The Continuum Hypothesis and the Universe: Reflections on Independence and Existence

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This paper explores some philosophical and mathematical relationships between the continuum hypothesis, the axiom of choice, logical independence within mathematics, the existence of a higher being, the size of the universe, and some of the objects that compose it. Different comparisons are made between the nature of various classes of infinities and their consistency with our universe, referencing the viewpoints of Cantor and Peirce.

Keywords: Actual infinity, potential infinity, Cantorian continuum, Peircean continuum.

Introduction

The problem of the size of the continuum, the first millennium problem proposed by David Hilbert in the famous 1900 conference and which George Cantor needed to be exactly the size of the second infinity, did not have a solution until much later thanks to the work of Kurt Gödel and Paul Cohen. However, the solution was not as expected: the size of the continuum is independent of the logical-mathematical axiom system being used, so its magnitude could be equivalent to the second infinity or another infinity without implying any contradiction within mathematics itself.

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What consequences beyond the simple mathematical fact could this answer bring? We know that Cantor was a believer and considered God as something absolute and incomparable in reason. However, if we were to view God as something more consistent and try to compare Him with the mathematical continuum, what could we derive from its independence? Could we find any meaning in the size of the universe? Could the mathematical continuum then be the best model to understand the confusing world of the philosophical continuum?

Development

The continuum hypothesis tormented Cantor for much of his life, which involved a considerable number of arguments aimed at attempting a proof. Many infinite subsets of real numbers possess the cardinality of the continuum, and almost any of these has a perfect subset, making it natural to think that the assumption is correct. However, this reasoning is not true when we consider mathematics augmented with the now known axiom of choice. As we know, for Cantor[2] it was natural to have a well-ordering in any set; that is, for him, it was always possible to order the elements of a set so that there is a first element, a property equivalent to the axiom. Therefore, the truth or falsity of the continuum hypothesis depends on how we want to view mathematics, that is,

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[2] We can think of a perfect set as one that has enough real numbers so that between any pair of them, there is always another number from the set.

the axioms under which we are developing our thoughts, and with that, the freedom to place said cardinal in the desired position within the sequence of alephs, except for certain, let's say, singularities.

On the other hand, the figure of a supreme being that is beyond human comprehension, a figure to which the existence of the universe and, especially, of human beings is attributed. This idea is quite natural and logical, since in the absence of sufficient scientific arguments or when understanding the world becomes complicated, the most common thing is to adopt the simplest idea as an axiom.

Under the assumption that there will always be unanswered questions, the necessity for the idea of a superhuman figure arises. If we assume that God exists, then He would necessarily have to be unique. What does this mean? If we thought of God as the mathematical continuum, we could derive a kind of continuum-God hypothesis. What element of the sequence could we assign to Him? Would this imply the loss of His uniqueness?

It is important to consider that $\aleph_2$ is the cardinality of the set of all real-variable functions, and among these, those that are continuous have a cardinality of $\aleph_1$. However, every mathematician knows that this is not necessarily what we want to convey. Quoting the philosopher Peirce, who offered a perspective on mathematics: "... the only thing they require is adequate internal coherence...". If the problem of the size of the continuum can be thought of, then we can reflect on the size of the continuum and its utility, which may not be reflected currently, but this does not predict the future (as an example, think of the application of number theory to cryptography and coding theory). The internal coherence of all these thoughts is natural and essential in this context.

How many infinities can exist? The set of natural numbers encapsulates the simplest and, at the same time, most complicated notion of what infinity can be. Viewed as a whole, it is a type of constructible infinity associated with the idea of how many objects there are in each place. However, as Cantor demonstrated, this is not the only type of infinity. Knowing that the whole is greater than the sum of its parts when that whole is something tangible, which we would call finite, and thinking about this notion when the objects are no longer so tangible, that is, when they are infinite, reveals that this statement is no longer necessary, but it is just a quality of infinity.

The size of the set of real numbers can be understood as the size of the power set of natural numbers. In this case, as in any type of nature set, we will always have that this is larger than the set itself. The postulate "To each point on the line corresponds a unique real number and reciprocally to each real number corresponds a unique point on the line," along with the fact that rational numbers do not correspond uniquely to these points, and thus real numbers fill in the missing gaps, induces us or aims to make us think about the continuum or something continuous. This is an object of ideal nature, undefined, indeterminate, perfect, innocuous, omnipresent, and in some sense that I will specify later, "empty", with the mathematical object called real numbers, whose form transcends beyond the rational numbers, which are incomplete in their mathematical and metaphysical sense by not occupying all the points on that line, as the Pythagoreans noted when trying to locate the diagonal of a square concerning the length of its sides. Within this framework and following Cantor, given the bijection between a line, any of its segments, a square, and other subsets of real numbers or their Cartesian products, the variable and indeterminate nature of the continuum stands out, whose contradiction is intrinsic in the primitive concept of point.

The process of taking a set, then calculating its power set to obtain a larger set, offers us, when this is infinite, an unlimited amount of infinities, each broader than the other. Cantor presents us with the sequence of alephs, which is nothing more than placing infinities in an order, and thus asking where the cardinality of the continuum goes, and what importance it would have for it to be the next infinity.

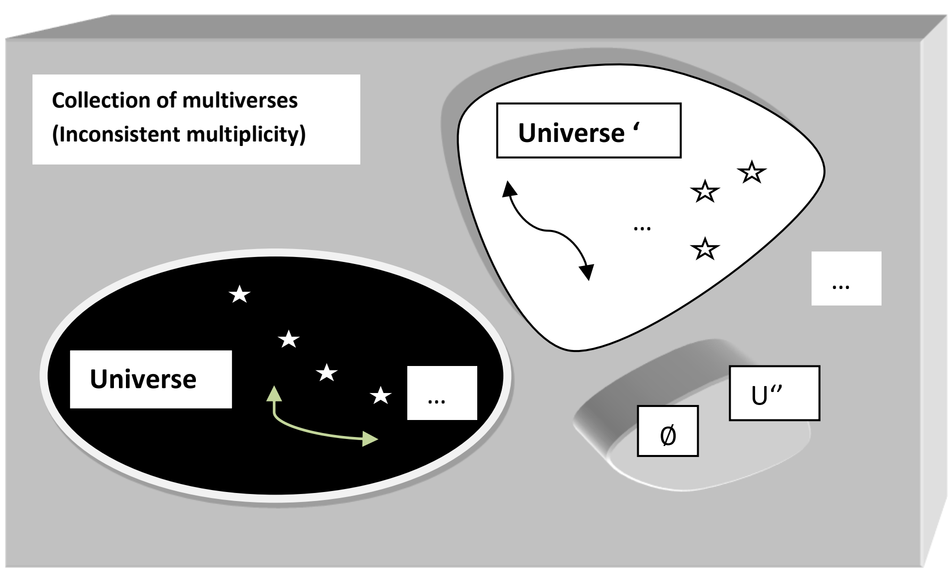
Can we associate the location of this with a way of thinking about the conception of the universe? If the universe is infinite, it is natural to think that the mathematical model for its cardinality is that of real numbers. Is it natural to think that stars are enumerable? Not necessarily finite. But is the universe finite? And if it is finite, finite with respect to what? Perhaps another type of infinite and possibly higher cardinal nature.

Is God one of the alephs, or is He of a completely superior nature to any of them? Is God singular, or should we consider the possibility that He is plural, without excluding the possibility that He is empty and, therefore, a quite natural and at the same time metaphysical description of the best way to describe the contradiction in man? If this set must be unitary, then there should be a single God, modulo "isomorphisms". If it is not, this should generate a quantity, at most, enumerable of them. But why at most enumerable? What does it mean for this collection to be continuous?

"All beings, whether finite or infinite, are defined and, except for God, can be determined by the intellect," is a beautiful phrase from Cantor that reflects the impossibility of making God rational. There will always be shadows and gaps around Him, making Him an inconsistent multiplicity, something that exists but is beyond the rational. Thus, God is incomparable to the continuum; however, an irrational number or, rather, a transcendental number in the real world, would not be as rational or real as an algebraic one? The difference lies in that these objects belong to reason, while God is simply absolute.

Consider the collection called God, which may be unitary or not. Within the answer's humanity seeks, God is a possible solution. But would He have the answers we would need? Would He possess all human knowledge, past, present, and future, like a continuum like the Akashic records? The question of who created God arises, being our creator, would He not need His own creator? For some, the answer is that we created Him. This would lead to a chain of gods, each being the creator of the other, as there will always be doubts (along with the additional question of which of them should be called our God). Therefore, the collection of gods must be at least enumerable. If we remain in the assumption that the universe is something absolute, I find no arguments to make this chain anything more than enumerable.

Now consider the universe in all its dimensions, that is, the collection that encompasses the galaxy and all the stars unknown to man, let's label it with the letter U. This object can be thought of as one and multiple at the same time, and we could call it the absolute universal set, as it includes every thinkable object, even those that exist only in our minds, regardless of their inconsistency. We face a natural dilemma: is the universe infinite or finite?



If we think of the universe as an unlimited object, it implies that it is not bounded by a higher nature. We know that the natural mathematical space to model this is $\mathbb{R}^3$, which has the same size $\mathbb{R}$. Now, let's consider the collection of stars. Is this a finite set? Better yet, consider any celestial body, understood as any object in space that occupies a place. If the collection of these objects is finite, the only real infinite object in the universal set would be the universe itself.

Our thoughts are also real and bring with them potential infinities, which we can think of as actual, but I am referring to a "tangible" infinite object. Although we will never be able to empirically verify its existence (because being infinite, its verification will depend more on the need for it to be so, following the principle of "it must be achieved with an instant act by which one suddenly stands outside the world of experience and human operations"[3]), this infinite would capture the notion of a continuum. In this sense, the infinite obtained from the universal set would be the only "real" infinite. Now, what position could we assign it in the sequence of alephs? Would it be $\aleph_0$ or $\aleph_1$?

As mentioned before, stars, if not finite, could deserve to have the first infinite cardinal. At first glance, this statement seems plausible. How could there be an amount of stars comparable to the set of real numbers, especially considering that the latter is dense, and density is a property of the continuum?

However, the reason why stars do not have the mentioned property is that we are imagining them within a larger collection and denying their density relative to that collection. It is like thinking of real numbers within a higher and unobservable object, like those times when atoms were

[3] Taken from Brouwer and Intuitionism by José Montesinos [4]

considered indivisible. We could then be imagining completing them with "meta-numbers" that, depending on the barrier from which we observe them, could be the famous infinitesimals (this last part following a style that we consider Peircean).

Although we must not forget how real numbers are constructed from rational numbers, it is important to recognize that, like many mathematical objects, we extend them out of the need for "symmetry" to make the theory more attractive and complete. An example of this is the notion that every vector space has a basis and the existence of an algebraic closure. It is crucial to highlight the use of Zorn's lemma and, consequently, the axiom of choice in these contexts.

As mentioned before, irrationals are not less rational than a negative integer literally. The conception of quantities beyond the first cardinal seems to belong more to our mind than to reality. Is the application of mathematics a model for reality, or could it be applied inversely?

Now, the problem of determining whether the stars have the mentioned cardinal is not so evident. Why couldn't they have a higher cardinal or a different aleph from the first? Suppose they have the cardinal $\aleph_1$? In that case, what structure would occupy the place of $\aleph_0$? Following this line of thought, we could assign the universe a cardinal $\aleph_n$ with n greater than one.

Whatever the choice of cardinal, observe that, in the end, it will result in the same as being an aleph. Applying the principles of generation would only lead us to the next question: finding the next object in the universe that models it. At this point, it would be best to remain in our mind and only consider these objects as possible formations of subsets of previously defined sets, i.e., the power set of the collection of stars. We must always be bounded by the universe, which restricts us from accepting something superior to it for the moment.

The assumption of assigning the stars the cardinal of the Cantorian continuum, $\aleph_1$ in ZFHC, would contradict the idea that the set of real numbers is an adequate model to express a continuum, as it would be natural to think of them as discrete. However, I want to emphasize that I am considering them as isolated structures, not dependent on the universe. On the other hand, this also shows why for Peirce, a true continuum is above any cardinal.

Now, extrapolating this, what importance would it have, from a purely mathematical or perhaps philosophical point of view, that there exists a single universe or that it transcends beyond science fiction the so-called multiverses or multiverses? Before addressing this, observe that, although some physicists might make sense of the expansion of the universe into a "meta-universe," which would be the collection of all these universes, it would force the creation of phenomena like the Big Bang theory. However, we could not predict what relationship would derive from the theory of everything or string theory in this context, especially when the existence of the Higgs boson is already being discussed.

If these multiverses were possible, regardless of arguments in physics, let's think of the alephs as representable objects of these. This idea, which I doubt is original, but I will give my interpretation, would assign each $\aleph_\alpha$ with α ordinal a universe U, thus having a function between the collection of alephs and the collection of universes, which as far as is known are not sets, and the latter could not be thought of as a universe in the same way that the collection of alephs is not an aleph. Thinking in Cantor's style, we would wish for this function to be bijective. Injectivity would be natural as it would count with an unlimited number of universes, but surjectivity would place us at the wake of a generalized continuum hypothesis.

We know that if every cardinal is an aleph, then every set can be well-ordered, and this would induce a kind of satisfaction in mathematics (dimmed by things like the Banach-Tarski paradox). This would translate into a certain order or symmetry in the given universe. We mark this order concerning phenomena so that the Big Bang would be the origin of everything, and within the universe, nothing could be thought of before it, eliminating internal doubts about how it occurred, as this would depend on the superior environment, the collection of universes.

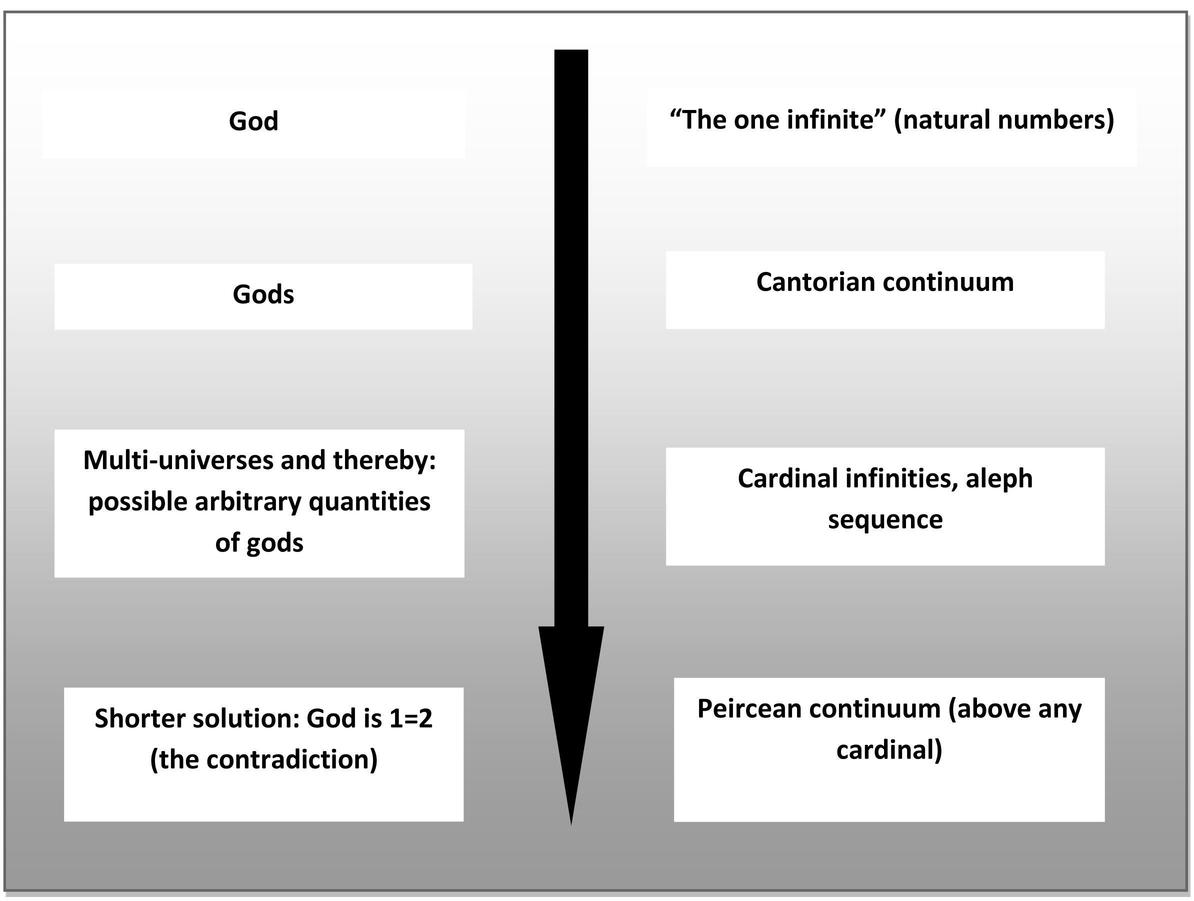
So, what position would our universe occupy in the $\aleph_\alpha$-sequence? Before addressing this, let's consider whether we could indeed assign it one of these cardinals. To do this, let's examine the internal structure of each universe and compare whether ours is “more or less regular” than the others.

In some representations of universes in science fiction, the structure is delineated by probability theory. Each decision taken would generate a possibility, and some of these possibilities would be more probable than others, depending on the rules of the given universe. These rules would be determined by previously calculated probabilities (in our universe, the first probability to calculate would be that of its own creation). Every move we make carries a decision, and each decision builds a new universe. In this context, we would have access to unimaginable amounts of universes.

This approach suggests that the multiplicity of universes can be understood in terms of constant branching of possibilities, where each branching represents an alternative reality based on the decisions made. However, assigning a specific cardinal from the $\aleph_\alpha$-sequence to our universe would depend on how this multiplicity is conceptualized and what mathematical or logical criteria would be applied for its classification.

Considering whether our universe occupies a special position in the $\aleph_\alpha$-sequence is an intriguing question. While the idea of universes generated by different possibilities and human decisions is fascinating, it is also important to explore the conditions that would allow the formation of life and how these universes could evolve over time.

By proposing to assign a finite ordinal arbitrarily to universes that do not meet adequate conditions for life or that collapse quickly, a way to distinguish between "special" universes and others that are not. Postulating that our universe is in a special position, represented by the ordinal $\aleph_\omega$, it may be an interesting choice. The restriction of the mathematical continuum, where its co-finality must be different from $\aleph_0$ according to König's results, raises questions about whether the Cantorian continuum truly models the continuum of our universe.



Conclusion

To conclude, the real world is full of limitations to our understanding. Human beings, being the only rational beings, seek to eliminate these limitations, and mathematics is the ultimate tool for this. However, mathematics also surprises and can exist with absolute freedom, taking paths and shortcuts where contradiction lurks. Nonetheless, there are many propositions that cannot be proven (based on axiomatics), and the continuum hypothesis is one of these undecidable propositions. No matter where one wants to place the cardinality of numbers, there is absolute freedom (except for matters regarding its co-finality). It is worth noting that the astonishing aspect lies in the fact that it can be demonstrated that something cannot be demonstrated, according to the author. Furthermore, the independence of this problem does not affect the development of more practical mathematics (although Gödel said that the cardinality of the real numbers should be $\aleph_2$ due to its relationships in functional analysis).

On the other hand, what is God? A precise definition of this cannot be given. It is not enough to say omniscient, benevolent, etc., as through the same mathematics one could demonstrate the existence of such a being. Therefore, the existence of God is something relative that generates more and more doubts. One argument is the existence of several gods, but this would be a problem with a response like the size of the continuum.

Similarly, considering the universe as limitless represents the ideal of a true continuum. It would possess points like planets or any celestial body, but these would only be small annexes to it. It's like thinking of a vacuum-sealed room, with no gravity and no objects inside; would it truly be empty? No, because it contains space inside (even if we have removed all points or objects). This would be a continuum and is empty in a sense (not mathematical since the idea of imagining an empty set is somewhat misleading). So, is the cardinality of this continuum that of the real numbers, or should it have no cardinality? The collection of stars is naturally discrete, but this cannot be proven since it is like a real current infinity. It could be biunivocal with the real numbers, or these are just a necessity for completing the rational numbers, so no collection of the real world can achieve its cardinality. However, the fact that this question can be asked entails the notion of points, and thinking of stars as points, and since they are neither connected nor perfect with respect to the universe, relating them to the real numbers implies that these do not fill what a continuum is. Therefore, is the continuum hypothesis a gap between the continuous and the discrete?

If the universe were bounded, there would be a higher collection. Then, the alephs or the unlimited amount of infinities are models for this event. The independence of the continuum hypothesis, what independence would it bring in these multi-universes, would structure some universes better than others, likewise if every cardinal were an aleph, it would show a beginning of creation of these and with-it countless doubts about God.

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