proof: (a) Start by expanding the right-hand-side of Def$\circ$ using Def<. This gives us

$$x \circ y \Leftrightarrow (\exists z)((\forall w)(w \circ z \rightarrow w \circ x) \wedge (\forall w)(w \circ z \rightarrow w \circ y))$$

which can be simplified to:

$$x \circ y \Leftrightarrow (\exists z)(\forall w)(w \circ z \rightarrow (w \circ x \wedge w \circ y))$$

The left-to-right direction of Def$\circ$ can thus be seen to be equivalent to OP.

(b) Now to prove the right-to-left direction. We proceed by conditional proof. Suppose there are some x, y and z such that $(\forall w)(w \circ z \rightarrow w \circ x \wedge w \circ y)$. (c) Instantiating the formula of (b) with z gives us $z \circ z \rightarrow z \circ x \wedge z \circ y$; but by ORefl, $z \circ z$; so $z \circ x$. (d) Instantiating (b) again with x gives us $x \circ z \rightarrow x \circ x \wedge x \circ y$; since $z \circ x$, we get $x \circ y$.

Refl is a theorem of O+. Proof: the result of applying Def< is a first-order tautology.

Trans is a theorem of O+. Proof: the result of applying Def< is a first-order tautology.

SSP is a theorem of O+. Proof: (a) Suppose $\neg x < y$, for a conditional proof of SSP. (b) By Def<, there is some z such that $z \circ x$ and $\neg z \circ y$. (c) So, by Def$\circ$ (which we have already proven is admissible in O+) there is some w such that $w < z$ and $w < x$; if we can now show that $\neg w \circ y$, then we have proven the consequent of SSP, completing our conditional proof. (d) $w < z$, so by Trans, every part of w is a part of z; at (b) we learned that $\neg z \circ y$; using Def$\circ$, that means that no part of z

Not every pre-ordering, and not every formal system proposed as a mereology,[16] admits Def<; such theories cannot be axiomatised with mereological overlap as a primitive (or not in the standard way). SPO is, in fact, the weakest mereology to have this feature — if Def< is added directly to Trans, Refl, and Def∘, the result is SPO.[17] This sheds new light on the significance of SSP. What SSP adds to Trans and Refl — what distinguishes a mereology from any common or garden pre-ordered set — is the ability to interdefine $<$ and $\circ$ in the standard way — the ability to uniquely recover the whole pre-ordering relation from just the facts about which pairs of elements have some element less than both in the pre-ordering.

It is also possible to give axiomatic bases for SO, SSO, and SSPO by extending O+ with axioms containing only $\circ$. SO may obtained by adding this axiom, the *extensional overlap principle* (EOP) to O+:[18]

$$(\forall z)(z \circ x \leftrightarrow z \circ y) \rightarrow x = y \tag{EOP}$$

In the context of Def<, EOP is equivalent to ASym.[19] So adding EOP to O+ produces SO.

is a part of y; therefore no part of w is a part of y. (e) So, by Def∘ $\neg w \circ y$; from (c) we got $w < x$; so $(\exists z)(z < x \wedge \neg z \circ y)$, completing the conditional proof.

ORefl is a theorem of SPO. Proof: the result of applying Def∘ is a consequence of Refl.

OSym is a theorem of SPO. Proof: the result of applying Def∘ is a first-order tautology.

OP is a theorem of SPO. Proof: (a) Suppose that $x \circ y$, for conditional proof. (b) Applying Def∘, there is some z such that $z < x$ and $z < y$. If we can now show that $(\forall w)(w \circ z \rightarrow (w \circ x \wedge w \circ y))$, then we have completed the conditional proof of OP. (c) Suppose, for conditional proof, again, that $w \circ z$. Since $w \circ z$, by Def∘, there is a v such that $v < w$ and $v < z$. (d) Since $z < x$ and $z < y$ from (b), by Trans, $v < x$ and $v < y$. (e) Therefore, by Def∘, $w \circ x$ and $w \circ y$ (f) Therefore, by conditional proof, discharging the supposition at (c), $(\forall w)(w \circ z \rightarrow (w \circ x \wedge w \circ y))$, which was all we need to finish the proof begun at (a).

[16]Varzi's (2009) theory MM does not, for example.

[17]Proof: the proof is the same as the proof that SSP is a theorem of O+ given earlier, since that proof did not appeal to the axioms of O+, except indirectly via the use of Trans and Def∘. That shows that the theory whose axioms are Tran, Refl, Def∘, and Def< has SSP as a theorem (and is thus an extension of SPO). That this theory is no stronger than SPO follows from the fact that Def< is admissible in SPO.

[18]And since GSP is already stated in terms of $\circ$, an axiomatic basis for SSPO may be obtained by adding GSP to O+; and a basis for SSO by adding both EOP and GSP to O+.

[19]Proof: applying Def< to ASym results in

$$(\forall z)(z \circ x \rightarrow z \circ y) \wedge (\forall z)(z \circ y \rightarrow z \circ x) \rightarrow x = y$$

This is logically equivalent to EOP.

3 Coincidence as a mereological concept

In section 2, I defined coincidence as mutual parthood; a move that, while popular, is also controversial. In this section, I defend that definition. I also address what appears to be a serious rival view of non-extensional mereology, stemming from the work of Peter Simons, and presented especially clearly by Achille Varzi, on which coincidence is not mutual parthood, and on which SSP is false.

I will begin by defending my definition of coincidence. First, I rescind my definition. Let *proper coincidence* be a name for a formally undefined concept that we all understand, the relation that holds between two things when they are, so to speak, made of the the same stuff. Let *strict parthood* be the also formally undefined relation that holds between *x* and *y* when *x* is part of *y* in an sufficiently strict sense as to exclude *x* and *y* being the same individual and to exclude *x* and *y* properly coinciding. I am going to assume that we have a grasp on those two concepts without any need for definition, and also that we have a grasp of the concept of *numerical identity*, of *x* and *y* being one-and-the-same thing.

Now I will introduce some defined terminology. *x* and *y coincide* iff either *x* and *y* properly coincide, or are numerically identical. (Only two things can properly coincide; everything improperly coincides with itself). Let *x* be a *general part* of *y* iff *x* is a strict part of *y*, *or x* and *y* coincide.

With "general part" so-defined, it can be proven that *x* and *y* coincide iff *x* and *y* are general parts of each other. I take it that (a) the relations of strict parthood, proper coincidence, and numerical identity are mutually exclusive; (b) that proper coincidence and numerical identity are symmetric;[20] (c) strict parthood is anti-symmetric. (a) is a matter of conceptual truth — it's part of what's meant by "strict parthood" that it exclude coincidence of any kind, part of what's meant by "*proper* coincidence" that it exclude identity; (b) likewise — if *x* and *y* are two things "made of the same stuff", then so are *y* and *x*; if *x* and *y* are one and the same, then so are *y* and *x*. (c) seems marginally less trivial than the others, but still a conceptual truth: I cannot find any coherent sense in which two things may be parts of each other without coinciding.[21]

[20] It's important, in this context, to distinguish *coincidence* (being "made of exactly the same stuff") from *constitution* (which may be ostensively glossed as "the relation that holds between a statue and the clay it is made of, but not vice versa"). Coincidence is necessary for constitution, but not sufficient. What it takes, over and above the clay's coinciding with the statue, for it to constitute the statue, is beyond the scope of this paper. See Thomson (1998) for a well-developed theory of constitution. It is of course, coincidence, and not constitution, that I am claiming to be symmetric.

[21] Perhaps Sanford's (1993) example of the Earth and the Aleph is a case in which we imagine two things to be parts of each other without coinciding. But I am not sure that this example is

Given the definition of general part, if x and y are general parts of each other, then x cannot be a strict part of y. Suppose, for reductio, that x and y are general parts of each other, and that x is a strict part of y . Then y is not a strict part of x, by (c); but y is a general part of x, so x and y must be either identical or properly coincident; but then by (b) x is either identical to or properly coincident with y; therefore by (a) x is not a strict part of y, contradicting our supposition. So if x and y are general parts of each other, then they coincide (i.e. are either identical or properly coincident). Conversely, if x and y coincide, they are general parts of each other (by the definition of general part).

So if "part" means general part, then coincidence is mutual parthood. What I meant by "part" (and by $<$) in section 2 was general part, and I will continue to mean that throughout the rest of this paper. So it was correct of me to define coincidence in the way I did in section 2.

Since, by definition of "coincide", two things properly coincide iff they coincide and are numerically distinct, our informal primitive, "proper coincidence" can be defined in terms of general part. If we write $x \approx y$ for "x properly coincides with y", then the definition could be formally written as follows:

$$x \approx y \Leftrightarrow x < y \wedge y < x \wedge x \neq y$$

That concludes the positive case for my definition of coincidence as mutual parthood. The remainder of this section discusses some senses of "part" other than general part that interact with "strict part" and "proper coincidence" in different ways. I use these to resolve a number of equivocations in the literature; and to resist what would otherwise appear to be a rival and inequivalent definition of coincidence.

3.1 The danger of equivocation

I have already described two senses of "part": strict and general part; I now introduce two more. Let x be a *classical part* of y iff x is a strict part of y or x and y are numerically identical. (Proper coincidents count as general parts of each other, but not as classical parts). Let x be a *non-identical part* of y iff x is a strict part of y or x and y properly coincide.

Discussions of extensional mereology frequently begin with a distinction between two senses of "part": an anti-reflexive sense, which is given the technical name "proper part"; and a reflexive sense, "proper-or-improper part". Conventionally, it is the latter that is meant by "part" in the literature on extensional

coherent, and it is certainly very controversial.

mereology. In a non-extensional setting, however, this distinction between a reflexive and an anti-reflexive sense of "part" is insufficient to remove all ambiguity. I've just described four different senses of "part": two reflexive (general and classical parthood) and two anti-reflexive (strict and non-identical parthood). It seems to me that failure to distinguish between the concepts of general and classical parthood and between strict and non-identical parthood is at the root of a large amount of equivocation and verbal disagreement in the literature.

Using the same premises I used to show that proper coincidence is definable as mutual general parthood, we can see that strict parthood is definable in terms of general parthood in a distinctive way: x is a strict part of y iff x is a general part of y but not vice versa. That is, if we write $x \ll y$ for x is a strict part of y:

$$x \ll y \Leftrightarrow (x < y) \wedge (\neg y < x) \tag{PP2}$$

Proof: Suppose that x is a strict part of y; then x is a general part of y, and y is neither a strict part of x (strict parthood is anti-symmetric) nor coincident with x (strict parthood excludes coincidence); therefore y is not a general part of x. Suppose that x is part of y and not vice versa; then x is not coincident with y (as coincidence is symmetric); therefore x is a strict part of y.

This is one common way of defining "proper part"; just as popular, however, is another, inequivalent way:[22]

$$x \lneq y \Leftrightarrow (x < y) \wedge (x \neq y) \tag{PP1}$$

The relation defined by PP1 is non-identical part. (I will now write $x \lneq y$ for x is a non-identical part of y). The difference between the two is simply that two properly coincident individuals count as non-identical parts of each other, but not as strict parts of each other.

PP1 and PP2 are often (when distinguished at all!) regarded as two rival definitions of one concept of "proper part" and arguments allowed to break out concerning which is the "correct" definition.[23] That, however, mistakes a verbal for a philosophical disagreement. PP1 and PP2 are both the correct definition of their respective definitienda. The trouble is that the usage of "proper part" in the literature does not decide which is meant.[24]

[22]For a discussion of the use of these definitions in the literature, see Cotnoir (2010, p. 398); the labels PP1 and PP2 are also his.

[23]Cotnoir (2010, p. 398-399), for example, argues that PP2 is the "correct" definition. In a sense, I agree with this — strict parthood is a more useful concept to define! But I think Cotnoir is fighting a losing battle if he wants to make everyone mean strict parthood by "proper part".

[24]In their very interesting paper (2012), Cotnoir and Bacon use "proper part" in yet a third sense, related to what I call non-identical parthood. Their intended meaning, to judge by their examples,

It's easy to understand how this has happened: if we assume that there is no proper coincidence, then every case of general parthood is a case of classical parthood (and vice versa) and every case of strict parthood is a case of non-identical parthood. Most formal work in mereology has been done in extensional mereology, where that assumption is in force. So it is unsurprising that these distinctions have been neglected. But we should be wary of assuming that a concept that is univocal in an extensional setting remains so in a non-extensional one.

I consider two case studies of the dangers of equivocation. The first is Peter Simons' weak supplementation principle (WSP), which is widely regarded as a trivial or analytic mereological truth,[25] and which is used as an axiom in Casati and Varzi's non-extensional Minimal Mereology (MM). (1999, p. 39) In Simons's formulation,[26] WSP says that if anything has a proper part, then it has another proper part disjoint from the first.

But what is meant by proper part here? Two versions of WSP, one using strict parthood, the other non-identical parthood, may be formulated as follows:

$$x \ll y \to (\exists z)(z \ll y \wedge \neg z \circ x) \tag{WSP1}$$
$$x \lneq y \to (\exists z)(z \lneq y \wedge \neg z \circ x) \tag{WSP2}$$

WSP1 is a theorem of SPO, but WSP2 is not (the model of figure 1 is a counter-model); WSP2 is, however, a theorem of SO, our minimal extensional mereology.[27] I believe that WSP1 is the trivial or analytic truth intended by Simons;

is that if x is a non-identical part of y, then x is a proper part of y. But not vice versa, because proper parthood is transitive, according to Cotnoir and Bacon, and neither anti-reflexive nor reflexive. Some individuals are Cotnoir-Bacon proper parts of themselves and not others. Moreover, their mereology allows for models that differ only in which individuals are proper parts of themselves; for example, they would draw a distinction between a model in which there is an unaccompanied atom which is a proper part of itself, and another model in which that very same unaccompanied atom is not a proper part of itself. This seems to me to be a distinction without a difference. Cotnoir and Bacon should strengthen their system to eliminate these pairs of models by adding an axiom to the effect that if x is a proper part of itself, then there is a y such that $x \neq y$ and x and y are mutually proper parts of each other. If they did that, then their "proper parthood" would be definable as follows: x is a Cotnoir-Bacon proper part of y iff $(x \lneq y) \vee (x = y \wedge (\exists z)(x < z \wedge z < x \wedge x \neq z))$. Special thanks to Andrew Bacon for discussion of this point.

[25]Considering a counter-model to WSP, Simons says "That goes against what we mean by part" (Simons 1987, p. 28)

[26]Casati and Varzi use a different formulation: if x is a proper part of y, then there is a z that is *part* of y and disjoint from x. (Casati and Varzi 1999, p. 39)(Varzi 2009) Given the definitions, Refl, and Trans, this formulation is equivalent to Simons. Simons's principle entails Casati and Varzi's (since every proper part of y is part of y). Casati and Varzi's entails Simons, since if z is disjoint from something that overlaps y, z and y cannot be identical and nor can they coincide; so if z is part of y, z is a proper part of y.

[27]Proof of WSP1. (a) Suppose for reductio that $x \ll y$, $(\forall z)(z \ll y \to z \circ x)$. (b) By the definition

WSP2 however, is inconsistent with the existence of any properly coincident individuals. Suppose that x and y properly coincide; the left hand side of WSP2 is satisfied, since properly coincident individuals are non-identical parts of each other. But the right hand side requires there to be something that is a non-identical part of y without overlapping x, which is impossible if x and y coincide.

The weak supplementation principle is supposed to be a triviality that would be a theorem even of a non-extensional mereology; however, it is easy to conflate it with a much stronger extensionality principle. It's hard to find this mistake being made in print,[28] though I have heard it made in person.

Second case study: Varzi's *extensionality of parthood* principle (EP), that "if x and y are composite objects with the same proper parts, then $x = y$" (2008, p. 108). This too is susceptible of two different formulations, depending on what is meant by "proper part":

$$(\exists z)(z \ll x) \land (\forall z)(z \ll x \leftrightarrow z \ll y) \rightarrow x = y \tag{EP1}$$

$$(\exists z)(z \lneqq x) \land (\forall z)(z \lneqq x \leftrightarrow z \lneqq y) \rightarrow x = y \tag{EP2}$$

The situation here is similar to the situation with WSP — one principle plays the role intended by its author (in this case EP1) and the other does not (EP2). EP2 is a theorem of SPO[29] and can be verified to be satisfied by the model of figure 1, in which coincidence occurs. Therefore, if extensionality is the impossibility of coincidence, then EP2 is not an extensionality principle. EP1, on the other hand, is not a theorem of SPO; it is not satisfied in either the models of figure 1 or figure 2, and it is a theorem of the minimal extensional mereology SO.[30]

of $\ll$ in terms of $<$, $\neg y < x$. (c) By SSP, then, there is some x such that $z < y$ and $\neg z \circ x$; if we can show that this z is a strict part of y, then we have a contradiction with (a). (d) $y \circ x$ (since $x \ll y$) but $\neg z \circ x$ (from (c), above); so y and z are not mereologically equivalent; therefore y and z do not coincide (recall that it is possible to prove in SPO that x and y coincide iff they are mereologically equivalent); therefore since $z < y$, $z \ll y$, so we have the contradiction noted at (c).

It is easy to prove that in SO (but not of course SPO) that $x \ll y \leftrightarrow x \lneqq y$; thus the proof above can be used to show that WSP2 is a theorem of SO.

[28]Varzi (2009) claims that Refl, Trans, and WSP entail ASym. He must be mistaken, though, since the model of figure 1 is a counter-model to this. I suspect that the proof he has in mind equivocates between WSP1 and WSP2 (the latter of which would indeed entail ASym in company with Refl and Trans).

[29]Proof: (a) Suppose that the antecedent of EP2 is satisfied by some x and y. (b) By PPP2, which we have already seen to be a theorem of SPO, $x < y$ and $y < x$; so x and y coincide. (c) However, by definition of $\lneqq$, $\neg x \lneqq x$, and by (a), $x \lneqq x \leftrightarrow x \lneqq y$ (instantiating $(\forall z)(z \ll x \leftrightarrow z \ll y)$ with x; so $\neg x \lneqq y$ (d) If x and y properly coincide, then $x \lneqq y$; therefore x and y do not properly coincide. (e) But as we saw at (b) they do coincide, so $x = y$.

[30]Proof: as with the proof of WSP2, it is easy to show that in SO, $x \ll y \leftrightarrow x \lneqq y$; thus the proof of EP2 can be used to show that EP1 is a theorem of SO.

I believe that EP1 is the extensionality principle that Varzi intended. In both cases, the charitable interpretation of "proper part" has turned out to be "strict part". But in applying this principle, Varzi equivocates.[31] He argues that certain arguments against extensionality fail, because if two things coincide (on the conception of coincidence I am defending) and are thus parts of each other, then they are proper parts of each other, and thus do not have all and only the same proper parts, so EP is true. (Varzi 2008, p. 116) This reasoning requires us to understand "proper part" as meaning non-identical part and EP to refer to EP2. Varzi is right that it would be bad news for the mutual parthood conception of coincidence if mutual parthood were consistent with extensionality. But it is not; mutual parthood is consistent with EP2 (a mereological triviality), not EP1 (Varzi's extensionality principle).

To summarise, here is a table showing all the senses of "part" that I have just introduced, together with their relationships to the primitive vocabulary:

		$x \ll y$	$x \approx y$	$x = y$
Is x a strict part of y?	$(x \ll y)$	yes	no	no
Is x a non-identical part of y?	$(x \lneq y)$	yes	yes	no
Is x a classical part of y?	$(x \lnapprox y)$	yes	no	yes
Is x a general part of y?	$(x < y)$	yes	yes	yes

The elementary logical properties of these concepts (on the assumption that $<$ satisfies the axioms of SPO) are shown here:[32]

Strict parthood	$(x \ll y)$	anti-reflexive, anti-symmetric, transitive
Non-identical parthood	$(x \lneq y)$	anti-reflexive, non-transitive (!)
Classical parthood	$(x \lnapprox y)$	reflexive, anti-symmetric, transitive
General parthood	$(x < y)$	reflexive, transitive

3.2 Classical parthood and the proper parts conception of coincidence

In the previous section, I mentioned Simons' weak supplementation principle, and Varzi's extensionality of parthood principle. These principles play an important

[31] It's with great reluctance that I make Varzi my stalking horse in various places in this paper. I am a great admirer of his work. If I had made as few logical errors as he has, while contributing as much of value to the literature, I would be very happy.

[32] The exclamation mark in the table draws attention to the fact that non-identical parthood is, unlike all the other senses of "part", non-transitive. Suppose that x and y properly coincide. Then $x \lneq y$ and $y \lneq x$ but not $x \lneq x$. Therefore $\lneq$ is not transitive.

role in what appears to be a rival approach to non-extensional mereology — an approach that could not be more different from the approach I took in section 2. On this rival approach, coincidence is not regarded as mutual parthood, but as having all and only the same *proper parts*. This approach originates in the early chapters of Simons's influential book *Parts*[33] and Varzi, in recent work, is its most explicit defender. The approach embodied in section 2 may be called the *mutual parthood conception* of coincidence; the rival view I am now describing the *proper parts conception* of coincidence.

Varzi is an extensionalist in mereology; and Simons' influence is via his presentation of extensional mereology. They do not explicitly define coincidence; their adherence to this proper parts conception appears in how they describe which aspects of their theories make those theories extensional. Simons describes extensionality thus: "by analogy with the extensionality principle of class theory, a principle which says that if individuals have the same parts they are identical. If 'part' meant '$<$', this would be trivial, so it must here mean 'proper part'" (1987, p. 28)[34] Simons goes on to say that an additional restatement is needed because mereological atoms have all and only the same proper parts. So the extensionality principle must say something like Varzi's EP: "If x and y are composite objects with [all and only] the same proper parts, then $x = y$" (Varzi 2008, p. 108) This suggests a definition of coincidence in terms of proper parts: x and y coincide iff x and y are composites and have all and only the same proper parts.

A distinctive feature of this approach is that it regards ASym as a trivial conceptual truth — "a minimal requirement which any relation must satisfy... if it is to count as parthood at all" (2008, p.110-111). ASym is not, on this view, a ban on coincidence, because coinciding objects are not parts of each other. SSP, on the other hand, is regarded as a principle of extensionality.[35] An appropriate axiom set for a non-extensional mereology, on this view, includes Simons's weak supplementation principle, together with the partial ordering axioms: Refl, Trans, ASym, WSP. Indeed, Casati and Varzi propose just this axiom set as their *Minimal Mereology* (MM). (1999, p. 39)

[33] The later chapters of *Parts* tell a slightly different story; in chapter 6, when discussing temporal mereology, Simons defines coincidence as mutual parthood, and allows for the rejection of (ASym). (Simons 1987, p. 180) Earlier on, however, he says that it is "trivial" that no two individuals have all and only the same parts (Simons 1987, p. 28) and at least gives the strong impression that for two individuals to coincide is for them to share all and only the same proper parts (in the context of stating principles that say that there are no such objects).

[34] Compare Varzi: "it is also crucial that [the principle of extensionality] is phrased in terms of proper parts, otherwise the principle would be trivially true" (2008, p. 108)

[35] Simons devised SSP as a principle that when added to the partial ordering axioms would yield a minimal extensional mereology. (Simons 1987, p. 31) Casati and Varzi makes similar use of it. (Casati and Varzi 1999, pp. 39-40)

MM and SPO are quite different theories, but there is a weird kind of symmetry between them. SPO allows the model of figure 1 but not the model of 2; MM vice-versa. A cynical third-party might suspect that the mutual parthood and proper parts conceptions of coincidence are debating a psuedo-question — perhaps either way of fitting coincidence into a mereological theory is equally good; perhaps the extensionality of SO is equally due to the presence of both ASym and SSP.

In fact, I think a dissolution of this "debate" is in order, but not of the kind imagined above. I believe that the mutual parthood and proper parts conceptions of coincidence are both correct, and that the apparent different between them is best explained by a terminological conflation of what I have called "general part" and "classical part".

I've already argued that if "part" means general part, then the mutual parthood conception of coincidence is provably correct; if "part" means general part, then it follows from the definition of "general part" that coincidence is mutual parthood. Defenders of the proper parts conception of coincidence appear to be disagreeing with this. However, if we interpret them as meaning *classical part* by "part", then what they say is consistent with SPO, and consistent with the definition of coincidence as mutual general parthood. Moreover, that interpretation makes good sense of three features of the proper parts conception that would otherwise be puzzling or seemingly unmotivated.

Let us write $x \lnsim y$ for "x is a classical part of y". Recall that x is a classical part of y iff x is a strict part of y or x and y are numerically identical.

$$x \lnsim y \Leftrightarrow (x \ll y) \vee (x = y) \qquad \text{(Def}\lnsim\text{)}$$

First, defenders of the proper parts conception are untroubled by the inequivalence between the two definitions of "proper part", PP1 and PP2. If "part" means classical part, then the varying definitions of "proper part" are in fact equivalent, and always define strict parthood. That is, both of the following definitions are admissible:

$$x \ll y \Leftrightarrow (x \lnsim y) \wedge (x \neq y) \qquad \text{(PP1*)}$$

$$x \ll y \Leftrightarrow (x \lnsim y) \wedge (\neg x \lnsim y) \qquad \text{(PP2*)}$$

That explains well the inattention to the differences between the definitions in the literature. It also makes good sense of these authors' use of WSP and EP — if "part" means classical part, then "proper part" must mean strict part, and so WSP and EP must be interpreted as WSP1 and EP1, which I have already argued is the best interpretation of those principles.

Simons makes "proper part" a primitive, and defines "*x* is part of or numerically identical to, *y*". (Simons 1987, p. 26) On the assumption that Simons means strict part by "proper part" (which he must do, given that he thinks that WSP is analytic), this definition makes "part" mean classical part, not general part.

Second, if "part" means classical part, then it is indeed "trivial" that no two things have all and only the same "parts". Let us call (ASym*) the principle that says that classical parthood is anti-symmetric:

$$x \lnsim y \land y \lnsim x \rightarrow x = y \qquad \text{(ASym*)}$$

ASym* is a theorem of SPO,[36] and is satisfied even in the model of figure 1. (A quick way to check Hasse diagrams for satisfaction of principles containing occurences of $\lnsim$ but no occurences of $<$ is to see whether they would satisfy the corresponding "unstarred" principles containing $<$, were all horizontal lines removed from the diagram). So coincidence cannot be defined as mutual classical parthood and extensionality cannot be regarded as the principle that no two things have all and only the same classical parts, for precisely the reasons that Simons and Varzi give.

Third, interpreting "part" as classical part makes sense of these authors' view that SSP entails extensionality. Let (SSP*) be the result of reinterpreting SSP replacing "part" by "classical part":

$$x \circ^* y \Leftrightarrow (\exists z)(z \lnsim x \land z \lnsim y)$$
$$\neg x \lnsim y \rightarrow (\exists z)(z \lnsim x \land \neg z \circ^* y) \qquad \text{(SSP*)}$$

SSP* is not a theorem of SPO; it fails to be satisfied in the model of figure 1. However, it is a theorem of SO.[37] The starred version of $\circ$ is needed because SSP is stated in terms of "overlap", which is stipulatively defined in terms of "part"; so if we want to see the effects on SSP of meaning classical part by "part", we must revise that stipulative definition too.[38]

[36] Proof: it is a consequence of (PP2) that $x \ll y \rightarrow \neg y \ll x$; then Asym* is easily proven via (Def$\lnsim$).

[37] Proof: it is easily proven that in SO, $x \lnsim y \leftrightarrow x < y$; in that context SSP* is equivalent to SSP.

[38] The difference between the starred and unstarred versions of $\circ$ is subtle: in all cases, if $x \circ^* y$, then $x \circ y$, but not vice versa. For $x \circ y$ but not $x \circ^* y$ to obtain, it must be that x and y be numerically diverse; that there be some z that both properly coincide with; and that there be nothing that is a strict part of both. The only way that that can happen is if x and y properly coincide but do not have any strict parts; that is, a pair of properly coincident mereological atoms would count as $\circ$ing but not $\circ^*$ing each other. Since other pairs of properly coincident individuals would count as $\circ^*$ing each other, $\circ*$ draws an arbitrary distinction between atoms and composites, and is thus not an especially useful notion of overlap.

A third version of the strong supplementation principle could be formulated, resembling SSP* except that $\circ^*$ is replaced by $\circ$. This, like SSP*, would be a non-theorem of SPO, but a theorem of SO.

In short, if we interpret Simons and Varzi as meaning classical part by their use of "part", then not only are the axioms of Casati and Varzi's minimal mereology theorems of SPO[39], but much of the surrounding philosophical discussion is consistent with the view that coincidence is mutual general parthood.

3.3 Atomic coincidence

I would now like to return to the possibility of coinciding mereological atoms that, as I noted earlier, SPO allows for. Earlier, I raised two issues about this. The first was the issue of whether it is intuitively plausible that atoms may properly coincide. I argued that it is no *less* plausible than proper coincidence of any other kind. The second was a more tricky conceptual issue, of whether two things that are parts of themselves, of each other, and of no other thing, should count as mereological atoms. Couldn't someone say that an atom is something that has no parts other than itself, and that our two coinciding "atoms" are therefore composite?

We are now in a position to resolve this second issue. It comes down to what we mean by "mereological atom". Atoms are normally defined as things that have no proper parts, but as we have seen, that is ambiguous. Do we mean that an atom is an individual that has no strict parts, or that an atom is an individual that has no non-identical parts? I think we should mean the former. Atoms are the fundamental building blocks of the realm of individuals. If two individuals properly coincide, have no strict parts, and are all there is, then they are atoms, for there is nothing more fundamental than either. That favours the definition of "atom" as "an individual having no strict parts", which in turn favours my interpretation of the model of figure 3.

The possibility of atomic coincidence in SPO is connected with an important fact about classical parthood. Though it is possible to formally define strict parthood and proper coincidence in terms of general parthood (which is what I do), it is not possible to adequately define proper coincidence, coincidence, or general parthood in terms of classical parthood or strict parthood.

[39] In our notation, the axioms are:

$$x \lnapprox x \qquad \text{(Refl*)}$$
$$x \lnapprox y \wedge y \lnapprox z \rightarrow x \lnapprox z \qquad \text{(Trans*)}$$
$$x \lnapprox y \wedge y \lnapprox x \rightarrow x = y \qquad \text{(ASym*)}$$
$$x \ll y \rightarrow (\exists z)(z \ll y \wedge \neg z \circ^{*} x) \qquad \text{(WSP*)}$$

Of these I have explicitly proven only ASym*. WSP* is an easy consequence of WSP1 (which we have already seen is a theorem of SPO) since $\neg z \circ x$ entails $\neg z \circ^{*} x$. The other proofs are also easy and left to the reader.

Consider the model of figure 3, in which there are two coinciding atoms, and another model, in which the same two atoms exist, but do not coincide, as shown in figure 4.

Figure 4: A model of SPO with two atoms but no coincidence

You may verify that the two atoms stand in all and only the same classical parthood and strict parthood relationships in both models. (In both models, both individuals have no strict parts; in both models, both individuals have themselves and nothing else as a classical part). However, the atoms coincide with and are general parts of each other in the model of figure 3 but not figure 4. Therefore coincidence and general parthood are indefinable in terms of classical parthood.

For this reason, it's not surprising that formal mereological theories using what I called the proper parts conception of coincidence hardly ever explicitly define coincidence. The closest they can come is with the definition I suggested in the previous section, that x and y coincide iff x and y are composites and have all and only the same proper parts. This explicitly restricts coincidence to holding between composites, which seems unmotivated; we have now seen that this restriction cannot be removed. All the more reason then, to define coincidence as mutual general parthood.

4 An argument for extensionalism

I remain a convinced extensionalist in mereology; though I can see how to play the non-extensionalist's game — and the main task of this paper has been to describe its rules — it seems to me that the philosophical reasons for playing that game are unconvincing. To debate the philosophical arguments for and against extensionality would be beyond the scope of this paper.

However, it does seem to me that there is a kind of formal argument for mereological extensionalism available, drawing on the conceptual background described in section 3. Most people who reject extensional mereology do so because they want to give some kind of explanation of the relation that holds between (e.g.) statues and the clay they are made of; persons and human bodies; committees and the fusions of their members. Non-extensionalists identify that relation with what I've called proper coincidence; and frequently say that two things coincide iff they are parts of each other; or iff they have the same parts.

But is this any real explanation at all? The mutual parthood conception of coincidence is an analysis of "coincidence" in terms of "general part"; but it is only made plausible by the stipulation that coincident individuals are to count as general parts of each other. The real primitive concepts of this paper are strict parthood and proper coincidence; coincidence was not, therefore, analysed in mereological terms, and I do not see how it can be.

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