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# Contradiction is Derivable from the Fixed Point Lemma

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## *1. Introduction*

One version of Curry's (1942) paradox is as follows. Define an "informal proof" of *p* as a series of English sentences that express a valid argument to the conclusion that *p*. Then, the initial observation is that we can informally prove the following sentence:

(C) If (C) is informally provable, then 1=0.

Assume provisionally (for conditional proof) that (C) is informally provable. By the meaning of (C), it is then informally provable that if (C) is informally provable, then 1=0. So, assuming that informal proofs are sound, this indicates if (C) is informally provable, then 1=0. Yet that in conjunction with our provisional assumption implies, by *modus ponens*, that 1=0. So by conditional proof, if (C) is informally provable, then 1=0.

However, the truth of (C) means that its informal provability implies that 1=0. Since the preceding *is* an informal proof of (C), it follows that 1=0, which is absurd.

This is a predicate formulation of the validity-Curry or "v-Curry" paradox (see Whittle 2004, Shapiro 2011, and especially Beall & Murzi 2013). Classical logicians have responded by restricting semantic terms like 'informal proof' so that it is not defined on sentences of its own language. ('Informal proof' is a semantic term, given that it was defined using the notion of validity, hence, truth.) Then, (C) will be ruled as a non-well-formed formula, thus thwarting any argument for its truth.

However, I will show that a variant on the paradox still results from the Diagonal Lemma, a.k.a. the Fixed Point Lemma. The new Curry paradox could conceivably be taken to show that Church's thesis is false, or even that Q, i.e., Robinson's (1950) arithmetic, is inconsistent. Yet we shall not take these possibilities seriously here. A different option would be to withhold Gödel numbers to certain formulae of Q. But in brief, this would mean that these formulae are direct counterexamples to the Fixed Point Lemma. And the falsity of the Lemma, of course, would upend many important proofs in metalogic, including the standard proofs of Gödel's incompleteness theorems (although not Gödel's own proof).[1] Thus, a satisfying solution to the paradox ultimately remains an open question.

## *2. An Informal Argument*

It is worth approaching the basic issue informally as a start, in order to subsequently grasp better the issue within Q. The informal version still employs an unrestricted semantic notion of an "informal proof;" hence, it is not entirely akin to the later argument (which will use only a syntactic notion of "derivability"). But since the reasoning is similar in some key respects, it will be useful to consider the informal argument as preparatory.

As a special case of an informal proof, let us speak of an informal proof "from $p$ to $q$;" this is an informal proof that starts with $p$ as the sole premise, and validly infers $q$ under that premise. Consider, then, the following statement:

[1] Gödel himself invokes not the Lemma itself, but only one instance of the Lemma; see Gödel (1931), pp. 188-189. Carnap (1934) seems to be the first to argue for the Lemma in its full generality. See Gaifman (2006) for an excellent analysis and discussion of the Lemma. For an introduction to Gödel and other matters relevant to this paper, I recommend Berto (2009).

(Mu) Haskell is an informal proof from (Mu) to '1=0' .

The name 'Haskell' in (Mu) has a denotation, but for the moment, we shall hold off on identifying what this is. First, we will show that (Mu) entails '1=0'.

Assume provisionally that (Mu) is true. Then, given what (Mu) says, it follows that Haskell is an informal proof from (Mu) to '1=0'. But the existence of such a proof is sufficient for the falsity of (Mu). So if (Mu) is true then it is false; hence, (Mu) entails '1=0'.

Note that this argument does not make any assumptions about what 'Haskell' denotes. It does not even assume that it denotes a proof. (Mu) *claims* that Haskell is a certain type of informal proof, but the truth of (Mu) was not assumed either (except provisionally) when arguing that (Mu) entails '1=0'.

At any rate, since (Mu) is the sentence 'Haskell is an informal proof from (Mu) to '1=0',' the preceding argument also suffices to show that the following Curry-like conditional is true:

(CC) If Haskell is an informal proof from (Mu) to '1=0', then 1=0.

With (CC) in hand, we can further note the existence of the following informal proof:

1. Haskell is an informal proof from (Mu) to '1=0'. [Premise]
2. 1=0 [From (CC), 1]

We may now reveal that the proof at 1-2 is the bearer of the name 'Haskell'. Observe that Haskell is an informal proof from (Mu) to '1=0', but Haskell is different from the earlier informal proof showing that (Mu) entails '1=0'. Indeed, since Haskell cites (CC) at line 2, and (CC) was established via the earlier proof, Haskell depends on the earlier proof for its cogency.

Yet now that we have specified which proof is Haskell, we are able to show that (Mu) is both true and false. Observe first that the falsity of (Mu) follows directly from the fact that there is some informal proof from (Mu) to '1=0'. Even so, we can now verify for ourselves that Haskell is an informal proof from (Mu) to '1=0'—and that is precisely what (Mu) claims. So (Mu) is also true. Contradiction.

Again, the classical logician is well-served here by restricting semantic notions like "informal proof" and "entailment." But it turns out that such notions are not necessary to re-create the Mu-paradox, as I shall now illustrate.

*3. A Metamathematical Argument*

Assume some Gödel numbering of the symbols, wff, and proofs of Q.[2] (In the remainder, "proofs" are purely syntactic objects—and to maintain clarity on this, I often refer to them simply as "derivations.") It is known that the relation "$x$ is a derivation from the members of $\Gamma$ to $\psi$" is decidable, where $\Gamma$ is an arbitrary set of wff and $\psi$ is any wff. Hence, as a special case, the relation "$x$ is a derivation from $\phi$ to $\psi$" is decidable, where $\phi$ is a single wff.

The decidability of the latter relation means, by Church's Thesis, that the following function $f$ is recursive. If $\phi$ and $\psi$ have Gödel numbers $^{\#}\phi$ and $^{\#}\psi$, respectively, then for any natural number $n$,

$$f(n, {}^{\#}\phi) = \begin{cases} {}^{\#}\psi & \text{if } n \text{ codes a derivation in Q from } \phi \text{ to } \psi; \\ 0 & \text{otherwise.} \end{cases}$$

[2] On Gödel numbering, see Gödel, *op. cit.*, pp. 178-179. Gödel's object of study was not Q but rather the Peano Axioms supplemented by the logic of *Principia Mathematica.* But Q is the weakest arithmetical theory in which undecidable sentences are known to exist; see Tarski, Mostowski, & Robinson (1953).

Since *f* is recursive, then by the Expressibility Lemma, a.k.a. the Strong Representability Theorem,[3] it follows that *f* is numeralwise expressible (strongly represented) in Q. This implies that, where *n* has numeral $\underline{n}$ and '$\vdash$' indicates derivability in Q, there is a formula '$D(x, y, z)$' such that:

(i) $\vdash \ulcorner D(\underline{n}, \underline{\#\phi}, \underline{\#\psi}) \urcorner$ if $f(n, \#\phi) = \#\psi$, and

(ii) $\vdash \ulcorner {\sim} D(\underline{n}, \underline{\#\phi}, \underline{\#\psi}) \urcorner$ if $f(n, \#\phi) \neq \#\psi$.[4]

Hence, an arithmetic formula $\ulcorner D(\underline{n}, \underline{\#\phi}, \underline{\#\psi}) \urcorner$ is provable in Q iff the metamathematical relation holds "*n* codes a proof in Q from $\phi$ to $\psi$." Conversely, $\ulcorner {\sim} D(\underline{n}, \underline{\#\phi}, \underline{\#\psi}) \urcorner$ is provable in Q just in case *n* does not code a proof in Q from $\phi$ to $\psi$. So in brief, there is an indication *within* Q for whether something is a derivation from $\phi$ to $\psi$.

Thus far, these matters should be uncontroversial. But now, consider the one-place formula $\ulcorner D(\underline{h}, y, \underline{\#1{=}0}) \urcorner$, where *h* is a specific positive integer, to be identified in a moment. By the Fixed Point Lemma, it follows that:

($FPL_\mu$) There is a sentence $\mu$ such that $\vdash \ulcorner \mu \equiv D(\underline{h}, \underline{\#\mu}, \underline{\#\text{'}1{=}0\text{'}}) \urcorner$.

We shall now demonstrate in addition that:

(Lemma) $\vdash \ulcorner \mu \supset 1{=}0 \urcorner$.

Proof of (Lemma): If we follow through with the proof of ($FPL_\mu$), it can be verified that $\mu$ is the sentence $\ulcorner D(\underline{h}, \text{diag}(\underline{\#\ulcorner D(\underline{h}, \text{diag}(x), \#\text{'}1{=}0\text{'}) \urcorner}), \underline{\#\text{'}1{=}0\text{'}}) \urcorner$, where '$\text{diag}(x)$' numeralwise

[3] Proposition V in Gödel, *op. cit.*, p. 186.

[4] Corner quotes in discussions of metamathematics are often used to denote the Gödel numbers of formulae. However, I am using '#' for this purpose instead. My use of corner quotes rather corresponds to Quine (1951) to indicate the concatenation of symbols inside the corners, after the metavariables have been replaced by expressions of the language. Thus, if $n = 1{+}2$, then $\ulcorner \underline{n} = 3 \urcorner$ is the sentence '$3 = 3$'. (The corner quotes clarify that derivations in Q are of *syntactic items*, rather than the propositions expressed by those items. I believe a lack of clarity here has caused some textbooks to err when presenting the proof of the Expressibility Lemma.)

expresses "the diagonalization of *x*."[5] The details in this are not crucial; it suffices to note that $\mu$ is free of quantifiers. As such, it is an elementary sentence of arithmetic; hence, $\mu$ is provable in Q if it is true—and similar remarks hold for $\ulcorner\sim\mu\urcorner$.[6] Since either $\mu$ or $\ulcorner\sim\mu\urcorner$ is true (classically speaking), it follows:

(QLEM$_{\mu}$) $\vdash \mu$ or $\vdash \ulcorner\sim\mu\urcorner$.

So basically, since $\mu$ is elementary, it is not an example of an undecidable sentence in Q. With this in mind, suppose first that $\vdash \mu$. Then by (FPL$_{\mu}$), $\vdash \ulcorner D(\underline{h}\ \underline{^{\#}\mu}, \underline{^{\#}\text{‘}1=0\text{’}})\urcorner$. Given the arithmetization above, this would mean that *h* codes a derivation from $\mu$ to '1=0', and such a derivation indicates that $\ulcorner\mu \supset 1=0\urcorner$ is a theorem.[7] Hence, if $\vdash \mu$, then $\vdash \ulcorner\mu \supset 1=0\urcorner$. On the other hand, if $\vdash \ulcorner\sim\mu\urcorner$, then by propositional logic, $\vdash \ulcorner\mu \supset 1=0\urcorner$. So either way, $\vdash \ulcorner\mu \supset 1=0\urcorner$, which is what (Lemma) says.

It is important that in the argument just given, it was not assumed that *h* actually codes a proof from $\mu$ to '1=0'. While this turns out to be correct (see below), the proof of (Lemma) does not presume this on pain of begging the question. Indeed, the proof did not even assume that *h* codes anything. Of course, $\mu$ "says" that *h* codes a proof from $\mu$ to '1=0', but $\mu$ was not assumed either (except provisionally) while arguing for (Lemma).

Let (CC*) be the formula $\ulcorner\mu \supset 1=0\urcorner$. (Lemma) tells us that (CC*) is a theorem of Q; hence, we know that the following derivation in Q exists:

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[5] I am guided here by the proof of the Fixed Point Lemma in Gaifman, *op. cit.*, p. 710.

[6] Briefly, the provability of true elementary sentences follows from the Expressibility Lemma and the fact that all elementary formulae in Q express a recursive relation. Cf. Gödel, *op. cit.*, p. 177 and p. 183.

[7] Just to be clear: If $\vdash \ulcorner D(\underline{h}\ \underline{^{\#}\mu}, \underline{^{\#}\text{‘}1=0\text{’}})\urcorner$, then *h* must code the relevant derivation, even though clause (i) of the arithmetization states the converse. For if $\vdash \ulcorner D(\underline{h}\ \underline{^{\#}\mu}, \underline{^{\#}\text{‘}1=0\text{’}})\urcorner$ and *h* failed to code the relevant derivation, then clause (ii) would imply that $\vdash \ulcorner\sim D(\underline{h}\ \underline{^{\#}\mu}, \underline{^{\#}\text{‘}1=0\text{’}})\urcorner$ and Q would be inconsistent.

1. $\mu$ [Premise]
2. $1{=}0$ [From 1 and (CC*)]

As a derivation in Q, it will have a unique Gödel number. Assume that this number happens to be *h*. Since the above is a derivation from $\mu$ to ‘1=0’, then by clause (i) of our earlier arithmetization, we know that $\vdash \ulcorner D(\underline{h}, \underline{{}^{\#}\mu}, \underline{{}^{\#}\text{‘}1{=}0\text{’}})\urcorner$. Moreover, this indicates by ($FPL_{\mu}$) that $\vdash \mu$. However, (Lemma) indicates that $\vdash \ulcorner{\sim}\mu\urcorner$. So Q is inconsistent.

The sentence $\mu$ effectively means “*h* codes a derivation from me to absurdity.” Since *h* indeed codes such a derivation, $\mu$ is true. But at the same time, such a derivation suffices for its falsity. So $\mu$ is inconsistent, and this can be captured within Q via the arithmetization. Yet since the existence of $\mu$ is guaranteed by the Fixed Point Lemma, the Fixed Point Lemma suffices for contradiction in Q.

*4. Some Clarifications*

It might be objected that, under a standard Gödel numbering of wff and proofs, it is not possible for a derivation to contain its own Gödel numeral. But this is what the derivation coded by *h* would be, for it begins with $\mu$, i.e. it begins with the sentence $\ulcorner D(\underline{h}, \text{diag}(\underline{{}^{\#}\ulcorner D(h, \text{diag}(x), {}^{\#}\text{‘}1{=}0\text{’})\urcorner}), \underline{{}^{\#}\text{‘}1{=}0\text{’}})\urcorner$. In fact, there are Gödel numberings where such things are possible (see Kripke ms.), but let that pass. One may use a functor instead of a numeral to formulate the new Curry-paradoxical sentence. For instance, suppose that *r* is a function such that $r(0){=}k$. The Fixed Point Lemma guarantees that:

($FPL_{\delta}$) There is a sentence $\delta$ such that $\vdash_{Q} \ulcorner\delta \equiv D(r(0), \underline{{}^{\#}\delta}, \underline{{}^{\#}\text{‘}1{=}0\text{’}})\urcorner$.

One could now argue much as before that the following is a theorem:

(CC†) $\delta \supset 1{=}0$

And the theorem secures the existence of the following derivation:

1. $\delta$ [Premise]
2. $1=0$ [From 1 and (CC†)]

If $r(0)=k$ happens to be the Gödel number for this derivation, it can be shown in a similar fashion that $\vdash \delta$ and $\vdash \ulcorner\sim\delta\urcorner$. And $\underline{k}$ would not occur in the very derivation coded by $k$, since $\delta$ would officially be the sentence $\ulcorner D(r(0), \text{diag}(\underline{\#\ulcorner D(r(0), \text{diag}(x), \#`1{=}0'\urcorner}), \underline{\#`1{=}0'})\urcorner$. But having said all that, I will for convenience continue to use $\mu$ as my example.

Some may have noticed a different argument suggesting that neither $\mu$ nor its negation is a theorem in Q. Briefly, we can show a contradiction results from either supposition. This might lead one to think that $\mu$ does not generate a paradox; we should rather say it is another point at which Q is incomplete.[8] But the bare existence of a sentence like $\mu$ is sufficient for contradiction. Hence, we would expect that one can show a contradiction under the supposition that $\vdash \ulcorner\sim\mu\urcorner$ or that $\vdash \mu$; after all, each supposition presumes that $\mu$ exists. Hence, while the incompleteness argument is valid, it hardly discredits the idea that m is pathological.

Even so, an intuitionist may try to reject that $\mu$ or $\ulcorner\sim\mu\urcorner$ is true, hence, reject ($QLEM_\mu$).[9] But this would mean $\mu$ as undecidable in Q. And since $\mu$ is an *elementary* sentence of arithmetic, this seems implausible. It would flout the idea that all elementary formulae in Q express a recursive function. Besides, the rest of us are already committed to

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[8] The incompleteness argument in brief is as follows: If $\vdash \ulcorner\sim\mu\urcorner$, then since $\mu$ is the antecedent of (CC*), we know that (CC*) is a theorem. As above, we can then show that $\vdash \mu$. ☒ On the other hand, if $\vdash \mu$, then as was shown during the proof of (Lemma), it follows that $\vdash \ulcorner\sim\mu\urcorner$. ☒

[9] The *loci classici* on intuitionism include Brower (1912) and Heyting (1930).

($QLEM_\mu$), for we are committed to the bivalence of elementary sentences (if not all sentences) of Q.

Consider also that a dialetheist response seems unhelpful.[10] The dialetheist might accept and deny $\mu$ itself—however, this would leave untouched the argument showing that a contradiction is derivable *inside* Q. In fact, this point is analogous to the problem the dialetheist faces regarding the v-Curry paradox. But granted, as with the v-Curry, perhaps the dialetheist could devise some other, congenial strategy for understanding the new Curry paradox.[11]

## *5. Toward a Solution?*

It is conceivable that the paradox indicates that Q really is inconsistent. But given the apparent banality of the Q-axioms, it is hard to take this seriously. A second possibility is that Church's Thesis is false: Even though "$x$ is a derivation from $\phi$ to $\psi$" is intuitively decidable, the function expressing this relation is not recursive, hence, not arithmetizable. However, Church's Thesis has tremendous utility in the field and is evidenced by the striking convergence between Turing computable functions, lambda-computable functions, and general recursive functions, *inter alia*.[12] So I should like to place the blame for the paradox elsewhere.

The strict analogue to the classical solution of v-Curry would be to blame the predicate '$D(x, y, z)$', i.e., the formula that arithmetizes "$x$ is a derivation from $\phi$ to $\psi$".

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[10] On dialetheism, see Priest (2006), Beall (2009).
[11] For a dialetheist discussion of the v-Curry, see Priest, *op. cit.*, ch. 6.
[12] See Church (1936), Turing (1937), Shepherdson and Sturgis (1963), etc. A good summary of these results is found in chapters 12 and 13 of Kleene (1952).

Accordingly, the predicate would be restricted so that it is undefined on Gödel numbers for sentences of its own language. But a moment's thought reveals that this does not make sense. In the first instance, 'D($x$, $y$, $z$)' is an arithmetical formula. As such, it is already defined on any ordered triple of natural numbers, regardless of whether those numbers are Gödel codes or not.

It also makes no sense to conclude that the predicate 'D(x, y, z)' must not exist in the arithmetical language. Here, a comparison with Tarski's (1933) indefinability theorem helps. Tarski's theorem indicates that if the truths of Q were arithmetically definable by a formula 'T($x$)', then the Fixed Point Lemma would secure that:

($FPL_L$) There is a sentence L such that $\vdash L \equiv \sim T(\underline{^{\#}L})$

From L, Tarski is able to derive a contradiction. Yet the existence of L is secured by the Fixed Point Lemma, *if* the language contains a formula T($x$) which arithmetically defines truth. So the conclusion is that no such formula exists in the language.

This conclusion is plausible given that 'T($x$)' does not express a recursive property. But 'D(x, y, z)' indeed expresses a recursive relation on natural numbers, given that it is intuitively decidable whether $x$ is a derivation from $\phi$ to $\psi$ (and given Church's Thesis). Thus, since every recursive relation is numeralwise expressible in Q, 'D($x$, $y$, $z$)' must be a formula of Q. Tarski's way seems unavailable to us.

But there is another solution which is similar in spirit. The blame for the paradox goes on the *code* for 'D($x$, $y$, $z$)', and the solution is to refuse to code this formula. After all, the Fixed Point Lemma requires this code in order to secure the existence of $\mu$. Still, there are equivalent formulae to 'D($x$, $y$, $z$)' which would also need to be denied a code. (Otherwise, these equivalent formulae would also secure the existence of $\mu$.) For example, 'D($x, y, z$)' is really just a special case of Gödel's proof predicate '$B(x, y)$'.[13] So a code would be denied either of '$B(x, y)$' or of any formula which, when conjoined with '$B(x, y)$', results in something equivalent to 'D($x, y, z$)'. And there are denumerably many formulae of the latter kind. One example would be a formula ⌜P!($x$, $\underline{^{\#}\phi}$)⌝, which arithmetizes the relation "$x$ codes a derivation whose sole premise is $\phi$." (It is clear that the relation is arithmetizable, given that it is decidable and given Church's Thesis.) Patently, 'D($x, y, z$)' is equivalent to '$B(x, z)$ & P!($x, y$)'; hence, if '$B(x, y)$' is not denied a code, then we must deny a code to the other conjunct. But by the same token, we would need to deny a code to a formula that arithmetizes "$x$ codes a derivation whose premises are exactly $\phi$ and $\tau$," where $\tau$ is some tautology. And the same applies to a formula that arithmetizes "$x$ codes a derivation whose premises are exactly $\phi$, $\tau$, and $\sigma$" where $\tau$ and $\sigma$ are distinct tautologies. Etc.

Regardless, these denumerable formulae all seem to depend on a coding of a formula ⌜Premise($x$, $\underline{^{\#}\phi}$)⌝ which arithmetizes "$x$ codes a derivation with $\phi$ as a premise." So rather than withholding a code to the proof predicate, a code might be denied of the premise predicate. But the bad news is this. Any formula of Q which lacks a code will be a direct counterexample to the Fixed Point Lemma. So on this approach, if the paradox does not falsify the Lemma directly, it will be falsified by counterexample.

For what it's worth, the incompleteness theorems seem provable even without the Fixed Point Lemma.[14] More broadly, withholding codes for 'D($x, y, z$)' and certain other

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[13] Cf. Gödel, *op. cit.*, p. 186.

[14] I am thinking especially of the incompleteness proof by Kripke (reported in Putnam 2000), where the undecidable sentence exhibited is "not at all 'self-referring'" (p. 55). Though as mentioned (n. 1), even Gödel did not invoke the Fixed Point Lemma; he used only one instance of the Lemma vis-à-vis the arithmetization of "there is no proof of $x$." Of course, if the provability predicate is denied a code, then not even this instance of the Lemma is available. Still, Gödel's own proof can be preserved if we deny a code to the premise predicate instead of the provability predicate, as explained above.

formulae may be less radical than rejecting Church's Thesis or embracing the inconsistency of Q.[15] Nevertheless, since the current solution fails to uphold the Fixed Point Lemma, one may wonder how much of a "solution" it is. It is for this reason that I cannot unreservedly endorse it. But admittedly, I am currently unable to discern a suitable alternative.

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[15] Granted, we might have wanted just to withhold a code for μ itself or withhold a code for the derivation coded by *h*, above. But if 'D(*x*, *y*, *z*)' has a code, then these things automatically have a code, thanks to the coding scheme for sentences and derivations in general. (Besides, here too there would be denumerably many sentences and derivations that would also need to be uncoded; consider μ conjoined with a tautology, μ conjoined with two tautologies, etc.)

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