Abstraction relations need not be reflexive*

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Abstract

Neo-Fregeans such as Bob Hale and Crispin Wright seek a foundation of mathematics based on abstraction principles. These are sentences involving a relation called the abstraction relation. It is usually assumed that abstraction relations must be equivalence relations, so reflexive, symmetric and transitive. In this paper I argue that abstraction relations need not be reflexive. I furthermore give an application of non-reflexive abstraction relations to restricted abstraction principles.

Neo-Fregeans such as Bob Hale and Crispin Wright (e.g. Hale 1987; Hale and Wright 2001; Wright 1983) seek a foundation of mathematics based on *abstraction principles*. These are sentences of the following form:

$$(\mathrm{AP}_{\sim}) \qquad \forall F \forall G [\S F = \S G \leftrightarrow F \sim G]$$

Here, § is an *abstraction operator*, which, when attached to a second-order term (such as a predicate or a second-order variable) results in a singular term, called an *abstract term*. The referent of an abstract term—if any—is an *abstract*. Finally, the relation ~ is the *abstraction relation*. The effect of an abstraction principle is to map each concept onto an abstract, such that two concepts have the same abstract if and only if they are related by ~.

One particular instance of an abstraction principle is *Hume's Principle* (HP), which states that the *number* of Fs is equal to the *number* of Gs iff the Fs and Gs can be put into a one-to-one correspondence. This abstraction principle has played a fundamental role in the neo-Fregean programme, since it serves as an axiom for

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arithmetic. That is, along with suitable explicit definitions it can be used to derive the Dedekind–Peano axioms.

Another—more infamous—abstraction principle is Frege's *Basic Law V* (BLV), which formed the basis of his theory of classes in Frege (1893). BLV states that the *extension* of a concept F is equal to the *extension* of the concept G iff exactly the same objects fall under F as fall under G. It is thus a natural candidate as an abstraction principle for set theory, since sets could be considered to be extensions of concepts. It is, alas, inconsistent, since it allows for the derivation of Russell's Paradox.

These abstraction principles are *second-order*, in that the abstraction operator maps concepts or properties to objects. It is worth noting that there are also *first-order* abstraction principles, where the abstraction operator maps objects to other objects. For example, Frege (1884) gives as an example what has become known as the *direction equivalence*, which associates lines with directions. The direction equivalence says that the direction of a line ℓ_1 is equal to the direction of a line ℓ_2 iff ℓ_1 and ℓ_2 are parallel. I will only consider second-order abstraction principles in what follows, but the same considerations will clearly apply to first-order abstractions as well.

In virtually every paper and book mentioning abstraction principles, it is claimed that the abstraction relation must be an equivalence relation, that is, symmetric, transitive and reflexive. The aim of this paper is to observe that this need not be the case; when the natural background logic of abstraction is taken into account, there is no need for reflexivity. Indeed, non-reflexive abstraction relations can serve a useful purpose.

1. Free logic

The main claim of this paper may seem, at first, clearly false. For there appears to be a perfectly good, simple proof that the abstraction relation must be reflexive. Simply reason left to right across the abstraction principle from the reflexivity of identity as follows:

Argument 1:

1. $\S F = \S F$ (reflexivity of identity)

2. $F \sim F$ (reasoning left to right across the abstraction principle)

3. $\forall F(F \sim F)$ (universal generalisation)

Alternatively, this argument could be phrased in the form of a *reductio*; assuming an instance of non-reflexivity (i.e. $F \nsim F$), one may derive the apparent absurdity that $\S F \neq \S F$.

Such a proof takes place in a standard classical logic. But there are good philosophical reasons for taking the natural background logic of abstraction not to be such a standard logic, but rather a *free* logic. This is a logic which removes the usual presumption that every term in a language refers, or, equivalently, the assumption that every functional expression denotes a *total* function, taking a well defined value for any given argument.

The reason that a free logic is appropriate is that one of the main claims made by neo-Fregeans is that abstraction principles may underwrite our knowledge of the existence of, say, numbers. But it would then plainly be begging the question to *presuppose* that number terms refer before even introducing an abstraction principle. Free logic removes this presupposition. Indeed, somewhat of a consensus on this issue appears to have emerged in recent years.[1]

Moreover, it is clear that neo-Fregeans desire a *negative* free logic.[2] This is one in which an atomic sentence involving a singular term may be true only if that singular term refers. In such a logic, it is possible to infer from an atomic sentence $A(t)$ the existential claim $\exists x(x = t)$. But in addition, since identity statements are atomic sentences, a negative free logic requires a restriction on their introduction; in order to assert $t = t$, one must first derive the claim that $\exists x(x = t)$.[3]

If a negative free logic were not adopted (resulting instead in a *positive* free logic), abstraction principles would be vacuous, since it could just be the case that

[1] So, for example, on behalf of the neo-Fregeans, see Hale and Wright (2003, 2008, 2009), and on behalf of their opponents, MacFarlane (2009); Rumfitt (2003); Shapiro and Weir (2000) and, to a lesser extent, Potter and Sullivan (2005).

[2] See the previously cited papers by Hale and Wright. But although it is clear that a negative free logic is desired by neo-Fregeans, it is highly contentious whether such a choice of logic is *justified*. So, for example, the majority of those opponents of neo-Fregeanism cited in the previous footnote claim that a negative free logic is not justified. This is not the topic of this paper, however, and I shall simply assume that the correct free logic is a negative one.

[3] For a more thorough introduction to free logic an its varieties, see Bencivenga (2002).

no abstract term refers. But, if the abstraction relation is reflexive, the existence of abstracts can be derived in a negative free logic as follows:

Argument 2:

1. $F \sim F$ (an instance of reflexivity of $\sim$).
2. $\S F = \S F$ (right to left across abstraction principle).
3. $\exists x(x = \S F)$ (since identity statements are atomic).

It can be noted that this is essentially the converse of argument 1.

Now, if we accept a negative free logic as the background for abstraction, there is no longer a need for the abstraction relation to be reflexive. For in a negative free logic, argument 1 is fallacious. The premise of that argument—that $\S F = \S F$ for any F—is not a logical truth, requiring as it does an additional premise that $\S F$ exists, which is just what is missing in a free logic. Likewise, the statement that $\S F \neq \S F$ is not an absurdity in a negative free logic; it simply amounts to the claim that $\S F$ does not exist.

So reflexivity of the abstraction relation is essentially tied to the existence of abstracts; the existence of $\S F$ entails that $F \sim F$ (by a simple modification of argument 1), and conversely, $F \sim F$ entails that $\S F$ exists (by argument 2). In the limiting case, the general reflexivity of the abstraction relation entails that every concept has an abstract—so that the logic is essentially non-free—and a non-free logic entails that every concept has an abstract.

But since there is no reason to adopt this limiting case for all abstraction principles *tout court*, the requirement typically made that abstraction relations are reflexive should be abandoned.[4]

II. Free logic and restricted abstraction principles

Further than simply being consistent, there are many natural cases where non-reflexive abstraction relations can be put to good use. In particular, this will be the case when there is a desire to *restrict* abstraction principles.

[4] A stronger case for the acceptability of non-reflexive abstraction operators could be given by giving a model-theoretic proof for the consistency of such principles. I will not go into the details of such a proof here, but merely note that it is relatively trivial to construct such models, by letting the function denoted by the abstraction operator be undefined for F such that $F \not\sim F$.

There are a number of cases in which it might be desired that not every concept has a corresponding abstract. Perhaps most notable is the case of BLV, where we may wish that some, but not all, concepts have a corresponding extension, on pain of contradiction. But there are also other abstraction principles which one may wish to restrict. So, for example, there is a natural abstraction principle for order types, which is inconsistent if not restricted. Or, one may want to restrict an abstraction principle for rational numbers in such a way as to rule out fractions with zero as a denominator.

It would be natural in such circumstances to turn to free logic. After all, for a restricted abstraction principle, the desired effect should be that certain concepts do not have corresponding abstracts, and thus that the abstraction operator is a partial function. Most approaches to restricting abstraction principles have, however, taken place in a non-free logic, and thus been constrained to make use of reflexive abstraction relations.[5] Most such approaches have made use of a technique popularised by Boolos (1989) in a restriction to BLV. He restricts BLV to concepts which are *small*, by which he means not equinumerous with the universe. He does so as follows:

$$\text{(NV)} \qquad \varepsilon F = \varepsilon G \leftrightarrow (\text{Small}(F) \vee \text{Small}(G) \to \forall x(Fx \leftrightarrow Gx))$$

He calls this abstraction principle *New V*.

The effect of New V is not that the non-small concepts fail to have abstracts. Instead, they all have the *same* abstract. The resulting restriction thus bears some resemblance to the Frege-Carnap *chosen object* approach to empty singular terms (c.f. Kaplan 1972). This approach first identifies a particular *null* object.[6] Then terms which would otherwise be taken to be non-denoting are stipulated to have the null object as their referent.

This method can be extended to abstraction principles more generally. Given an abstraction principle with relation ~, we can restrict it to concepts which satisfy

[5]One possible exception is Wright (2001), who considers how the direction principle might be restricted so as to rule out directions for objects which are not lines. He considers that the abstraction operator may only be partially defined, since 'unsuitability of either object to be parallel to anything, then by the same token they are not *self*-parallel' (p. 314). He does, however, go on in the same paper to assume that abstraction operators must be equivalence relations.

[6]This use of the word 'null' should be distinguished from the common use of 'null set' to mean the empty set. To avoid ambiguity, I will always use 'empty set' to refer to a set with no members, and 'null object' to refer to a null object in the sense of the chosen object theory.

a formula $\phi(F)$ as follows:

$$(\text{NAP}_{\sim,\phi}) \qquad \S F = \S G \leftrightarrow (\phi(F) \vee \phi(G) \rightarrow F \sim G)$$

Assuming that $\sim$ is an equivalence relation, the resulting abstraction relation is also an equivalence, on the proviso that $\sim$ is a congruence with respect to ϕ—that is, if $\phi(F)$ and $F \sim G$, then $\phi(G)$.[7]

Again the effect is that non-ϕ concepts are all mapped to the same abstract. This approach has been essentially the standard approach to restricted abstraction. So, various restrictions of BLV along these lines have been made with varying choices of ϕ; some alternatives to smallness which have been suggested are *double*-smallness (roughly, smaller than some concept which is smaller than the universe) (e.g. Hale 2000, 2005) and *definiteness* (e.g. Shapiro 2003; Shapiro and Wright 2006). Likewise, this method of restriction has been used for other abstraction principles. For example, Cook (2003) makes use of such a technique to restrict an abstraction principle for order types (so as to avoid the Burali-Forti paradox) and Shapiro (2000) does the same for an abstraction principle for rational numbers (to rule out cases where the denominator would be 0).

The results of this method of restriction are to some extent rather unnatural. They require a seeming superfluous null object, and in some cases a hierarchy of objects based on this null object (so, for example, in the case of set theory, there are abstracts which correspond to sets formed out of the null object and so on). Moreover, in theories which make use of multiple abstraction principles, there may be a need for there to be a null object for every type of abstract introduced.

These problems are reasonably superficial, in that it is a relatively trivial matter to restrict quantifiers to 'genuine' abstracts—i.e. those that do not involve the null object in any way. So, for example, in set theory, this would require restricting to objects which are neither the null object nor sets with the null object in their transitive closure. And no technical problems arise if we take this approach (see

[7] *Proof:* Reflexivity follows immediately from the reflexivity of $\sim$, and symmetry follows immediately from the commutativity of disjunction together with the symmetry of $\sim$. For transitivity, suppose that $\phi(F) \vee \phi(G) \rightarrow F \sim G$ and $\phi(G) \vee \phi(H) \rightarrow G \sim H$. Now suppose that $\phi(F) \vee \phi(H)$, with the intention of showing $F \sim H$. Suppose $\phi(F)$. Then $F \sim G$ and hence (since $\sim$ is a congruence) $\phi(G)$. Then $G \sim H$ and we have $F \sim H$ by the transitivity of $\sim$. A similar argument goes if we assume instead that $\phi(H)$.

Boolos (1989) for details in the case of NV). Nonetheless, it would be desirable to avoid them if it is possible to do so.

And indeed, in the presence of a free logic, there is a very natural alternative to the Boolos method, by allowing for an abstraction relation which is non-reflexive. Since partial functions are admissible in a free logic, the aim is to construct an abstraction principle whose effect is that the abstraction operator denotes a partial function which is undefined for non-ϕ concepts. Such an abstraction principle is available as follows:

$$(\text{FAP}_{\sim,\phi}) \qquad \S F = \S G \leftrightarrow (\phi(F) \wedge \phi(G) \wedge F \sim G)$$

The abstraction relation in this case is clearly transitive and symmetric, assuming that $\sim$ is.[8] But it is not reflexive in general. In particular, if $\neg\phi(F)$, then $\neg(\phi(F) \wedge \phi(F) \wedge F \sim F)$. It is, however, reflexive for ϕ concepts.

What is the effect of this? It is straightforward to prove that, in a negative free logic, the following follows from $\text{FAP}_{\sim,\phi}$:[9]

$$(1) \qquad \forall F(\phi(F) \leftrightarrow \exists x(x = \S F))$$

That is, it does exactly what is required of it.

The result of this abstraction principle is, I claim, more natural than that of the Boolos method. It does precisely what might be expected of an abstraction principle restricted to ϕ concepts; to concepts which are ϕ, it assigns abstracts according to the original abstraction relation. To concepts which are not ϕ, it assigns no abstracts at all. So, in the case of set theory, a free logic based restriction will assign no abstract to the Russell concept and (for most choices of restriction) will assign no abstract to the universal concept. There is no need for a number of strange objects as there are in the Boolos case. Thus, if there is a need or desire to restrict an abstraction principle for whatever purpose, a free-logical restriction should be preferred to the Boolos-style restriction.

Finally, it is important to say something about the relationship between FAP- and NAP-style restriction, and to anticipate an objection to the FAP-style which

[8] *Proof*: Symmetry follows immediately from the commutativity of conjunction together with the symmetry of $\sim$. Transitivity follows immediately from the transitivity of $\sim$.

[9] *Proof*: First note that the new abstraction relation holds between a concept F and itself iff $\phi(F)$. Then (1) follows immediately from this and the observation in the previous section that reflexivity coincides with abstract existence.

may arise from this relationship. The answer is that, although very similar, the free logic method is slightly weaker logically. The reason is that, as long as there is at least one concept which is provably non-ϕ, the Boolos method allows one to prove that there is a null object. Depending on the nature of ϕ, this may provide one with a method to prove that certain other concepts are ϕ, and so on. Indeed, Boolos makes essential use of such a bootstrapping technique in his proof that New V allows for a certain amount of set theory: It can easily be shown that the universal concept (that given by the formula $x = x$) is not small, and that the empty concept (that give by the formula $x \neq x$) is small. They thus have distinct abstracts, which then means that singleton concepts (e.g. one given by a formula $x = a$ for some a) are not small, and so on. Such a method is unavailable for the free logical approach.

It can however be shown that the supply of such an object, or something similar, is all that the Boolos method provides over and above the free-logical approach. In particular, if we add to $\mathrm{FAP}_{\sim,\phi}$ a statement asserting that there is at least one *urelement*—an object which is not the abstract of any concept—then the resulting systems are of the same logical strength. (See the appendix for full details).

But this lack of logical strength should not be considered a weakness of this method of restriction, anymore than the relative weakness of one abstraction principle to another should count as a weakness in general. Different abstraction principles will correspond to different kinds of abstract object, or perhaps to different conceptions of a kind of abstract object; their relative logical strength will thus correlate with the relative strength of the corresponding conception. Suppose, for example, that we are presented with an abstraction principle *A* which is logically stronger than HP (perhaps an abstraction principle for set theory). This does not mean that we should abandon HP in favour of *A*, together with some (perhaps unnatural) definitions of arithmetical vocabulary in the language of *A*. A strong case can be made for HP accurately representing a particular conception of the concept of (cardinal) number, whereas *A* may not, and this is reason enough to make use of HP as an axiom for arithmetic in place of *A*.

So too should we not abandon an FAP-style restriction in favour of a possibly logically stronger NAP-style restriction. If I am right that FAP more naturally and accurately represents the result of restricting an abstraction principle, then it is really the case that this weakness correctly represents the weakness of the corresponding conception. The corresponding NAP-style restriction, by contrast, does not represent such a restriction, but instead represents a different kind of abstract object altogether—one which includes a null object—which simply bears a resemblance to the restricted principle.

Absent any particular reason to prefer the conception of objects implicit in NAP-style restrictions, one should thus prefer the FAP-style version. And in cases where the NAP-style restriction *is* required (such as when NAP is stronger than FAP), it should be kept in mind that the kind of abstract objects involved do not simply result from a restriction of the base class, but involve also a sometime peculiar null-object. That is, it is not simply a restriction that is required, buy a wholly different abstraction principle, related to the original principle some way other than as a restriction.

III. Conclusion

There have been two main aims of this paper. The first is to argue that there is no need for abstraction relations to be reflexive. Such a restriction on abstraction principles should thus be dropped. The second has been to point out a novel method of producing abstraction principles, by harnessing non-reflexive abstraction operators. This method of restriction, I claimed, is more natural than the predominantly used method in the literature, and thus should be considered as the default option for restriction.

A. Proof of interpretability.

For the purposes of showing the relationship between FAP and NAP, I shall make three assumptions about the nature of the restriction. These are:

a) The restriction is non-vacuous, so that $\exists F \neg \phi(F)$.

b) $\sim$ is a congruence with respect to ϕ.

c) ϕ does not contain the abstraction operator §.

These assumptions hold for all examples of restriction in the literature. (a) and (c) could be dispensed with, but this would result in additional complication for little gain. (b) is required if Boolos-style restrictions are to be consistent (otherwise, the new abstraction relation may not be transitive).

First, let us set up languages and theories. We will have separate languages for each abstraction principle to make it clear that the abstraction operator in each case means something different.

Let $\mathcal{L}_1$ be the second-order language whose only non-logical constant is the abstraction operator $\S_1$. Let T_1 be the $\mathcal{L}_1$-theory which results from the addition of $\text{NAP}_{\sim,\phi}$ to non-free second-order logic.

Let $\mathcal{L}_2$ be the second-order language whose non-logical constants are the abstraction operator $\S_2$ and constant c (which will denote an urelement). Let T_2 be the theory which result from the addition to negative free second-order logic both $\text{FAP}_{\sim,\phi}$ and the following sentence:

$$\psi \overset{\text{df}}{=} \exists x(x = c) \wedge \forall F(\S_2 F \neq c)$$

ψ says that 'c' refers (which is required since the logic is free) and that c is not the abstract of any concept.

Then, the claim of this appendix is that T_1 and T_2 are mutually interpretable. That is, there is a translation $\tau_1 : \mathcal{L}_2 \rightarrow \mathcal{L}_1$ such that, for any formula ϕ of $\mathcal{L}_2$, if $T_2 \vdash \phi$ then $T_1 \vdash \tau_1(\phi)$. And conversely, there is a translation $\tau_2 : \mathcal{L}_1 \rightarrow \mathcal{L}_2$ such that for any formula ϕ of $\mathcal{L}_1$, if $T_1 \vdash \phi$ then $T_2 \vdash \tau_2(\phi)$.

(I will from this point on omit the subscripts on NAP and FAP, with the assumption that the relation is always $\sim$, and the restriction is always ϕ.)

1.1. Interpreting T_2 in T_1

To show that T_1 interprets T_2, we need a translation $\tau : \mathcal{L}_2 \rightarrow \mathcal{L}_1$. This will take the form of definitions of $\S_2$ and c in the language $\mathcal{L}_1$. The definition of $\S_2$ will be a formula $\sigma(F, x)$ such that $T_1 \vdash \forall F \forall x \forall y(\sigma(F, x) \wedge \sigma(F, y) \rightarrow x = y)$. The definition of c will be a formula $\gamma(x)$ such that $T_1 \vdash \forall x \forall y(\gamma(x) \wedge \gamma(y) \rightarrow x = y)$. This then induces a translation by treating each occurrence of $\S_2 F$ and c as definite descriptions $(\iota x)\sigma(F, x)$ and $(\iota x)\gamma(x)$ respectively, which are then eliminated in the usual Russellian manner.[10]

[10]Note that only a uniqueness requirement is placed on these definitions, but no existence requirement. This is because the theory which is being interpreted is in a free logic so the constant may fail to refer and the function may fail to be total.

The following definitions suffice:

$$\sigma(F,x) \stackrel{\text{df}}{=} x = \S_1 F \wedge \phi(F)$$
$$\gamma(x) \stackrel{\text{df}}{=} \exists F(\neg\phi(F) \wedge x = \S_1 F)$$

It is easy to check that the required uniqueness conditions obtain.

We need to show that, for any theorem of T_2, T_1 proves its translation. However, it is well known that for translations of this sort, it is sufficient to show that this holds for the axioms of T_2, that is, for FAP and ψ. We can show this as follows:

For FAP, first note that:

$$\begin{aligned} \tau(FAP) &= \tau\big[\forall F\forall G(\S_2 F = \S_2 G \leftrightarrow \phi(F) \wedge \phi(G) \wedge F \sim G)\big] \\ &= \forall F\forall G(\tau(\S_2 F = \S_2 G) \leftrightarrow \phi(F) \wedge \phi(G) \wedge F \sim G) \\ &= \forall F\forall G(\exists x(\sigma(F,x) \wedge \sigma(G,x)) \leftrightarrow \phi(F) \wedge \phi(G) \wedge F \sim G) \end{aligned}$$

Now, to prove the left to right direction, suppose that $\sigma(F,x) \wedge \sigma(G,x)$. Thus $\phi(F)$, $\phi(G)$ and $\S_1 F = x = \S_1 G$. So, from the left to right direction of NAP, $F \sim G$, and we have the right hand side of $\tau(FAP)$ as required.

For the right to left direction, suppose that $\phi(F) \wedge \phi(G) \wedge F \sim G$. Let $x = \S F$. Then, immediately, $\sigma(F,x)$. In addition, the right hand side of NAP clearly obtains (since $\phi(F) \vee \phi(G)$ and $F \sim G$), so, by the right to left direction of NAP, $\S G = \S F = x$, and so $\sigma(G,x)$ as well. Hence we have $\exists x(\sigma(F,x) \wedge \sigma(G,x))$ as required.

For ψ, first note that:

$$\begin{aligned} \tau(\psi) &= \tau(\exists x(c = x) \wedge \forall F(c \neq \S_2 F)) \\ &\equiv \exists x\gamma(x) \wedge \forall F\forall x(\gamma(x) \rightarrow \neg\sigma(F,x)) \end{aligned}$$

The first conjunct follows fairly immediately from the assumption that the restriction is not vacuous, together with the fact that, for NAP-style restrictions, every concept is still assigned an abstract.

For the second conjunct, suppose for contradiction that $\gamma(x)$ but $\sigma(F,x)$. $\gamma(x)$ is $\exists F(\neg\phi(F) \wedge x = \S_1 F)$. Let G be such a concept, so that $\neg\phi(G)$ and $x = \S_1 G$. $\sigma(F,x)$ is $\phi(F) \wedge x = \S_1 F$. Thus $\S_1 F = \S_1 G$, and so, by NAP, $F \sim G$. But since $\phi(F)$ and $\neg\phi(G)$, this contradicts the assumption that ϕ is a congruence.

So, we have it that T_1 proves the translations of FAP and ψ. It follows that T_1 proves the translation of any theorem of T_2 by a simple but tedious induction (where the base case is what has just been proved). Thus T_2 is interpretable in T_1.

1.2. Interpreting T_1 in T_2

The procedure is the reverse of the previous direction. This time we need a translation $\tau : \mathcal{L}_1 \to \mathcal{L}_2$. In this case, there is just the abstraction operator which requires definition. This definition will take the form of a formula $\sigma(F, x)$ such that $T_2 \vdash \forall F \exists x \forall y(\sigma(F, y) \leftrightarrow y = x)$. In contrast to the interpretation of T_2 in T_1, there is an existence requirement here, since the logic of T_1 is not free.

A suitable σ is the following:

$$\sigma(F, x) \stackrel{\text{df}}{=} (\phi(F) \wedge x = \S_2 F) \vee (\neg\phi(F) \wedge x = c)$$

Existence and uniqueness are simple to prove. Again, to show that this is an interpretation, it is sufficient to show that $T_2 \vdash \tau(NAP)$. First note that:

$$\begin{aligned}\tau(NAP) =& \tau\big[\forall F \forall G(\S_1 F = \S_1 G \leftrightarrow \phi(F) \vee \phi(G) \to F \sim G)\big]\\ =& \forall F \forall G(\exists x(\sigma(F, x) \wedge \sigma(G, x)) \leftrightarrow (\phi(F) \vee \phi(G) \to F \sim G))\end{aligned}$$

To prove the left to right direction: Assume that the left hand side holds, so that there is x such that $\sigma(F, x) \wedge \sigma(G, x)$. To show the right hand side, suppose that $\phi(F) \vee \phi(G)$. Without loss of generality, suppose $\phi(F)$. Thus, $x = \S_2 F$, and, by ψ, $x \neq c$. Thus, since $\sigma(G, x)$, $\phi(G)$ and $\S_2 F = \S_2 G$. So, by the left to right direction of FAP, we thus have $F \sim G$ as required.

To prove the right to left direction: Assume that the right hand side holds, so that $\phi(F) \vee \phi(G) \to F \sim G$. Now, either $\phi(F)$ or $\neg\phi(F)$. In the former case, by our assumption, $F \sim G$, and so, since $\sim$ is a congruence, $\phi(G)$. So $\phi(F) \wedge \phi(G) \wedge F \sim G$, and thus by the right to left direction of FAP, $\S_2 F = \S_2 G$. Let $x = \S_2 F$, and then $\sigma(F, x) \wedge \sigma(G, x)$ as required.

Suppose instead that $\neg\phi(F)$. Again, since $\sim$ is a congruence with respect to ϕ, $\neg\phi(G)$. Then let $x = c$, and we have $\sigma(F, x) \wedge \sigma(G, x)$ as required.

So T_2 proves the translation of NAP. As before, a simple but tedious proof by induction will show that T_2 proves the translation of any theorem of T_1 and hence T_1 is interpretable in T_2.

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References

Bencivenga, Ermanno (2002). "Free Logics". In: *Handbook of Philosophical Logic, 2nd Edition*. Ed. by D. M. Gabbay and F. Guenthner. Vol. 5. Kluwer, pp. 147–196 (cit. on p. 3).

Boolos, George (1989). "Iteration again". In: *Philosophical Topics* 17, pp. 5–21 (cit. on pp. 5, 7).

Cook, Roy T. (2003). "Iteration one more time". In: *Notre Dame Journal of Formal Logic* 44.2, pp. 63–92 (cit. on p. 6).

Frege, Gottlob (1884). *The Foundations of Arithmetic*. translated J.L. Austin. Northwestern University Press (cit. on p. 2).

— (1893). *The Basic Laws of Arithmetic*. Translated Montgomery Furth. University of California Press (cit. on p. 2).

Hale, Bob (1987). *Abstract Objects*. Blackwell (cit. on p. 1).

— (2000). "Abstraction and set theory". In: *Notre Dame Journal of Formal Logic* 41.4, pp. 379–398 (cit. on p. 6).

— (2005). "Real Numbers and Set theory–Extending the Neo-Fregean Programme Beyond Arithmetic". In: *Synthese* 147.1, pp. 21–41 (cit. on p. 6).

Hale, Bob and Crispin Wright (2001). *The Reason's Proper Study: Essays toward a Neo-Fregean Philosophy of Mathematics*. Oxford: Clarendon Press (cit. on p. 1).

— (2003). "Responses to commentators". In: *Philosophical Books* 44.3, pp. 245–263 (cit. on p. 3).

— (2008). "Abstraction and Additional Nature". In: *Philosophia Mathematica* 16.2, pp. 182–208 (cit. on p. 3).

— (2009). "Focus restored: Comments on John MacFarlane". In: *Synthese* 170.3, pp. 457–482 (cit. on p. 3).

Kaplan, David (1972). "What is Russell's Theory of Descriptions?" In: *Bertrand Russell: A Collection of Critical Essays*. Ed. by D. Piers. New York, pp. 227–244 (cit. on p. 5).

MacFarlane, John (2009). "Double vision: two questions about the neo-Fregean program". In: *Synthese* 170.3, pp. 443–456 (cit. on p. 3).

Potter, Michael and Peter Sullivan (2005). "What Is Wrong with Abstraction?" In: *Philosophia Mathematica* 13.2, pp. 187–193 (cit. on p. 3).

Rumfitt, Ian (2003). "Singular terms and arithmetical logicism". In: *Philosophical Books* 44.3, pp. 193–219 (cit. on p. 3).

Shapiro, Stewart (2000). "Frege meets Dedekind: a neologicist treatment of real analysis". In: *Notre Dame Journal of Formal Logic* 41.4, pp. 335–364 (cit. on p. 6).

— (2003). "Prolegomenon to any future neo-logicist set theory: abstraction and indefinite extensibility". In: *The British Journal for the Philosophy of Science* 54.1, pp. 59–91 (cit. on p. 6).

Shapiro, Stewart and Alan Weir (2000). "'Neo-Logicist' Logic is not Epistemically Innocent". In: *Philosophia Mathematica* 8.2, pp. 160–189 (cit. on p. 3).

Shapiro, Stewart and Crispin Wright (2006). "All Things Indefinitely Extensible". In: *Absolute Generality*. Ed. by Agustin Rayo and Gabriel Uzquiano. Oxford: Oxford University Press, pp. 255–304 (cit. on p. 6).

Wright, Crispin (1983). *Frege's Conception of Numbers as Objects*. Aberdeen: Aberdeen University Press (cit. on p. 1).

— (2001). "Is Hume's principle analytic?" In: *The Reason's Proper Study: Essays toward a Neo-Fregean Philosophy of Mathematics*. Oxford: Clarendon Press, pp. 307–332 (cit. on p. 5).