Studies in History and Philosophy of Science 40 (2009) 281–289

Contents lists available at ScienceDirect

Studies in History and Philosophy of Science

journal homepage: www.elsevier.com/locate/shpsa

Signs, toy models, and the a priori: from Helmholtz to Wittgenstein

Lydia Patton

Department of Philosophy, Virginia Tech, 229 Major Williams Hall (0126), Blacksburg, VA 4061, USA

ARTICLE INFO

Article history:
Received 12 February 2008
Received in revised form 11 December 2008

Keywords:
A priori
Empiricism
Ludwig Wittgenstein
Rudolf Carnap
Hermann von Helmholtz
Logic

ABSTRACT

The Marburg neo-Kantians argue that Hermann von Helmholtz's empiricist account of the a priori does not account for certain knowledge, since it is based on a psychological phenomenon, trust in the regularities of nature. They argue that Helmholtz's account raises the 'problem of validity' (*Gültigkeitsproblem*): how to establish a warranted claim that observed regularities are based on actual relations. I reconstruct Heinrich Hertz's and Ludwig Wittgenstein's *Bild* theoretic answer to the problem of validity: that scientists and philosophers can depict the necessary a priori constraints on states of affairs in a given system, and can establish whether these relations are actual relations in nature. The analysis of necessity within a system is a lasting contribution of the *Bild* theory. However, Hertz and Wittgenstein argue that the logical and mathematical sentences of a *Bild* are rules, tools for constructing relations, and the rules themselves are meaningless outside the theory. Carnap revises the argument for validity by attempting to give semantic rules for translation between frameworks. Russell and Quine object that pragmatics better accounts for the role of a priori reasoning in translating between frameworks. The conclusion of the tale, then, is a partial vindication of Helmholtz's original account.

1. Introduction

In what follows, I will evaluate the influence of Helmholtz's sign theory, which pairs an empiricist account of the origin of geometry in experience with a Humean account of a priori principles as based on a habitual belief in the regularity of experience. In response to Helmholtz, Hermann Cohen and Ernst Cassirer of the Marburg neo-Kantian school argue that the sign theory does not answer the problem of the validity of a priori principles as warrants for scientific claims. I will present the epistemological theory that Wittgenstein and Hertz developed to interpret Helmholtz's use of signs in response to the problem of validity—the *Bild* or picture theory.[1] I will argue that the *Bild* theory gives significant support to the argument that the a priori has two parts: the specification of mathematical and logical possibility, and the construction of experiments to test these possibilities.

As Howard (1996), Heidelberger (1998), and Leroux (2001) have argued, Hertz's *Bild* theory is an ancestor of Carnap's formal semantics (Howard), of Wittgenstein's account of the sentence–world relation (Heidelberger), and of the semantic approach to the philosophy of science of Sneed, van Fraassen, and Suppe (Leroux). One central result of the *Bild* tradition endures, the definition of necessity in terms of the relation of a given theory to its models. However, as Quine, Duhem, and Schlick argue, the validity and necessity of a priori mathematical inferences is defined only within a given model, and is not independent of the physical theory in which the inference is made. Early on, Carnap attempted to make a semantic argument that all such differences are merely linguistic, and that each theory differs only in its manner of expression. Carnap tried to make such an argument by giving translation rules, first syntactic, and then, beginning in 1938, semantic. But Quine and Russell observe that a semantic account is not necessary, and that it involves Carnap in difficulties. Quine and Russell argue that Carnap's semantic account should be replaced with a pragmatic one, in terms of the expectations that rational agents have given certain commitments, and the commitments that such agents are willing and not willing to give up. Thus, by the 1950s, there was a return to Helmholtz.

E-mail address: critique@vt.edu

[1] *Bild* is translated variously in the literature as 'picture', 'image', or 'model'. A *Bild* should not be understood as a visual or mental image or picture, however. I tend to use the German word, to underscore the fact that Hertz and Wittgenstein use the same terminology.

0039-3681/$ - see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.shpsa.2009.07.004

2. The problem of validity

The problem of validity (*Gültigkeitsproblem*) was raised by Hermann Lotze, Hermann Cohen, Alois Riehl, Bruno Bauch, and Heinrich Rickert[2] against naturalist accounts of the a priori. There were significant differences in the philosophical reasons for raising the problem of validity: Bauch did so for logicist reasons, Cohen for neo-Kantian ones.[3] However, the common ground is that all these philosophers argue that naturalism alone does not establish the justification for the assumption, used in making inductive inferences, that objects and events experienced will continue to display the same observed regularities. The problem is a variant of the problem of induction, since establishing the validity of a principle of inference, based on regularity, involves making an inference beyond objects as experienced.[4]

As Robert DiSalle notes, Helmholtz's own answer to the problem of validity was to argue that the a priori is 'something that Kant never intended it to be: a species of psychological adaptation to regularities in the external world'.[5] Helmholtz refers to the principle on which he bases his own inferences as the 'transcendental law of causality', which he interprets as a belief that observed regularities are stable, for example, that like causes are followed by like effects. Helmholtz thought that some elements of geometry were assumptions necessary to making inductive inferences—in particular, the assumption that rigid bodies that are observed to move without deformation will continue to do so. Helmholtz observes that without that assumption, it does not seem possible to construct a system of geometry that accounts for our experience, and so the assumption is necessary to constructing a geometrical system. But that does not mean, for Helmholtz, that we have a logical warrant to accept the assumption as a necessary truth.

Helmholtz's account raises the question of whether valid principles of knowledge are valid because they are presuppositions necessary to make sense of our actual knowledge, or because they are rationally justified a priori.[6] Helmholtz himself did not see the second option as a possibility. However, even if it may not be possible to give a universal rationalist justification for a priori principles, it is possible to ask why the principles of a particular inference are valid for *that* inference, and then to extend it to like cases. The development of the picture or *Bild* theory, a precursor of model theory that influenced Einstein, Hilbert, and Schrödinger, made answering this question more urgent, because it allowed for validity to be defined within a given picture or model. This approach presages the method in formal semantics of treating valid inferences within a theory as valid relative to the class of models the theory implicitly specifies.

Two of the most influential *Bild* theorists, Helmholtz's student Heinrich Hertz and (in his early work) Ludwig Wittgenstein, were preoccupied with the problem of validity. In what follows, I will sketch the history of Hertz's and Wittgenstein's answers to the question of the validity of a priori cognition, both of which go well beyond Helmholtz's psychological account, but which make use of Helmholtz's empiricist sign theory. First, in this section, I will sketch Helmholtz's argument that, while spatial measurement is derived from experience, determining spatial and causal relations depends on postulating lawlike regularities of experience. Helmholtz argues that such determinations require trust in lawlike postulates as principles of organization of a system of signs. Second, I will show how Helmholtz's use of these postulates as rules in his fluid mechanics contributed to the methodology of Hertz's and Wittgenstein's picture or *Bild* theory, which gives a preliminary answer to the question of the validity of a priori demonstrations using signs. I will show how the *Bild* theory supports the claim that it is rational to believe some a priori claims about constraints on the states of a system, even claims based on counterfactual underlying stipulations. Finally, I will assess the consequences for the *Bild* theory of Russell's and Quine's objection that our warrant for belief in these claims is pragmatic, not semantic.

The most potent challenge to Kant's notion of the apriority of space and causality in the mid-nineteenth century comes from Helmholtz, who argued that geometry is empirical since it is based on spatial measurement, and causality is based on trust in the regularities of nature. In the case of geometry, as Helmholtz argues in 'On the origin and significance of the axioms of geometry':

> any comparative estimation of magnitudes, or measurement of spatial relationships, starts from a presupposition about the physical behavior of certain bodies, whether of our own body or of applied measuring instruments. This presupposition may incidentally have the highest degree of probability and be in the best agreement with all physical circumstances otherwise known to us, but it still goes beyond the domain of pure spatial intuitions.[7]

Helmholtz's argument seemed to rule out an a priori status for geometry. However, as DiSalle has observed, Helmholtz's view presents an interesting dichotomy:

> while Helmholtz certainly understood geometry to be an empirical science in some sense, he also recognized its status as a formal deductive structure that stands independently of its intuitive or sensory content.[8]

For Helmholtz, spatial relations are constructed partly through inference from experience and partly in response to expectations about the regularity of experience.

In 'The facts in perception', perhaps his best known epistemological essay, Helmholtz argues that spatial relations cannot be derived immediately from sensation. Sensation presents us with signs of the sensations themselves, which do not necessarily resemble the objects of those sensations. For instance, the analysis of stereoscopic vision shows that what appears to us as a single image is in fact two retinal images resolved into one. Perspective can distort size in our everyday experience: when you are looking at the moon in the night sky, you can cover it with your finger. According to Helmholtz's explanation of the physiology of perception, such properties as configuration and size are not derived directly from sensation. Instead, perception (*Wahrnehmung*) goes beyond sensation to present us with objects constructed within a system of signs, according to an inductive postulate of lawful regularity:

> By moving the touching finger along the objects, one comes to know the sequence in which their impressions offer themselves. This sequence shows itself to be independent of whether one touches with one finger or another. It is moreover not a uniquely determined sequence, whose elements one must

2 Amongst many others.

3 For a book length discussion of the problem of validity and its history, see Liebert (1914).

4 In his *Logik*, Hermann Lotze has a fascinating discussion of this as the problem of how to establish the validity of Platonic ideas without trying to establish their being or independent existence. See Liebert (1914), pp. 204 ff.

5 DiSalle (1993), pp. 505–506.

6 This question would become a key point of disagreement between Helmholtz and Poincaré. See Cassirer (1978 [1946]), pp. 44 ff.

7 Helmholtz (1977a [1870]), p. 19.

8 DiSalle (1993), pp. 499–500.

always go through, forwards or backwards, in the same order in order to get from one to another; thus it is not a linear sequence, but a surfacelike 'one beside another', or in Riemann's terminology a second-order manifold. That all this is so is easily seen.[9]

Helmholtz argues that we base our inference that an object of experience has certain spatial properties on the regular sequences or orderings of our impressions. For Helmholtz, it is reasonable to infer from these regularities that an object in fact has the spatial properties it appears to have.

Helmholtz argues that we construct spatial determinations by learning to interpret the signs that correspond to sensations from our nerve endings. Helmholtz observes, 'to the sensations from spatially distinct nerve endings correspond various determinate *Localzeichen* [literally "place signs"], whose spatial meaning is learned by us'.[10] Our sensations give us signs, not direct copies of their objects. These signs may have as little resemblance to their objects as the written name 'Winston Churchill' has to Churchill in the flesh.[11] Perceptual space is a general *Localzeichen* that relates sensations to each other.[12] According to Helmholtz's explanation of the physiology of perception, even such relations as separation in space and relative spatial position are not drawn directly from experience, but are constructed in the sign system, which presupposes a postulate of lawlike regularity.[13] That our experience conforms to law is the warrant that it is experience of actual phenomena: 'The lawlike is therefore the essential presupposition for the character of the actual'.[14] The general law of causality is for Helmholtz a 'transcendental' law, an a priori condition for constructing any theory that corresponds to actual objects, that is, objects that can be present in the sign system.[15] This law, as is suggested by the word 'transcendental', cannot be derived from experience.[16] It is based on our 'trust' (*Vertrauen*) that like causes will be followed by like effects, and that observed regularities will continue to obtain.[17]

Helmholtz argues that we must trust to the 'lawlikeness of everything that happens' to construct any theory of nature.[18] Measuring the spatial properties of an object will consist of making judgments of the congruence between the measuring standard and the object under investigation, by placing a meter stick along the surface of a table, for instance. Such judgments of congruence depend on the assumption of the 'lawlikeness of everything that happens'. As I am moving the meter stick along the surface of the table, the meter stick should not shrink in comparison to the table, or the measurements will not be valid. Helmholtz argues that since we can represent causal relations directly in a geometrical system, we have access to 'the lawlike order in the realm of the actual', though 'only as portrayed in the sign system of our sense impressions'.[19] The system requires a postulate that regular or 'lawful' relations obtain. If my meter stick were to shrink and to extend randomly with respect to the table as I measure the table, I would not be able to arrive at a stable value for the size of the table.

Helmholtz argues that if the philosopher or scientist trusts to the postulate that observed regularities are regularities in nature, most of the content previously annexed to the a priori turns out to originate in experience. Helmholtz replaces Kant's pure intuitions of space and time and a priori pure principles with trust (*Vertrauen*) that observed regularities will continue to obtain. The content of judgments based on spatiotemporal concepts and causal laws is constructed entirely from features of observed phenomena within the sign system. The a priori thus shrinks to our trust or belief that the regularity of experience to which we adapt our cognition maps on to what Helmholtz calls the 'actual' (*wirklich*) in nature, that is, 'that which lies behind the change of appearances and acts upon us'.[20]

The Marburg school of neo-Kantianism reacted to Helmholtz's research by defending a limited notion of Kant's a priori. Hermann Cohen, the founder of the school, began his career in the 1870s by responding to debates over whether Kant's a priori could be naturalized. In an appreciation of Cohen's work in 1912, Ernst Cassirer argued that the value of Cohen's philosophy is in posing the question of the validity of the principles of judgment that Helmholtz uses, for example the principles of organization of the sign system. Cassirer remarks that Helmholtz, despite himself, gave a naturalist interpretation of the principles for constructing objects in the sign system:

> Even when it was most prevalent, 'naturalism' as a *metaphysical* view never achieved unlimited dominance. In the circles of speculative philosophy, Schopenhauer's idealist doctrine was opposed to it, and in research circles, most notably, Helmholtz's epistemological research (which was, again, linked deliberately to Kant) was opposed to it. But one can see the power exercised by the methodology of naturalism even in these conflicts, even where one thought the real content of its worldview had been surmounted. True, from his metaphys-

[9] Helmholtz (1977b [1878]), p. 127.

[10] Helmholtz (1968 [1869]), p. 57. Here, Helmholtz cites Hermann Lotze as the source for his notion of *Localzeichen*.

[11] Compare Helmholtz (1903), pp. 41 ff.

[12] Helmholtz's empiricist sign theory was influenced by Mill's theory of induction. However, Helmholtz revises Mill's view significantly. For a detailed account of Mill's influence on Helmholtz, see Hatfield (1991), pp. 200 ff.

[13] It may be profitable to compare Helmholtz's theory to Ruth Millikan's modern sign theory. In 'Biosemantics', Millikan argues, 'If it really is the function of an inner representation to indicate its represented, clearly it is not just a natural sign, a sign that you or I looking on might interpret. It must be one that functions as a sign or representation *for the system itself*' (Millikan, 1989, p. 284). Millikan contends that the function of a 'sign' (an aspect of a representation) is determined by its role in a '*system* of representation' (ibid., p. 287). This account makes a useful distinction between signs' role in constructing representations in response to stimuli and their function as part of a physical module that is an inherited adaptation. Millikan argues for a naturalized account of how representations arise, but argues that in the process of representing, innate dispositions that have been selected for are supplemented by abilities developed while learning to represent phenomena. Helmholtz's view is based as well on this cooperation between innate and learned dispositions.

[14] Helmholtz (1977b [1878]), p. 140.

[15] Ibid., p. 142.

[16] Helmholtz does not use the word 'transcendental' in Kant's sense, that cause and effect is a pure principle of the understanding. By calling causality 'transcendental' Helmholtz means that causal relations cannot be represented directly as empirical relations between objects or events as sensed, but only as ordering principles of the 'sign system' with which we interpret our sensations and which makes experience of causal relations, and the spatial relations described above, possible. Thus, Helmholtz argues, we can acquire knowledge of the lawful order of our representations without arguing that the laws are pure principles with a priori content. 'I need not explain to you that it is a *contradictio in adjecto* to want to represent the real (*das Reele*) or Kant's *Ding an sich* in positive terms, without absorbing it into the form of our manner of representation. This is often discussed. What we can attain, however, is an acquaintance with the lawlike order in the realm of the actual, admittedly only as portrayed in the sign system of our sense impressions' (ibid., p. 141).

[17] 'Every inductive inference is based on trusting that an item of lawlike behaviour, which has been observed up to now, will also prove true in all cases which have not yet come under observation. This is a trust in the lawlikeness of everything that happens' (ibid., p. 142).

[18] Ibid., p. 142.

[19] Ibid., p. 141.

[20] Ibid., p. 140. For an explanation of what Helmholtz means by the 'actual in nature', see Heidelberger (1998), pp. 10 ff.

> ical heights Schopenhauer looked down genteelly on the 'men of the crucible and of the alembic', but nonetheless in his epistemology he used, utterly naïvely and without critical scrutiny, the *language* that natural science, and in particular physiology, had constructed. In fact, *Helmholtz* made this language incomparably sharper and more precise; but even he used it far beyond the limits within which it is valid in a strict sense, and within which alone it possesses a real meaning beyond the metaphorical. The entire doctrine of the a priori appears from now on as a mere extension of a certain individual result of natural science.[21]

The 'individual result' to which Cassirer refers is Helmholtz's supervisor Johannes Müller's argument in physiology, that to each sensation corresponds a specific 'sense energy', or configuration of a nerve ending, that is a sign of the actual object. As we have seen, Helmholtz gives this view an epistemological backing by means of the theory of signs (*Zeichentheorie*). In the Marburg neo-Kantian tradition Cassirer was describing, and to which he belongs, the 'puzzle of cognition' is understood as the question of how a priori cognitions can be valid: the *Gültigkeitsproblem*, as described above. Cassirer argues that Helmholtz uses the language of physiology beyond its valid domain of application. After all, Cassirer observes, demonstrating that a principle of organization of a sign system is the actual principle that a given physiological theory uses to account for our actual perceptions does not answer the epistemological question of the validity of that cognition. How do we prove that the ordering principles of the sign system are the necessary relations between phenomena or events?

> Thus, all these efforts effectively take on the color of the very systems they are fighting. They search in vain to analyze the whole critically, meanwhile taking one element of the cognition of nature as fixed. The a priori truths, understood in terms of 'type classification', become a particular class of psychophysical 'actualities', and thus inevitably are classified under, and subordinated to, the conditions for cognition of actuality, rather than being able to ground them and analyze them independently. Likewise, whether phenomenal actuality is interpreted as 'a product of the brain' or, in seemingly refined usage, as 'a product of representation', the mere concept of a 'product' still begs the entire question from the standpoint of epistemology.[22]

It is odd to hear a neo-Kantian speak of analysing the a priori independently of its role as one of the 'conditions for cognition'. However, Cassirer is not arguing that the a priori exists independently of our knowledge, that is, he is not arguing for mathematical or logical Platonism. Instead, Cassirer argues that if a philosopher can analyse a mathematical or logical rule only as it helps to achieve a *particular* result, in a particular, given domain, that philosopher cannot answer the epistemological questions: whether that rule is a necessary condition for that result, and whether it is valid to apply to other domains.

3. Signs and the a priori

In the above presentation of the problem of validity, I have emphasized Helmholtz's psychological argument, what Cassirer calls his 'naturalism', about the justification of the principles of organization of the sign system. However, in his theory of fluid dynamics, Helmholtz uses a priori reasoning about the principles of the sign system to resolve a conflict at the level of mathematical explanation.[23] This practical result presages the later *Bild* theoretic approach to the problem of validity: the argument that a priori reasoning can yield meaningful claims about the relationship between the logical and mathematical rules of a *Bild* and the valid results or theorems that can be proven within that system. However, this approach does not give logic and mathematics enough independence to satisfy the Marburg school, as we will see later.

In fluid dynamics, Helmholtz was faced with a set of mathematical equations based on inconsistent a priori assumptions. Euler's equations require fluids to be continuous, while Lagrange's equations depended on the hypothesis that fluids are made up of discrete particles.[24] To solve the problem, Helmholtz introduced concepts that are in use still. Assuming that a given fluid is free of viscosity, and perfectly continuous, Helmholtz showed how its motion could be determined precisely using two conceptual notions. The first is the notion of the 'vorticity' of a fluid, or its average rotation within a region. The second is a 'vortex line', which is the tangent to the vorticity of a fluid at a single point. Using these concepts, Helmholtz was able to formulate a single set of laws of motion for non-viscous fluids, and to prove that all the results derivable from Euler's and Lagrange's systems can be derived from his laws as well.[25] Helmholtz's laws of motion are still in use, because they describe fluid motion correctly. However, Helmholtz's laws require fluids to be perfect continua. Since Helmholtz's laws are still in use, it is an open secret that, while the classical laws of fluid motion require fluids to be perfect continua, our best empirical evidence reveals that fluids are granular, that is, are made up of discrete particles.[26]

Helmholtz did not introduce these concepts to oppose atomic theory—in fact, he famously supported that theory.[27] But for Helmholtz concepts such as vortex lines are manipulations of signs, and signs do not have to mimic the properties of real objects. Reasoning using signs is a tool to investigate objects, but signs are not copies of the objects themselves. In 'The facts in perception', Helmholtz argues that the fact that we visualize objects close up as larger than identical objects farther away is due to the fact that objects in perspective are signs. A physiologist may demonstrate mathematically how we construct an image using perspective. When that physiologist is faced with evidence that two objects differentiated by perspective in a visual image are identical, why should he abandon his explanation of the relation between the objects as perceived? It is still correct. According to Helmholtz's epistemology, properties such as separation in space and distance are relations between signs. So if our best mathematical description of fluids is that they are perfect continua, why should a physicist be worried if presented with evi-

---

21 Cassirer (2005 [1912]), p. 114.

22 Ibid.

23 Much of the scholarship on Helmholtz's theory of the a priori has focused on his account of geometrical axioms. Moritz Schlick, in his Notes to Helmholtz's *Epistemological writings*, argues that Helmholtz's related theories of epistemology and of geometry are the most enduring elements of his work. Schlick argues that Helmholtz's theory of fluid motion, for instance, is already consigned to a purely historical interest. (See Schlick's notes to Helmholtz, 1977a, on p. xxxiv and following). However, I will argue here that Helmholtz's work on fluid dynamics is directly relevant to his epistemology.

24 See, e.g., Emanuel (2000), pp. 8 ff.

25 These laws are in use today, in the following form from a modern textbook:

1. Fluid particles originally free of vorticity remain free of vorticity.
2. Fluid particles on a vortex line remain on a vortex line, so that vortex lines move with the fluid.
3. The strength of the vorticity is proportional to the length of the vortex line. (Fuhs et al., 1999, p. 736).

26 See, e.g., Smith (1998), pp. 254 ff., and Emanuel (2000), pp. 8 ff.

27 As Kelvin emphasizes in his Introduction to Königsberger (1906).

dence that they are in fact granular? Her theory manipulates signs of fluids, which do not have to resemble the fluids themselves.

As Cassirer points out, though, it is difficult to see how to evaluate whether our scientific claims are justified, if the sign theory countenances inconsistent underlying hypotheses about the actual objects under investigation. In this case, Helmholtz's laws of fluid motion require the a priori postulate that fluids are perfect continua, and our best theory of the makeup of those fluids is that they are granular. Surely, given Helmholtz's own empiricist epistemological views, it would be better for Helmholtz's fluid laws at least to approach more closely an accurate description of the phenomena. Here is a key problem for Helmholtz's psychological justification for the a priori postulate of regularity as governing the sign system. A scientist can construct a theory that does not map on to the known properties of the objects considered independently, but can argue that the theory nonetheless describes the phenomena as represented in the sign system. In that case, it is reasonable to ask: what is our justification for constructions in the sign system? Helmholtz's answer is that we must trust that the regularities of nature that that system describes map on to regularities in nature. The philosophers who pose the problem of validity find this answer unsatisfying.

Helmholtz's actual practice in his theory of fluid dynamics presages the *Bild*-theoretic answer to the problem of validity: defining the relationship between a given principle or law and the results that can be derived within a theory as necessary within that theory. Helmholtz shows that Euler's and Lagrange's formulations of fluid dynamics can be reformulated to have identical deductive consequences—he shows, in other words, that there is a minimal theory that captures all the results of Euler's and Lagrange's formulations, and that resolves the conflicts between the two. But this is a proof that the results obtain, not only in Euler's and Lagrange's formulations of fluid dynamics, but in any consistent formulation that is a model of Helmholtz's laws. In other words, Helmholtz's proof is an attempt at a proof (which would be given, now, in model-theoretic terms) that there is a *necessary*, not just a psychological, relationship between his laws and the provable theorems of fluid dynamics.

Helmholtz has a profound influence on the subsequent development of the picture or *Bild* theory constructed by Helmholtz's pupil Heinrich Hertz and later by Ludwig Wittgenstein. Helmholtz's epistemology, and his way of resolving problems in science as exemplified by his laws of fluid dynamics, pose an early challenge to the analytic–synthetic distinction. This is a challenge that Helmholtz himself did not make explicitly. However, the development of the *Bild* theory treats much that previously was considered analytic as in fact synthetic: a priori reasoning using logic and mathematics, in particular. The *Bild* tradition replaces analytic 'truths' with rules of construction of a *Bild*.

The roots of the *Bild*-theoretic undermining of the analytic–synthetic distinction are found in Helmholtz's theory, in his theory of signs and even in his fluid dynamics as sketched here. Constructions in the sign system are based on a postulate of regularity, for which we have no a priori warrant, only the need for some such postulate to construct a theory. For instance, the proposition 'Fluids are perfect continua' is neither an analytic truth nor an inference from observation in Helmholtz's theory. It is a postulate of regularity, a presupposition necessary to constructing Helmholtz's theory that makes the observed behaviour of fluids comprehensible. As such, when it is used to make predictions in the theory, the proposition *says* something about the observed phenomena: for instance, the postulate of continuity predicts that we can find the vorticity, or average rotation, of the fluid in any region, however small.[28] Two aspects of Helmholtz's view carry over to Hertz and Wittgenstein: that our theories depict actual objects, instead of being direct copies of those objects; and that the fundamental epistemological relation is between a construction in a sign system and the actual objects.

While writing the *Tractatus*, Wittgenstein read Hertz's *Principles of mechanics*, which was influenced by Helmholtz's sign theory, but which interprets that theory in a *Bild*- or picture-theoretic framework. Hertz, like Helmholtz, argues that spatial and temporal relations can be represented only in a sign system. However, for Hertz, judgments about relations between objects are constructed within a *Bild*, variously translated as 'symbol' or 'picture' in the literature, although *Bild* can also have the sense of 'model'. A *Bild* for Hertz is related to Helmholtz's sign system, but for Hertz, a *Bild* is constrained by its fundamental mathematical principle, which will be different for distinct *Bilder*, and by the basic logical notions and conceptual primitives of the *Bild*.[29]

Like Helmholtz's sign systems, Hertz's *Bilder* depend on the specification of signs. In his Introduction to the *Principles of mechanics*, Hertz argues that

> We form for ourselves symbols (*innere Scheinbilder*) or representations of objects; and the form we give them is such that the necessary consequents of the symbols in thought are always the images of the necessary consequents in nature of the things pictured.[30]

What guarantees the mapping between necessary consequence in thought and necessary consequence in nature? In the first Part of the *Principles*, Hertz argues that for any given system, we can give an a priori account of the possible geometrical configurations of that system, as well as the possible transformations of the system from one state to the next.[31] This account is arrived at by deduction from the basic principle of the system, e.g., the principle of least action, plus the basic logical and mathematical axioms and the fundamental notions (for example space, time, mass). Experience will answer the question of whether our a priori constraints, which predict the next observed configuration, capture the observed effects.

For Hertz, it is possible, then, to show *why* our scientific explanation of the consequences of a given mechanical experiment describes actual relations. If the system is constructed properly and the experiment is successful, we can show that the experiment, when plugged in to the possible configurations established a priori, rules out at least some of the other possible configurations of the system. This yields a way to test the *Bild* in experience: if the relations within the *Bild* contradict the observed relations, the *Bild* is 'incorrect'. Further, if the *Bild* is logically inconsistent, it is not 'permissible'. However, two *Bilder* can be correct and permissible, but

[28] The vorticity of a fluid, as Helmholtz defines it, is its average rotation in a chosen area of the fluid, represented by a vector. The vortex lines of a fluid in a region are the tangents to the vorticity vector of the fluid in that region. The postulate of continuity also allows Helmholtz to argue that when vortex lines are drawn around the boundary of a region of a fluid, and when the area around each is taken to a minimum, the vortex lines will converge on a single vortex filament. Vortex filaments are now called vortices.

[29] Hertz observes that the Hamiltonian formulation of mechanics uses a distinct mathematical principle from the Newtonian formulation. According to Hertz, the basic mathematical principle of Newtonian or 'classical' mechanics is a generalization of Newton's third law of motion, and the basic principle of Hamiltonian mechanics is a variational principle from which Hamilton derives the Euler-Lagrange equations. These principles dictate the methods used within the two *Bilder*. Newtonian mechanics maps attractive and repulsive forces between particles, usually with differential equations. Hamiltonian mechanics uses integral sums to evaluate the relationship between the potential and kinetic energy of a system (Introduction to Hertz (1956 [1894])).

[30] Ibid., p. 1; translation revised. For accounts of the relationship between Hertz, semantics, and theories of causality, see Heidelberger (1998), Howard (1996), and Leroux (2001).

[31] 'The subject-matter of the first book is completely independent of experience. All assertions made are a priori judgments in Kant's sense' (Hertz (1956 [1894]), §1).

can postulate distinct relations as the actual relations, that is, can depict the same results differently. In that case, Hertz appeals to the *Zweckmäßigkeit* or 'fitness to the purpose' of the *Bild*. Fitness to the purpose is a relative criterion: usually, in the progress of science, choosing between two correct *Bilder* will be a matter of demonstrating that one *Bild*, and not the other, models the phenomena univocally (*eindeutig*), by coordinating a single relation within the *Bild* to a single, corresponding relation in nature. If we can show that a model is univocal, Hertz argues, we have constructed an argument that a necessary relation in the model is necessary in nature.[32]

Conceiving of the a priori as a set of possible configurations of a system is the link between Hertz and Wittgenstein.[33] For Hertz, the a priori configurations are the possible geometrical configurations of a system of mechanics; for Wittgenstein, they are the possible logical configurations of a Tractarian *Bild*, usually translated as 'picture'.[34] As Wittgenstein puts it, 'we make *Bilder* of facts for ourselves', where a *Bild* is a 'model of actuality (*Wirklichkeit*)', and 'the total actuality is the world'.[35] (Wittgenstein, like Helmholtz, uses the term 'actuality', not 'reality', to describe states of affairs as modelled by the sign system.) For the Wittgenstein of the *Tractatus*, *Bilder* are logical means, composed of propositions, of depicting states of affairs. Elementary propositions depict the possibilities of states of affairs:

> The possible truth conditions of the elementary propositions refer to the possibilities of the existence or nonexistence of states of affairs.[36]

A name 'is not to be analyzed further by any definition: it is a primitive sign (*Urzeichen*)'.[37] A logical *Bild* can look like a truth table. A truth table contains names (P and Q, for example) and connectives (i.e., if ... then, and, or). A set of truth tables for all the connectives can exhibit formally all the logical possibilities associated with the primitive names, such as P and Q, for those connectives. For Hertz, the basic principle and the axioms of a *Bild* are the rules for determining the possible configurations of a system. For Wittgenstein, logic gives the rules for determining the truth conditions, and thus the meaning, of a proposition.

For Wittgenstein, a *Bild* refers to the world if and only if its internal form, the relations between its elements, is identical with the actual form.

> The form of depiction (*Abbildung*) is the possibility that things are related to each other as are the elements of the *Bild*.[38]

> What every *Bild*, of whatever form, must have in common with actuality, in order to depict it—correctly or falsely—is logical form, that is, the form of actuality.[39]

So if a *Bild* shows that 'If A then B' is true, then it is a correct depiction if 'If A then B' picks out an actual state of affairs. In other words, to trace Hertz's influence, we can construct our *Bilder* so that if 'If A then B' is a necessary consequence in thought, and it can be depicted as such in an 'experiment' (see below), then we have a warrant to argue that it is a necessary consequence in nature.

Making judgments is then a matter of depicting necessary relations based on the prior specification of the possibilities and their configuration in the propositions of the *Bild*. As Anthony Kenny remarks, in his notebooks from 1914 Wittgenstein describes the relationship between experiment and depiction as follows:

> In the proposition a world is as it were put together experimentally. (As when in the law-court in Paris a motor-car accident is represented by means of dolls, etc.). This must yield the nature of truth straight away (if I were not blind).[40]

In court, Wittgenstein saw a reconstruction of a traffic accident, in which lawyers used toy models to represent a lorry and a car that had hit each other. To determine who was at fault, the lawyers reconstructed the accident, and showed what the relative positions of the lorry and the car would have been. Wittgenstein pointed out that, although the toy cars are artificial, their relative positions depicted necessary relations. Only the possible configurations of the cars depicted the accident. Perhaps the toy models were made of plastic rather than steel, for instance. Nonetheless, it is true that if a manipulation of the toy cars (the experiment) shows that one driver was at fault, the reconstruction is a *valid* reason to accuse that driver of causing the accident.[41] In the case of a valid reconstruction, the lawyers can demonstrate to anyone's satisfaction—who is not 'blind'—that it would have been impossible, given the relative positions of the toy cars, for the other car to have caused the accident. For Wittgenstein, the reconstruction of the accident creates 'a world' that depicts experimentally the logical possibilities for the event.

The toy cars and the fluid lines are manipulations of signs, in the sense that they depict regularities in the behaviour of signs. According to the *Bild* theory, constructions in the sign system depict or symbolize objects, they do not mirror or copy the properties of the objects. Helmholtz chose to use vortex lines to symbolize the tangent vectors to the rotation of a fluid in a region. Helmholtz and generations of physicists following him were able to prove theorems about fluid motion using the notions of vortex lines and vorticity. They were able to depict 'a world', in Wittgenstein's sense, that includes vortex lines and vorticity. According to Hertz's and Wittgenstein's theories, Helmholtz et al. are able to prove that there is a warrant to think that the relations between vortex lines and vorticity that are necessary in the sign system are necessary in nature, instead of assuming or trusting that observed regularities are actual regularities in nature. That does not imply an endorsement of the claim that vortex lines are real features of objects, any more than the toy cars in Wittgenstein's court were the real cars involved in the accident. But it is not a good enough defence on the part of the driver who was at fault to claim that, since the cars used in the court reconstruction were not made of the same

[32] Two points are relevant here. First, as an anonymous referee for this journal pointed out, it is unclear whether there are any actual cases that meet Hertz's stringent requirements for representing necessary connections in a *Bild*. After all, few if any experiments rule out *all* alternative interpretations—this is part of Duhem's point in 'Physical theory and experiment'. It is an interesting further question whether Hertz thought necessary relations could be represented in a *Bild*, or whether representing such relations was what Kant calls a regulative ideal. Second, that there is a warrant for a claim that necessary relations in a *Bild* are necessary relations in nature does not mean that that warrant is infallible or indefeasible, of course.

[33] 'The absolute division in Hertz's and Helmholtz's philosophies of science between a priori 'spatial' structures and their *a posteriori* contents is replicated in Wittgenstein's analysis of language and cognition' (Hyder, 2002, p. 47).

[34] Again, I will not translate *Bild* as 'picture' here, however, because it is too easily confused with a mental or visual image. Instead, I use the German, because this emphasizes that Hertz and Wittgenstein use the same term.

[35] Wittgenstein (1922), 2.1, 2.12, 2.063; all translations from this work are my own.

[36] Ibid., 4.3.

[37] Ibid., 3.26.

[38] Ibid., 2.151.

[39] Ibid., 2.18.

[40] Wittgenstein (1961), p. 7, entry from 29 September 1914; cited in Kenny (2006), pp. 42–43, and discussed at pp. 44 ff.

[41] Note that the claim that the reconstruction gives a valid reason to accuse a driver of being at fault does not mean that it is the only possible evidence, or that it is not defeasible.

material as the actual cars, the reconstruction is invalid. Similarly, a sceptic who argues that the laws of fluid dynamics must be invalid because they require the counterfactual stipulation that fluids be perfect continua would not be able to prove her case.[42]

Within his framework, Helmholtz was able to prove that 'The strength of the vorticity (rotation) of a fluid is proportional to the length of the vortex line'. Helmholtz himself argues that the law cannot be derived directly from experience. Hertz adds that such laws must have an a priori warrant to be valid, in this case, because the law depends (1) on the counterfactual assumption that fluids are perfect continua and (2) on Helmholtz's specification of the signs 'vorticity' and 'vortex line', which are not derivable directly from experience. Helmholtz argues that reasoning using (1) and (2) is acceptable because these hypotheses are merely principles for inductive inference, and we cannot construct any theory of fluid dynamics without assuming some such principle.

Hertz and Wittgenstein argue that there is good evidence for the claim that, given a certain specification of fundamental notions (signs) and mathematical and logical principles, we can make warranted claims a priori about the possible configurations of a given system. The signs and rules do not need to be true or even meaningful. Wittgenstein, in particular, argues that names are primitive signs that are not logically analysable, and that logical and mathematical rules are either senseless or nonsense. Nonetheless, while a priori claims about possible configurations in the *Bild* theory do not have the thick content of the Kantian synthetic a priori, and in particular do not license claims that subjects have pure intuitions of space and time, they are instances of a priori reasoning that go beyond the systematisation of previous mathematics.

Using Helmholtz's laws to achieve fruitful results in science does not require arguing that it is an a priori truth that fluids are perfect continua. Further, inferences based on Helmholtz's laws must be demonstrated in Kant's 'tribunal of experience', or Wittgenstein's court of experiment, to be valid. The laws are valid principles of inference, but not a priori truths about the fluids themselves. The independent properties of the fluids themselves are not evaluated in the sign theory, only the signs that we use to refer to the fluids, and the lawlike relations that hold between successive representations of the fluids. The use of a priori reasoning in the *Bild* theory is to constrain our elaboration of the possible configurations of that theory.

Helmholtz's law is not a claim about the real properties of fluids, nor is it meant to be. It is a rule governing the possible transformations of a dynamical system from one configuration to another. Without some such rules, proving theorems within the system would not be possible. Within the sign theory framework, without which it is not possible to reconstruct Helmholtz's reasoning, mathematical principles govern the possible transformations of elements of the theory from one configuration to another. For instance, it is possible to map the possible moves from one configuration of a Rubik's cube to another. A mathematical rule governs the possible transformations between one state, or configuration, of the system to another. Given a mathematical rule, and a set of states of affairs, for example, two configurations of the faces of the cube, we can specify the possible transformations or paths from one configuration to another within the system. However, the mathematical rule does not make any verifiable statement about the properties of the cube.

The difference between Helmholtz's approach and the *Bild*-theoretic approach of Hertz and Wittgenstein is that, for Helmholtz, our expectation that one state will follow another is based on trust that observed regularities in nature are actual regularities. Hertz and Wittgenstein respond that *within* a given system, this trust can be shown to be based on a demonstrable correspondence, between a relation in the *Bild* and a relation in nature, for Hertz, or between the logical form of a proposition and actual relations in the world, for Wittgenstein.

4. Syntax, semantics, and pragmatics

By the beginning of the twentieth century, the *Bild* theory had replaced Helmholtz's sign theory. Early on, the *Bild* theory answers the problem of validity by arguing that there is a necessary relation between a given set of a priori laws or postulates and the theorems provable within a *Bild*. Cohen and Cassirer cite Hertz's *Bild* theory approvingly, as a sound answer to the problem of validity. However, both argue that Hertz's and Wittgenstein's version does not give enough independence to the mathematical and logical rules that constitute relations in the *Bild*.[43] The Marburg school argument, to which Cassirer subscribed early on, was that since a priori relations are necessary to constructing a system of natural science, they generate or constitute objects insofar as we can know them, and thus statements about such relations are warranted a priori.

But such arguments for independence became progressively more difficult to support. The development of the *Bild* theory, along with the progress of science over the twentieth century, undermines the argument that a fixed set of a priori principles of logic or mathematics are necessary to the constitution of a given set of objects or concepts. Over the twentieth century, mathematicians and scientists delighted in giving more and more proofs that the same results could be accounted for using distinct a priori assumptions. In that case, it is difficult to argue for any necessary stable content for the a priori. It is possible to show that the relation between theoretical framework and scientific result is necessary within a given model, but not that the model itself is necessary to capturing the result.

The *Bild* theory itself is not challenged by these results. Hertz and Wittgenstein argue that logical and mathematical rules are valid only for making determinations within a given *Bild*. Such rules do not have any validity outside a *Bild*. In particular, a rule valid in one *Bild* may not be valid in another. For instance, Hertz argues that Newton's law of universal gravitation, expressed in terms of an inverse square force between point masses, has no sense in a Hamiltonian framework based on energy transfer. However, the question of how to define mathematical and logical propositions as inter-theoretically valid remained of great philosophical interest. As Friedman (2000) explains, early in his career Carnap picks up a project related to the Marburg defence of the independent validity of mathematical and logical relations.[44] Carnap's analysis of logical syntax is an attempt to show that such differences in content are merely linguistic, for example, that the content is stable, but that the language, the means of expressing the content, changes.

By 1938, however, Carnap saw the difficulties with the syntactic approach, and began to develop a semantic analysis of translation between languages. In 1945, Russell, in an appreciation of the his-

[42] There are many examples of the use of toy models, like Wittgenstein's toy cars, in the history of science. Faraday's model of the atom and Maxwell's cylindrical model of the electromagnetic ether are two of the most famous. These are not formal models in the contemporary, semantic sense. Maxwell used rotating cylinders to depict his notion of the relation of electromagnetic forces. Maxwell did not argue that electromagnetism is made up of rotating cylinders in reality. Instead, he argued that the rotating cylinder toy model depicts necessary relations between electromagnetic forces, just as Wittgenstein's toy cars depict what really happened in an accident. Larry Laudan gives several similar examples in his classic paper 'A confutation of convergent realism' (Laudan, 1981).

[43] For instance, Cassirer argues that while the axioms of a particular geometry may not be a priori universal truths, we must choose *some* group of transformations to construct any physical theory. As Friedman (2000) observes, there is some common ground between Cassirer's and Carnap's approaches (see, e.g., pp. 70 ff.).

[44] See, e.g., ibid., pp. 74 ff.

tory of logical positivism until that time, commented on Carnap's shift from the early theory that logical truth depends only on syntax, to a later view that logical truth depends on semantics.[45] Russell characterized the shift as follows:

> It was always obvious that 'empirical truth' must be defined semantically: there is no syntactical difference between 'I had coffee for breakfast' and 'I had tea for breakfast', but on most days one of these is true and the other false. A sentence of pure logic, on the other hand, can, given the rules of syntax, be known to be true (or false) by its form. Nevertheless, Carnap now holds that even in pure logic it is in virtue of its significance that a sentence is true (or false), and that therefore, even in this sphere, it is semantics, not syntax, that is involved.[46]

Part of the reason for Carnap's shift can be seen already in the history of the *Bild* theory. You can prove the same theorems in Helmholtz's *Bild* as in Euler's and Lagrange's theories. But Euler's and Lagrange's theories are inconsistent with each other. So in this case, we can construct a *Bild* that allows us to prove equivalents of all the theorems of both given theories, but that does not incorporate all the analytic a priori claims of the two theories—if it did, it would be inconsistent. So how can we say that there is any necessary, analytic content to the a priori? More and more cases of this type built up in science, the coordinate free formulations of general relativity being a particularly significant instance.

Early on, Carnap argues that reformulations of the same results in distinct linguistic frameworks are only syntactically distinct. He proposes that he can remove the apparent conflicts between linguistic frameworks by means of syntactic analysis and translation rules. By 1938, as Carnap writes to Quine, he has run into problems in inter-translating frameworks using only syntactic, extensional logic. Influenced by Tarski, Carnap develops a semantic, intensional approach, which quantifies over sentential variables. For instance, such an analysis can yield sentences such as (expressed in relatively ordinary language): 'It is not the case that there exists an x such that x = "The cat's being alive" at time t'. Here 'x' is meant to be analogous to a Wittgensteinian state of affairs, the referent of an elementary proposition. In his discussion with Quine, Carnap argues that such sentences are found in ordinary language:

> I doubt whether sentence designata are not referred to in everyday language. The customary grammar forbids also predicates in noun position, but it replaces them by corresponding nouns e.g. 'red' by 'redness;' in the same way it replaces sentences in noun position by corresponding expressions as e.g. 'my arrival' or 'his being opposed to this'. In any case I think the question should be discussed much more and my view about it is not at all definitive.[47]

Carnap's idea is that semantic analysis of how sentences of a scientific theory depict their content can yield transformation rules between frameworks. The idea is that a logical sentence refers to a proposition, which is the stable content of the sentence that remains the same, independent of the framework in which it is expressed. Semantic analysis of the logical sentences of a framework, Carnap argues in the late 1930s and early 1940s, will yield translation rules based on this stable propositional content.

Russell and Quine object to this move on similar, though distinct grounds. Russell argues that Carnap's introduction of sentential variables into semantics complicates his ontology:

> A name, such as 'Churchill' or 'Stalin' or 'Big Ben', means a certain person or thing, but a sentence does not mean a definite object. If I say 'Churchill is Prime Minister', I am not asserting or naming some entity, 'Churchill's premiership'. We cannot therefore treat the significance of sentences as something closely analogous to the significance of proper names. If 'proposition' is to mean 'significance of a sentence', we have on our hands the problem of deciding what this can be.[48]

On similar grounds, Quine objects that Carnap is complicating his ontology unnecessarily. Here Quine objects that Carnap is violating Ockham's razor: Carnap is adding an entity, a proposition, as the referent of a sentence. Such entities, Quine observes, are not necessary for classical mathematics.

> Indeed, the latter [bound sentential variables] could even be introduced by contextual definition, if they would be convenient, so they would still have no status at the primitive level ... Your argument that everyday language does after all support sentence-designata must, I guess, be admitted (contrary to my previous argument); but I suppose I am affected mainly by Ockham's razor.[49]

Quine argues that 'convenience' and simplicity (Ockham's razor) should drive the choice whether or not to allow sentence designata into one's semantics. Quine admits that if they were necessary, he would allow for sentence designata, though reluctantly. In 1938, Carnap thought sentence designata were necessary to giving translation rules, and so he defended them, though not robustly.

As Carnap says in his letter to Quine (cited above), his views on semantics in 1938 were not definitive. In 'Empiricism, Semantics, and Ontology', Carnap distinguishes between questions internal to and external to a linguistic framework. Carnap argues that the relation *within* a theory between a priori propositions and the results of the theory is meaningful and determinate. The choice between frameworks, including the choice to employ a framework at all, is practical.[50] Carnap retreats somewhat, from the goal of accounting for the propositions of natural science as inter-translatable, to giving a semantic account of a subset of those propositions that are common to linguistic frameworks.

Carnap's mature distinction between internal and external questions is a clear way to frame this early logical empiricist answer to the problem of validity. The problem of validity, says Carnap, is a problem that it makes sense to pose only within a given linguistic framework. Questions of the validity of an a priori relation are legitimate, and meaningful, within that linguistic framework. Here Carnap is in agreement with Hertz and Wittgenstein, that outside a given framework, questions of the validity of the principles of construction of that framework have no meaning. According to Carnap, the choice of linguistic framework, or even the choice of whether to use language to refer to things at all, is practical.[51] Here is the legacy of Helmholtz in logical empiricism. The decision to use signs to construct objects in scientific language

---

[45] I am grateful to Clark Glymour for pointing out to me the significance of Russell's essay in this connection.

[46] Russell (1997 [1945]), pp. 152–153.

[47] Quine & Carnap (1990), p. 245, letter from Carnap to Quine, 11 February 1938.

[48] Russell (1997 [1945]), p. 153.

[49] Quine & Carnap (1990), p. 248, letter from Quine to Carnap, 15 February 1938.

[50] As Pincock (2007) has emphasized, here Russell and Carnap disagree (Pincock, 2007, pp. 127 ff.). For Russell, pragmatics is independent of a given linguistic framework, whereas for Carnap all analysis takes place within some such framework. (For instance, analysis of a mental state that motivates someone to accept a theory might take place within empirical psychology).

[51] Carnap (1950), §2.

is the only part of the system of science not determined by experience. It is appropriate, then, to give Helmholtz the last word:

> Here the only valid advice is: have trust and act![52]

Acknowledgements

I would like to acknowledge very gratefully Clark Glymour, Gary Hatfield, Don Howard, David Hyder, Jean Leroux, Robert Richards, and two anonymous referees for this journal, all of whom suggested new directions for my research in this area and clarifications of my project in the paper, which I hope I have used to good effect. I have profited from conversations with Robert DiSalle, Scott Edgar, Sanford Goldberg, Michael Hallett, Jeremy Heis, Alan Kim, James Klagge, Alison Laywine, David McCarty, Paul Pietroski, Christopher Pincock, Josef Stern, Abraham Stone, and Jamie Tappenden about aspects of my project in this paper. An earlier version of the paper was presented at the reading group of the University of Notre Dame History and Philosophy of Science Program, at which Don Howard, Elise Crull, Anja Jauernig, Lynn Joy, and Phillip Sloan, amongst others, contributed to a very productive discussion of the project.

References

Carnap, R. (1950). Empiricism, semantics, and ontology. *Revue internationale de philosophie, 4*, 20–40.
Cassirer, E. (1978). *The problem of knowledge: Philosophy, science and history since Hegel* (W. H. Woglom, & C. W. Hendel, Trans.). New Haven & London: Yale University Press. (First published 1946)
Cassirer, E. (2005). Hermann Cohen and the renewal of Kantian philosophy (L. Patton, Trans.). *Angelaki*, *10*, 95–108. (First published 1912 as Hermann Cohen und die Erneuerung der Kantischen Philosophie. *Kantstudien*, *17*, 252–272)
DiSalle, R. (1993). Helmholtz's empiricist philosophy of mathematics. In D. Cahan (Ed.), *Hermann von Helmholtz and the foundations of nineteenth century science* (pp. 498–521). Berkeley: University of California Press.
Emanuel, G. (2000). *Analytical fluid dynamics* (2nd ed.). Boca Raton: CRC Press.
Friedman, M. (2000). *A parting of the ways: Carnap, Cassirer, and Heidegger*. Chicago & La Salle, IL: Open Court.
Fuhs, A. E., & Shetz, J. A. (Eds.). (1999). *Fundamentals of fluid mechanics*. New York: John Wiley.
Hatfield, G. (1991). *The natural and the normative: Theories of spatial perception from Kant to Helmholtz*. Cambridge, MA: The MIT Press.
Heidelberger, M. (1998). From Helmholtz's philosophy of science to Hertz's picture-theory. In D. Baird, R. I. G. Hughes, & A. Nordmann (Eds.), *Heinrich Hertz: Classical physicist, modern philosopher* (pp. 9–24). Boston Studies in the Philosophy of Science, 198. Dordrecht: Kluwer Academic.
Helmholtz, H. (1903). *Vorträge und Reden, Bd. 1*. Braunschweig: Friedrich Vieweg und Sohn.
Helmholtz, H. (1968). Über das Ziel und die Fortschritte der Naturwissenschaft. In idem, *Das Denken in der Naturwissenschaft* (pp. 31–61). Darmstadt: Wissenschaftliche Buchgesellschaft. (First published 1869)
Helmholtz, H. (1977a). On the origin and significance of the axioms of geometry. In idem, *Epistemological writings: The Paul Hertz/Moritz Schlick centenary edition of 1921, with notes and commentary by the editors* (R. Cohen, & Y. Elkana, Eds.; M. Lowe, Trans.) (pp. 1–38). Boston Studies in the Philosophy of Science, 37. Dordrecht: D. Reidel. (First published 1870)
Helmholtz, H. (1977b). The facts in perception. In idem, *Epistemological writings: The Paul Hertz/Moritz Schlick centenary edition of 1921, with notes and commentary by the editors* (R. Cohen, & Y. Elkana, Eds.; M. Lowe, Trans.) (pp. 115–185). Boston Studies in the Philosophy of Science, 37. Dordrecht: D. Reidel. (Talk first given in 1878)
Hertz, H. (1956). *The principles of mechanics* (R. S. Cohen, Ed.; D. E. Jones, & J. T. Walley, Trans.). New York: Dover. (First published 1894)
Howard, D. (1996). Relativity, *Eindeutigkeit*, and monomorphism: Rudolf Carnap and the development of the categoricity concept in formal semantics. In R. Giere & A. Richardson (Eds.), *Origins of logical empiricism* (pp. 115–164). Minnesota Studies in the Philosophy of Science. Minneapolis & London: University of Minnesota Press.
Hyder, D. (2002). *The mechanics of meaning: Propositional content and the logical space of Wittgenstein's Tractatus*. Berlin: Walter de Gruyter.
Kenny, A. (2006). *Wittgenstein* (rev. ed.). Oxford: Basil Blackwell.
Königsberger, L. (1906). *Hermann von Helmholtz*. New York: Dover.
Laudan, L. (1981). A confutation of convergent realism. *Philosophy of Science, 48*, 19–49.
Leroux, J. (2001). 'Picture theories' as forerunners of the semantic approach to scientific theories. *International Studies in the Philosophy of Science, 15*, 189–197.
Liebert, A. (1914). *Das Problem der Geltung*. Kantstudien Ergänzungshefte, 32. Würzburg: 'Journalfranz' Arnulf Liebing.
Millikan, R. (1989). Biosemantics. *Journal of Philosophy, 86*, 281–297.
Pincock, C. (2007). Carnap, Russell, and the external world. In M. Friedman, & R. Creath (Eds.), *Cambridge Companion to Carnap* (pp. 106–128). Cambridge: Cambridge University Press.
Quine, W. V. O., & Carnap, R. (1990). *Dear Carnap, dear Van: The Quine–Carnap correspondence and other material* (R. Creath, Ed.). Berkeley & Los Angeles, CA: University of California Press.
Russell, B. (1997). Logical positivism. In idem, *The collected papers of Bertrand Russell: Last philosophical testament, 1943–1968* (J. Slater, & P. Köllner, Eds.). London: Routledge. (First published 1945)
Smith, P. (1998). Approximate truth and dynamical theories. *British Journal for the Philosophy of Science, 49*, 253–277.
Wittgenstein, L. (1922). *Tractatus logico-philosophicus* (C. K. Ogden, Trans.). London: Routledge & Kegan Paul. (First published 1921 as Logisch-Philosophische Abhandlung. *Annalen der Naturphilosophie*, *14*(3/4))
Wittgenstein, L. (1961). *Notebooks 1914–1916*. Oxford: Basil Blackwell.

[52] Helmholtz (1977b [1878]), p. 142.