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# INTRODUCTION
# **Philosophy of mathematics**

## ***Why philosophy of mathematics?***

An entire book could be written about why the philosophy of mathematics exists. Indeed it has, by the distinguished philosopher Ian Hacking, with the title *Why Is There Philosophy of Mathematics At All?* To cut Hacking's long story short, there are two answers. The *epistemological* answer is based on the experience of doing mathematics. When we prove a mathematical proposition, be it a simple fact such as that 37 is prime or a sophisticated conjecture such as Fermat's Last Theorem,[1] we use only pure thought. We do not perform experiments; we do not investigate the historical record; we do not even need to look at the world before us. We could in principle determine that 37 is a prime number with our eyes shut, indeed without relying on any sensory information, using only our powers of reasoning. So how can pure thought give rise to mathematical knowledge? This question was stressed by both Plato in the early 4th century BC and, more than two millennia later, by Kant in the 18th century. In fact, Kant's entire philosophy grows out of his answer to it, as in a way does Plato's.

The other reason the philosophy of mathematics exists is *metaphysical*, the question now being: how is it possible for the subject matter of mathematics to relate to the physical world? The objects of mathematics – numbers, geometric shapes, functions, sets – seem to hold universally, yet the objects themselves are nowhere to be found. However far, deep and wide you search our universe, you will not come across the number 2. What you will find are pairs of objects, though not the abstract quantity they have in common. So if not in space, where are numbers? Can non-physical entities even exist? Plato had an answer to these questions in keeping with his epistemology, and his pupil Aristotle had another. In fact, there would be no better way to explain the key difference between Plato's and Aristotle's philosophies than to begin with their disagreement over the nature of mathematics.

In sketching why the philosophy of mathematics exists, we have mentioned three great philosophers: Plato, Aristotle and Kant. This is no coincidence. Coeval with philosophy itself, the philosophy of mathematics lies at its core. As soon as one begins to think about the nature of reality, the question of how mathematics fits into it becomes not just natural, but pressing.

## ***Historical division***

The five volumes you have before you comprise the most significant contributions to philosophical thought about mathematics. Although I have consulted several people formally and informally, the final selection is entirely mine. My working criterion for inclusion has been this. Suppose an English-speaking philosopher of mathematics were stuck on a desert island and could take with her only about 70 articles or book excerpts, totalling no more than 2,000 pages. What would she take?

To answer the question, one must appreciate why contemporary philosophy of mathematics is different from its pre-20th-century antecedents. One reason is that mathematics has become integral to science since the 17th century. Science has over the past few centuries become increasingly mathematical; indeed the fundamental science of nature, physics, is today

[1] The claim that there are no solutions in positive integers $x, y, z$ to $x^n + y^n = z^n$ for integer $n > 2$. Posed by Pierre de Fermat in 1637, a proof had to wait until Andrew Wiles (assisted by Richard Taylor) finally produced one in 1995.

recognised as a branch of applied mathematics. The second reason is that mathematics underwent a transformation in the course of the 19th century: starting the century as a traditional-looking science of quantity, it ended it a radically transformed abstract theory of structure. The final factor in the transformation of the philosophy of mathematics is the rise of modern logic. Developed by Frege, Cantor and others in the late 19th century, modern logic pervades contemporary mathematics, philosophy and computer science. It has had an immeasurable impact on the philosophy of mathematics, as on philosophy more generally.

In light of this history, we may broadly divide the historical evolution of the philosophy of mathematics into three periods. The first period is from its birth circa the 6th century BC up to the late 19th century, before mathematics had fully evolved into its modern form and before modern logic had been created. The second is the so-called golden age of the philosophy of mathematics, from the late 19th century to the mid-20th century. In this period, several important philosophies of mathematics were developed which in their scope, precision and mathematical detail surpassed anything previously seen. That leaves 'contemporary' philosophy of mathematics as the third and final (and ongoing) period. The period begins towards the middle of the 20th century, after modern mathematics and logic had evolved into roughly the form they take today.

The structure of the anthology reflects this tripartite division. Volume I traces some key moments in the evolution of the philosophy of mathematics. We begin with Plato, the earliest philosopher whose extant body of work represents a coherent philosophy of mathematics. We conclude with a capstone to the historical evolution of mathematics and its philosophy during this first period: Albert Einstein's 1921 lecture, which depicts the new mathematical landscape. Volume II picks up the story and showcases the principal philosophies of mathematics that emerged in reaction to 19th-century developments, from logicism to logical empiricism via intuitionism and formalism. Volumes III, IV and V are then devoted to contemporary issues. Each of them presents key works, mainly from the past 60 years, on the three principal topics that animate contemporary debate. Volume III is on the foundations of mathematics, set theory and structuralism. The selected papers investigate the metaphysics of modern mathematics and ask whether set theory is in some sense the true metaphysics of mathematics. Volume IV is concerned with the nature of mathematical justification, in particular proof. The papers in Volume V deal with the relation of mathematics to the natural world: what philosophical lessons can we draw from the fact that mathematics is applied in science?

### *Selection*

Although historically and thematically wide-ranging, the collection does not pretend to either impartiality or comprehensiveness. I am all too aware that it reflects biases and predilections, however much I may have tried to mitigate them. Omissions of important material, both historical and contemporary, are unavoidable. A conscious example of a gap is the philosophy of mathematical practice, a movement only about a couple of decades old. The movement's guiding thought is that the philosophy of mathematics should examine the detail of what mathematicians do; as a (usually intended) consequence, it distances itself from more traditional metaphysical and epistemological concerns. This movement is not much represented here, partly because to my mind it has not yet reached maturity. That said, several of the works in later volumes could be so labelled – e.g. Penelope Maddy's magisterial contributions – since they are concerned with the fine grain of mathematical experience.

My selection criterion also imposes a utilitarian constraint on historical work. Historical work in the philosophy of mathematics is included only to the extent that it resonates today, that it offers ideas, techniques or arguments for the contemporary philosopher of mathematics. A concomitant of this approach is that the summary of these five volumes sometimes lapses into

Whig history. For our representative desert-island philosopher of mathematics, the value of a historical source is sometimes that it expresses the germ of an idea that was to flower centuries later.

Another potential omission concerns non-Western philosophy of mathematics. The constraint that my desert-island philosopher is a speaker – reader – of English excludes a great deal of untranslated material in other languages. It also creates a presumption in favour of 'Western' philosophy of mathematics, since this is the tradition best represented in the English language and in English-speaking culture. The constraint that the material must be a contribution to the *philosophy* of mathematics, as opposed to mathematics itself (which of course has an immensely rich 'non-Western' tradition) or other modes of reflection about mathematics – theological, sociological, educational and so on – also biases the collection in favour of, dropping scare quotes now, the Western tradition. I have searched for important non-Western texts in the philosophy of mathematics, but in the end was unable to include any. Unhappily, I found that the translated work of many candidate authors falls between two stools. Umar Khayyām, for example, produced much important mathematics; witness, say, his commentary on Euclid, in which he tries to prove the parallel postulate, predating Saccheri's work by many centuries. Khayyām also produced much work of philosophical and theological importance. But I could not find any English-language writings of his that satisfied me as falling under the heading of the philosophy of mathematics as (perhaps narrowly) understood today. Other writers are not sufficiently known to merit inclusion. My intended readership would not be well served by texts unknown to the majority of today's philosophers of mathematics, students and professionals alike. I have construed my remit of canon compilation more descriptively than normatively, and have accordingly aimed to report rather than to reconfigure the philosophical conversation through the ages. It would of course be a mistake to think that all later philosophers in this anthology read earlier ones, to try to draw a simplistic narrative arc in which the future always logically builds on the past. Nevertheless, there is a strong unity to the tradition here represented, which ostensively defines the philosophy of mathematics as understood by a contemporary English-speaking philosopher.

### *Prior philosophy of mathematics collections*

Of the prior English-language collections, the most famous are probably the two editions of Paul Benacerraf and Hilary Putnam's *Philosophy of Mathematics*, published in 1964 and 1983 respectively. The 1983 edition covers a good deal of the most important literature from the late 19th century to the 1970s, and significantly overlaps with some of the works in the present anthology, in particular those in Volume II. Both these collections – mine and Benacerraf and Putnam's – also overlap significantly with Jean van Heijenoort's *From Frege to Gödel*, which, as its title indicates, focusses on the golden age and which was published in 1967. Other noteworthy collections include the historically wider-ranging *From Kant to Hilbert*, edited by William Ewald (1998); the state-of-the-art anthology *Philosophy of Mathematics Today*, edited by Matthias Schirn (1998); and the shorter set of readings *Philosophy of Mathematics*, focusing on the second half of the 20th century and edited by W.D. Hart in 1996. An advantage of the present collection is that its coverage is historically wider than any of the before mentioned; with the partial exception of the two Ewald tomes, it also goes deeper, and it is naturally more up to date.

### *Volume I*

The first volume offers a historical introduction to the philosophy of mathematics, taking in some emblematic positions. We start with selections from Plato, Aristotle and Kant, as well as some representative early modern thinking about the nature of mathematics. We conclude with

19th-century selections from Mill and Dedekind. The coda is Einstein's 1921 lecture, which surveys the 19th-century revolution in mathematics, focusing on geometry in particular.

However little it might otherwise have in common with Plato's doctrines, Platonism in mathematics derives its name from Plato's claim that the objects of mathematics are not spatiotemporal. In the selections from the Platonic dialogues we do indeed find their author's conviction that mathematical Forms are immutable and outside space and time. The focus in these excerpts, however, is more epistemological than it is metaphysical. Plato propounds his theory of recollection (*anamnesis*) according to which knowledge of the Forms (in particular, of mathematics) derives from our soul's acquaintance with them before physical birth.

The passages also reveal an evolution in Plato's thought: Plato's mature view in *The Republic* that the mathematics of his day does not amount to proper knowledge goes against those views expressed in the *Meno* and *Phaedo*.

The passages from Aristotle pinpoint his fundamental disagreement with his teacher, Plato. In the *Metaphysics* selection we find Aristotle's general view that Forms are not independent of their instances: the bad does not exist apart from bad things, as he puts it in Book IX. When it comes to mathematics itself, Aristotle's view is subtle and defies simple characterisation. But roughly speaking, on his metaphysics, what mathematicians investigate are certain properties of (real, instantiated) objects considered in the abstract – the attributes of things *qua* quantitative and continuous, as he puts it in Book XI. Although it would be anachronistic to call Aristotle an empiricist in the philosophy of mathematics, this line of thought certainly influenced later empiricists, including John Stuart Mill, as we shall see. Some of the excerpted passages (e.g. from Book III of the *Physics* or the selection from *De Anima*) also articulate Aristotle's view that the infinite is potential rather than actual. In contemporary terms, the idea is that a sequence such as 0,1,2, … is infinite not in virtue of its terms existing as an actually realised infinite totality, but rather in virtue of being indefinitely extensible, meaning that one can continue the sequence beyond any given term. This potentialist conception of the infinite was to inspire late 19th- and 20th-century constructivists in the philosophy of mathematics, in particular the intuitionists Brouwer, Heyting and Dummett featured in Volume II.

Although Descartes was a great mathematician and a great philosopher, his output in the philosophy of mathematics proper was meagre. The reason for including the short passage from the *Discourse on Method* here is that it expresses the foundationalist ideal in epistemology and by implication in the philosophy of mathematics. The precept to not accept anything as true unless it is evident has had an important influence in the history of mathematics and its philosophy, in particular in the choice of axioms and discussions of their status. The precept leads directly to some of the foundationalist programmes of the late 19th and early 20th centuries, covered in Volume II. In Locke and Berkeley we find classic expressions of early modern empiricist views of the nature of mathematics. Locke in particular distinguishes intuitive, demonstrative and other types of knowledge. The upshot for mathematics is that demonstrative mathematical knowledge consists in the perception of the agreement or disagreement of ideas by means of proof. The Lockean passages are also the *locus classicus* of their author's conviction that human knowledge is derived 'without the help of any innate impressions'—the so-called *tabula rasa* (blank slate) theory of the human mind. Mathematics was one of Bishop Berkeley's lifelong interests, and he returned to it on several occasions. In the *Principles of Human Knowledge*, Berkeley cleaves mathematics in two, offering differing accounts of geometry and arithmetic. Later historians of mathematics have seen in Berkeley's account of arithmetic one of the first manifestations of a formalist philosophy of mathematics. There is certainly much evidence for this interpretation, as witnesses Berkeley's claim that '[i]n arithmetic therefore we regard not the things but the signs, which nevertheless are not regarded for their own sake, but because they direct us how to act with relation to things and dispose rightly of them' (*Principles of Human Knowledge*, §122). In his discussion of geometry, Berkeley anticipates his famous critique of the calculus in the *Analyst* (1734).[2]

Leibniz's ideas about mathematics are scattered throughout his writings; it is a pity that he never synthesised them into a systematic whole. I have selected a few short pieces mainly with a view to illustrating Leibniz's key idea of a *universal characteristic,* which he believed was 'a calculus more important than those of arithmetic and geometry'. The idea of a universal language into which mathematics – more ambitiously, all factual discourse – could be transcribed, and all its problems mechanically solved, is often traced back to these passages. It is an idea that flowered in the late 19th and early 20th centuries in the works of the logicist and formalist schools. In a broad sense it still animates contemporary mathematical logic and other formalisation programmes, even if we have learnt to live with the fact that there is no decision procedure for mathematics (see the discussion of Volume IV).

If Diderot's *Encyclopedia* of 1751 is the great literary project of the Enlightenment, the *Preliminary Introduction* to it may be regarded as its manifesto. The author of the *Preliminary Introduction*, Jean d'Alembert, a noted 18th-century mathematician, devotes several of its pages to mathematics and its place in the worldview of the French Enlightenment *philosophes*. Though 21 centuries separate Aristotle and d'Alembert, their philosophies are remarkably similar. According to the Enlightenment philosopher, the human mind via an operation of abstraction considers physical objects as divested of various sensible properties. The residual shaped extensions are then the subject matter of geometry; arithmetic and algebra arise as the sciences of the laws governing the shaped extensions' relationships. The account of mathematical applications follows on naturally. We see, then, that in 1751 it was still tenable to understand mathematics as in some sense the quantitative science of space and time. That was about to change very soon.

The monumental *Critique of Pure Reason* is Immanuel Kant's answer to this question: how is synthetic a priori knowledge possible? By an analytic proposition, Kant understands one of the form 'All *A*s are *B*' in which the concept of an *A* includes that of being a *B*, such as 'All bodies are extended' in Kant's own example; a synthetic proposition is one that is not analytic.[3] A proposition that is a priori knowable is, broadly speaking, a proposition that is knowable in a way that does not require sense experience for its justification. Kant thought that the paradigm of synthetic a priori knowledge was mathematics, and that this created a major philosophical problem. How can one know something that is *not* a matter of conceptual necessity without relying on sense-experience?

The *Critique of Pure Reason*, a magnificent testament to the reach and depth of philosophical thought, represents Kant's attempt to answer this question. To explain how synthetic a priori knowledge is possible, Kant hypothesised that it is we who bring space, time, force, action and motion into our experience, and that we do so a priori—without the aid of any previous experience. In his terminology, we cognise the world using two distinct faculties. One is the faculty of sensibility, which moulds the raw matter of sensory experience into spatiotemporal form. The other is the faculty of understanding, which brings the intuitions produced by the sensibility under pure concepts. The so-called forms of intuition (due to the faculty of sensibility) and the pure categories (due to the faculty of understanding) are the means by which we organise the raw sensory input of a world that cannot be known in itself into a coherent

[2] Berkeley's differentiated philosophy of mathematics may also be found in his later *Alciphron* (1732).

[3] Today, those who accept the analytic-synthetic distinction, questioned by Quine in the 20th century (see Volume V), would characterise analytic truths as true in virtue of meaning, and synthetic truths as true but not solely in virtue of meaning.

law-governed spatiotemporal reality. The exercise of this capacity in the absence of sensory data is the domain of pure mathematics, and explains how we can know synthetic propositions a priori. A notable consequence of Kant's conviction that space is a construct of the human sensibility is that space *must* conform to the laws of Euclidean geometry. This implication of Kant's position was confuted by later developments in mathematics and physics, as Albert Einstein's lecture at the end of this volume makes clear. Readers new to Kant's philosophy of mathematics may wish to start with the excerpt from Kant's 1783 *Prolegomena*, which introduces Kant's thinking about mathematics, before moving on to the passages from the *Critique of Pure Reason*.

John Stuart Mill's *A System of Logic*, published in 1843, is a high-water mark in the history of empiricism. In a nutshell, and to simplify somewhat, Mill construes mathematics as what we today would call natural science. Mathematical knowledge is a form of empirical or scientific knowledge, only more general. Indeed, as Mill sees it, all so-called deductive sciences are in fact inductive. As for the objects of mathematics, they are ultimately physical; for example, the proposition that $1 + 2 = 3$ expresses the physical fact that conjoining a 1-membered collection and a 2-membered collection yields a 3-membered collection (of physical objects).

Dedekind's 'On the nature and meaning of numbers' answers a natural question that had arisen towards the end of the 19th century. Over the course of the century, analysis had been 'arithmetised', and in particular real numbers (rational numbers and irrational numbers such as $\sqrt{2}$ or $\pi$) had been shown to be constructible from natural numbers (0,1,2,3. . .) by set-theoretic means. So the question becomes this: how does one account for natural numbers themselves? Are they fundamental, or are they also constructed somehow? Dedekind in this essay provides his own answer to this question, and set outs the first axiomatisation of arithmetic, about 2,200 years after Euclid's axiomatisation of geometry. This axiomatisation is usually erroneously named after Peano, who elaborated it (fully acknowledging Dedekind as its source).

We end Volume I with Einstein's 1921 lecture. This lecture is less an original piece of philosophy than a summary of the new way of thinking about geometry in particular and mathematics in general that had emerged at the start of the 20th century. According to this new, and today still orthodox, conception, pure geometry is concerned with mathematical spaces (typically set-theoretic structures of a certain sort). In geometry we deduce consequences from stipulated axioms. In particular, any collection of objects constitutes a geometry so long as it satisfies the axioms in question. As far as pure geometry is concerned, Euclidean space is thus just one space among many: it satisfies the axioms of some geometries, but not those of others. As a consequence, there is no sense in asking which is the 'correct' pure geometry. In contrast, applied or physical geometry is about *our* actual space. Its methods are the methods of physics, and the geometric structure of our spacetime is a question inseparable from other physical ones, addressed by theories of gravitation and electromagnetism. In answer to this question, Einstein points out that physical space is most likely curved and therefore non-Euclidean.

### *Volume II*

On display in the second volume are some of the jewels of the golden age of the philosophy of mathematics. We divide this era into four broad movements: logicism, intuitionism, formalism and logical empiricism.

The volume begins with Frege, often regarded as the first 'modern' philosopher of mathematics. Frege was also famously a logicist about arithmetic and analysis. Logicism is the thesis that mathematics is logic. Logicism about a particular branch of mathematics takes its objects, if any, to be logical objects; mathematical knowledge (of that branch) to be logical and its truths (falsehoods) to be logical truths (falsehoods). In his *Foundations of Arithmetic* of

1884, Frege tried to argue for his brand of logicism both positively and negatively: negatively by arguing that previous non-logicist accounts of arithmetic and analysis were flawed, and positively by arguing that arithmetic – and, more summarily, analysis – are logical. He sought to establish the conclusion that arithmetical truths are a certain kind of logical truth by demonstrating that (i) the axioms of arithmetic are logical, and (ii) inference steps preserve the property of being a logical truth.

Frege's official logicist strategy in the *Foundations of Arithmetic* is based on *Basic Law V*, a principle about extensions (his word for what we would now call sets) not stated as such in *Foundations of Arithmetic*, and which in modern notation would be formulated as:

$$\text{Ext}\,(F) = \text{Ext}\,(G) \leftrightarrow F \equiv G,$$

where '$F \equiv G$' means that every instance of an $F$ is an instance of a $G$ and vice versa. For example, the extension of the concept of a crow is the same as the extension of the concept of a raven if and only if any instance of a crow is an instance of a raven; since as a matter of fact crows and ravens don't share all their instances (the two concepts are not extensionally equivalent), the concepts have different extensions. This logicist strategy occupies sections 68–83 of *Foundations of Arithmetic*. In this book Frege says surprisingly little about extensions; his theory of extensions comes later, in *Basic Laws of Arithmetic* (Volume I: 1893; Volume II: 1903). The Fregean samples in Volume II present his positive philosophy of mathematics in *Foundations of Arithmetic*, and the key ideas, definitions and early proofs in *Basic Laws of Arithmetic*, using the recent first full English translation of the text. We conclude the Fregean sections with Bertrand Russell's letter to Frege, in which he informed Frege of the inconsistency of *Basic Law V*, followed by Frege's remarkable reply. This dramatic exchange speaks for itself and marks the passing of the logicist torch from Frege to Russell and A.N. Whitehead.

Russell and Whitehead circumvented the inconsistency of *Basic Law V* by formalising a ramified 'type theory', which employs propositional functions in place of Frege's extensions. Each object of the theory is assigned a logical type, and a propositional function can only take arguments from a lower type in the hierarchy. In particular, a function can never take arguments of the logical type to which it belongs; the implication is that constructions such as the set of all sets that are not members of themselves (the 'Russell Set') are inadmissible in the logic of *Principia Mathematica*. Despite the historical importance of *Principia Mathematica*, the theory of propositional functions is mathematically unwieldy, and has been for the most part eschewed by mathematicians in favour of set theory (see Volume III).

The last three articles in the logicist section are critical pieces. The first, by (Jules) Henri Poincaré, sometimes known under its alternative title, 'The last efforts of the logisticians', is a take-no-prisoners polemic targeted at logic's role in mathematics in general and logicism in particular. Poincaré develops a circularity objection against logicism. This argument has several strands, the most prominent of which are perhaps these: how can logic found arithmetic if recursive definition is required to state the logical principles and rules on which arithmetic is supposedly founded? And how can we can convince ourselves that a logicist (or any other) system is consistent without an argument that relies on the principle of mathematical induction, thereby rendering the epistemology of logicism reliant on that of arithmetic?

Moving forward in time, we include a late 20th-century article by Crispin Wright, the best-known contemporary neo-logicist. Strictly Fregean logicism is not viable, as we saw, since its theory of extensions is inconsistent. The publication of Crispin Wright's *Frege's Conception of Numbers as Objects* in 1983 led to a resurgence of interest in a broadly Fregean logicism based on *Hume's Principle*. *Hume's Principle* states that

$$\text{Number}(F) = \text{Number}(G) \leftrightarrow F \sim G,$$

where $F \sim G$ abbreviates the claim that there is a correspondence from the $F$s to the $G$s that is both one-one and onto, i.e. it associates each $F$ to a single $G$ in such a way that no two $F$s are associated with the same $G$ and every $G$ is associated with some $F$. Thus the number of crows equals the number of ravens if there is a one-one and onto correspondence between the crows and the ravens. In light of the fact that *Hume's Principle* is consistent and sufficient for deriving the basic principles of arithmetic in a second-order setting, neo-Fregeans have argued that it is the correct foundational axiom for logicism about arithmetic. Crispin Wright advocates this position in his 1997 article and summarises the (then) state of play. The 20th-century American logician and philosopher George Boolos[4] provides a penetrating critique of this revived version of Frege's logicism. Although a fair amount of ink has been spilled over neo-Fregeanism since the late 1990s, we have scarcely moved the opposing positions elegantly staked out by Wright and Boolos, at least with regard to the main philosophical points.

The next section presents readings by proponents of intuitionism, a radical philosophy of mathematics dreamt up in the first half of the 20th century by L.E.J. Brouwer, a brilliant Dutch mathematician also known for his contributions to topology. Brouwer's difficult prose is set alongside that of his disciple Arend Heyting. The third entry in this trio, by Michael Dummett, is a late 20th-century swansong to intuitionism. It represents Dummett's impressive, though ultimately unsuccessful, attempt to revive intuitionism by shifting its foundation from metaphysics to the philosophy of language. So what is intuitionism? In a succinct formulation, the intuitionist's credo is this:

> Mathematics deals with constructions. In particular, the truth of a statement consists in the existence of a construction establishing it, and the falsity of a statement consists in the existence of a construction establishing its absurdity. There is no logical reason to assume that for any statement there is a construction establishing its truth or absurdity. Hence neither bivalence nor the law of excluded middle is assumed. Constructions are not to be identified with proofs in some formal system. In addition, the completed infinite is an illusion: infinities are only potential. An infinite sequence is one that may be indefinitely iterated rather than one all of whose members are in some illusory sense predetermined.

This foundational philosophy leads intuitionists to reject not just standard mathematics but also standard logic. In particular, intuitionists do not accept the principle that either $A$ is the case or *not*-$A$ is the case (the law of excluded middle). Although intuitionism has not been adopted by the mathematical community, it marks an exciting moment in the philosophy of mathematics. For the first time in history a philosophy of mathematics motivated a systematic alternative to mainstream mathematics.

The next two articles are on formalism, usually regarded as the third of the 'big three' golden-era philosophies of mathematics alongside logicism and intuitionism. 'On the infinite' is emblematic of Hilbert's mature, instrumentalist and part-formalist philosophy, which gave rise to a mathematical research programme known as Hilbert's Programme. Hilbert divided mathematics into two: finitary (or real or contentual) and ideal (or infinitary) mathematics. The finitary portion consists of the basic parts of arithmetic; the rest of mathematics makes up the ideal portion. The finitary/infinitary distinction was intended by Hilbert to mirror the observation/theory distinction in natural science. Infinitary mathematics is an instrument for deriving

[4] Not to be confused with the 19th-century English logician, philosopher and mathematician George Boole.

finitary truths and has no intrinsic value of its own. Showing by finitarily acceptable methods that ideal mathematics is consistent—or, potentially more strongly, that it does not prove any false finitary statements—was the programme's goal. Hilbert's programme is generally thought to have been scuppered by Gödel's incompleteness theorems, proved in his famous 1931 article that opens Volume IV.[5] Gödel's second incompleteness theorem implies that finitary mathematics cannot even prove its own consistency, never mind its consistency with ideal mathematics appended. Bill Tait's 'Finitism' from 1981 further elucidates the nature of Hilbert's programme, brings to bear on it logical methods not available in Hilbert's time and imbues it with deeper philosophical clarity and precision.

Volume II ends with some selections from logical empiricists. A.J. Ayer's *Language, Truth and Logic* is a classic of popular philosophy. It more than makes up for what it lacks in originality – the book was the young Ayer's dispatches from philosophical Vienna – with its style and verve. In Chapter IV, Ayer sets himself the task of accounting for necessary truths, in particular 'the truths of formal logic and mathematics', from empiricist principles. Rejecting Mill's empiricist account and Kant's characterisation of mathematical truths as synthetic (both expounded in Volume I selections), Ayer proposes that mathematics is made up of analytic truths. According to Ayer, analytic truths are the product of linguistic convention, and so to deny a mathematical truth is to contradict oneself. It also follows that mathematical truths are devoid of factual content. Moritz Schlick makes a similar point in his entry, with more stress on the anti-Kantian implications of the theory and a greater focus on the nature of geometry. The final article is Rudolf Carnap's 'Empiricism, Semantics and Ontology', which arguably represents a more mature version of logical empiricism. Carnap in this article draws a now famous distinction between internal and external questions. To paraphrase Carnap, questions of the existence of certain entities *within* a framework are internal; questions concerning the existence of a system of entities *as a whole* are external questions. Thus to ask whether there is a prime number between $10^{10}$ and $10^{10} + 10$ is to pose an internal question, but to ask whether there are numbers at all—to ask that question in a way that does not admit a trivial answer such as 'Yes, of course, since 0 and 1 and 2 etc. exist'—is to pose an external question. Typically mathematicians ask internal questions, whereas philosophers are interested in external questions. According to Carnap, internal mathematical questions, if answerable at all, are answerable using the rules of the framework, i.e. mathematics. In contrast, external questions such as 'Do numbers exist?' are practical rather than theoretical. Fundamentally, what such an external question asks is whether we should adopt the framework of standard mathematics. Carnap's distinction between internal and external questions was later criticised by W.V. Quine in 'Two Dogmas of Empiricism' (reproduced in Volume V) and elsewhere.

## *Volume III*

The topic of the third volume is the foundations of mathematics. Set theory is widely acknowledged to be a foundation for mathematics, by mathematicians and philosophers of mathematics alike. But a foundation in what sense? The third volume presents answers to this question, examines the justification for the axioms of set theory, and also considers arguments for alternative foundations of mathematics.

We begin with Ernst Zermelo's 1908 article, on whose system today's standard version of set theory, Zermelo-Fraenkel-Choice (ZFC), is based. ZFC effectively consists of the axioms

[5] But see my 'Mathematical Instrumentalism, Gödel's Theorem and Inductive Evidence', *Studies in the History and Philosophy of Science* 42 (2011), pp. 140–9, for a re-evaluation of that conclusion from an instrumentalist perspective.

Zermelo introduced in this paper, plus a later addition by Fraenkel (the Axiom Scheme of Replacement) and a more precise understanding of what Zermelo meant by a 'definite property'. Although this work is not in the philosophy of mathematics proper, it earns its place in the volume as a cornerstone of contemporary metaphysics of mathematics.

Paul Benacerraf's 'Mathematical Truth' in a sense defines contemporary philosophy of mathematics. Benacerraf in this article presents a dilemma for any philosophy of mathematics. Anyone who takes mathematical language at face value and interprets it as akin to the rest of language faces an epistemological problem: how do we know mathematics? Alternatively, if one opts for a plausible epistemology of mathematics, then the semantics of mathematics – the interpretation of its claims – becomes problematic. The other piece by Benacerraf, the third entry in this volume, can be seen as a response to the second article, extracted from Quine's 1960 book *Word and Object*. Quine in this chapter develops a philosophical justification for taking the objects of mathematics to be sets. His celebrated pursuit of ontological economy – preferring to posit as few entities as possible to get the job done – finds crisp articulation in these pages. Benacerraf's response in 'What Numbers Could Not Be' is to take issue with the idea that numbers could be discovered to have been sets all along. Benacerraf's famous paper, which set the scene for the structuralist philosophies of mathematics of the past 50 years, consists of two arguments. The first is supposed to establish that numbers are not sets, the second that numbers are not objects. From the second conclusion, Benacerraf apparently infers a form of structuralism about arithmetic. Although the second argument is somewhat opaque, the first is clear. Take the number 2. Arithmetic can be interpreted in set theory in different ways. On the von Neumann interpretation, 2 is construed as {Ø, {Ø}}, the set consisting of the empty set and its singleton. On the Zermelo interpretation, 2 is construed as {{Ø}}, the singleton of the singleton of the empty set. But 2 cannot be both {Ø, {Ø}} and {{Ø}}, since these two sets are distinct. Hence, Benacerraf concludes, the number 2 is not a set. The argument generalises, apparently showing that one cannot reduce any part of mathematics to set theory. Although critics have generally not found the argument convincing,[6] they are nevertheless agreed that Benacerraf has set an important challenge for the idea that all of mathematics is set theory in disguise.

Chapter 2 of Penelope Maddy's *Realism in Mathematics* follows on neatly from Benacerraf's 'Mathematical Truth'. Maddy in these pages attempts to directly answer Benacerraf's challenge by arguing on the basis of neurological and developmental evidence that we can, in fact, perceive certain types of sets. The penultimate piece in the volume, by the Cambridge philosophers Alex Oliver and Timothy Smiley, picks up on Zermelo's remark in the first entry that, properly speaking, the empty set is not a set. Basing their arguments on ideas developed in their study of the plural idiom of natural language,[7] they give Zermelo's thought a run for its money and thereby propose a novel metaphysics of set theory.

Eliminative structuralism is Charles Parsons' label in 'The Structuralist View of Mathematical Objects' for the version of structuralism that construes mathematical theorems as universally quantified claims conditional on axioms. The basic idea is that instead of interpreting '1 + 2 = 3' as an assertion about numbers, one should interpret it (roughly) as 'in any structure instantiating the axioms of arithmetic, the function representing addition takes the second element (the first being the element representing zero) and the third element to the fourth element'. By the axioms of arithmetic what is usually meant is some version of Peano's

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[6] My own diagnosis of where it goes wrong can be found in 'Reducing Arithmetic to Set Theory', in Ø. Linnebo & O. Bueno (eds), *New Waves in Philosophy of Mathematics* (Palgrave Macmillan, 2009), pp. 35–55.

[7] See their *Plural Logic* (Oxford University Press, 2013).

axioms; the account is eliminative because it avoids reference to mathematical entities. The problem with this simple version of structuralism—perhaps the one intended by Benacerraf in the positive part of 'What Numbers Could Not Be'—is twofold. First, if there are no structures instantiating the axioms of arithmetic, then all arithmetical statements come out vacuously true; this is because a claim of the form 'For all *A*s it is the case that. . .' is true if there are no *A*s. Second, if such structures exist only contingently, arithmetical statements will only be contingently true. Yet arithmetic, it seems, is necessary.

The sixth, eighth and ninth entries in Volume III – by Hilary Putnam, Geoffrey Hellman and David Lewis respectively – represent different reactions to these problems. The Putnam and Hellman pieces should be read in conjunction, as the latter is a development of the former. Hellman's Putnam-inspired idea is to reformulate eliminative structuralism so that it interprets a claim such as '1 + 2 = 3' as 'necessarily, in any structure instantiating the axioms of arithmetic, the function representing addition takes the second element (the first being the element representing zero) and the third element to the fourth element'. The supplementary hypothesis underpinning the account is that it is possible for some structure to instantiate the axioms of arithmetic. Hellman's 1989 book, *Mathematics Without Numbers*, from which the introduction and first chapter are here reproduced, develops an account along these lines, generally known as 'modal-structuralism', in great detail and sophistication. This work can be seen as a high point of the philosophical unfolding of the slogan that mathematics is the study of abstract structure. A different fork in the structuralist road is taken by David Lewis in 'Mathematics is Megethology', 'megethology' meaning the theory of size. Lewis interprets mathematics structurally without a modal element, and avoids the vacuity and contingency problems by positing an appropriately large amount of necessary abstract objects. The article by Parsons and the second one by Hellman, the eighth and tenth entries, are authoritative surveys of the varieties of structuralism in contemporary philosophy of mathematics.

The final piece, by Øystein Linnebo and Richard Pettigrew, is *sui generis* in the context of this volume and anthology. The orthodox picture of mathematics that emerged in the middle of the 20th century is this: mathematics is about sets; ZFC set theory, or some extension-cum-refinement of it is the correct theory of sets and the rest of mathematics (e.g. group theory or number theory) is interpreted structurally within set theory. One challenge to this picture comes from category theory. Created by Samuel Eilenberg and Saunders Mac Lane in the 1940s, category theory unifies a good deal of mathematics in one fell swoop. By taking a bird's-eye view of the subject, it perspicuously describes conceptual connections and recurring patterns. According to some, category theory's importance also lies in the fact that it suggests an alternative picture to orthodox set-theoretic foundationalism. In their article, Linnebo and Pettigrew briefly survey some of the various category theories in the literature and present one – Lawvere's ETCS – with a philosophical audience in mind. Their article dissects the notion of a foundation for mathematics, presents the by-now standard arguments for and against category-theoretic foundations, and pinpoints the philosophical issues trenchantly. This final piece leaves us with a more sophisticated understanding of the pressures set theory faces as a foundation for mathematics.

### *Volume IV*

Mathematics is thought to differ from the empirical sciences (physics, chemistry and so on) in two important ways. As stressed in the readings in Volume III, the first difference is ontological: the objects of mathematics are standardly thought to be abstract, whereas the empirical sciences are typically concerned with concrete entities (as well as abstract ones). The second presumed difference is epistemological: whereas the empirical sciences employ inductive (as well as deductive) methods, mathematics employs only deductive methods.

Mathematics, in short, is based on proof. Volume IV concentrates on this apparently distinctive aspect of mathematics.

The volume begins with two articles by the Austrian-born logician Kurt Gödel. Although Gödel was not the founder of logic, nor even of modern logic (Frege, if any single person, deserves that title), his incompleteness results of 1931 are undoubtedly one of its highest achievements. The first entry in Volume IV is accordingly Gödel's original 1931 article, in which he proved his first incompleteness theorem and sketched the argument for the second. In a concise, fairly informal formulation that incorporates a further improvement by J. Barkley Rosser, the first incompleteness theorem states that in any consistent logical system whose set of theorems may be mechanically generated (i.e. generated using some algorithm) and which is strong enough to carry out basic mathematics, there is at least one proposition *A* expressible in the language of the system such that *A* is neither provable nor refutable in the system. In other words, to put it even more informally, any decent logical system for mathematics is incomplete. The second incompleteness theorem affirms (roughly) that any such system will also be unable to prove its own consistency. Prior to Gödel, mathematicians such as Leibniz and Hilbert had hoped that a single axiomatisation of mathematics might suffice to prove all mathematical truths and to refute all mathematical falsehoods. Their great hope had been the *decidability* of mathematics: find a single decision procedure – an algorithm or mechanical procedure of some sort – such that any given mathematical problem could be fed into it and, after a finite amount of time and with no further input, the procedure would terminate and return the answer Yes if the statement is true and No if the statement is false. One of the consequences of Gödel's first incompleteness theorem is that no such decision procedure exists for mathematics, at least given the Church-Turing Thesis, which equates the informal notion of a decision procedure with a precise mathematical one.

Gödel's 'What is Cantor's Continuum Problem?' can be read as his own reaction to the incompleteness results. Although his first theorem implies that completeness in mathematics cannot be attained within a single system, it does not exclude the possibility that any mathematical question of interest can be answered in some system or other. In fact, this is precisely what Gödel proposes in his 1964 article by arguing that there is a coherent way to extend the axioms of set theory so as to progressively increase their deductive power. It was Gödel's earnest hope that this large cardinals programme (as it has come to be known) would allow us to settle any mathematical question using some system in the hierarchy of progressively stronger set theories. By Cantor's Continuum Problem, by the way, Gödel means the Continuum Hypothesis, first conjectured by Georg Cantor, the founder of set theory, in the 1870s. It is a consequence of Cantor's development of set theory that infinities come in different sizes (known as alephs). The Continuum Hypothesis is the conjecture that the size of the real numbers is the next one up from the smallest infinite size (aleph-null), that of the natural numbers. It has been known since the 1960s that the Continuum Hypothesis is neither provable nor refutable in ZFC set theory.

Daniel Isaacson's 'Arithmetical Truth and Hidden Higher-Order Concepts' is another philosophical reaction to Gödel's first incompleteness theorem. Gödel's theorem implies that the standard predicate logic formalisation of arithmetic, known as Peano Arithmetic (PA),[8] is incomplete. It is noteworthy, however, that the statements in its language that PA is unable to prove or refute tend to be of a 'metamathematical' character.[9] The classic example is the Gödel sentence, intuitively interpretable as 'I am not provable in the system PA'. The Gödel sentence

[8] Erroneously, as noted in our discussion of Dedekind.

[9] A statement that is neither provable nor refutable in the system is known as an undecidable statement. The term derives from the – not always accurate – idea that provability in a system amounts to decidability.

is metamathematical rather than straightforwardly mathematical because it is a statement which, on this understanding, is about a system of arithmetic rather than about numbers themselves. Isaacson argues that though incomplete in the logician's sense, PA is complete in an epistemologically important sense: it proves all and only the truths of arithmetic that are directly arithmetical rather than metamathematical. As Isaacson puts it more precisely himself, it proves all and only 'those truths which can be perceived as true directly from the purely arithmetical content of a categorical conceptual analysis of the notion of natural number'. Along with other contributions to our anthology, his article exemplifies what the philosophy of mathematics can achieve when it marries philosophical sensitivity with a deep understanding of the relevant mathematics.

Euclid's *Elements* (c. 300 BC) came to be seen as an ideal of mathematical method: in mathematics we start with self-evident axioms, and via watertight deductive reasoning derive theorems. This ideal still holds sway today, with the difference that axioms are no longer required to be self-evident. Reflecting on the Euclidean paradigm, one might naturally wonder: where do the axioms come from? In particular if, as some of the pieces in Volume III argue or presuppose, set theory is the foundation of modern mathematics, where do its axioms come from? George Boolos in 'The Iterative Conception of Set' addresses precisely this question. Boolos argues that the iterative conception of set justifies most, but not all, the ZFC axioms. According to the iterative conception, sets are formed in stages. At stage zero, the set containing all previous elements, i.e. the null set, is formed. At any finite stage, any sets formed at earlier stages are collected into a set. After the finite stages come the infinite ones, corresponding to the transfinite ordinals (ordinals beyond the finite ones), at which further sets are formed in like fashion. Boolos' article is one of the first systematic explanations of precisely how the iterative conception underlies the axioms of contemporary set theory.

An even more sustained analysis of the provenance of the ZFC axioms may be found in Penelope Maddy's 'Believing the Axioms I'.[10] In this article Maddy goes through the various proposed justifications for the axioms of set theory with a fine-tooth comb. She draws an important distinction between *intrinsic* justification, according to which an axiom is evident or obviously flows from an intuitive conception of set, and *extrinsic* justification, whereby an axiom is justified in terms of its consequences. In her later 'Does $V = L$?', Maddy investigates why most set theorists reject the so-called Axiom of Constructibility (whose most concise statement is $V = L$, $V$ being the universe of sets and $L$ the universe of constructible sets). Despite its name, this principle is a candidate for axiomhood rather than an axiom proper. In her article, Maddy diagnoses the debate between a proponent of the Axiom of Constructibility (e.g. Gödel, for a brief period) and its many detractors as ultimately a choice between the methological maxims she calls Definabilism and Combinatorialism respectively. The two Maddy articles display methodology at its best and pioneered recent interest in the fine detail of mathematical justification.

A concern with the detail of mathematics as it is actually done is very much evident in Mic Detlefsen's 'Purity as an Ideal of Proof'. Roughly speaking, a mathematical argument is pure to the extent that it does not involve any extraneous ideas or method; for example, despite the theorem's name, by dint of taking a detour outside algebra the usual proofs of the Fundamental Theorem of Algebra are impure.[11] Detlefsen outlines some of the history of philosophical thinking about purity in mathematical argumentation and purity's significance.

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[10] The sequel (as well as part of the article in question) is concerned with the justification for proposed extensions to ZFC.

[11] The theorem states that every non-constant polynomial of one variable over the complex numbers has a complex root – in other words, the complex numbers are an algebraically

The inclusion of the two chapters on the Church–Turing thesis has a dual purpose. First, it serves as a brief introduction to the Church–Turing thesis, which equates an informal notion – that of an effectively or algorithmically computable function on the natural numbers – with a mathematically precise one. Of the many mathematically equivalent ways to state the thesis, perhaps the clearest is that a function is effectively computable if and only if it is computable by a Turing Machine. Precisely because it claims that a non-mathematical concept is coextensive with a mathematical concept, the Church–Turing thesis is usually thought incapable of mathematical proof. In the second of the excerpted chapters by Peter Smith, we see him mounting an argument to the effect that the Church–Turing thesis is actually susceptible to proof.

Mathematics is unique among all the areas of human inquiry in that it relies almost exclusively on proof for its justification. This prompts several questions: why is there such an intimate relationship between mathematics and deductive reasoning? What is so special about proof — what are its epistemological virtues? The works by Imre Lakatos, Don Fallis and the present author all address these questions. Lakatos's classic 1976 article contrasts a Euclidean mathematical theory with a 'quasi-empirical' theory. As sketched earlier, a Euclidean theory is top down: truth flows from the axioms down to the theorems. A quasi-empirical theory, in contrast, is (at least in part) bottom up: truth flows from some particular theorems up to the axioms. The axioms of a quasi-empirical theory are thus justified extrinsically, at least in part. The main contention of Lakatos's article is as simple as it is striking: mathematics is quasi-empirical rather than Euclidean.

Don Fallis's contribution compares one particular probabilistic method, which he calls probabilistic DNA proof, with generally accepted mathematical methods such as deductive proof. Fallis argues that there is no epistemically important difference between mathematicians' orthodox methods and probabilistic DNA proof. This radical conclusion challenges part of mathematics' self-image; if correct, it shows that some of the methodological distinctions drawn by mathematicians are epistemically unmotivated. The last paper, by the present author, is very much in the same vein. Mathematicians never claim to know a proposition unless they think that they possess a proof (or proof sketch) of it. For all their confidence in the truth of the Riemann Hypothesis,[12] they maintain that, strictly speaking, the hypothesis will become known only until such time as someone has proved it. My paper marshals arguments against this strict conception of mathematical knowledge. It denies that knowledge of mathematics must be deductive, thereby chipping away at the epistemological dimension of the presumed divide between empirical science and mathematics.

### *Volume V*

The fifth and final volume consists of papers concerned with the lessons the philosophy of mathematics should conclude from the fact that mathematics is applied in science. More

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closed field. It was first proved in the early 19th century. The proofs of the theorem usually take a detour via analysis.

[12] The Riemann Hypothesis states that the nontrivial zeros of the zeta function all lie on the line with real part ½, known as the critical line. The Riemann zeta function $\zeta(s)$ is the analytic continuation of $\sum_{n=1}^{\infty} \frac{1}{n^s}$ to $\mathbb{C}\backslash\{1\}$; its trivial zeros are $-2, -4, -6, \ldots$ . The conjecture was first proposed by Bernhard Riemann in 1859. Although the Riemann Hypothesis remains unproven to this day, it is now known that at least the first $1.5 \times 10^9$ non-trivial zeros of the zeta function satisfy the hypothesis, and that millions of later zeros all lie on the critical line. The importance of the conjecture for core parts of mathematics, in particular number theory, can hardly be overstated.

precisely, it is concerned with the indispensability argument, famously expounded by the American philosophers W.V. Quine and Hilary Putnam in the second half of the 20th century. A classic version of the argument takes something like this form: we should believe in the existence of entities indispensably invoked in successful scientific theories; mathematical objects (numbers, functions, sets, etc.) are indispensable in this way; therefore we should believe in the existence of mathematical objects. This is roughly the form the argument takes in Putnam's *Philosophy of Logic*, here reprinted.[13] A version of the argument may also be educed from Quine's 'Two Dogmas of Empiricism', included in this collection as much for its implicit statement of the indispensability argument as for its general philosophical importance.

One response to the indispensability argument is to deny its second premiss, that mathematical objects are indispensable to successful science. Many nominalists in the philosophy of mathematics, who reject the existence of abstract entities, take this route.[14] Interestingly, Quine himself had been a nominalist prior to writing 'Two Dogmas of Empiricism'. Indeed, in 'Steps Towards A Constructive Nominalism', he and Nelson Goodman had attempted to rewrite the mathematical parts of science, assuming that if mathematics contains reference to objects, then these objects must be abstract. Yet try as he might, Quine found that he could not get the nominalisation programme to work for non-elementary parts of science. His apparent failure to nominalise science convinced him that it could not be done at all. As a result, he embraced the second premiss of the indispensability argument and recanted nominalism. The nominalist baton was picked up three decades later by Hartry Field, who reckoned that Quine had given up too soon. With a little more perseverance, Field maintained, one can nominalise all of science. The selections from Field take the next steps following those of Quine and Goodman along the hard path to nominalism. On the resulting conception of pure mathematics, mathematical sentences we typically take to be true are not in fact literally true. As Field himself puts it, '2 + 2 = 4' is not literally true, but only true according to the fiction of mathematics, in much the same way in which 'Oliver Twist lived in London' is not literally true but true only in Dickens's novel. In contrast, a statement such as 'Charles Dickens lived in Doughty Street in London' is literally true.

George Boolos in 'Nominalist Platonism' is not directly concerned with the indispensability argument. Yet as his title hints, his aim is also to nominalise a part of discourse, or rather to explain why a part of discourse hitherto construed as platonist is not in fact committed to sets. The focus of Boolos's interest is plural quantification, which he argues should be admitted as a primitive logical notion. Take the sentence, 'There are some Frenchmen such that Arnaud is one of them'. We could paraphrase it as 'Arnaud is a Frenchman' and formalise the latter in standard, non-plural, logic as *Fa*. But what if we would like to formalise the sentence with as little paraphrasis as possible? A traditional option has been to resort to sets and read the sentence as 'There is a set of Frenchmen of which Arnaud is a member'. This interpretation, observe, commits us to the existence of a particular set. A pluralist proposal along the lines Boolos proposes would instead lightly regiment the sentence as 'There are some things such that they are Frenchmen and Arnaud is one of these things' and then formalise it directly in plural logic. A highlight of the paper is Boolos's use of plural quantification to interpret monadic second-order logic.

The next two entries, by Elliott Sober and Penelope Maddy respectively, offer critiques of the indispensability argument. Chapters 4 to 6 of Mark Colyvan's *The Indispensability of Mathematics* respond to these critiques from the perspective of an indispensabilist platonist. In

[13] The subtle differences between the two writers' formulations will not matter here.

[14] Nominalism in the philosophy of mathematics should not be confused with nominalism in metaphysics, which is usually understood as the rejection of properties or universals. The two forms of nominalism are independent in principle and in practice.

his article, Sober argues that the indispensability of mathematics to natural science is precisely why the success of scientific theories does *not* justify taking mathematical truths to be literally true. More precisely, Sober emphasises that the empirical success of a theory is always relative to (actual or imagined) alternatives – for example, relativistic as opposed to classical physics. If mathematics is truly indispensable to scientific theory, then all the alternatives will quantify over mathematical objects, and therefore the existence of such objects is no more confirmed by our most successful scientific theories than it is by our least.

The extract from Penelope Maddy's *Naturalism in Mathematics* is built around her case study of modern atomic theory. From about 1860 onwards the atomic theory in chemistry had proved itself so successful that it had become indispensable to science, Maddy claims. Yet in spite of all the evidence in its favour it was still viewed with suspicion, because the evidence for the atomic theory, though extensive, remained fairly indirect. In 1905 Einstein produced a mathematical analysis of Brownian motion. This spurred the French physicist Jean Baptiste Perrin to perform experiments to determine the mass and dimensions of atoms. Perrin's experiments from 1908 to 1913 produced direct evidence for the existence of atoms. The experiments' success led to the widespread acceptance of atomism. Maddy concludes from this episode that the indispensability and empirical success of a scientific theory are insufficient for scientists to literally believe it. She then carries over the moral to mathematics: just as the scientific indispensability of late 19th-century atomic theory was insufficient reason to regard it as true at the time, so we should not take the indispensability of mathematics as sufficient grounds for its truth.

Indispensability platonists other than Colyvan have responded to Field, Sober and Maddy. In their *A Subject with No Object*, John Burgess and Gideon Rosen offer an assessment and critique of contemporary nominalism in the philosophy of mathematics. Their excellent book has both a philosophical and a more technical dimension. In our selection we focus on the more philosophical side of things. In these sections Burgess and Rosen offer strong arguments against the philosophical theories of knowledge, justification and reference that typically motivate nominalism. They roll back the platonist-nominalist debate to metaphilosophical territory, showing how and why it turns on the norms governing philosophical theory choice. Alan Baker, in 'Are there Genuine Mathematical Explanations of Physical Phenomena?', opens another front in the platonist defence against the nominalist offensive. A popular response to the indispensability argument by post-Fieldian nominalists has been to argue that the mathematics used in scientific explanations of natural phenomena, though perhaps indispensable in the logician's sense, is not truly explanatory. A slightly revised version of the indispensability argument then insists that we should only believe the parts of scientific theories genuinely responsible for successful scientific explanation. Applying this argument, Baker contends that there *are* genuine mathematical explanations in science. His clever case study of periodical cicadas has attracted much attention, and Baker himself has become a key figure in the second wave of the indispensability debate.

The last entry strives to draw a fundamental distinction. Suppose one agrees with Quine and Putnam that the applications of mathematics in science are sufficient reason to believe that mathematics is true. Does it follow that they are sufficient reason to believe in the existence of mathematical objects? In this final contribution, I suggest that the answer may well be no. By bringing to light this nuance in the indispensability argument, I hope to point the way to future research on the topic.

A.C. Paseau (editor)