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Intuitionism, Justification Logic, and Doxastic Reasoning

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by

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APPROVAL

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Abstract

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In this Dissertation, we examine a handful of related themes in the philosophy of logic and mathematics. We take as a starting point the deeply philosophical, and—as we argue, deeply Kantian—views of L.E.J. Brouwer, the founder of intuitionism. We examine his famous first act of intuitionism. Therein, he put forth both a critical and a constructive idea. This critical idea involved digging a philosophical rift between what he thought of himself as doing and what he thought of his contemporaries, specifically Hilbert, as doing. He sought to *completely separate* mathematics from mathematical language, and thereby logic. In chapter 3, we examine the philosophical foundations for this separation. Artemov Artemov (2001) articulates what we might think of as *constructive propositional reasoning* in a formal system that augments classical propositional logic with a theory of proofs. In doing this, instead of using just one type of object to characterize *constructive reasoning*, he uses two; propositions and proofs. In chapter 4, we explore the extent to which it might make sense to think of *classical propositional reasoning* as instead a theory that has two types of objects in the Artemov style. In chapters 5 and 6, we examine two specific case studies; we look at two philosophical phenomena that admit of formal characterizations and then propose those. In both cases, we focus on predicate style treatments of modality.

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Contents

Chapter 1

Introduction

1.1 Introduction

In this Dissertation, we examine a handful of related themes in the philosophy of logic and mathematics. We take as a starting point the deeply philosophical, and—as we argue, deeply Kantian—views of L.E.J. Brouwer, the founder of intuitionism. We examine his famous first act of intuitionism. Therein, he put forth both a critical and a constructive idea. This critical idea involved digging a philosophical rift between what he thought of himself as doing and what he thought of his contemporaries, specifically Hilbert, as doing. He sought to *completely separate* mathematics from mathematical language, and thereby logic. It is because of this critical aspect that it is strange to think of Brouwerian and Dummettian intuitionism as members of the same genus at all! This is the focus of chapter 3.

In chapter 3, we discuss Brouwer's eight-fold enumeration of stages in which what begins as an intuitive process descends to eventually become symbolic manipulation devoid of content (Brouwer, 1907, pp. 94-95, 173-175). Even if we leave behind some of Brouwer's specific ideas about what reasoning should be like, we can start from the hypothesis that it has some sort of intuitive character and the business of proposing and creating logical systems is to capture that. The challenge, then, is to provide a formal articulation of that idea. In the propositional case, Heyting's *Intuitionistic Propositional Calculus*, or IPC, is one such attempt at responding to this challenge. Artemov Artemov (2001) takes up this challenge as well. He articulates what we might think of as *constructive propositional reasoning* in a formal system that augments classical propositional logic with a theory of proofs. In doing this, instead of using just one type of object to characterize *constructive reasoning*, he uses two; propositions and proofs. In chapter 4, we discuss the extent to which an Artemov-style proposal can be refitted for a *classical propositional reasoning*. The reader will note that, of course, classical propositional reasoning already has a formal articulation in CPC. We explore the extent to which it might make sense to think of *classical propositional reasoning* as instead a theory that has two types of objects.

In chapters 5 and 6, we examine two specific case studies; we look at two philosophical phenomena that admit of formal characterizations and then propose those. In chapter 5, we look at Gödel's Disjunction, the claim that either the power of the human mind surpasses that of any machine or that there are absolutely unsolvable problems. We follow previous attempts at studying this thesis in formal systems of arithmetic augmented with modalities for an epistemic feature that in some way represents the power of the human mind. Those approaches have traditionally treated that modality as an operator. We depart from those approaches, though, and treat that modality as a *predicate*. We discuss some candidate systems.

Then, in chapter 6, we provide a response to Mannourry's 1927 challenge through the Dutch Mathematical Society to axiomatize intuitionistic arithmetic. While the accepted answer to this has been Heyting Arithmetic HA, we suggest instead that an epistemic feature ought to be added to capture what Brouwer had in mind. We propose one such system.

1.2 Logical Preliminaries

In the first Chapter, we introduce the logical preliminaries that we will refer to throughout this Dissertation. We introduce two propositional logics, a handful of modal logics, two justification logics, and then the modal extensions of arithmetic that we will discuss in Chapters 5and 6.

1.3 Kantian Brouwer

L.E.J. Brouwer famously argued that mathematics was *completely separated from formal language*. His explanation for why this is so leaves room for interpretation. Indeed, one might ask: what sort of philosophical background is required to make sense of the strong anti-linguistic views of Brouwer? In this chapter, we outline some possible answers to the above. We then present an interpretation that we argue best makes sense of Brouwer's first act.

1.4 Justification Logic

Artemov, building upon a tradition beginning with Kolmogorov and Gödel, developed a paradigm for understanding Constructive Reasoning in terms of classical proofs. In 1933, Gödel Gödel (1933) showed that Intuitionistic Propositional Logic could be interpreted in $\mathsf{S4}$ by prefixing every subformula of an intuitionistic propositional formula with the modal operator $\Box$. Artemov Artemov (2001), in 2001, then showed that the modalities of $\mathsf{S4}$ could be realized with explicit proof terms. A consequence of this was, then, the interpretability of Intuitionistic Propositional Logic into Artemov's Logic of Proofs. This provided a novel way of understanding constructivism. Indeed, Kolmogorov-Gödel-Artemov constructivism flies in the face of the usual understanding of Constructive Reasoning as being distinguished from Classical Reasoning in terms of its theory of truth.

When Classical Reasoning is formally presented, it is usually done so in terms of Classical Propositional Logic. Constructive Reasoning, also, is usually presented in terms of Intuitionistic Propositional Logic. Call this approach the truth paradigm. But, just as Artemov showed that we can understand Constructive Reasoning of a theory of justification, can it correspondingly make sense to think of *Classical Reasoning* in terms of a theory of justification? That is, can we present a justification paradigm account of Classical Reasoning? In this Chapter, we examine the extent to which we can understand a Kolmogorov-Gödel-Artemov-style picture of Classical Reasoning. We present one such justification-based account of Classical Reasoning. The traditional truth paradigm account of Classical Reasoning leads to the well-known paradoxes of material implication. We show that the justification account of Classical Reasoning avoids this problem.

1.5 Gödel's Disjunction

In this Chapter we investigate epistemic predicates in extensions of arithmetic. We use as our case study Kurt Gödel's 1951 thesis that either the power of the human mind surpasses that of any finite machine or there are absolutely unsolvable problems. Because Gödel also claimed that his disjunction was a mathematically established fact, we must ask: what sort of syntactical object should formalize *human reason*?

In this Chapter, we lay the foundations for a *predicate* treatment of this epistemic feature. If we were to do this with an operator, we will see, we would be unable to prove a Gödel sentence for that epistemic feature. The predicate approach, on the other hand, allows for the proof of a corresponding Gödel sentence. We begin with a very general examination of the Gödel sentence in the arithmetical context. We then discuss two systems of modal predicates over arithmetic. The first, called Coreflexive Arithmetic or CoPA, extends PA

with a coreflective modal predicate but does not contain a consistency statement. The second, called Doxastic Arithmetic, or DA, has as its characteristic feature the consistency statement but does not contain coreflection or its instance, the 4 axiom. We examine the logical properties of, motivations for, and criticisms of both systems. We close with a brief comparison of the systems in the context of Gödel's Disjunction.

1.6 Intuitionistic Arithmetic

L.E.J. Brouwer famously took the subject's intuition of time to be foundational and from there ventured to build up mathematics. Despite being largely critical of formal methods, Brouwer valued axiomatic systems for their use in both communication and memory. Through the Dutch Mathematical Society, Gerrit Mannoury posed a challenge in 1927 to provide an axiomatization of intuitionistic arithmetic. Arend Heyting's 1928 axiomatization was chosen as the winner and has since enjoyed the status of being the *de facto* formalization of intuitionistic arithmetic. We argue that axiomatizations of intuitionistic arithmetic ought to make explicit the role of the subject's activity in the intuitionistic arithmetical process. While Heyting Arithmetic is useful when we want to contrast constructed objects with platonistic ones, Heyting Arithmetic omits the contribution of the subject and thus falls short as a response to Mannoury's challenge. We offer our own solution, Doxastic Heyting Arithmetic, or DHA, which we contend axiomatizes Brouwerian intuitionistic arithmetic.

Chapter 2

Logical Preliminaries

In this chapter, we introduce the logical preliminaries that we will refer to throughout this dissertation. We will discuss three sorts of systems, modal propositional, justification propositional, and first-order modal. We introduce them in turn.

2.1 Propositional Logics

As a base for the logics in 2.3.1 and 2.4.1, we introduce the following two propositional logics. These are in the following propositional language:

Definition 2.1.1 *Language of Propositional Logic*

For atoms A and formulas F and G:

$$\bot \; | A \; | \neg F \; | F \vee G \; | F \wedge G \; | F \rightarrow G$$

are all formulas in the language of propositional logic.

$\bot$ is the logical constant falsum. Here $A, B, C, \ldots$ are atomic propositional formulas. $\neg$ is the unary connective symbol for negation. $\vee$, $\wedge$, $\rightarrow$ are the symbols for disjunction, conjunction, and conditional, respectively. We use $\neg F$ as an abbreviation for $F \to \bot$ and abbreviate $\vee$ and $\wedge$ as usual.

We refer to the following as bases for other logics we consider. First, we have Classical Propositional Logic:

Definition 2.1.2 *Classical Propositional Logic* CPC

1. $F \to (G \to F)$
2. $(F \to (G \to H)) \to ((F \to G) \to (F \to H))$
3. $(F \wedge G) \to F$
4. $(F \wedge G) \to G$
5. $F \to (G \to (F \wedge G))$
6. $F \to (F \vee G)$
7. $G \to (F \vee G)$
8. $(F \to H) \to ((G \to H) \to ((F \vee G) \to H))$
9. $\bot \to F$
10. $\neg\neg F \to F$
11. *Modus Ponens*

The Deduction Theorem holds for all the logics we consider. It is provable in the familiar way as follows:

Theorem 2.1.3 *Deduction Theorem*

$\vdash F \to G \Leftrightarrow F \vdash G$

From left to right, if $\vdash F \to G$, then, if we assume F, by Modus Ponens we will prove G.

From right to left, if we assume $F \vdash G$, there are three cases. If $G = F$, then $F \to G$ is just $F \to F$, which is provable from no assumptions. If G is an axiom, then G is provable without F and $G \to (F \to G)$ is provable by axiom 1. Then, $F \to G$ follows by Modus Ponens. The final case is that G follows by Modus Ponens, from X and $X \to G$. $(X \to G) \to (F \to (X \to G))$ is an instance of axiom 1. By Modus Ponens on this and $X \to G$, we get $F \to (X \to G)$. Now, $(F \to (X \to G)) \to ((F \to X) \to (F \to G))$ is an instance of axiom 2. By Modus Ponens on this and what we obtained from the previous Modus Ponens, we get $(F \to X) \to (F \to G)$. Since $\vdash X$ and $X \to (F \to X)$ is another instance of axiom 1, we get $F \to X$, by Modus Ponens. With that, our instance of axiom 2, and one more Modus Ponens, we get $\vdash F \to G$.

Definition 2.1.4 *Intuitionistic Propositional Logic* IPC

1. $F \to (G \to F)$
2. $(F \to (G \to H)) \to ((F \to G) \to (F \to H))$
3. $(F \land G) \to F$
4. $(F \land G) \to G$
5. $F \to (G \to (F \land G))$
6. $F \to (F \lor G)$
7. $G \to (F \lor G)$
8. $(F \to H) \to ((G \to H) \to ((F \lor G) \to H))$
9. $\bot \to F$

10. Modus Ponens

It is worth observing that in this presentation of classical and intuitionistic propositional logic, CPC and IPC are identical, except for the double negation axiom $\neg\neg F \to F$.

2.2 Modal Propositional Logics

The propositional modal logics we discuss are in the following language:

Definition 2.2.1 *Language of Modal Logic*

For atoms A and formulas F and G:

$$\bot \mid A \mid \neg F \mid F \vee G \mid F \wedge G \mid F \to G \mid \Box F$$

are all formulas in the language of modal logic.

The reader will note that this is just the language from Definition 2.1.1, extended with the unary operator $\Box$. The modality $\Box$, of course, has a number of interpretations including *metaphysical necessity*, *knowledge*, and *belief*.

2.3 Principles

The modal logics we discuss begin from bases of CPC or IPC and introduce various principles and rules. The first two we consider are *Distribution*, or the K principle, and the rule *Necessitation*. *Distribution* is the following principle:

$$\Box(F \to G) \to (\Box F \to \Box G)$$

This says that $\Box$ distributes over conditional. In other words, if $\Box$ holds of a conditional, and it holds also of the antecedent of that conditional, then it holds of the consequent.

The rule *Necessitation* is the following:

$$\vdash F \ \Rightarrow \vdash \Box F$$

This says that if F is provable, then $\Box F$ is provable. It is worth noting that this *only* applies to things that are provable in the logic under discussion. Necessitation does not apply to any additional assumptions.

A modal principle that will come up quite a bit in Chapters 6 and 5 is the *Consistency*, or D, principle. It is the following:

$$\neg \Box \bot$$

This says that it is not the case that $\bot$ is necessary, or known, or believed, depending on the interpretation of $\Box$. In the context of normal modal logics, this corresponds to the seriality condition on the accessibility relation in Kripke frames.

Due to the definition of $\neg$ as implication to $\bot$, *Consistency* can be rewritten as follows:

$$\Box\bot \rightarrow \bot$$

Looking at it like this, *Consistency* is but one instance of another famous modal principle, *Factivity*. *Factivity*, or the T axiom, is the following principle:

$$\Box F \rightarrow F$$

This says that if it is necessary that F, then F holds. In other words, $\Box F$ is *factive*. In the context of Kripke models, this corresponds to the reflexivity condition on the accessibility relation.

An interesting principle is the converse of the above. This principle is called *Cofactivity*, it is:

$$F \to \Box F$$

If we limit the above to antecedents that are $\Box$'d formulas only, we get the $\mathsf{4}$ axiom, also known as *Positive Introspection*:

$$\Box F \to \Box \Box F$$

This corresponds to the transivity condition on the accessibility relation.

There is also the *Negative Introspection* principle, or $\mathsf{5}$ axiom. This is the following:

$$\neg \Box F \to \Box \neg \Box F$$

This corresponds to the Euclidean condition on Kripke frames.

The B axiom is the following:

$$F \to \Box \neg \Box \neg F$$

This corresponds to the symmetry condition on Kripke frames.

2.3.1 Modal Logics

The modal logics we will consider are the following:

Definition 2.3.1 *Modal Logic* K

1. *Rules and Axioms of Classical Propositional Logic;*

2. $\Box(F \to G) \to (\Box F \to \Box G)$;

3. *If* $\vdash F$ *then* $\vdash \Box F$.

Definition 2.3.2 *Modal Logic* D

1. *Rules and Axioms of Classical Propositional Logic;*

2. $\Box(F \to G) \to (\Box F \to \Box G)$;

3. $\neg \Box \bot$;

4. *If* $\vdash F$ *then* $\vdash \Box F$.

Definition 2.3.3 *Modal Logic* T

1. *Rules and Axioms of Classical Propositional Logic;*

2. $\Box(F \to G) \to (\Box F \to \Box G)$;

3. $\Box F \to F$;

4. *If* $\vdash F$ *then* $\vdash \Box F$.

Definition 2.3.4 *Modal Logic* $\mathsf{4}$ *(or* $\mathsf{K4}$*)*

1. *Rules and Axioms of Classical Propositional Logic;*

2. $\Box(F \to G) \to (\Box F \to \Box G)$;

3. $\Box F \to \Box \Box F$;

4. *If* $\vdash F$ *then* $\vdash \Box F$.

Definition 2.3.5 *Modal Logic* B

1. *Rules and Axioms of Classical Propositional Logic;*

2. $\Box(F \to G) \to (\Box F \to \Box G)$;

3. $F \to \Box \neg \Box \neg F$;

4. *If* $\vdash F$ *then* $\vdash \Box F$.

Definition 2.3.6 *Modal Logic* S4

1. *Rules and Axioms of Classical Propositional Logic;*

2. $\Box(F \to G) \to (\Box F \to \Box G)$;

3. $\Box F \to F$;

4. $\Box F \to \Box\Box F$;

5. *If* $\vdash F$ *then* $\vdash \Box F$.

Definition 2.3.7 *Modal Logic* S5

1. *Rules and Axioms of Classical Propositional Logic;*

2. $\Box(F \to G) \to (\Box F \to \Box G)$;

3. $\Box F \to F$;

4. $\Box F \to \Box\Box F$;

5. $F \to \Box\neg\Box\neg F$;

6. *If* $\vdash F$ *then* $\vdash \Box F$.

We also consider the intuitionistic versions of these. These are defined as follows:

Definition 2.3.8 *Modal Logic* iK

1. *Rules and Axioms of Intuitionistic Propositional Logic;*

2. $\Box(F \to G) \to (\Box F \to \Box G)$;

3. *If* $\vdash F$ *then* $\vdash \Box F$.

Definition 2.3.9 *Modal Logic* iD

1. *Rules and Axioms of Intuitionistic Propositional Logic;*
2. $\Box(F \to G) \to (\Box F \to \Box G)$;
3. $\neg \Box \bot$;
4. *If* $\vdash F$ *then* $\vdash \Box F$.

Definition 2.3.10 *Modal Logic* iT

1. *Rules and Axioms of Intuitionistic Propositional Logic;*
2. $\Box(F \to G) \to (\Box F \to \Box G)$;
3. $\Box F \to F$;
4. *If* $\vdash F$ *then* $\vdash \Box F$.

Definition 2.3.11 *Modal Logic* i4 *(or* iK4*)*

1. *Rules and Axioms of Intuitionistic Propositional Logic;*
2. $\Box(F \to G) \to (\Box F \to \Box G)$;
3. $\Box F \to \Box \Box F$;
4. *If* $\vdash F$ *then* $\vdash \Box F$.

Definition 2.3.12 *Modal Logic* iB

1. *Rules and Axioms of Intuitionistic Propositional Logic;*
2. $\Box(F \to G) \to (\Box F \to \Box G)$;
3. $F \to \Box \neg \Box \neg F$;

4. *If* $\vdash F$ *then* $\vdash \Box F$.

Definition 2.3.13 *Modal Logic* iS4

1. *Rules and Axioms of Intuitionistic Propositional Logic;*

2. $\Box(F \to G) \to (\Box F \to \Box G)$;

3. $\Box F \to F$;

4. $\Box F \to \Box\Box F$;

5. *If* $\vdash F$ *then* $\vdash \Box F$.

Definition 2.3.14 *Modal Logic* iS5

1. *Rules and Axioms of Intuitionistic Propositional Logic;*

2. $\Box(F \to G) \to (\Box F \to \Box G)$;

3. $\Box F \to F$;

4. $\Box F \to \Box\Box F$;

5. $F \to \Box\neg\Box\neg F$;

6. *If* $\vdash F$ *then* $\vdash \Box F$.

Artemov and Protopopescu introduce their Intuitionistic Epistemic Logics in Artemov and Protopopescu (2016). Artemov and Protopopescu's *Intuitionistic Epistemic Logic of belief* is the following:

Definition 2.3.15 *Modal Logic* IEL−

1. *Rules and Axioms of Intuitionistic Propositional Logic;*

2. $\Box(F \to G) \to (\Box F \to \Box G)$*;*

3. $F \to \Box F$*.*

Artemov and Protopopescu's *Intuitionistic Epistemic Logic of knowledge*:

Definition 2.3.16 *Modal Logic* IEL

1. Rules and Axioms of Intuitionistic Propositional Logic;

2. $\Box(F \to G) \to (\Box F \to \Box G)$*;*

3. $\neg \Box \bot$*;*

4. $F \to \Box F$*.*

2.4 Justification Logics

We will also discuss Justification Logics. Justification Logic was introduced by Sergei Artemov in Artemov (1995) and Artemov (2001) as a way of making explicit the modalities in modal logic. We follow the presentation of Artemov (2001) here. Consider, for example, the modal formula:

$$\Box F \to \Box F$$

Does this express the simple tautology that $\varphi \to \varphi$ or does this express something about the connection between the modalities? Explicit modal logic allows us to make the distinction between these senses.

We will use the following language for the Justification Logics we discuss:

Definition 2.4.1 *Justification Logic Language*

Justification terms are defined as follows for variables x*, constants* c *and terms* t *and* s*:*

$$x \mid c \mid t + s \mid t \cdot s \mid !t \mid ?t$$

For atoms A, formulas F and G, and justification terms t, formulas are defined as follows:

$$\bot \mid A \mid \neg F \mid F \vee G \mid F \wedge G \mid F \rightarrow G \mid t : F$$

Now, instead of modalities we have proof terms that can be fixed to formulas of any length. We read $t : F$ as t is a proof of F. There are two types of simple proof terms; proof constants and proof variables. There are also operations on our proof terms. The operation $+$ expresses that a proof is found in the sum of the terms. Consider the modal formula we looked at above. We could translate this into either:

$$t : F \rightarrow t : F$$

or

$$t : F \rightarrow t + s : F$$

The first is just the tautology $\varphi \rightarrow \varphi$, of course. The second expresses that if t is already a proof of F, then the sum of t and any other s is a proof of F. This principle is called *Sum*, we will see that it is in all of our Justification Logics.

Does the converse hold? That is, should it be the case that $t + s : F \rightarrow t : F$? Not necessarily. Imagine that the proof of F draws on both t and s but is present in its entirety in neither.

There is another proof function for application $\cdot$, and related principle, *Application*:

$$t : (F \rightarrow G) \rightarrow (s : F \rightarrow (t \cdot s) : G)$$

This gives an explicit sense to Modus Ponens. The idea is that $t \cdot s$ expresses in the language the Modus Ponens step we would do to conclude G from a proof t of the conditional and a proof s of the antecedent.

There is an explicit version of the modal Reflection axiom. This is also known as Reflection. If it is ever ambiguous, we disambiguate by calling this *Explicit Reflection*. This is the following principle:

$$t : F \rightarrow F$$

This says that if something is a proof of F, then F is true.

Positive proof checker is the axiom relating to the symbol !, pronounced "bang." It is the following axiom:

$$t : F \rightarrow !t : t : F$$

The idea is that if t is a proof of F then there is a proof, $!t$, that verifies that t is in fact a proof of F. This is, of course, the explicit version of positive introspection or the 4 axiom.

Negative proof checker is an axiom for the ? symbol. It is:

$$: F \rightarrow ?t : \neg t : F$$

This is the explicit version of negative introspection. negative proof checker

2.4.1 Logics

There are explicit justification logics corresponding to the modal logics mentioned above. We will, though, only focus on a few systems for the purposes of this Dissertation. These are the ones corresponding to S4 and S5.

Sergei Artemov's Logic of Proofs (LP) is the following Artemov (2001)[p. 8]:

Definition 2.4.2 *The Logic of Proofs*

1. *Axioms of Classical Propositional Logic;*

2. $t : (F \to G) \to (s : F \to (t \cdot s) : G)$ *Application;*

3. $t : F \to (t + s) : F$ *and* $t : F \to (s + t) : F$ *Sum;*

4. $t : F \to F$ *Reflection;*

5. $t : F \to !t : t : F$ *Proof Checker;*

6. *An Axiomatically Appropriate Constant Specification;*

7. *Modus Ponens.*

Building upon a Classical Propositional base with Modus Ponens, we begin by adding the Application and Sum axioms. We then add the Explicit Reflection axiom and the Proof Checker axiom.

At this point, the above looks quite a bit like an explicit verison of S4 without Necessitation. For this, we add an *axiomatically appropriate constant specification.* A Constant Specification is a set of formulas of the form $c_n : c_{n-1} \ldots c_1 : A$, where A is an axiom. Whenever $c_n : c_{n-1} : \ldots A$ is in our constant specification, so is $c_{n-1} : \ldots A$. An axiomatically appropriate constant specification is one where every axiom, including new axioms in the constant specification has a justification. There are other types of constant specifications.

An axiomatically appropriate constant specification is essentially the explicit version of axiom necessitation. That is, instead of having modal necessitation on all of our axioms with $\Box$ we have some justification constant c_a for each of our axioms a. We then inductively prove that this holds not just for axioms but for everything provable, this is *Constructive Necessitation* for LP, which is just the explicit version of the full necessitation of modal logic.

Where a *ground proof polynomial* is one that does not contain proof variables, it is well-known that the following holds Artemov (1995) and Artemov (2001)[p. 10]:

Theorem 2.4.3 *Constructive Necessitation (Artemov 1995)*

If LP $\vdash F$ *then* LP $\vdash p : F$ *for a ground proof polynomial* p*.*

The reasoning for this is a straightforward induction on provability. If F is an axiom, the claim holds by our axiomatically appropriate constant specification. If F follows by modus ponens, and the theorem holds for G and $G \to F$, then there is a proof t of G and a proof s of $G \to F$. Using Application and modus ponens, we then get $t \cdot s : F$.

We will examine also the Justification Logic JS5. This corresponds to the modal logic S5. It is the following:

Definition 2.4.4 JS5

1. *Axioms of Classical Propositional Logic;*
2. $t : (F \to G) \to (s : F \to (t \cdot s) : G)$ *Application;*
3. $t : F \to (t + s) : F$ *and* $t : F \to (s + t) : F$ *Sum;*
4. $t : F \to F$ *Reflection;*
5. $t : F \to !t : t : F$ *Proof Checker;*
6. $\neg t : F \to ?t : \neg t : F$ *Negative Introspection;*
7. *An Axiomatically Appropriate Constant Specification;*
8. *Modus Ponens.*

The logic JS5 also enjoys an explicit version of necessitation (due to Pacuit (2006) and Rubtsova (2006)):

Theorem 2.4.5 *(Pacuit 2006, Rubtsova 2006) If* JS5 $\vdash F$ *then* JS5 $\vdash p : F$ *for a ground proof polynomial* p*.*

2.5 First-Order Arithmetic

We will focus on a couple specific extensions of Classical and Intuitionistic First-Order Logic. Specifically, we will look at systems of formal arithmetic.

Definition 2.5.1 *Language of First-Order Arithmetic*

The language of first-order arithmetic consists of terms generated as follows:

$$\mathbf{0} \mid x \mid \mathbf{s}(t_1) \mid t_1 + t_2 \mid t_1 \cdot t_2$$

Given terms t_1, t_2, the equality symbol $=$, and formulas F and G, we generate formulas as follows:

$$\bot \mid t_1 = t_2 \mid \exists x F \mid \forall x F \mid F \rightarrow G \mid F \wedge G \mid F \vee G \mid \neg F$$

The symbols $\mathbf{s}$, $+$, and $\cdot$ are the symbols for *successor*, *addition*, and *multiplication*, respectively. $\exists$ and $\forall$ are symbols for the existential and universal quantifiers, respectively.

We will use $\ulcorner F \urcorner$ as the arithmetical code of F. We introduce $t_1 \neq t_2$ as an abbreviation for $\neg(t_1 = t_2)$.

For our purposes, consider any Hilbert-style axiom system for Classical First-Order Logic with equality. Peano Arithmetic is the following:

Definition 2.5.2 *Peano Arithmetic*

1. *Rules and axioms of* FOL *with equality;*

2. $\forall x(\mathbf{s}(x) \neq \mathbf{0})$*;*

3. $\forall x \forall y(\mathbf{s}(x) = \mathbf{s}(y) \rightarrow x = y)$*;*

4. $\forall x(x + \mathbf{0} = x)$*;*

5. $\forall x \forall y(x+\mathbf{s}(y) = \mathbf{s}(x+y))$;

6. $\forall x(x \cdot \mathbf{0} = \mathbf{0})$;

7. $\forall x \forall y(x \cdot \mathbf{s}(y) = x \cdot y+x)$;

8. $F(\mathbf{0}) \wedge \forall x(F(x) \to F(\mathbf{s}(x))) \to \forall x F(x)$.

The intuitionistic counterpart of the above is Heyting Arithmetic, or HA. This consists of Peano's arithmetical axioms and a base of first-order intuitionistic logic with equality. We make use of the following list of axioms of intuitionistic predicate logic from Troelstra and van Dalen (Troelstra and van Dalen, 1988, pp. 68, 48).

Definition 2.5.3 IQC *with Equality*

1. $(F \wedge G) \to F$ *and* $(F \wedge G) \to G$;

2. $F \to (G \to (F \wedge G))$;

3. $F \to (F \vee G)$ *and* $F \to (G \vee F)$;

4. $(F \to H) \to ((G \to H) \to ((F \vee G) \to H))$;

5. $F \to (G \to F)$;

6. $(F \to (G \to H)) \to ((F \to G) \to (F \to H))$;

7. $\bot \to F$;

8. $F(x/t) \to \exists x F$ *(where* t *is free for* x *in* F*)*;

9. $\exists x(F \to G) \to (\exists y F(x/y) \to G)$ *(where* x *is not free in* G*, and* y *is* x *or* y *is not free in* F*)*;

10. $\forall x F \to F(x/t)$ *(where* t *is free for* x *in* F*)*;

11. $\forall x(G \to F) \to (G \to \forall y F(x/y))$ *(where x is free in G, and y is x or y is not free in F);*

12. $x = x$;

13. $(A(x) \land x = y) \to A(y)$;

14. *Modus Ponens;*

15. $\Gamma \vdash F \Rightarrow \Gamma \vdash \forall x F$ *(where x is not free in Γ).*

(1)-(7) are the axioms of Intuitionistic Propositional Calculus. (8)-(11) are the first-order axioms. (12) and (13) are the equality axioms, and (14) and (15) are our *Modus Ponens* and Universal Quantifier Rules.

Heyting Arithmetic (HA) is the following:

Definition 2.5.4 *Heyting Arithmetic*

1. *Rules and axioms of* IQC *with equality;*

2. $\forall x(\mathbf{s}(x) \neq \mathbf{0})$;

3. $\forall x \forall y(\mathbf{s}(x) = \mathbf{s}(y) \to x = y)$;

4. $\forall x(x+\mathbf{0} = x)$;

5. $\forall x \forall y(x+\mathbf{s}(y) = \mathbf{s}(x+y))$;

6. $\forall x(x \cdot \mathbf{0} = \mathbf{0})$;

7. $\forall x \forall y(x \cdot \mathbf{s}(y) = x \cdot y+x)$;

8. $F(\mathbf{0}) \land \forall x(F(x) \to F(\mathbf{s}(x))) \to \forall x F(x)$.

Note that (8) is schematic.

2.6 Arithmetical Systems with Modalities

The last sort of system we will look at are arithmetical systems with modalities. We will look at both cases where that modality is treated as an operator and treated as a predicate. Within a single system we will never have a modality treated as *both* an operator and a predicate. For this reason, we just use the symbol $\Box$ for both. This will be clear from context. When $\Box$ is affixed to formulas, as in $\Box F$, it is the symbol for the operator. When it is affixed to terms, as in $\Box(x)$ or $\Box(\ulcorner F \urcorner)$, it is the symbol for the predicate. We make use of Boolos brackets $[\ ]$ as well.[1] The following is the language for both systems treating the modality as an operator and as a predicate:

Definition 2.6.1 *Language of Modal Arithmetic*

Terms are built up as follows:

$$\mathbf{0} \mid x \mid \mathbf{s}(t_1) \mid t_1 + t_2 \mid t_1 \cdot t_2$$

For terms t_1 and t_2 and formulas F and G, formulas are generated as follows:

$$\bot \mid t_1 = t_2 \mid \exists x F \mid \forall x F \mid F \rightarrow G \mid F \wedge G \mid F \vee G \mid \neg F \mid \Box[F] \mid \Box F$$

The new element here are of course the predicate $\Box(x)$ and the operator $\Box$.

There are a handful of logical systems that we will talk about to illustrate various points in this dissertation. These are the following. First, we have Stewart Shapiro's *Epistemic*

[1] Where $su(\mathbf{i}, j, k)$ is the substitution function. As Boolos presents them, they are the following (Boolos, 1995a, p. 45). Provided a coding where $\ulcorner F \urcorner$ is the Gödel number for F, we read it as "substitute numeral $\mathbf{i}$ for the j^{th} variable in the formula encoded with k." Where F is a formula with m free variables $v_{k_1} \ldots v_{k_m}$, where $k_1 < \ldots k_m$, $\Box[F]$ is:

$$\Box(su(\mathbf{k_m}, v_{k_m}, \ldots, su(\mathbf{k_2}, v_{k_2}, su(\mathbf{k_1}, v_{k_1}, \ulcorner F \urcorner)) \ldots))$$

There are interesting questions about quantifying in, see (Boolos, 1995a, p. 225-6), though these are not immediately relevant to the issue at hand and so we set them aside for the time being.

Arithmetic, or EA: EA is the following system:

Definition 2.6.2 EA

1. *Rules and axioms of Peano Arithmetic;*
2. $\Box(F \to G) \to (\Box F \to \Box G)$;
3. $\Box F \to F$;
4. $\Box F \to \Box\Box F$;
5. $\vdash F \;\Rightarrow \vdash \Box F$.

It is easy to see that this is just PA extended with the modal logic S4. We will consider also the intuitionistic version of EA, call it *Epistemic Heyting Arithmetic*, or EHA:

Definition 2.6.3 EHA

1. *Rules and axioms of Heyting Arithmetic;*
2. $\Box(F \to G) \to (\Box F \to \Box G)$;
3. $\Box F \to F$;
4. $\Box F \to \Box\Box F$;
5. $\vdash F \;\Rightarrow \vdash \Box F$.

We will consider a version of PA extended with a cofactive, *predicate style*, modality. This, we call *Cofactive Peano Arithmetic*, or CoPA:

Definition 2.6.4 CoPA

1. *Rules and Axioms of* PA*;*

2. $\Box[F \to G] \to (\Box[F] \to \Box[G])$;

3. $F \to \Box[F]$.

We consider also the intuitionistic version of this, called *Cofactive Heyting Arithmetic*, or CoHA:

Definition 2.6.5 CoHA

1. *Rules and Axioms of* HA;

2. $\Box[F \to G] \to (\Box[F] \to \Box[G])$;

3. $F \to \Box[F]$.

A big theme of this dissertation is Doxastic Arithmetic, PA extended with a KD predicate, and Doxastic Heyting Arithmetic, HA extended in the same way. Doxastic Arithmetic, or DA, is the following:

Definition 2.6.6 *Doxastic Arithmetic* DA

1. *Axioms and rules of* PA;

2. $\Box[F \to G] \to (\Box[F] \to \Box[G])$

3. $\neg \Box (\ulcorner \mathbf{0} = \mathbf{1} \urcorner)$

4. $\mathsf{DA} \vdash F$ *then* $\mathsf{DA} \vdash \Box[F]$

Here, and throughout, $\mathbf{1}$ is defined as $s(\mathbf{0})$. The intuitionistic system, Doxastic Heyting Arithmetic, or DHA, is the following:

Definition 2.6.7 *Doxastic Heyting Arithmetic* DHA

1. *Axioms and rules of* HA;

2. $\Box[F \to G] \to (\Box[F] \to \Box[G])$

3. $\neg \Box (\ulcorner \boldsymbol{0} = \boldsymbol{1} \urcorner)$

4. $\mathsf{DHA} \vdash F$ *then* $\mathsf{DHA} \vdash \Box[F]$

This is the intuitionistic version of Doxastic Arithmetic, introduced in Peluce (2018) and Peluce (2020). We follow naming conventions introduced there.

Chapter 3

Brouwerian Intuitionism

3.1 Two Aspects of the First Act

Following Michael Dummett, it would not be hard for someone to identify intuitionism with the acceptance of intuitionistic logic. Indeed, Dummett begins his "The Philosophical Basis of Intuitionistic Logic" (Dummett, (1975, p. 97):

> The question with which I am here concerned is: What plausible rationale can there be for repudiating, within mathematical reasoning, the canons of classical logic in favour of those of intuitionistic logic? I am, thus, not concerned with justifications of intuitionistic mathematics from an eclectic point of view, that is, from one which would admit intuitionistic mathematics as a legitimate and interesting form of mathematics alongside classical mathematics: I am concerned only with the **standpoint of the intuitionists themselves**, namely that classical mathematics employs forms of reasoning which are not valid on any legitimate construal of mathematical statements (save, occasionally, by accident, as it were, under a quite unintended reinterpretation). Nor am I concerned with exegesis of the writings of Brouwer or of Heyting: the question is what forms of justification

> of intuitionistic mathematics will stand up, not what particular writers, however eminent, had in mind. And, finally, I am concerned only with **the most fundamental feature of intuitionistic mathematics, its underlying logic**, and not with the other respects (such as the theory of free choice sequences) in which it differs from classical mathematics. [Bold ours]

Perhaps the first thing one learns of L.E.J. Brouwer is his critical stance toward the use of logic in mathematics. Obviously, if the most fundamental feature of intuitionistic reasoning were underlying logic, Brouwer would not be an intuitionist.

Apart from this negative point about what, contra Dummett, intuitionism *is not*, we set aside further discussion of *intuitionism itself*. In this chapter, we explore the philosophical basis of Brouwer's intuitionism. Specifically, we focus on the source of his critical stance toward logic.

Brouwer's aversion to formal methods is codified in his *first act of intuitionism*. For example, we have the statement in his *Cambridge Lectures* (1946-50) (Brouwer, 1981, p. 4):

> FIRST ACT OF INTUITIONISM Completely separating mathematics from mathematical language and hence from the phenomena of language described by theoretical logic, recognizing that intuitionistic mathematics is an essentially languageless activity of the mind having its origin in the perception of a move of time. This perception of a move of time may be described as the falling apart of a life moment into two distinct things, one of which gives way to the other, but is retained by memory. If the twoity thus born is divested of all quality, it passes into the empty form of the common substratum of all twoities. And it is this common substratum, the empty form, which is the basic intuition of mathematics.

Above we see both the critical aspect of the first act and a creative one. The critical aspect is of course that mathematics is separated from mathematical language, and thereby logic. The creative aspect, on the other hand, has to do with the way mathematics itself is generated in accordance with—what Brouwer calls—*the perception of the move of time*.

While not referred to as "the first act," it is not hard to trace both of these themes backward in Brouwer. For example, in Brouwer's 1912 "Intuitionism and Formalism"—which, of course, *is* a sustained criticism of formalism—we have a clear statement of the creative aspect (Brouwer, 1912, p. 85):[1]

> [N]eo-intuitionism considers the falling apart of moments of life into qualitatively different parts, to be reunited only while remaining separated by time as the fundamental phenomenon of the human intellect, passing by abstracting form its emotional content into the fundamental phenomenon of mathematical thinking, the intuition of bare [twoity].

"Neo-intuitionism" is, of course, what Brouwer calls his own view in his 1912 paper to contrast his position with the French intuitionists.

We find many explicit statements of the creative aspect around Brouwer's 1907 dissertation. Consider the following passage that was rejected from Brouwer's 1907 dissertation van Dalen (2013)[p. 87]:

> The primeval phenomenon is simply the intuition of time, in which the iteration of 'thing-in-time, and one more thing' is possible, but in which (and this is a phenomenon outside mathematics) a sensation can resolve into constituent qualities, such that a single moment of life is lived as a sequence of qualitatively distinct things... One can, however, restrict oneself to the simple observation of those sequences as such, independent of the emotional content, that is from the

[1] Here, and throughout this dissertation, we standardize "two-one-ness", "two-ity", and "twoity" to just *twoity*.

> various gradations of frightfulness and desirability of that which is observed in the outer world. (Restriction of the attention to intellectual contemplation).

We find a statement of the critical aspect in Brouwer's dissertation (Brouwer, 1907, p. 92):

> [Logistics] can teach us nothing about the foundations of mathematics, because it remains irrevocably separated from mathematics...

How strong is the critical aspect of the first act intended? In other words, exactly what sort of separation is he invoking? Most simply, the distinction between formal and mathematical reasoning could be one of degree or quality.

First, Brouwer could have the view that construction *outstrips* formal language in some sense. This would allow that very simple formal reasoning counts as mathematical reasoning, but then when a certain level of complexity is achieved, that formal reasoning loses its mathematical character. Call this the *permissivist interpretation*, because it allows for *some* formal reasoning to count as mathematical reasoning.

Second, Brouwer could think that uses of formalism are just in principle different from mathematics itself. On this reading, even very simple formal reasoning corresponding mathematical reasoning still would not be mathematical reasoning. Call this the *impermissivist interpretation*, because, of course, on this interpretation *no* formal reasoning counts as mathematical reasoning.[2]

Briefly, we note that Brouwer is completely comfortable with the use of formalism for aiding 1) communication and 2) memory. Indeed, the previous quote from Brouwer's dissertation continues (Brouwer, 1907, p. 92):

> [Logistics] can teach us nothing about the foundations of mathematics, because it remains irrevocably separated from mathematics; on the contrary, in order to

[2] We will discuss Brouwer's *Enumeration of Stages which are confused in the logical treatment of mathematics* in the following section.

> maintain an existence on its own account, i.e., to safeguard itself against contradictions, [logistics] must reject all its own special principles and **acquiesce to be a faithful, automatic, stenographic copy of *the language of mathematics*, which itself *is not mathematics*, but no more than a defective expedient for men to communicate mathematics to each other and to aid their memory for mathematics**.

This is compatible with both the permissivist and impermissivist interpretations; that *formalism* could be used to aid communication and memory is independent of whether or not *formal reasoning* counts as *mathematical reasoning*.

In sections 3.2 and 3.3 we discuss two arguments in favor of the permissivist reading. The first has to do with very simple reasoning. There are passages in Brouwer that suggest that using formal methods in very simple cases one can *get the right answer*. Depending on how we understand this, this might be taken to support a permissivist interpretation. The other argument has to do with one way of reading Brouwer's claim that the set of mathematical theorems is denumerably unfinished. We argue that both of these fail to make a good case for the permissivist reading of Brouwer.

In section 3.4, we present Michael Detlefsen's interpretation of Brouwerian intuitionism. Detlefsen provides an impermissivist reading of Brouwer. We argue, however, that Detlefsen fails to justify the sharp distinction between mathematical reasoning on one hand and logical and linguistic reasoning, on the other. Then, in section 3.5 we present an interpretation of Brouwer that we argue provides a philosophical ground for an impermissivist reading. In section 3.6, we discuss the view of the continuum resulting from our interpretation of Brouwer.

3.2 Permissivist Brouwer: Simple Reasoning

There seem to be two main reasons to consider a permissivist interpretation of the critical aspect of Brouwer's first act. The first has to do with what Brouwer has to say about simple reasoning. In "The Unreliability of the Logical Principles," Brouwer writes (Brouwer, 1908, p. 109):

> Thus there remains only the more special question: 'Is it allowed, in purely mathematical constructions and transformations, to neglect for some time the idea of the mathematical system under construction and to operate in the corresponding linguistic structure, following the principles of *syllogism*, of *contradiction* and of *tertium exclusum*, and can we then have confidence that each part of the argument can be justified by recalling to mind the corresponding mathematical construction?' Here it will be shown that this confidence is well-founded for the first two principles, but not for the third.

The thought here is that *syllogism*, Brouwer gives the example of hypothetical syllogism here, and reasoning according to the principle of non-contradiction will never lead one astray. That is, if one reasons linguistically in this way, they will be able to reconstruct the relevant construction if need be.

Of course, this does not, however, hold in general for reasoning according to the principle of excluded middle. This is what gets explained as the intuitionistic unsoundness of the principle of excluded middle; it is simply not intuitionistically true that every object *can be shown* to have property P or that it *can be refuted* that this is so.

Where $s(x)$ stands for "the successor of x," and other symbols have their usual meaning, consider the following HA proof of $1 + 1 = 2$ as a paradigm example of simple reasoning:

1. $\forall x \forall y(x + s(y) = s(x + y))$ - HA axiom;

2. $\forall y(1 + s(y) = s(1 + y))$ - Universal Instantiation on 1;

3. $1 + s(0) = s(1 + 0)$ - Universal Instantiation on 2;

4. $1 + 1 = 2$ - Rewriting 3.

One reading of the above quote would suggest that perhaps examples of simple reasoning like the above could count as properly mathematical reasoning.

While it is correct that Brouwer thinks that in simple cases formal and mathematical reasoning can *converge* on the correct answer, he makes clear that syntactic transformations or linguistic reasoning, are simply distinct from real mathematical reasoning. In his eight step "Enumeration of the stages which are confused in the logical treatment of mathematics," the first four are the following (Brouwer, 1907, p. 94/173):

> 1. The pure construction of intuitive mathematical systems which, if they are applied, are turned outward in life by taking a mathematical view of the world.
>
> 2. The linguistic parallel to mathematics: mathematical speaking or writing.
>
> 3. The mathematical study of language: we notice logical linguistic structures, raised according to principles of ordinary logic or from its extension by the logic of relations, i.e. logistics, but the elements of these linguistic structures accompany mathematical structures or relations.
>
> 4. Forgetting the sense of the elements of the logical figures mentioned just now, and imitating the construction of these figures by a new mathematical system of *second* order, provisionally without a language parallel to the construction; this is the system of logisticians, which, if it is in the least generalized by a free extension, is very well pervious to contradiction, unless Hilbert's precautions are taken; these precautions form the main content of Hilbert's paper.

We have a four step descent above. We go from the first stage which has to do with real mental mathematical construction, to the second which has to do with speaking or writing about that construction. The third stage is the very earliest that has to do with what we think of as logical reasoning.Stages 7 and 8, in particular, are notable for our purposes. Stage 7 involves "Forgetting the sense of logical figures, and imitating their construction in anew mathematical system," Stage 8 centers upon consistency (Brouwer, 1907, p. 95/175). The HA reasoning above falls into this category. Then the fourth stage has to do with studying that which was delineated in the third stage.

Brouwer himself emphasizes that there is something important in the shift from 2 to 3, (Brouwer, 1907, p. 95/175):

> [E]ven the stages mentioned above, from the third on, are deprived of mathematical significance. Mathematics has its place only in the first; in practice it cannot remain aloof from the second, but this stage remains an unconscious non-mathematical act.

We can present this in the following table:

Table 3.1: First Four Stages of Brouwer's Enumeration

	Level 1 Mental Construction	Level 2 Language of Math	Level 3 Logic	Level 4 Metamathematics
Mathematical?	Yes	No	No	No
Linguistic?	No	Yes	Yes	Yes
About?	-	Level 1	Level 1	Level 3

The permissivist interpretation is thus not supported by the passages about simple reasoning. Indeed, Brouwer would claim that just because a first individual reasoning at stage 1 and a second individual reasoning at stage 3 might communicate the same thing (at stage 2), it does not follow that their actions were in fact the same; the individual at stage 1 is interacting with *the objects themselves* while the individual at stage 3 is dealing with *an* *abstraction from* the *language of mathematics* from stage 2—that which is used for in aiding communication and memory in (Brouwer, 1907, p. 92).

Brouwer makes this point especially clear in discussing the principle of non-contradiction. While he thinks that reasoning with the principle of non-contradiction will never lead to an output that is wrong, contradiction is a linguistic representation of *the inability of a construction to proceed further* (Brouwer, 1907, p. 73) He writes (Brouwer, 1907, p. 73):

> At the point where you announce the contradiction, I simply perceive that the construction no longer *goes*, that the required structure cannot be embedded in the given basic structure.

While in a case like this both the intuitionist and the formalist would agree that reasoning in some sense stops, Brouwer's commitment is based on the *inability to manipulate a construction* itself and not on the arrival at F and its negation for some F.

Brouwer even goes so far as to allow for the possibility of two different communities agreeing at stage 1 of the enumeration but having mutually incompatible steps to stage 2 and beyond (Brouwer, 1907, pp. 73-74):

> [I]t is easily conceivable that, given the same organization of the human intellect and consequently the same mathematics, a different language would have been formed, into which the language of logical reasoning, well known to us, would not fit. Probably there are still peoples, living isolated from our culture, for which this is actually the case. And no more is it excluded that in a later stage of development the logical reasonings will lose their present position in the languages of the cultured people.

How would this happen? The idea would be that because of some contingent features of the development of different communities, even if there is genuine convergence at stage 1, because of the ways that language developed, those ideas get expressed differently at stage 2. Indeed, when logic is viewed as that which has its origin in the second stage, it is easy to see why Brouwer would liken it to *ethnography* (Brouwer, 1907, p. 74). Indeed, this accounts for the reason for which a Brouwerian today might acknowledge that *both* Heyting's Intuitionistic Propositional Logic and Artemov's Logic of Proofs Artemov (2001) express the propositional transitions of constructive reasoning.

Now, if a mathematical reasoning *admits of* contradictory formalizations, mathematical reasoning seems to be essentially independent of formal reasoning. Here the issue with formal reasoning is not that it invariably "falls short" but rather that mathematical reasoning is independent from language in some strong sense. In other words, this example seems to suggest the *impermissivist interpretation* instead.

3.3 Permissivist Brouwer: Inexhaustibility

In this section, we discuss another argument in favor of the permissivist reading of Brouwer. This has to do with Brouwer's claim that the set of mathematical theorems was *denumerably unfinished* van Atten (2017) (see also (van Atten, 2004, p. 7-8), for discussion).

Brouwer defined *denumerably unfinished sets* as follows (Brouwer, 1907, p. 82):

> [Those sets such that] we can never construct in a well-defined way more than a denumerable subset of [that set], but when we have constructed such a subset, we can immediately deduce from it, following some previously defined mathematical process, new elements which are counted in the original set. But from a strictly mathematical point of view this set does not exist as a whole . . .

The thought here is that as soon as one thinks one has proved every mathematical theorem, they immediately realize that there is a further one left to prove. Strictly speaking, the set does not exist as a whole, because it is always under construction.

This idea was influential to Gödel. Wang reports, from Carnap's diary, on the influence of Brouwer on Gödel (Wang, 1987, p. 84):

> Gödel talked 'about the inexhaustibility of mathematics [...] He was stimulated to his idea by Brouwer's Vienna lecture. Mathematics is not completely formalizable. He appears to be right.'

It's not hard to see a *Brouwerian* philosophical justification for such a view. Carl J. Posy presents one such account (Posy, 1988, pp. 310-311):

> The idea of a finitely (or recursively) axiomatized formal system is the modern counterpart of Kant's notion of conceptual description: finitely recognizable properties whose conjunction and consequences characterize a domain of objects. And the idea of a formal language, with defined notions built on a fixed and enumerated collection of base concepts, fits the same mold. With this translation—the linguistic as the conceptual—we can indeed see the Kantian origin for Brouwer's claim that codified language cannot suffice to express the entire intuitive content of our mental life and our scientific knowledge. Intuition will always outrun linguistic description. This is Brouwer's constant view, and it underlies his rejection of mathematical formalization.

Posy's reading emphasizes the connection to Kant's notion of *conceptual description*. Notably, these are *finitely recognizable*. The problem is that, due to inexhaustibility or incompleteness phenomena, "intuition will always outrun linguistic description" (Posy, 1988, pp. 310-311). This, for Posy, is the ground for the critical aspect of Brouwer's first act.

With respect to Brouwer's claim (Brouwer, 1907, p. 82) that denumerably finished sets of theorems never can capture the set of mathematical theorems, which is denumerably *un*finished, Posy's idea is that what grounds the critical aspect of Brouwer's first act is that the set of mathematical theorems always outruns formal theorems.

With that in mind, we point out that we can find unambiguous statements of the *inexhaustibility of science* in contemporaries of Brouwer. Indeed, Michael Friedman discusses such a view in the context of the dialogue between the Marburg and Southwest Schools. In *Substance and Function* (1910), Ernst Cassirer writes (Friedman, 2000, pp. 78-9):

> That this function [of empirical cognition] does not arrive at an end in any of its activities, that it sees rather behind every solution that may be given to it a new task, is certainly indubitable. Here, in fact, "individual [*individuelle*]" reality confirms its fundamental character of inexhaustibility. But, at the same time, this forms the characteristic advantage of true scientific relational concepts: that they undertake this task in spite of its incompleteability in principle. Every new postulation, in so far as it connects itself with the preceding, constitutes a new step in the *determination* of being and happening. The individual [*Das Einzelne*] determines the direction of cognition as infinitely distant point.

For Cassirer, *science as a whole* is such that it can never be completed. "Incompleteability in principle" just is what is valuable about scientific concepts.[3]

We argue that Brouwer held a similar view of science. For Brouwer, science has to do with the classifying and cataloguing of "causal sequences and phenomena" (Brouwer, 1912, pp. 123-124). In the section of his dissertation about the value of the 'explanation' of phenomena, Brouwer writes (Brouwer, 1907, p. 58):

> Let us remark further that it can never be said afterwards that an explanation, which served its purpose in extending the region of known sequences by means of induction, was shown to be incorrect. For, in that case, a clash with experience proves no more than that on the strength of the explanation a field of induction was opened which was *too large*.

[3] Indeed, Hintikka uses such a view of science to argue against a the KK principle in epistemic logic (Hintikka, 1970, pp. 148-9).

We see that when an apparently good explanation goes wrong, it was not, strictly speaking, *incorrect*. Instead, it was correct with respect to a limited domain; it went wrong in failing to tell *the whole* story.

We saw in his *Enumeration of the stages which are confused in the logical treatment of mathematics* that Brouwer contrasts mathematical reasoning with interpreting and codifying that reasoning. The former, of course, has to do with real construction. The latter Brouwer compares to *ethnography* (Brouwer, 1907, p. 74). Indeed, in his "Unreliability of Logical Principles" (Brouwer, 1908, p. 108):

> Moreover, the function of the logical principles is not to guide arguments concerning experience subtended by mathematical systems, but to describe regularities which are subsequently observed in the language of the arguments.

Logic here is a science just as much as linguistics. What the logician looks for are regularities in the ways that people talk about mathematics.

We can thus return to Posy's interpretation that the ground for the critical aspect of Brouwer's first act is that "intuition will always outrun linguistic description" (Posy, 1988, pp. 310-311). The idea here was that intuition's *outrunning* of language is what explains the critical aspect of Brouwer's first act. We showed that, though, a sort of *outrunning* of explanation by phenomena was a feature of Brouwer's more general philosophy of science. If Posy is correct, then, Brouwer's first act is a consequence of his philosophy of science. This, it seems, gets the priorities wrong; indeed, if this is so, it would seem that Brouwer's philosophy of science is actually the *Zeroeth Act of Intuitionism*.

3.4 Detlefsen's Brouwer

In his "Brouwerian Intuitionism," Michael Detlefsen presents an impermissivist intuitionism. For Detlefsen, it is a feature of his interpretation of Brouwer that it does not rely on a traditionally Brouwerian metaphysical picture (Detlefsen, 1990, pp. 502, 525, 532). A virtue of Detlefsen's account is that it can be seen as providing a way of rejecting Dummett's thesis from (Dummett, (1975, p. 97) that what is most fundamental to intuitionism is its logic, while respecting to some extent the anti-metaphysical views expressed by Dummett in that paper. Ultimately, however, we suggest this is the problem with Detlefsen's Brouwerian intuitionism.

Detlefsen contrasts what he calls *classical epistemology* with *non-classical epistemology*. Classical epistemology, Detlefsen writes (Detlefsen, 1990, p. 509):

> [E]mphasizes the *contentual* ingredient of knowledge, and de-emphasizes the matter of its *cognitive mode*.

Here the way one gets to a conclusion matters less than *that* they got to that conclusion. So, if one gets to the conclusion that $1 + 1 = 2$, it does not matter if they did it by reasoning abstractly about twoity or if they did so by syntactically manipulating axioms. Clearly, logic is granted a significant role in classical epistemology because the manner of transition between knowledge contents is less important than the knowledge contents themselves.

Brouwerian epistemology, on the other hand, is a paradigm example of non-classical epistemology. Detlefsen reads the first act as a Brouwerian *antidote* to the mistake of classical epistemology (Detlefsen, 1990, p. 514). Detlefsen writes (Detlefsen, 1990, p. 515):

> According to Brouwer, mathematics is essentially a form of introspective constructional *activity* or *experience* whose growth or development thus cannot proceed via the logical extrapolation of its content (as classical epistemology maintains), but rather only by its *phenomenological* or *experiential* development—that is to say, its extension into *further experience* of the same epistemic kind.

The thought is, then, that for Brouwer logical extensions of mathematical knowledge are deficient because they fail to preserve that *experiential* or *phenomenological* mode of knowledge.

Logical reasoning, on Detlefsen's interpretation, *cannot* count as mathematical reasoning because *mere* convergence of content is not the same as the phenomenological development of that content. In this sense, mathematics will be independent of logic. Mathematics will be independent *even if* logic could somehow be rectified to account for *inexhaustibility* features, and rectified to preserve *soundness* by dropping, for example, the principle of excluded middle—that logic would "still serve only to *identify* those propositions that are capable of intuitionistic justification—which is a very different thing (and epistemically inferior to) actually supplying such a justification" (Detlefsen, 1990, p. 520).

What principle justifies Detlefsen's Brouwer in separating mathematical practice from logical language? Detlefsen grounds his reading in the claim (Detlefsen, 1990, p. 521):

> This tenet is the deceptively simple, though in truth quite radical, idea that mathematics, in its essence, is a form of mental activity.

Mathematics is characteristically and *in essence* a form of mental activity.[4]

But if *being essentially mental* is what explains the separation of math from logic in the first act, the following question remains. *Are there other essentially mental activities, or, is math unique in that it is the only essentially mental activity*?

In the first case—if we must grant that there are *other* essentially mental activities—we risk losing the importance of mathematics for Brouwer. Indeed, we do not want to commit Brouwer to the claim that *all* characteristically mental activities are independent of language in the sense required by the first act. Had Brouwer written more, on this interpretation we would expect a fist act of imagining, dreaming, and (a priori) philosophizing as they too seem essentially mental. Of course this does not work as an interpretation of Brouwer.

In the second case, we need to say why it is that mathematics is essentially mental.

[4] That activity still has structure, though that structure is non-logical. Detlefsen writes, "[mathematical thought] is to be thought of as a body of actions organized by a scheme of actional connections reflecting some sort of practical disposition to pass from one *act* to another, rather than a body of truths organized by a network of logical relations" (Detlefsen, 1990, p. 523).

While Detlefsen mentions repeatedly that he wants to provide a picture of intuitionism free of Brouwerian assumptions, he also gestures toward more substantial Brouwerian principles (Detlefsen, 1990, p. 524), but this not quite in the context of the principle in question. In response to the question of how individual proof activities come to constitute a global whole, Detlefsen appeals to a characteristically Brouwerian explanation (Detlefsen, 1990, p. 524):[5]

> Perhaps Brouwer's singling out of the unfolding of the bare notion of [twoity] in the mind at perfect rest, with no 'sinful' designs on the conquest of nature, and no 'cunning' or even 'playful' attempts to manipulate the stream of inner experience, can be seen as bearing on such a concern [that of the unity of the global configuration of proof activities]: those proof-activities which are dispositionally related to other proof-activities in such a way as to grow into the right sort of global practice are those of the mind at perfect (causal-manipulatory) rest, with no designs on causal dominion over nature or even over one's own stream of inner mathematical experience.

Detlefsen's explanation as one of how a mathematical life, taken as a global configuration of proof-activities, is constituted from individual proof activities has quite a bit of plausibility as an explanation of Brouwer. Indeed, we do not question the merit of this account for *that* purpose.

Detlefsen, however, does not delve further into the Kantian ground that would philosophically justify the critical aspect of the first act. Indeed, this is against his stated goal of providing an account of Brouwer free from such metaphysical assumptions. While this is a strength in that it presents an account of Brouwerian epistemology only relying upon the experience intensivity of mathematics, it is a weakness if we cannot explain just what is so special about mathematical experience. In either case, we are left wanting an explanation

[5] Again, we note that we standardize terminology to *twoity*.

for why *Brouwer should hold the strong separation between mathematics and logic*; in other words, *what could justify an impermissivist interpretation of the critical aspect of the first act*?

3.5 Kantian Brouwer

We argued that arguments for permissivist readings of the critical aspect of Brouwer's first act fail. We suggested that convergence of simple mathematical and formal reasoning does not entail that one counts as the other. We also argued that inexhaustibility, if it is to explain Brouwer's *first act*, justifies that separation in a philosophy of science, which seems to put the cart before the horse. We also argued that while Detlefsen's account explained what such a separation between mathematical and logical reasoning would look like, it did not sufficiently justify the strong separation between mathematics and language required for an impermissivist reading of the first act.

In this section, we provide a philosophical basis for an impermissivist reading of Brouwer. An interpretive problem immediately arises. While Brouwer maintained his commitments to the creative and critical aspects of the first act (though they were not always unified under that banner), other parts of his view changed over time. Since, as we saw, he developed his anti-linguistic views early in his career, we elect to focus on early Brouwer.

Recall the creative aspect of the first act. The thought is that one has an experience, say, of waiting for the train and then getting on the train.[6] They subtract away all particulars from that experience, until they reach what Brouwer calls *the intuition of bare twoity*.

It is natural to thus think about what Brouwer thinks this intuition of time is like. A way

[6]Note that Brouwer distinguishes between scientific and intuitive time. *Intuitive time* is the time that constitutes the only *a priori* element of science, while *scientific time* is the system introduced for cataloguing phenomena (Brouwer, 1907, p. 61/99, fn. 2). He points out that scientific time is *introduced* while intuitive time is, of course, not. The sort of connection suggested by our discussion, that of being able to abstract away from the more scientific to the intuitive, seems presupposed. Indeed, the *intuitive time* is what is *a priori*.

to make this precise is to ask: is time itself perceived here or is time meant as the condition of perception (and so what is referred to is actually a form of perception)? Indeed, these choices correspond to the two main figures that Brouwer is read in light of: Husserl and Kant. Call the view that we can perceive the flow of time itself the *Husserlian view*. Let the view that time is the condition for such perceptions (though not itself perceived) be the *Kantian view*.

In this section we show that a Kantian background can provide a philosophical basis for the critical aspect of Brouwer's thought.[7] We then show that there is reason to read, at least early, Brouwer in this Kantian manner.

We must begin, however, by noting two important differences between Brouwer and Kant. First, if creation according to the intuition of time is interpreted in the Kantian sense, Brouwer will be using *the intuition* here to really mean *awareness of the form of perception*. Indeed, on this view it is not *time itself* that is perceived but rather the form of time as it is inherent in particular moments. Second, for Kant logic characterizes the rules of *a priori* reasoning (see Lu-Adler (2018) and Buroker (2019)). For Brouwer, of course, logic does not have this privileged position. Instead, we have a picture where *a priori* reasoning is mathematical. The science of logic, for Brouwer, is comprised of drawing generalizations from people's linguistic behavior. It would not be hard to find other ways in which Brouwer differs from Kant (indeed, Brouwer is explicit about how he disagrees with Kant with respect to the relation between Euclidean geometry and experience (Brouwer, 1907, pp. 70-1)).

Consider now what language would do on such a view for Brouwer. Language here would associate linguistic objects with objects of a given domain of objects. In either case, we are associating objects with objects; we are fully in the realm of the phenomenal. If the inner objects of mathematics are *actually created* according to time understood in the Kantian sense, then where do we put the *the generation itself* of these mental objects? Clearly, if

[7] A similar style of proposal underlies Bentzen's diagrammatic interpretation in Bentzen (2023).

time is taken to be the condition of possibility of objects, then mathematical creation is quite unlike logical activity. Mathematical creation involves real generation, while logic has to do with the association of existing objects (again, as we saw in stage 3 of (Brouwer, 1907, p. 95), the laws of logic are generalizations drawn from mathematical language, and is thus twice-removed from construction itself). In other words, if time is taken to be the condition of possibility of objects itself and so the boundary condition for phenomena, the process of generation itself is not phenomenal.

The Kantian view thus makes mathematical generation in principle different from other types of mental generation. We see that it provides a philosophical basis for Detlefsen's tenet that mathematical thinking is in essence a mental activity. On the other hand, creation according to time in the Husserlian sense can at best say that mathematical mental creation is different from, say, logic by degree, insofar as it is a type of coming to be according to a (admittedly privileged) phenomenon.

We now examine reasons for thinking we ought to read early Brouwer in this Kantian manner. Brouwer, in the concluding subsection "Summary of the relation between mathematics and experience" of his dissertation, writes (Brouwer, 1907, p. 70):

> Mathematics develops out of its basic intuition in a self-multiplication guided by an entirely free choice. The only synthetic judgements a priori generally, are therefore those which are obtained as possibilities of mathematical constructions by virtue of the basic intuition of time.

In this short subsection, he explicitly references not only Kant but the Transcendental Aesthetic. This provides an obvious clue to look to what Kant writes (Kant, 2001a, A34/B50):

> Time is the formal condition, *a priori*, of all appearances whatsoever. . . Since all representations, whether they have for their objects outer things or not, belong in themselves, as determinations of the mind, to our inner state; and since this

> inner state stands under the formal condition of inner intuition, and so belongs to time, time is an *a priori* condition of all appearances whatsoever.

Here, of course, we find a clear articulation of the view of time suggested by our interpretation. It is this, we suggest, that grounds the critical aspect of Brouwer's first act. The interpretive picture of Brouwer we presented was that (early) Brouwer based his criticism of formal methods on the Kantian view of time. The early position lets us make especially clear, however, the sense of freedom which Brouwer invokes. While there is a sense in which logic can get at already generated *constructions*—when they are phenomenal, for example as held in memory—logic does not characterize *construction itself*, insofar as it is something free in the transcendental sense. This freedom is best characterized by Rilke (Rilke, 1982, p. 159):

> Look: trees do exist; the houses
> that we live in still stand. We alone
> fly past all things, as fugitive as the wind.
>
> - Rilke, *Second Elegy*

3.6 Early Brouwer on the Continuum

The Kantian interpretation of Brouwer emphasizes that time is a form of intuition though not itself phenomenal. This conflicts with established interpretations of Brouwer (see van Stigt (van Stigt, 1990, p. 151), for example). Indeed, perhaps the best philosophical basis for later Brouwer's analysis of the continuum is in terms of Husserlian phenomenology van Atten (2007).

Even in early Brouwer, there are claims that read most naturally as claims about the perception of the flow of time. In his dissertation, Brouwer writes "The simplest causal sequences which man can perceive have in reality for their mathematical substratum only *time* as a one-dimensional intuitive continuum; it does not matter that in them no other objects, that is invariants, than time itself, appear" (Brouwer, 1907, p. 64). While a non-Kantian Brouwer would allow that time itself feature into perception and thus provide a straightforward interpretation of these passages, a Kantian account will need to say something more; a Kantian interpretation of Brouwer would need to explain just what all these apparent references to the perception *of time* are doing.

Our task is thus twofold: we must find a way to justify a reinterpretation of the text at (Brouwer, 1907, p. 64), for example, where time *appears* and then must say something about how that interpretation could form a consistent whole. First, how can we reconcile an interpretation on which time is not itself perceived with passages that could suggest the opposite? There are passages that suggest that, instead of time itself, empirical moments are perceived and then abstracted away from. Indeed, what we called the creative aspect of the first act of intuitionism tells us that van Dalen (2013)[p. 87]:

> The primeval phenomenon is simply the intuition of time, in which the iteration of 'thing-in-time, and one more thing' is possible, but in which (and this is a phenomenon outside mathematics) a sensation can resolve into constituent qualities, such that a single moment of life is lived as a sequence of qualitatively distinct things. . . One can, however, restrict oneself to the simple observation of those sequences as such, independent of the emotional content, that is from the various gradations of frightfulness and desirability of that which is observed in the outer world. (Restriction of the attention to intellectual contemplation).

We have an experience and then abstract away from all emotional content. At last we are left with just *twoity*. This is the bare twoity of *thing in time and thing again*, the *empty form of all two-ities*. The process described here seems to be one where empirical time is perceived, and processed in some way, to arrive at the basic intuition.

This is also suggested by discussion in Brouwer's 1908 "The Unreliability of the Logical Principles." Having just argued that logic is not reliable in the case of wisdom or religious truth, Brouwer begins to discuss the reliability of logic in the mathematical case. He writes (Brouwer, 1908, p. 108):

> The question remains whether the logical principles are firm at least for mathematical systems exempt of any living sensation, i.e. systems constructed out of the abstraction of repeatable phenomena, out of the intuition of time, void of living content, out of the basic intuition of mathematics.

Here too it seems the way one gets to the intuition of time is by abstracting away from lived experience. In order to arrive at the basic intuition of mathematics, one begins with experience *and then* subtracts away particularities.

We suggest, then, that in the passages about the continuum what is being referred to as *perceived* is strictly speaking, empirical. By some process akin to the subtraction of emotional content in the case of twoity, we suggest then that the Kantian Brouwer reaches an awareness of the continuum.

There is a question about why experience provides an intuition of continuity as opposed to, say, an intuition of density. This is answered by the way Brouwer thinks about discreteness. In his 1912 "Intuitionism and Formalism," Brouwer writes (Brouwer, 1912, p. 128):

> [The] basal intuition of mathematics, in which the connected and the separate, the continuous and the discrete are united, gives rise immediately to the intuition of the linear continuum, i.e., of the "between," which is not exhaustible by the interposition of new units and which therefore can never be thought of as a mere collection of units.

Here discreteness and continuity presuppose one another; they are dual concepts. Imagine someone who abstracts away from every particularity to reach the "bare twoity". The above passage says that in order for this to be a genuine twoity, the between cannot be thought of as intermediate units of the same sort as "thing in time and thing again", but rather that *in between* must be thought of as having its own *sui generis* character. The reasoning is that if it did not have its own unique character, we would not be able to arrive at the discrete pair in the first place, for there would be infinitely many other discrete partners our attention would focus on.

We must now answer the question raised by discussion in van Atten's "Kant and Real Numbers," of how Brouwer (on our interpretation) gets around Kant's lack of access to reals van Atten (2012). Van Atten's argument is, roughly, that since Kantian synthesis of the imagination can only produce finitary images, incompletable processes never result in an image for the Kantian. Since real numbers are defined in terms of infinite sequences, Kant therefore cannot have access to them. It is natural then to think that given our early Brouwer is constrained by this sort of Kantian view, that he could not possibly have access to choice sequences as paradigmatic incomplete objects.

From the above, it is clear that Brouwer has a response to this. Exactly *how that response should be articulated* is a further interpretive question. There are two options. The first is that because, as we saw above, discreteness and continuity are equiprimordial (Brouwer, 1912, p. 128), Brouwer is committed to the possibility of an infinitary image, specifically, that of the continuum contrasted with the bare twoity. Indeed, when Brouwer claims that the discrete and continuous are equiprimordial (Brouwer, 1912, p. 128), he seems to be saying that the basic conceptual distinction is not finite and infinite but instead discrete and continuous. The second is that Brouwer maintains the Kantian view of image but holds that it is not an image that one has of the continuum but rather an awareness of discreteness and continuity as aspects of the form of perception. The thought here is that continuity is thus just as accessible as discreteness.

Indeed, the above passage from (Brouwer, 1912, p. 128) suggests that there is *one process* through which we arrive at awareness of continuity. Through the process of abstraction, we arrive at both *twoity* and the *between*; one could not have a twoity without an *inexhaustible between* and one could not have a between without the *discrete elements of twoity* as bookends. It is through the *single* process of abstraction away from emotional qualities of moments that we reach the form of discreteness and continuity.

3.7 Conclusions

In this chapter we discussed the philosophical basis for Brouwer's famous rejection of the use of formal methods in mathematics. First, we had to get clear on just what Brouwer's view was. While he indeed held that the principle of excluded middle was intuitionistically unsound, the unsoundness of certain logical principles, we argued, was not the philosophical basis of his complete separation of mathematics and language. Instead, we argued, this was a consequence thereof. We put forth that instead Brouwer's strong separation of mathematics from language had to do with a deeply held Kantian view of time. From this, we sketched how early Brouwer was able to make sense of reasoning about continuity.

Chapter 4

Justification Classicism

4.1 Truth, Knowledge, and Justification

One might think of logic broadly as the study of reasoning. The following question then arises: *what* does reasoning—and thereby logic—have to do with? That is, when we reason, what sorts of things are we reasoning about? Sergei Artemov distinguishes three competing paradigms in the development of logic Artemov (2020). These, in turn, provide three competing answers to the above. There is the *truth paradigm*, on which reasoning has to do with the behavior of *truth* and *truths*. There is the *epistemic paradigm*, under which reasoning will have to do also with the behavior of *knowledge*. The epistemic paradigm also involves truth insofar as *knowledge* involves the concept of truth. And, lastly, there is the *justification paradigm*, on which reasoning will have to do with *justification* and truth, to the extent that *justification* involves the concept of truth.

Truth, *knowledge*, and *justification* correspond to three levels of precision in logical language. If truth, for example, is the answer to the question we began with then we might think *non-modal language* is *prima facie* a sufficiently precise language to articulate the sorts of distinctions we want to make in the study of reasoning. If instead reasoning has to do with *knowledge*, propositional language will be too coarse-grained.[1] It is natural thus to think that *modal language* is the ideal level of precision at which to present our axioms, given that the behavior of knowledge and truth is what we aim to characterize with those axioms. Finally, if *justification* is taken to be the relevant concept, we will need an even more precise language. That is, we will make use of the *language of justification logic* or *explicit modal logic*.

Artemov has shown that for the purposes of modelling epistemic situations, the third paradigm is preferable (see Artemov (2011), Artemov and Fitting (2019), and Artemov and Fitting (2021)). But how do these paradigms fare as answers to the question of the proper subject of reasoning? The distinction between Classical and Constructive Reasoning was first posed by L.E.J. Brouwer, who sought to radically revolutionize mathematics.[2] He characterized his opposition as *Classical*. Today this distinction has entered common parlance. We want an account of the subject of reasoning to have a way of explaining types of reasoning. Hence, one way of measuring how well the paradigms answer the question of the proper subject of reasoning is by seeing how well they can account for the difference between *Classical Reasoning* and *Constructive Reasoning*.

The *truth paradigm* presents the following difference between Classical and Constructive Reasoning. Classical Reasoning, on this view, has to do with the behavior of *classical truth*. Within this paradigm, Classical Reasoning is formalized by non-modal logic, that is, either CPC or Classical First-Order Logic. For our purposes, we consider the propositional case since here relevant distinctions already arise. On this view, Constructive Reasoning, on the other hand, has to do with *constructive truth*. This is formalized by Intuitionistic

[1] In the intuitionistic setting, however, the behavior of double negation allows us to make this distinction. It is natural to understand F not preceded by two double negations as "F has an intuitionistic proof" and $\neg\neg F$ as "F is consistent." This is used in the modal context as well in Artemov and Protopopescu's Intuitionistic Epistemic Logic (IEL) Artemov and Protopopescu (2016). In IEL F is stronger than $\Box F$, which is stronger than $\neg\neg F$.

[2] see Carl Posy's (Posy, 2020, p. 2) for a genealogy of our use of *classical* and *constructive* in the context of mathematics, beginning with Brouwer's use of "classical mathematics" in his Brouwer (1908).

Propositional Calculus IPC.

The *epistemic paradigm* captures Constructive Reasoning by making use of Gödel's 1933 Gödel (1933) interpretation of IPC in the modal logic S4. On this view, Constructive Reasoning is articulated by S4-*knowledge*, that is, knowledge that distributes over the conditional, is factive, and satisfies positive introspection. Through a well-known argument, S5 is known to relate to CPC as S4 relates to IPC. Within the epistemic paradigm, classical logic is then articulated by the concept of S5-*knowledge*. This is like S4-knowledge except that it satisfies negative introspection as well.

Within the *justification paradigm* we can also distinguish between Classical and Constructive Reasoning. In his 2001 "Explicit Provability and Constructive Semantics," Sergei Artemov—building upon a line of reasoning going back to Kolmogorov and Gödel—famously presented a *justification paradigm* account of Constructive Reasoning Artemov (2001).[3] On this view, the relevant concept is *proof*, specifically, proof as articulated by the axioms of his system the Logic of Proofs or LP. Building upon Gödel's 1933 interpretation of intuitionistic logic in the modal logic S4, Artemov showed that the modal logic S4 could be realized in his more expressive Logic of Proofs LP.[4] In doing so, he provided a formal account of BHK semantics. In his 2020 "Manifesto of Justification Logic," Artemov writes (Artemov, 2020, p. 2):

> In particular, Justification Logic realizes Gödel's aforementioned suggestion of modeling constructive reasoning in classical logic augmented by an explicit representation of proofs. This led to a formalization of the paradigmatic constructive semantics offered by Brouwer, Heyting, and Kolmogorov.

[3] Let us distinguish between the epistemic and justification paradigms, on one hand, and the logics that are representative of them on the other. On the *logics*, one might consider justification logic just explicit modal logic or alternatively modal logic to be simplified justification logic. The relation between the paradigms will be determined between how we think knowledge (without a witness) relates to explicit justification.

[4] Artemov also provided an interpretation of LP in terms of arithmetic proofs Artemov (2001).

> [...] Justification Logic is a vibrant and lively field which—due to its foundational contributions—emerges as a basic logical paradigm of the present.

Artemov's Logic of Proofs beckons us to a paradigm of the *present* (without making a claim to eternal validity of this paradigm). Further, the Logic of Proofs provides an account of Constructive Reasoning that operates fully within a classical picture of truth. This last point is of note. Hence, what makes Constructive Reasoning constructive is not the properties of constructive truth but instead the *architecture* of proof over a background theory of classical truth (Artemov, 2001, p. 8):

1. If t is a proof of $F \to G$ and s is a proof of F, then t applied to s is a proof of G;

2. If t is a proof of F, then t extended with any other proof is a proof of F;

3. If t is a proof of F, then F is true;

4. If t is a proof of F, then the proof check of that fact is a proof that t is a proof of F;

Proof distributes to the consequent of the conditional when applied to a proof of the conditional's antecedent. It also has the property that it can be extended. That is, if someone produces a proof of F then anything extending that proof is also a proof of F. Proof is *factive*; that is, if F is proved then F is true. Lastly, proof checks are themselves a form of proof. That is, a check that t is a proof of F is itself a proof *that* t is a proof of F.

The *justification logic* corresponding to $\mathsf{S5}$—($\mathsf{JS5}$)—and its applications to formal epistemology has been discussed in the literature (see Pacuit (2006) and Rubtsova (2006)). The contribution of this chapter is not to the formal epistemological applications of $\mathsf{JS5}$. Our suggestion is instead that from the following:

$$\textit{Classical Reasoning} \hookrightarrow \mathsf{CPC} \hookrightarrow \mathsf{S5} \hookrightarrow \mathsf{JS5}$$

we can learn something about what Classical Reasoning really is. Because Classical Reasoning is fleshed-out in terms of CPC, which can be Gödel-interpreted in S5, which can then be in turn realized in JS5, we suggest something new can be learned about Classical Reasoning itself.

To the knowledge of the author, while the corresponding *justification logic* has been much discussed, a justification paradigm account of Classical Reasoning in this sense has not yet been given. In this chapter, we provide a justification paradigm account of Classical Reasoning. On our view the relevant concept is not *proof*, as it is in Artemov's account of Constructive Reasoning, but is instead *explicit justification*. Just as, in the propositional case, classical truth extends constructive truth, we suggest that explicit justification extends Artemov's notion of *proof* with the following:

> If t is *not* an explicit justification of F, then there is an explicit justification that t is not an explicit justification of F.

We will show that just as *proof* is the justification paradigm concept taking the place of *constructive truth*, *explicit justification* can do the work of *classical truth*. In doing so, we articulate the classical portion of Artemov's justification paradigm:

	Key Concept	Language	Logic
Truth Paradigm Constructivism	Constructive Truth	Propositional	IPC
Truth Paradigm Classicism	Platonistic Truth	Propositional	CPC
Epistemic Constructivism	S4-Knowledge	Modal	S4
Epistemic Classicism	S5-Knowledge	Modal	S5
Justification Constructivism	Proof	Explicit Modal	LP
Justification Classicism	Explicit Justification	Explicit Modal	JS5

Now, which pair provides the best account of the difference between constructive and Classical

Reasoning?[5] Most will find the truth paradigm intuitive. Indeed, when we first teach students about Classical Reasoning we do so by presenting CPC. It would thus not be difficult for a student to conclude that there was something conceptually privileged about CPC. At this point, we hear the non-classical logician suggestively asking:

> Why do we assume that *classical* logic is conceptually privileged?

The non-classical logician will have no shortage of shortcomings of classical logic to bring up at their disposal. Indeed, they may here put forth that while many aspects of Classical Reasoning are well-motivated, the philosophical public outside of philosophical logic has been misled into a privileging of Classical Reasoning for historical, sociological, and—importantly—*non-philosophical* reasons. And they are not wrong to look deeper into the reasons for classical logic's prominence. We ask a similar question:

> Why do we assume that *non-modal(explicit or not)* logic is conceptually privileged?

Indeed, it is worth mentioning the historical precedent set by Frege in his *Begriffsschrift* (see (Kneale and Kneale, 1962, p. 548) and (Fitting and Mendelsohn, 1998, pp. 4-5) for discussion). Just as non-classical logicians can genealogize the origin of the *classical bent*, one might do something similar for the birth of the *non-modal (explicit or not) bent*, that is, what we have called the *truth paradigm*.

With this preliminary motivation in mind, we will provide stronger support for the claim that the truth paradigm falls short in capturing the difference between Constructive and non-Constructive reasoning. We focus our question slightly to the following: does the *truth paradigm* or the *justification paradigm* provide a better account of the difference between classical and Constructive

[5] Note that, of course, in order to focus on relevant distinctions, this picture simplifies some things. Within what we called *Truth Paradigm Constructivism*, for example, there are a handful of different philosophical positions. On our view, this includes the traditional versions of the Brouwerian picture which explains Constructive Reasoning in terms of constructive truth (see Brouwer (1912)), the Dummettian view on which conditions of assertion instead play the central role (see Dummett ((1975)), and it also includes the presentations of intuitionistic logic understood in terms of Fine's truth maker semantics Fine (2014). Fine's approach to intuitionistic logic—and Van Fraassen's Van Fraassen (1969) previously developed similar approach to classical logic—is particularly interesting as it makes use semantic ideas that go beyond traditional Truth Paradigm approaches. These approaches introduce nuance on the semantic side, without introducing the sorts of syntactic-nuance available in modal and justification logic.

Reasoning?[6] We argue that the justification paradigm fares better than the truth paradigm because it does not fall prey to the paradoxes of material implication. It is well-known that the classical and constructive versions of the truth-paradigm suffer from the paradoxes of material implication. These are a family of provable conditionals—of CPC, and some of IPC—in which the antecedent does not relate to the consequent. We show that the justification paradigm avoids these paradoxes.

4.2 Artemov's Logical Foundations of Justification Constructivism

In section 4.1, we presented three paradigms within logic: the *truth paradigm*, the *epistemic paradigm*, and the *justification* paradigm. We narrowed our focus to the *truth paradigm* and the *justification paradigm*. We asked the question: which can better present the difference between constructive and Classical Reasoning? The truth paradigm accounts of constructive and Classical Reasoning have been well-known in logic for the last century. Indeed, it is nearly assumed that when we mention classicism or constructivism we do so within the foundational picture of the truth paradigm.

In this section, we outline Artemov's *justification paradigm* account of Constructive Reasoning. In the next section, we present an Artemov-style *justification paradigm* account of Classical Reasoning. Gödel embedded Heyting's Intuitionistic Propositional Calculus (IPC)—which was itself intended as a representation of L.E.J. Brouwer's ideas on intuitionistic truth—in the modal logic S4 Gödel (1933). He did so by showing that if one prefixed each subformula of a theorem of IPC with a modal operator, then the resulting formula would be provable in S4.[7] Where F^* is the result

[6]The reader will excuse us for not considering further the epistemic paradigm. While we will talk about modal logic—on the way to justification logic—our interest is instead what the truth paradigm and the justification paradigm can tell us about Classical and Constructive reasoning. The limits of the epistemic paradigm have been well-studied by Artemov and the justification logic community. Indeed, the modal paradigm can be viewed as a limited version of the explicit modal paradigm insofar as loosely everything that the modal paradigm can do can be replicated in the explicit modal paradigm.

[7]This method of translation is initially due to Orlov. See Došen (1992) for discussion. Of course, there are other translations that work as well.

of syntactically prefixing a $\Box$ before every subformula in F. For example,

$$(A \to B)^* = \Box(\Box A \to \Box B)$$

Gödel ($\Rightarrow$) Gödel (1933) and McKinsey and Tarski ($\Leftarrow$) McKinsey and Tarski (1944) showed:

Theorem 4.2.1 $\mathsf{IPC} \vdash F \Leftrightarrow \mathsf{S4} \vdash F^*$

This laid the foundations for what we called the *epistemic paradigm* account of Constructive Reasoning. Gödel's 1933 interpretation gives a reading of constructive truth in terms of "proof"[8] understood in terms of a modality characterized by S4 axioms Gödel (1933).

Artemov examined these concepts in the language of explicit modal logic. A way to see why one would want to do this at all is to notice the polysemy of:

$$\Box F \to \Box \Box F$$

when $\Box$ is read as proof. One reading is:

> If t is a proof of F, s is a proof that t is a proof of F.

another is:

> If t is a proof of F, there is a check that t is a proof of F.

Artemov showed that there is a *realization* of S4 in the Logic of Proofs (Artemov, 2001, p. 25). A *realization* is a replacement of modalities in a modal logic formula by proof terms. A *normal realization* is one that populates negative occurrences of modalities with proof variables in the context of an axiomatically appropriate constant specification. Artemov thus proved Artemov (2001):[9]

Theorem 4.2.2 *If* $\mathsf{S4} \vdash F$ *then* $\mathsf{LP} \vdash F^r$ *for a normal realization* r

[8] But not, of course, *provability* (Gödel, 1933, p. 301).

[9] The other direction is much simpler. It makes use of a mapping from every proof term to a modality, known as the *forgetful projection*.

In doing so, Artemov formally introduces the existential reading of modality. It is this step from non-explicit modality to explicit modality that is of central interest for our project.[10] The formal picture we have is thus:

$$\textit{Constructive Reasoning} \hookrightarrow \mathsf{IPC} \hookrightarrow \mathsf{S4} \hookrightarrow \mathsf{LP}$$

What is the philosophical account here? Artemov writes (Artemov, 2001, p. 7):

> This confirms Kolmogorov's assumption of 1932 that intuitionistic logic IPC is the calculus of proofs (solutions to problems) in classical mathematics . . . and achieves the original objective by Gödel [1933] to define IPC via the classical notion of proof.[11]

The philosophical picture is thus one where Constructive Reasoning is characterized in terms of two types of objects: classical truths, on one hand, and LP proofs—which represent classical mathematical proofs—on the other.

4.3 The Logical Foundations of Justification Classicism

In this section, we provide the logical foundations for a justification paradigm account of Classical Reasoning. Recall that here, instead of the concept of *proof* over a classical theory of truth, we make use of the concept of *explicit justification* over a classical theory of truth.

[10] Artemov took this a step further by then using his explicit modal logic characterization of S4—and thereby also of IPC and the BHK clauses—to show that LP can be embedded into arithmetic by making use of proof predicates. This ultimately thus provides an account of BHK in terms of the well-known machinery of specific arithmetical proofs.

[11] Artemov's account shows the ways in which Kolmogorov's thinking in terms of problems diverged from that of Heyting. Kolmogorov emphasized that (Kolmogorov, 1932, p. 153):

> The fact that *I* have solved a problem is a purely subjective one of no general interest in itself. Logical and mathematical problems, however, possess a special property of *universal validity of their solutions*, that is, if *I* have solved a logical or a mathematical problem, then *I* can present the solution in a commonly accepted way, and this solution must *necessarily* be recognized as being correct, although this necessity is of a somewhat ideal nature since the reader is assumed to have adequate qualification.

It is difficult to see how the checkability aspect of proofs would be represented purely propositionally. As we discussed, we lose the potential for polysemy in the modal case.

With the Gödel-Artemov foundations of constructivism in mind, it is not difficult to see that logically a similar story holds in the Classical case:

$$\mathsf{CPC} \hookrightarrow \mathsf{S5} \hookrightarrow \mathsf{JS5}$$

That is, CPC embeds into S5 which can be realized in JS5. First, the argument that classical tautologies embed into S5 has been well-known in the community for years (indeed, during the author's education the following was assigned as homework):

Theorem 4.3.1 $\mathsf{CPC} \vdash F \Leftrightarrow \mathsf{S5} \vdash F^*$

The relevant case to consider is just $\neg\neg F \rightarrow F$. The Gödel translation of this is $\Box(\Box\neg\Box\neg\Box F \rightarrow \Box F)$. Indeed, $\Diamond\Box F \rightarrow \Box F$ is an S5 validity; rewritten, this is $\neg\Box\neg\Box F \rightarrow \Box F$. Necessitation and distribution yields $\Box\neg\Box\neg\Box A \rightarrow \Box\Box A$. In S5 any number of modalities are equivalent, we get $\Box\neg\Box\neg\Box A \rightarrow \Box A$. One more necessitation yields our desired result.

For $\Leftarrow$, we observe that S5 axioms stripped of $\Box$ modalities are classical tautologies. We check that if $F \rightarrow G$ and F are tautologies then G is as well.

Realization theorems for S5 were proved by Artemov-Kazakov-Shapiro (1999) Artemov et al. (1999), Pacuit (2006) Pacuit (2006), Rubtsova (2006) Rubtsova (2006), and Fitting (2011) Fitting (2011). Here we consider the following version of realization:

Theorem 4.3.2 S5 *Realization*

If $\mathsf{S5} \vdash F$ *then* $\mathsf{JS5} \vdash F^r$ *for a normal realization* r.

The above furnishes the logical portions of our justification paradigm account of Classical Reasoning. That is, we have:

$$\textit{Classical Reasoning} \hookrightarrow \mathsf{CPC} \hookrightarrow \mathsf{S5} \hookrightarrow \mathsf{JS5}$$

Here, instead of stopping after the first arrow and focusing on truth, we can look onward beyond S5 knowledge to JS5. This then gives us a way of understanding our starting point, the idea of Classical Reasoning, in terms of *explicit justification*. Just as Artemov with LP is able to provide a justification paradigm account of Constructive Reasoning, we suggest that a justification paradigm account of non-Constructive Reasoning should be one centered upon the concept of *explicit justification.*

Explicit Justification allows that if t is not a justification for F, then something is a justification for that fact. The truth paradigm approach locates the main difference between classical and constructive approaches in the theory of truth presupposed by the respective camps. The justification paradigm approach locates this difference instead in the theory of *justifications.* In the justification paradigm, for the Artemov-style constructivist, justifications are limited to proofs. For the classicist, justifications include also those achieved through negative introspection, which expresses a form of omniscience.

4.4 Another Route to Justification Classicism

There is another route by which we might provide a justification paradigm account of Classical Reasoning. We do so by noting Melvin Fitting's 1970 (Fitting, 1970, p. 530):

Theorem 4.4.1 $\mathsf{CPC} \vdash F \Leftrightarrow \mathsf{S4} \vdash F^*$, *for suitably defined Fitting Translation* *.

The translation Fitting uses is as follows:

Definition 4.4.2 *Fitting Translation:*

1. *For atoms A,* $(A)^* = \Box\Diamond A$;
2. *For* F^* *and* G^*,
   (a) $(\neg F)^* = \Box\Diamond\neg(F^*)$;
   (b) $(F \wedge G)^* = \Box\Diamond((F^*) \wedge (G^*))$;
   (c) $(F \vee G)^* = \Box\Diamond((F^*) \vee (G^*))$;
   (d) $(F \rightarrow G)^* = \Box\Diamond((F^*) \rightarrow (G^*))$.

This translation arose from Fitting's dissertation Fitting (1969). His dissertation treated, among other things, how Cohen's independence results could be shown using Kripke models. Fitting recounted the history of his development of the $\Box\Diamond$-translation to the author in personal communication Fitting (2022):

> It was known that no inner model (such as the constructible sets) could be used to show the independence results. My idea was to look at Cohen forcing as if it were constructing an inner model, but an inner model in an intuitionistic sense. Then one could extract classical independence using the double negation embedding. It was clear to many at the time that there was a connection between Cohen's work and intuitionistic logic. I did the details.
>
> Now, in Cohen's book, section 4 of chapter IV, he gives a construction in Lemma 4 that can be used to extract a classical model from his forcing construction. At some point I realized that something much like that construction could be used to prove a box diamond embedding for S4, corresponding to the double negation intuitionistic embedding, and that led to my JSL paper Fitting (1970).
>
> ...
>
> In Smullyan and Fitting (1996) [with former thesis advisor Smullyan], instead of using intuitionistic logic, I used the modal logic S4, which I think presents things much more clearly. The connection with classical logic is then via the box diamond embedding (which is given a different proof in the book from the earlier semantic one).

Now, in providing this translation, Fitting articulates what we have called an *epistemic paradigm* reading of Classical Reasoning. On this view, instead of the concept of the *classical truth* of F, we make use of "S4-knowledge that it is not the case that there is S4-knowledge of $\neg F$." In other words, classical truth is replaced with (S4)-knowledge of the consistency of F.

We can again make use of Theorem 4.2.2, Artemov's S4/LP Realization Theorem, to take this a step further to the justification paradigm. This then gives us a picture of Classical Reasoning in terms of Artemov's concept of proof:

$$\textit{Classical Reasoning} \hookrightarrow \mathsf{CPC} \hookrightarrow \mathsf{S4} \hookrightarrow \mathsf{LP}$$

Here the second arrow is by Fitting's Theorem and the third is by the usual S4 Realization Theorems.

Consider a tautology F. In the truth paradigm, reasoning about F is reasoning about F's truth. In the Gödelian epistemic paradigm, we are reasoning about knowledge when we reason about F^G, where F^G is the Gödel translation of F. In the Fitting style epistemic paradigm, we are reasoning about knowledge of consistency when we reason about F^*. The Fitting-style justification paradigm would translate, for example, the arbitrary atom A as:

$$\Box\Diamond A$$

Or $\Box\neg\Box\neg A$. We would then realize this as:

$$t : \neg x \neg A$$

We can read this as "there is a proof t that it is not the case that the proof variable x is a proof of $\neg A$." In this sense, we can think of this as *proof of consistency*.

4.5 The Justification Paradigm and the Paradoxes of Material Implication

It is well-known that Classical Propositional Calculus falls prey to the paradoxes of material implication. Indeed, this is a problem for the *truth paradigm* generally as it is easy to see that Intuitionistic Propositional Calculus also falls prey to some of those paradoxes. In this section, we introduce seven such paradoxes that arise within CPC. We show which paradoxes still arise for IPC.

We discuss the criterion that this runs afoul of, the *stay on topic principle*, and its propositionally based formal articulation in terms of the *propositional-variable sharing principle*. We propose an alternative criterion, the *justification variable sharing* We then show that for any paradoxical implication $F \to G$, there is a realization r where the Gödel-translated formula $[F^* \to G^*]^r$ is provable in $\mathsf{JS5}$.

We suggest that the *truth paradigm*'s focus on the propositional (specifically, non-modal—explicit or otherwise) level of precision engenders paradoxes of material implication. These formulas are problematic, broadly, because they fail to satisfy conditions of relevance. Mares gives the example Mares (2020):

> If the moon is made of green cheese, then it is raining in Equador or it is not raining in Equador.

What goes wrong here? Mares explains Mares (2020):

> [T]here is a formal principle that relevant logicians apply to force theorems and inferences to "stay on topic". This is the variable sharing principle. The variable sharing principle says that no formula of the form $A \to B$ can be proven in a relevance logic if A and B do not have at least one propositional variable (sometimes called a proposition letter) in common and that no inference can be shown valid if the premises and conclusion do not share at least one propositional variable.

One way of formally articulating the *stay on topic principle* is by the *variable sharing principle*. Note that there are two principles here. There is the general principle that inferences stay on topic and then there is the formal interpretation in terms of the *variable sharing principle*. We will propose a justification paradigm formal interpretation of the *stay on topic principle*. But first, there are a handful of examples of paradoxes of material implication:

1. $B \to (A \vee \neg A)$;

2. $A \to (B \to A)$;

3. $\neg(A \to B) \to A$;

4. $\neg A \to (A \to B)$;

5. $((A \to B) \wedge (C \to D)) \to ((A \to D) \vee (C \to B))$;

6. $(A \to B) \vee (B \to C)$;

7. $((A \wedge B) \to C) \to ((A \to C) \vee (B \to C))$.

While particular logicians may disagree about the paradoxicality of some of these, it is not hard to find those who take each particular one to be paradoxical. If we look to CPC as the paradigm formal articulation of Classical Reasoning, it is implausible not to run into these paradoxes. Indeed, it is precisely this feature of the *truth paradigm*—it's focus on the propositional level of precision—that seems responsible for philosophical logicians' inquiry into subsystems of CPC. If instead we focus on justification, we do not have this same problem.

Our strategy is to propose instead the *justification sharing principle* as a formal articulation of the stay on topic principle. This is the following:

> **Justification Sharing Principle:** Conditional inferences are on topic if the antecedent and consequent share *justification terms*.

The following argument is a first attempt toward our goal. Indeed, it is easy to see that for an arbitrary provable conditional, JS5 will connect the antecedent and consequent with a proof term. Consider the following:

1. $\mathsf{CPC} \vdash F \to G$

2. $\mathsf{JS5} \vdash F \to G$

3. $\mathsf{JS5} \vdash h : (F \to G)$, for some ground proof term h

4. $\mathsf{JS5} \vdash x : F \to (h \cdot x) : G$

The last step follows by standard JS5 reasoning using the application axiom. Now, the arbitrary theorem:

$$x : F \to (h \cdot x) : G$$

will still violate the relevance logician's *principle of propositional variable sharing*. It does not, however, violate the justification-approach account of relevance, the *justification term sharing principle*, for the antecedent justification is a part of the consequent justification. We can read the above as:

> If x is a very general reason to accept F, then a reason for which $F \to G$ is provable—i.e., a ground term—applied to x is a reason for G.

Put another way, we would be misguided to focus on the fact that CPC proves conditionals that tie together unrelated antecedents and consequents. Instead, we should note that it proves that those pieces are related but only in the most general logical way. Explicit justification instead *explains* the way in which relevance obtains. Our thesis is that the "if then" of non-Constructive Reasoning is characterized not by the material implication $F \to G$ but by the explicit implication $t : F \to f(t) : G$.

What about cases of the paradoxes of material implication that arise in the more expressive language of explicit modal logic? The problem with the argument we presented is that even in the more expressive language of justification logic, the formulas F and G are still only at the level of propositional precision. We want to be able to do more than just dress the outside of propositional formulas. It can be proved that:

Theorem 4.5.1 *If* CPC $\vdash F \to G$*, then for some realization r with shared justification variables,* JS5 $\vdash [F^* \to G^*]^r$*.*

The reader will recall that the * translation of F requires us to prefix a $\Box$ before every subformula of F. By the definition of *, we know that X^* will always begin with $\Box$. So, we write $\Box F'$ for F^* and $\Box G'$ for G^*, for appropriate S5 formulas F' and G'. The outermost box is always recoverable using necessitation.

We need the following lemma:

Lemma 4.5.2 *If* $\mathsf{CPC} \vdash F \to G$*, then* $\mathsf{S5} \vdash F' \to G'$*.*

We prove the above by contrapositive. We suppose that $\mathsf{S5} \nvdash F' \to G'$. Then there is an $\mathsf{S5}$ model $\mathcal{M} = \{W, \mathcal{R}, \Vdash\}$ with a $u \in W$ where $u \nVdash F' \to G'$. Without loss of generality, we assume that $\mathcal{M}$ consists of one equivalence class, and hence for all atomic formulas p, $\Box p$ is true everywhere in $\mathcal{M}$ or $\Box p$ is false everywhere in $\mathcal{M}$. Note that since we are considering Gödel translations of propositional formulas, each propositional letter occurs boxed.

Everywhere in $F' \to G'$, for each propositional variable p, we replace each boxed occurrence of that variable $\Box p$ with a fresh propositional letter p'. The resulting formula, $F'' \to G''$, is an $\mathsf{S5}$ formula with fresh propositional variables. We then collapse model $\mathcal{M}$ into the singleton $\mathsf{S5}$ model $\mathcal{M}''$ with $W'' = \{u\}$, preserving evaluations in u.

The following claim proves that F'' and G'' preserve their truth values in $\mathcal{M}''$. This will mean that F'' is true and G'' is false in $\mathcal{M}''$, by our hypothesis for Lemma 4.5.2. [1] For each subformula X'' of $F'' \to G''$, and for each $u \in \mathcal{M}$, $u \Vdash X''$ iff $\mathcal{M}'' \Vdash X''$ We prove this by induction on formula. In the atomic case when $X = A$, $w \in \mathcal{M}$, $w \Vdash A$ iff $w \in \mathcal{M}''$ $w \Vdash A$. This is because each world at $\mathcal{M}$ has the same valuation of atoms and $\mathcal{M}''$ is the version of $\mathcal{M}$ collapsed into one world.

The induction steps are straightforward. Stripping all boxes of F'' and G'' yields the purely propositional formulas F''' and G''' with fresh propositional variables p'. We now need the following: [2] For each subformula X'' of $F'' \to G''$, $\mathcal{M}'' \Vdash X''$ iff $\mathcal{M}'' \Vdash X'''$ We prove this by induction on X. If X is an atom then $X'' = X'''$ and so the lemma holds.

We look at two induction steps. Let $X'' = Y'' \to Z''$. Then $X''' = Y''' \to Z'''$. We assume the lemma holds for the pairs Y'' and Y''' and Z'' and Z'''. Truth functionally, then, the lemma holds for X'' and X'''.

Let $X'' = \Box(Y'')$. By induction hypothesis, we assume that the lemma holds for Y, hence $w \Vdash Y''$ iff $w \Vdash Y'''$. Because the model $\mathcal{M}''$ is a singleton model, $w \Vdash \varphi$ iff $w \Vdash \Box\varphi$ for any φ. Since X''' is $\Box(Y'')$ stripped of $\Box$, $X''' = Y'''$. Therefore, $w \Vdash \Box Y''$ iff $w \Vdash Y''$ iff $w \Vdash Y'''$, which is X'''. Hence, $F''' \to G'''$ is false in $\mathcal{M}''$ and $F''' \to G'''$ is not derivable in CPC. Since F''' and G'''

are just the original F and G with renamed propositional variables (p' instead of p, for each p), $F \to G$ is not derivable in CPC either.

We have that if $\mathsf{CPC} \vdash F \to G$, then $\mathsf{S5} \vdash F' \to G'$. We take $F' \to G'$ and realize it in JS5 in an arbitrary way to get $(F')^r \to (G')^r$. So, $\mathsf{JS5} \vdash (F')^r \to (G')^r$. By Constructive Necessitation, $\mathsf{JS5} \vdash h : ((F')^r \to (G')^r)$ for a ground term h. Since $\mathsf{JS5} \vdash h : ((F')^r \to (G')^r)$, by application, we have $\mathsf{JS5} \vdash x : (F')^r \to [h \cdot x] : (G')^r$, which is a normal realization of $F^* \to G^*$ in JS5.

We have shown that if $\mathsf{CPC} \vdash F \to G$, then for some realization r with shared justification variables, $\mathsf{JS5} \vdash [F^* \to G^*]^r$. This means that, for every classically provable conditional—paradoxical or not—there is a proof of its normal realization with shared proof variables. This shows that each paradoxical conditional of CPC can be realized in a non-paradoxical way in JS5.

4.6 Realization and Relevance

While the structure of relevance is not visible in propositional logic alone, when we move to the language of explicit modal logic this deep structure manifests itself. Instead of looking to smaller and smaller fragments of classical logic, we suggest that we turn toward more and more expressive extensions of classical logic.

The way in which realization accounts for relevance has to do with the meaning of modality in the explicit modal logic context. Traditionally, modality is read with a universal force; it is necessary that F if F holds in *all* possible worlds. Kripke semantics reflects this intuition. While this seems correct as an account of *metaphysical* necessity, *necessity* is said in many ways. The explicit modal logic revolution turns the Kripkean paradigm on its head; here, a proof term has an *existential* force. This was, indeed the sort of reading Gödel gave S4 modalities in his Gödel (1933).

Since we deal with *classical logic* and S5, we extend Kolmogorov's reading of implication as functional dependency between antecedent and consequent beyond its original application.[12] We

[12]Kolmogorov's discussion is found in Kolmogorov ((1925) and Kolmogorov (1932). van Dalen discusses Kolmogorov's contribution in van Dalen (2004b). See also Došen for an overview of Orvlov's early work on relevance logic Došen (1992) in the context of the contemporaneous axiomatizations of intuitionistic logic.

suggest only that the functional account of implication understood through the prism of realization accounts for *relevance connections* in classical logic.

Consider the modal conditional $\Box F \to \Box G$. If these modalities both have an existential sense, we first get:

$$\exists x \text{ ``}x \textit{ proves } F\text{''} \to \exists y \text{ ``}y \textit{ proves } G\text{''}$$

It makes sense to then realize negative occurrences of modalities with proof variables standing for general proof terms. But a proof in a negative position and a proof in a positive position have different meaning. We convert this to $\forall x(\text{``}x \textit{ proves } F\text{''} \to \exists y \text{ ``}y \textit{ proves } G\text{''})$. We then Skolemize the existential quantifier to get:

$$\forall x(\text{``}x \textit{ proves } F\text{''} \to \text{``}f(x) \textit{ proves } G\text{''})$$

where $f(x)$ is a function of x.

The proposal of this chapter is that Skolemization is what ensures *relevance*. The formulas, F and G can have nothing to do with one another in terms of propositional variables. But the realization of $\Box F \to \Box G$ will connect $(\Box F)^r$ with $(\Box G)^r$ in terms of relevance because the first modality is realized as an argument for the second.

For reasons of space, we limit our discussion to the first of the aforementioned paradoxes. It should be clear to the reader how, with some effort, this analysis can be applied with its full power generally to paradoxes of material implication. Consider $B \to (A \vee \neg A)$. While propositions A and B can be completely independent, the above is nonetheless easily provable in classical logic.

We return to Mares' example Mares (2020):

> The moon is made of green cheese. Therefore, either it is raining in Ecuador now or it is not.

Indeed, what seems wrong with the above is that B can have nothing to do with A and $\neg A$.

We begin by Gödel translating $B \to (A \vee \neg A)$ into:

$$\Box(\Box B \to \Box(\Box A \vee \Box \neg \Box A))$$

This formula has different realizations in JS5. It suffices for our purposes to present one such meaningful (normal) realization where the relevance connections of $\to$ are emphasized.

We begin by replacing the modalities in $\Box B$ and in $\Box A$ with proof variables:

$$\Box_1(y : B \to \Box_2(x : A \vee \Box_3 \neg x : A))$$

Modality 3 can be thought of as occurring by negative introspection. Hence, here we set $r(\Box_3) = ?x$:

$$\Box_1(y : B \to \Box_2(x : A \vee ?x \neg x : A))$$

Since $\mathsf{JS5} \vdash B \to (x : A \vee ?x : \neg x : A)$, by constructive necessitation we have that there is a ground term g such that:

$$\mathsf{JS5} \vdash g : [B \to (x : A \vee ?x : \neg x : A)]$$

Using an instance of application and modus ponens, we get:

$$\mathsf{JS5} \vdash y : B \to (g \cdot y) : (x : A \vee ?x : \neg x : A)$$

This only leaves:

$$\Box_1[y : B \to (g \cdot y) : (x : A \vee ?x : \neg x : A)]$$

Modality 1 is a final constructive necessitation on our whole formula, so we set $r(\Box_1) = h$ for some ground term h. This yields:

$$h : [y : B \to (g \cdot y) : (x : A \vee ?x : \neg x : A)]$$

We can now return to Mares' example:

> We have a specific reason ((h)) to think that [if something provides a reason (y) for the moon being made of green cheese, then there is a specific reason ($f(y) = (g \cdot y)$) that (either something provides a reason (x) that it is raining in Ecuador or there is a specific reason ($?x$) for which nothing provides a reason for ($\neg x$) it being raining in Ecuador)].

Now, in both the cases of:

$$B \rightarrow (A \vee \neg A)$$

and:

$$h : [y : B \rightarrow (g \cdot y) : (x : A \vee\ ?x : \neg x : A)]$$

we have violations of the *propositional variable sharing principle.* Indeed, our translation did not change the propositional variables of our initial formula. In a deeper sense, however, $h : [y : B \rightarrow (g \cdot y) : (x : A \vee ?x : \neg x : A)]$ satisfies the desiderata of *staying on topic.*

Here the conditional connects a variable over reasons (y) and a specific reason g, to get $(g \cdot y)$. Moreover, that specific reason takes into account the aforementioned variable. What is asserted here is something about the structure of proof variables; they are genuinely general and range over specific bits of evidence. The conditional says that if something justifies that the moon is made of green cheese, then we really could build a specific justification dependent on that that supported the claim that either something justifies that it is raining in Ecuador or that something provides counterevidence to the claim that something justifies it is raining in Ecuador. Again, this is a claim about the general structure of evidence, in this sense, justification logic provides a *framework for relevance.*

4.7 Conclusions

In this chapter we have discussed the following taxonomy:

	Key Concept	Language	Logic
Truth Paradigm Constructivism	Constructive Truth	Propositional	IPC
Truth Paradigm Classicism	Platonistic Truth	Propositional	CPC
Epistemic Constructivism	S4-Knowledge	Modal	S4
Fitting-Epistemic Classicism	S4-Knowledge of Consistency	Modal	S4
Epistemic Classicism	S5-Knowledge	Modal	S5
Justification Constructivism	Proof	Explicit Modal	LP
Fitting-Justification Classicism	Proof of Consistency	Explicit Modal	LP
Justification Classicism	Explicit Justification	Explicit Modal	JS5

We focused on the comparison between the truth paradigm and the justification paradigm. We asked: which could better account for the difference between Classical and Constructive Reasoning? Our answer was that the justification paradigm did, because the truth paradigm engenders the paradoxes of material implication. We showed how, in the justification paradigm, any paradoxical conditional has an explicit and non-paradoxical counterpart. In other words, we recover the elements of relevance that were hidden by the conditional's initial formalization in CPC.

We put forth that paradoxicality arises in CPC because the language is not sensitive enough to display the structure of relevance. We translate to S5 due to the natural reading of S5 as a logic of knowledge and its connections with CPC, and then we populate occurrences of $\Box$ with explicit modalities in order to reveal elements of relevance that were buried in the coarseness of propositional language. We present this as an advantage of the justification paradigm over the truth paradigm.[13]

In what sense does the justification paradigm avoid the paradoxes of material implication if JS5, for example, still proves those paradoxical conditionals? The justification paradigm avoids the paradoxes insofar as for each such conditional, a Gödel interpreted and justification term Realized

[13] The truth paradigm constructivist might point out that something of what they thought was philosophically important was lost in translation to the language of modal logic and then to explicit modal logic. Intuitionistic Justification Logic (see Artemov and Iemhoff (2007) and Dashkov (2009)) is a promising candidate to account for both the truth paradigm constructivist's desire for a constructive theory of truth and our proposal that the justification paradigm can avoid paradoxes of material implication.

formula could be found.

What was the aim of this chapter with respect to the articulation of the justification paradigm? Consider the familiar picture:

$$\textit{Constructive Reasoning} \hookrightarrow \mathsf{IPC} \hookrightarrow \mathsf{S4} \hookrightarrow \mathsf{LP}$$

Artemov's contribution to the above was twofold. Of course, he provided the realization of S4 and the arithmetical semantics for LP. Also, using the right side of the above to help understand the left, he introduced a different account of what *Constructive Reasoning* itself had to do with. While the technical background in the classical case has been in place for years, the interpretation of *Classical Reasoning* itself in the justification paradigm has not. We aimed to provide this interpretation.

We consider one final question. In section 4.1, we introduced Negative Introspection as a part of our interpretation of *Classical Reasoning*. But what role does Negative Introspection play?

5. If t is *not* an explicit justification of F, then there is an explicit justification that t is not an explicit justification of F.

This is arguably an important *part* of the complete formulation of the *Principle of Sufficient Reason*, the ancient philosophical thesis that *everything* has a reason.[14] Indeed, if something fails to be a reason, the Principle of Sufficient Reason says there is a reason for why it fails in this way. This is an omniscience principle in explicit modal logic. The thought here is that explicit justification—understood as the classical counterpart to Artemov's constructive *proof*, characterized by LP—is far reaching. Even when an explicit justification fails, there is an explicit justification for why the first one failed.

[14] See Lovejoy (1963) for discussion of the Principle.

Chapter 5

Gödel's Disjunction

5.1 Gödel's Disjunction

In 1951 Kurt Gödel gave the 25^{th} Josiah Willard Gibbs Lecture entitled "Some basic theorems on the foundations of mathematics and their implications."[1] Therein he argued for his famous disjunction (Gödel, 1951, p. 310):

> Either. . . the human mind (even within the realm of pure mathematics) infinitely surpasses the powers of any finite machine, or else there exist absolutely unsolvable diophantine problems of the type specified.

Now, a *diophantine problem*—one that only accepts integer solutions—is precisely what is referenced in Hilbert's 10th problem. Both disjuncts, Gödel contended, were "decidedly opposed to materialistic philosophy" (Gödel, 1951, p. 311). Further, he claimed the disjunction was *inevitable* and *a mathematically established fact* (Gödel, 1951, p. 310).

Gödel's Disjunction has been heavily discussed and expounded upon since 1951 (see, for example, Lucas (1961), Penrose (1989), Penrose (1996), and Horsten and Welch (2016)). Before *anything* is proved about the relation between human and machine capabilities and absolute unsolvability, we have to make a decision about how our formal tools will relate to the philosophical notions employed

[1] For historical information on this lecture, see George Boolos Boolos (1995b).

in the disjunction. While both sides of the disjunction contain concepts wanting of elucidation (see Shapiro (1998) and Shapiro (2016)), since Gödel preferred the first of his two disjuncts,[2] a natural starting point is with the concept of the power of the human mind. For economy of communication, we will gloss this as *reason*.[3]

The primary way that human reason has been formally treated in this is as an *operator*.[4] Furthermore, it has been ascribed the very specific properties of a system developed by Stewart Shapiro Shapiro (1985) and William N. Reinhardt Reinhardt (1985). Peter Koellner, for example, discusses that system in this context (see Koellner (2016), Koellner (2018a), and Koellner (2018b)). Shapiro's *Epistemic Arithmetic*, or EA, is just PA with a K axiom for the modal operator, a Necessitation rule, the $\mathsf{4}$ axiom, and T. More simply, EA is PA extended with an $\mathsf{S4}$ operator, as presented in Definition 2.6.2, which we discuss in more detail in chapter 2. EA thus has the advantage of containing a system already much employed in the study of knowledge and with well-known ties to proof and intuitionistic logic.

While EA has been used fruitfully to study Gödel's Disjunction, the operator approach more generally is not without drawbacks. A standard objection to the treatment of modality as an operator in this context has to do with interaction with quantifiers and arithmetical predicates. Halbach, Leitgeb and Welch point out that with a predicate $P(x)$ we can express, (Halbach et al., 2003, p. 272):[5]

$$\forall x(P(x) \rightarrow Pr(x))$$

where $Pr(x)$ is the arithmetical predicate for Gödelian provability. If instead we have an operator O, we sacrifice the ability to tightly treat connections between our epistemic feature and the provability predicate in this way.

[2] See Hao Wang (Wang, 1996, p. 185) for discussion.

[3] The gloss of *reason* on *the power of the human mind* is only meant to be general enough to include the glosses that Gödel uses including *understanding* (Gödel, 1951, p. 310), *the workings of the mind* (Gödel, 1951, p. 311), *the passage of judgements* and *mathematical knowledge* (Gödel, 1951, p. 322).

[4] Most treatments of modality do so in this way, a notable exception being Johannes Stern's Stern (2016).

[5] See also Stern (2016) for extended discussion of predicate treatments of modality.

Further, when we treat the epistemic feature as an operator we forsake the possibility that diagonalization might teach us something new about the concept thus modelled. Consider, for example, W.V.O. Quine's discussion of Gödel's first Incompleteness Theorem (Quine, 2018, p. 17):

> Gödel's discovery is not an antinomy but a veridical paradox. That there can be no sound and complete deductive systematization of elementary number theory, much less of pure mathematics generally, is true. It is decidedly paradoxical, in the sense that it upsets crucial preconceptions. We used to think that mathematical truth consisted in provability.
>
> Like any veridical paradox, this is one we can get used to, thereby gradually sapping its quality of paradox. But this one takes some sapping. And mathematical logicians are at it, most assiduously.

The *quality of paradox* of Gödel's First Incompleteness Theorem has to do with the role it played in upsetting the hope of some for a sound and complete formal characterization of arithmetic. Luckily, as Quine points out, the quality of paradox is diluted with time and understanding, and new fields of inquiry are opened (Quine, 2018, p. 17). To return to our epistemic feature, if we treat it as an operator, we shy away from the possibility of learning something about the formalized concept by analyzing a provable Gödel sentence in this way.

The focus of this Chapter is not the cost of the operator approach but rather the potential benefit of the alternative. But, because predicate treatments of modality in the arithmetical context will allow us to prove Gödel sentences for those predicates, the modal properties we grant our predicate will need to be limited. In section 5.2, we reason generally about the Gödel sentence in extensions of arithmetic. It is here that we face a fork. Provided the arguably well-justified normality assumptions, we can either disallow an epistemic consistency statement (represented by the D axiom, propositionally: $\neg \Box \bot$) or require one, and thereby limit other axioms (most notably coreflection $F \to \Box F$ and its instance $\mathsf{4}$, $\Box F \to \Box \Box F$). In section 5.3, we discuss the first option. We prove some logical properties of such a system, and then discuss advantages and disadvantages of that system. In section 5.4, we examine the second possibility. We again introduce a paradigmatic example of such a system, explore some of its logical properties, and then mention some philosophical motivation and criticism.

5.2 Extending Arithmetic

Shapiro's Epistemic Arithmetic treats its epistemic feature as an *operator*. If we are to try the alternate approach, we must ask: what sort of *predicate* style approaches are available to us? We will make use of the language of arithmetic extended with a unary predicate $\Box(x)$. While we use the single symbol throughout, the specific properties we ascribe to $\Box(x)$ will be clear from context. For generality, sometimes we will reason in ways that do not essentially rely on first-order properties of $\Box(x)$ and accordingly simplify notation. This too will be clear from context.

We make use of the language outlined in Definition 2.6.1.

Lastly, we must emphasize that the topic of discussion is *idealized* human reason. Since we are discussing idealized human reason, we can limit our consideration to normal modalities, ones governed by the K axiom ($\Box(F \to G) \to (\Box F \to \Box G)$) and Necessitation rule (that if $\vdash F$ then $\vdash \Box F$). The justification for K is that it encodes an attractive relation between reason and Modus Ponens. The justification for Necessitation is that it encodes a form of soundness with respect to the underlying arithmetic. The argument is that since whatever given system we work with will extend arithmetic, Necessitation in that extension yields that arithmetically provable formulas fall under the extension of $\Box(x)$. Because we want to leave open the possibility for intuitionistic interpretation of our arguments, we reason in intuitionistically admissible ways where possible.

We can now observe that because Robinson Arithmetic Q is sufficient for the fixed point lemma, Q taken along with a KD4 unary predicate $\Box(x)$ will be inconsistent.

Lemma 5.2.1 KD4 + Q is inconsistent.[6]

It suffices to show that iKD4 + $G \vdash \bot$.

1. $\Box\bot \to \bot$ - D.

[6] Versions of this argument and some others here were originally developed in Peluce (2018).

2. $A \rightarrow \neg \Box A$ - Half of G.

3. $\Box(A \rightarrow \neg \Box A)$ - Necessitation on 2.

4. $\Box A \rightarrow \Box\neg \Box A$ - K reasoning on 3.

5. $\Box A \rightarrow (\Box A \wedge \Box\neg \Box A)$ - Propositional reasoning on 4.

6. $\Box A \rightarrow (\Box \Box A \wedge \Box\neg \Box A)$ - 4 reasoning on 5.

7. $\Box A \rightarrow \Box(\Box A \wedge \neg \Box A)$ - K reasoning on 6.

8. $\Box A \rightarrow \Box\bot$ - 7.

9. $\Box A \rightarrow \bot$ - 1 and 8.

10. $\neg \Box A$ - 9.

11. $\neg \Box A \rightarrow A$ - Other half of G.

12. A - Modus Ponens with 10 and 11.

13. $\Box A$ - Necessitation on 12.

14. $\bot$ - 10 and 13.

That the above should hold makes sense. The Hilbert-Bernays-Löb derivability conditions have the modal properties of Necessitation, K and 4, respectively (see, for example, (Smith, 2013, p. 258)). Since D is the generalized version of the consistency statement $\neg Pr[\bot]$ interpreting $\Box$ as provability, by the Second Incompleteness Theorem, we should anticipate Lemma 5.2.1.

Just as we provided a general way of reasoning about the derivability conditions in Lemma 5.2.1, we can also prove a general (and intuitionistically admissible) version of Richard Montague's 1962 Theorem that truth with *only* the properties of a T modality is not definable Montague (1962). While his investigation is intended as a strengthening of Tarski's Theorem and conducted in that context, reasoning propositionally shows us that the argument—for the most part—requires only very general modal reasoning.

Lemma 5.2.2 KT + Q is inconsistent.

It suffices to show that iK + T + Necessitation + $G \vdash \bot$.

1. $A \rightarrow \neg \Box A$ - Half of G.
2. $\Box(\neg \Box A \rightarrow A)$ - Other half of G with Necessitation.
3. $\Box A \rightarrow A$ - T.
4. $\Box(\Box A \rightarrow \neg \Box A)$ - From 1 and 3 propositionally, with Necessitation.
5. $\Box(\Box A \rightarrow (\Box A \wedge \neg \Box A))$ - From 4 by iK reasoning.
6. $\Box\neg \Box A$ - From 5.
7. $\neg \Box A$ - T on 6.
8. $\Box\neg \Box A \rightarrow \Box A$ - K reasoning on 2.
9. $\Box A$ - Modus Ponens on 6 and 8.
10. $\bot$ - 7 and 9.

A corollary of Lemmas 5.2.1 and 5.2.2, then, is that there is no interpretation of a predicate $\Box(x)$ with these properties in PA:

Corollary 5.2.3 *There is no interpretation of* $\Box(x)$ *in* PA*, where* $\Box(x)$ *is understood explicitly as a* KD4 *or* KT *predicate.*

Assume that there were such an interpretation. By the fixed point lemma, we would be able to construct a $G = A \leftrightarrow \neg \Box [A]$ such that PA $\vdash A \leftrightarrow \neg \Box [A]$. By Lemmas 5.2.1 and 5.2.2, respectively, this would mean that PA $\vdash \bot$. But PA is not inconsistent. So, there is no such interpretation in PA of $\Box(x)$ as a KD4 or T predicate.

Corollary 5.2.4 *There is no interpretation of* $\Box(x)$ *in* PA*, where* $\Box(x)$ *is understood explicitly as an* EA *predicate.*

This follows from the above and the fact that EA contains both $\mathsf{KD4}$ and KT.

Theorem 5.2.5 *For every normal modal logic* L*, where* $\mathsf{LD4} \nVdash \bot$ *or* $\mathsf{LT} \nVdash \bot$*, there is no* A *such that* $\mathsf{L} \vdash A \leftrightarrow \neg \Box A$

Using the proofs of Lemmas 5.2.1 and 5.2.2, we can see that if there were such an A, then $\mathsf{LD4} \vdash \bot$ or $\mathsf{LT} \vdash \bot$, respectively. This, however, would contradict our assumption that $\mathsf{LD4} \nVdash \bot$ and $\mathsf{LT} \nVdash \bot$.

What is the upshot of this discussion? Provided that we are treating our modality as a predicate we must look elsewhere than Epistemic Arithmetic. Since our system will prove a Gödel sentence for our epistemic feature, we must be cautious when ascribing modal properties to that feature. Given that we endorse K and Necessitation as accounting for the sort of idealization involved in thinking of human reason in Gödel's sense, the key decision has to do with the acceptance or rejection of the consistency statement. In the following sections, we contrast two systems that differ in preciesly this respect.

5.3 Conclusiveness

In his "Minds, Machines, And Mathematics," David Chalmers puts forth a predicate-style extension of arithmetic. Since we take K and Necessitation as our base, the characteristic feature in his system is $\mathsf{4}$. As we will see, $\mathsf{4}$ is associated with conclusiveness in the literature. Before turning to this, let us present Chalmers' system making use of $\Box(x)$ instead of $B(x)$:

Definition 5.3.1

1. *Axioms and rules of* PA*;*
2. $\Box[F \to G] \to (\Box[F] \to \Box[G])$*;*
3. $\vdash F$ *then* $\vdash \Box[F]$*;*
4. $\Box[F] \to \Box[\Box[F]]$.

The first item is our classical arithmetical base. The second and third formalize the idealization arguably built into the relevant sense of *human reason*. The fourth item is the 4 axiom. In the context of knowledge, this is known as the KK axiom.

Lemma 5.2.1 tells us that we cannot add D, the consistency statement, $\neg\Box[\bot]$, to Chalmers' system. But we can strengthen our system in a different manner. To see just how we might do so, let us take a brief detour to the intuitionistic context. The Intuitionistic Epistemic Logic of Knowledge was introduced by Sergei Artemov and Tudor Protopopescu (2016) to treat the relationship between intuitionistic truth, knowledge, and classical truth Artemov and Protopopescu (2016). It is the system presented in Definition 2.3.16. Observe that we need not posit Necessitation for IEL's modality $\Box$, since it will be derivable. While it is easy to see that $\Box F \to \neg\neg F$ is classically equivalent to $\Box F \to F$, these come apart in the intuitionistic context. (4) is therefore known as *intuitionistic reflection*. Given the characteristic IEL coreflection axiom $F \to \Box F$ intuitionistic reflection is equivalent to D, that is, $\neg \mathsf{K}\bot$. Insofar as the reasoning in Lemma 5.2.1 is intuitionistically admissible, we can see the following holds as well:

Corollary 5.3.2 $\mathsf{IEL} + G \vdash \bot$

This follows from Lemma 5.2.1 and the observation that IEL contains iKD4.

Reasoning similarly, we continue that:

Corollary 5.3.3 *There is no interpretation of $\Box(x)$ in Heyting Arithmetic (*HA*), where $\Box(x)$ is understood explicitly as an* IEL *predicate.*

If there were, then HA would prove a Gödel sentence for the predicate that interprets IEL's modality. We again reason from the fact that IEL contains (intuitionistic) KD4 and that $\mathsf{HA} \nvdash \bot$ to the conclusion that there is no such interpretation is possible.

Given Corollary 5.3.3 of course there is also no such interpretation in PA. For Artemov and Protopopescu's Intuitionistic Epistemic Logic of *Belief*, however, things are different. The Intuitionistic Epistemic Logic of Belief, or IEL^-, is simply the system introduced in Definition 2.3.15 without intuitionistic reflection (4). Insofar as this does not contain a version of (intuitionistic) D, it is natural to think we might be able to provide a predicate treatment of it in the arithmetical context. Furthermore, coreflection has the 4 axiom as an instance, therefore a predicate treatment of IEL^- over PA will contain Chalmers' system. Consider the extension of arithmetic, which we call CoPA for *Coreflective* PA, from Definition 2.6.4 and its intuitionistic version, CoHA, presented in Definition 2.6.5.

While systems like CoPA and CoHA have been considered in the literature, these have been extensions CoHA. One close example is Nik Weaver's system P Weaver (2013a) and Weaver (2013b). The most striking way in which P is an extension of CoHA is that it contains as an axiom $\Box[F \vee G] \to (\Box\ulcorner F \urcorner \vee \Box[G])$, which Artemov and Protopopescu have shown is provable in neither IEL nor IEL^- (Artemov and Protopopescu, 2016, p. 282). Weaver's axiomatization in (Weaver, 2013a, p. 4) also contains the Barcan formula, which may be questionable from the intuitionistic perspective. We discuss this in 6.6.

What sort of properties does this system have? We begin by noting that the epistemic feature of CoPA respects proof. That is, if t is the code of a proof of F in CoPA, then we will have that $\Box[F]$. By "proof" here we mean *Gödelian proof*; we symbolize that t is a proof of F as $\mathit{Proof}(t, \ulcorner F \urcorner)$. More specifically, this says that t encodes a finite sequence of axioms, formulas, and rule applications thereon of our system.

Theorem 5.3.4 Cofactive Modality Respects Proof

For each t, $\mathsf{CoPA} \vdash \mathit{Proof}(t, \ulcorner F \urcorner) \to \Box[F]$

Either $\mathit{Proof}(t, \ulcorner F \urcorner)$ is true or it is not. If $\mathit{Proof}(t, \ulcorner F \urcorner)$ is true, then it holds that t is a code for a proof of F. But then $\mathsf{CoPA} \vdash F$. By CoPA's derived Necessitation rule, it follows that $\mathsf{CoPA} \vdash \Box[F]$, and so $\mathsf{CoPA} \vdash \mathit{Proof}(t, \ulcorner F \urcorner) \to \Box[F]$. If $\mathit{Proof}(t, \ulcorner F \urcorner)$ is not true, then $\neg \mathit{Proof}(t, \ulcorner F \urcorner)$ is true. Because $\neg \mathit{Proof}(t, \ulcorner F \urcorner)$ is a provably primitive recursive formula, it follows that $\mathsf{CoPA} \vdash \neg \mathit{Proof}(t, \ulcorner F \urcorner)$. For any G, then, $\mathsf{CoPA} \vdash \mathit{Proof}(t, \ulcorner F \urcorner) \to G$. Thus, $\mathsf{CoPA} \vdash \mathit{Proof}(t, \ulcorner F \urcorner) \to \Box[F]$.

Theorem 5.3.5 Provable Σ Completeness of Cofactive Modality

For each Σ sentence σ, $\mathsf{CoPA} \vdash \sigma \to \Box[\sigma]$

By Theorem 5.3.4, we know that CoPA $\vdash$ $Proof(t, \ulcorner\sigma\urcorner) \rightarrow \Box[\sigma]$. Because it holds that CoPA $\vdash$ $\sigma \rightarrow Proof(t, \ulcorner\sigma\urcorner)$ (see, for instance, Boolos' (Boolos, 1995a, p. 46-9)), we have that $\Box(x)$ is Σ complete.

Theorem 5.3.6 Fixed Point Lemma for Cofactive $\Box$

For some CoPA*-formula G, it holds that* CoPA $\vdash G \leftrightarrow \neg \Box [G]$.

The General Fixed Point Lemma is provable in exactly the same way it is in PA. We then take $\neg \Box (x)$ as a formula with one free variable to get the characteristic CoPA Gödel sentence.

We will now show that CoPA is consistent. We can see that this will hold because CoPA has an interpretation in PA. Since Chalmers' system is contained in CoPA, his system will be consistent as well.

Definition 5.3.7 *An arithmetical interpretation of* CoPA *in* PA *is a pair $(P(x), {}^*)$ in which $P(x)$ is an arithmetical predicate such that for any* PA*-formulas F and G it holds that:*

1. PA $\vdash P[F \rightarrow G] \rightarrow (P[F] \rightarrow P[G])$
2. PA $\vdash F \rightarrow P[F]$.

   And * is a mapping from the language of CoPA *to that of* PA *such that:*

3. $F^* = F$*, for each $\Box$-free formula;*
4. * *commutes with Boolean connectives and quantifiers;*
5. $(\Box[F])^* = P[F^*]$.

Lemma 5.3.8 CoPA *has an interpretation in* PA

Consider the predicate $Form(x)$ defining formulas in arithmetic. It is easy to see that:

1. PA $\vdash Form[F \rightarrow G] \rightarrow (Form[F] \rightarrow Form[G])$

2. $\mathsf{PA} \vdash F \rightarrow \mathit{Form}[F]$.

Thus interpreting $P(x)$ as $\mathit{Form}(x)$, we see that CoPA has an interpretation in PA.

Lemma 5.3.9 *Let* $(P, ^{*})$ *be an interpretation of* CoPA *in* PA*. For any* CoPA*-formula* F*, if* $\mathsf{CoPA} \vdash F$ *then* $\mathsf{PA} \vdash F^{*}$*.*

This is proved by induction on derivability in CoPA.

From the above it follows that:

Corollary 5.3.10 *Consistency of* CoPA

$\mathsf{CoPA} \nvdash \bot$

There are natural questions about the possibility of extending CoPA. Two candidate formulas come to mind. Where $Pr(x)$ is the Gödelian proof predicate, consider the following:

$$\mathit{Pr}[F] \rightarrow \Box[F] \tag{5.1}$$

or

$$\Box[F] \rightarrow \mathit{Pr}[F] \tag{5.2}$$

First, (5.1) is clearly consistent as it holds when $\Box(x)$ is interpreted as $\mathit{Form}(x)$. On the other hand (5.2) would be less than ideal. From coreflection and (5.2), we get $F \rightarrow \mathit{Pr}[F]$ for any F. Half of the Gödel sentence for $Pr(x)$, is $\neg \mathit{Pr}[A] \rightarrow A$. Reasoning propositionally, we have $\neg \mathit{Pr}[A] \rightarrow \mathit{Pr}[A]$, then $\neg \mathit{Pr}[A] \rightarrow (\mathit{Pr}[A] \wedge \neg \mathit{Pr}[A])$, then $\neg\neg \mathit{Pr}[A]$. The contrapositive of the other half of the Gödel sentence is $\neg\neg \mathit{Pr}[A] \rightarrow \neg A$, we then have $\neg A$. By Necessitation, then, we have $\mathit{Pr}[\neg A]$, and then $\mathit{Pr}[\neg A] \wedge \mathit{Pr}[A]$, and by K reasoning the provable internal inconsistency $\mathit{Pr}[\bot]$.

Why would one consider a system like CoPA to represent idealized human reason? Well, in one sense the axiom $F \rightarrow \Box[F]$ and its instance $\Box[F] \rightarrow \Box[\Box[F]]$ certainly seem to represent a sort of *idealization*. The question is thus: is this the sort of idealization we want?

There are standard objections to the 4 axiom from the earliest discussions of epistemic logic. Jaakko Hintikka takes 4 to represent a sort of *conclusivity* of knowledge Hintikka (1962). He takes it to be a favorable condition when knowledge is understood as such, but emphasizes that this sense of knowledge is perhaps too strong Hintikka (1970). In a much-quoted passage from "'Knowing That One Knows' Reviewed," Hintikka writes (Hintikka, 1970, pp. 148-9):

> At this point, one might try a Popperian ploy and ask, in an appropriate tone of voice: 'Why aim at conclusiveness in the first place? What philosophers and scientists should aim at is new information, new knowledge, and for this purpose the very idea of a "discussion-stopper" concept—such as the strong sense of knowledge in which the KK-thesis holds was found to be—is not only useless but positively harmful.' Basically, I agree with the attitude thus expressed. In addition to being able to use all the relevant deductive and inductive modes of reasoning, we scarcely also need a notion "to seal up the conclusion to which ratiocination has brought me" (to use Cardinal Newman's words slightly out of their original context). Popper may even be right in connecting philosophers' preoccupation with the conclusiveness of one's knowledge with a quest of religious certainty rather than with a rational (scientific) quest of information. The KK-thesis relied heavily on the requirement (suitably interpreted) that our knowledge be conclusive. Now it certainly seems much more important to find methods of continuing once's quest of information and one's dialogues with others than ways of concluding them. The purpose which the strong sense of knowledge would serve is indeed somewhat suspect.

The strength of $\mathsf{4}$ is also its weakness; it is far from obvious that reason, even with idealization, should be governed by $\mathsf{4}$. A similar point can be made for the cofactive axiom in the classical context.[7] We have that if F holds, then it falls within the grasp of the agent and if the agent does not grasp F, then F fails. This presumes an excessive amount of confidence for even a picture of idealized reason.

[7] In the *intuitionistic* context, it has a well-justified meaning. The IEL formula $F \to \mathbf{K}F$ says that if F has an intuitionistic proof, then it is known. Bare knowledge, however, is not sufficient for a proof, take the case of testimony from a trusted source. For this reason the converse does not hold generally.

5.4 Consistency

The literature on non-Gödelian proof predicates offers plenty of examples of specific proof predicates with built-in consistency. The system known as *Doxastic Arithmetic* (2018), or DA, considers (normalized) consistent proof predicates in the general setting Peluce (2018). The axioms and rules of DA are presented in Definition 2.6.6.

The first item on our list is self-explanatory. The second and third are idealization assumptions shared with the system CoPA examined in section 5.3 (item (2) is listed, (3) is derivable). The characteristic feature of Doxastic Arithmetic is $\neg\Box[\bot]$, which, in the Gödelian context, reads "it is not the case that a contradiction falls under the power of the human mind." More colloquially, this says that *idealized reason is internally consistent.*

At this point we are in the position to prove some properties of DA. First, as we saw with CoPA, the $\Box(x)$ of DA also respects proof.

Theorem 5.4.1 DA $\Box$ **Respects Proof**

For each t, DA $\vdash \mathit{Proof}(t, \ulcorner F \urcorner) \to \Box[F]$

The proof of this is similar to that of Theorem 5.3.4.

Theorem 5.4.2 Provable Σ Completeness of DA $\Box$

For each Σ sentence σ, DA $\vdash \sigma \to \Box[\sigma]$

By Theorem 5.4.1, DA $\vdash \mathit{Proof}(t, \ulcorner \sigma \urcorner) \to \Box[\sigma]$ Since DA $\vdash \sigma \to \mathit{Proof}(t, \ulcorner \sigma \urcorner)$, it follows that $\Box(x)$ is Σ complete.

Theorem 5.4.3 Fixed Point Lemma for DA $\Box$

For some DA*-formula G, it holds that* DA $\vdash G \leftrightarrow \neg\Box[G]$.

The argument is the same as in Theorem 5.3.6.

We can also see that DA is consistent by the following argument. We begin by defining an arithmetical interpretation * of DA in PA.

Definition 5.4.4 *An arithmetical interpretation of* DA *in* PA *is a pair* $(B(x),\ ^*)$ *in which* $B(x)$ *is an arithmetical predicate such that for any* PA*-formulas* F *and* G *it holds that:*

1. $\mathsf{PA} \vdash B[F \to G] \to (B[F] \to B[G])$
2. $\mathsf{PA} \vdash F$ *then* $\mathsf{PA} \vdash B[F]$
3. $\mathsf{PA} \vdash \neg B[\bot]$

   And * *is a mapping from the language of* DA *to that of* PA *such that:*

4. $F^* = F$*, for each* $\Box$*-free formula;*
5. * *commutes with Boolean connectives and quantifiers;*
6. $(\Box[F])^* = B[F^*]$.

We are now in the position to prove that DA has an interpretation in PA and thereby see that it is consistent.

Lemma 5.4.5 DA *has an interpretation in* PA.

Consider now one of the systems with non-Gödelian proof predicates. Examples include Sol Feferman's system F (see Feferman (1960) and Albert Visser's Visser (1998), p. 173-8, esp. 174). Feferman's system has a predicate $\Delta(x)$ which satisfies the following:

1. $\mathsf{PA} \vdash \Delta[F \to G] \to (\Delta[F] \to \Delta[G])$;
2. $\mathsf{PA} \vdash F$ then $\mathsf{PA} \vdash \Delta[F]$;
3. $\mathsf{PA} \vdash \neg\Delta[\bot]$.

Thus, interpreting $B(x)$ as $\Delta(x)$, we see that DA in fact has an interpretation in PA.

Lemma 5.4.6 *Let* $(B,\ ^*)$ *be an interpretation of* DA *in* PA*. Then, for any* DA*-formula* F*, if* $\mathsf{DA} \vdash F$*, then* $\mathsf{PA} \vdash F^*$.

By induction on derivability in DA.

With this, we can conclude the consistency of DA:

Corollary 5.4.7 Consistency of DA

$\mathsf{DA} \nvdash \bot$

An advantage of DA is that it underlies the class of arithmetical provability predicates with built-in consistency. These systems are studied extensively in Visser (1998) and Shavrukov (1994). Because DA is not tied down to any one specific arithmetic interpretation of $\Box(x)$, it provides an abstract and general way of reasoning about those systems. Following our previous discussion, we examine two natural candidates for extension connecting $\Box(x)$ with Gödelian provability:

$$Pr[F] \to \Box[F] \tag{5.3}$$

or

$$\Box[F] \to Pr[F] \tag{5.4}$$

We can see that here (5.3) makes the resulting version of DA inconsistent, because $\vdash \neg \Box [\bot] \to \neg Pr[\bot]$ and $\vdash \neg\Box[\bot]$ will yield that $\vdash \neg Pr[\bot]$. This would mean that the Gödelian consistency of our extension of DA would be internally provable. By Gödel's Second Incompleteness Theorem, this would then yield the inconsistency of our reference system. On the other hand, (5.4) is consistent. Both the Rosser and Feferman provability predicates satisfy (5.4), see Visser (1998).

How does DA fare as a formal characterization of idealized human reason? Allowing for Gödelian uses of "perception," let us turn to an illustrative passage and footnote of Gödel. In discussion of his Second Incompleteness Theorem, Gödel argues (Gödel, 1951, p. 309):

> For, [the Second Incompleteness Theorem]*makes it impossible that someone should set up a certain well-defined system of axioms and rules and consistently make the following assertion about it: All of these axioms and rules I perceive (with mathematical*

> *certitude) to be correct, and moreover I believe that they contain all of mathematics.* If someone makes such a statement he contradicts himself. For if he perceives the axioms under consideration to be correct, he also perceives (with the same certainty) that they are consistent. Hence he has a mathematical insight not derivable from his axioms.

Gödel's claim here would seem to straightforwardly preclude the possibility of attributing anything like a consistency statement to formal representations of idealized human reason.

Gödel's— at least in the 1951 lecture—diagnosis of the situation looks grim. While he does allow for a *sort* of perception of consistency, he thinks it is only such that would "mean that the human mind (in the realm of pure mathematics) *is* equivalent to a finite machine" (Gödel, 1951, p. 309-10). Let us briefly outline his argument. First, by "perception with mathematical certitude", Gödel has in mind consideration of axioms and rule applications *all at once* (Gödel, 1951, p. 309). As we see above he ties this sort of perception with *derivability.* He reasons about a hypothetical mathematically certain insight of consistency. He concludes that there could not be one because—insofar as such an insight is, by hypothesis, mathematically certain—this would mean that that insight was derivable or somehow grounded in a derivation, which it cannot be. For our purposes, let us think of this as *formal certitude.*

In a footnote to the above quoted passage, Gödel contrasts a different possibility (Gödel, 1951, p. 309):

> [11] If he only says "I believe I shall be able to perceive one [axiom] after the other to be true" (where their number is supposed to be infinite), he does not contradict himself.

Gödel allows that reason perceive the correctness of axioms and rule applications *one after the other*, but not *all at once*; insofar as he does allow for a sort of perception of correctness, he makes room for one of consistency. The thought is emphasized, "I wish to point out that one may conjecture the truth of a universal proposition (for example, that I shall be able to verify a certain property for *any* integer given to me) and at the same time conjecture that no general proof for this fact exists" (Gödel, 1951, p. 313).

While the above might make us optimistic that Gödel allows for a way to save a sort of *mathematical* certitude of consistency, he quickly rids us of that hope. He first distinguishes between mathematics proper—all *true* mathematical propositions—and subjective mathematics, by which he means *provable* mathematics. He again gestures toward an inductive consistency, "[t]he assertion [that all axioms and rule applications are correct] could at most be known with empirical certainty, on the basis of a sufficient number of instances or by other inductive inferences" (Gödel, 1951, p. 309). It is this situation that would render the mind "equivalent to a finite machine." He diagnoses the mathematical aversion to such methods as "due to the very prejudice that mathematical objects somehow have no real existence" and continues "If mathematics describes an objective world just like physics, there is no reason why inductive methods should not be applied in mathematics just the same as in physics" (Gödel, 1951, p. 313).

We can summarize Gödel's argument as follows. The only possible types of certitude for consistency perceptions are either formal and empirical. Empirical certitude, while attainable, falls short of what we really want, and formal certitude is impossible. A plethora of questions will arise at this point for the contemporary reader. What role do arguments for and proofs of the consistency of arithmetic play?[8] Is the widespread conviction of consistency a *mere* conviction?

If we would ascribe some role other than that of mere empirical verification to these arguments and proofs and think the widespread conviction of consistency is something more than a *mere* conviction, it seems we must reject the 1951 claim that *either* consistency is grasped with *formal* certitude, understood in Gödel's sense, or consistency is merely an empirical certitude. The force of arguments and proofs of consistency and the thought that our conviction is no mere conviction push us to look for an ignored third between the formal and empirical alternatives. We search for a *mathematical* certitude that is not formal in the sense that it is divorced from that which can be carried out in a formal system and not inductive/empirical in the sense that mathematical and empirical objects are simply different sorts of things. In the words of Rilke (Rilke, 1982, p. 161):

> If only we too could discover a pure, contained,

[8] Recently, see McCall (2014) and Artemov (2019).

human place, our own strip of fruit-bearing soil

between river and rock.

While the claim that *there is* this ignored third is itself a philosophical claim that demands an example of such an account—and this falls beyond the scope of this paper—Doxastic Arithmetic can be seen as the system characteristic of the optimism that such an account is attainable.

5.5 Conclusions

In this Chapter we explored possibilities for formalization of *human reason* as a predicate in the context of Gödel's Disjunction as opposed to treating it as an operator, as is usually done. The latter half of this Chapter was devoted to exploring the two broad options that arise for extending a system of arithmetic with a normal modality epistemic feature treated as a predicate.

We began with an examination of the Gödel sentence in predicate extensions of arithmetic, investigating which sorts of extensions would lead straightforwardly to inconsistency. There were two primary directions available to extend from a normalized modal predicate extension of arithmetic. The first was to the system we called CoPA that extended PA with an IEL^- style modal predicate. The second was by extending arithmetic with a consistency predicate, in the direction of Doxastic Arithmetic or DA. We discussed philosophical motivations for and criticisms of both possibilities.

We close with a final comparison of the merits of coreflection and the consistency axiom in the context of a predicate style formalization of human reason. Hintikka argued that 4 was associated with conclusiveness of reason; we pointed out that coreflection trumps 4 in this respect. On the other hand, consistency was associated with the absence of a conflict internal to human reason about arithmetic. While this may have merit, it is a view that demands further development. We saw in Lemma 5.3.8 that CoPA has an interpretation with the predicate $Form(x)$, and therefore CoPA has a sigma interpretation. We can see also, however, that DA has no sigma interpretation.

Theorem 5.5.1

There is no interpretation of DA*'s predicate* $\Box(x)$ *as a sigma predicate.*

If there were a sigma interpretation of DA, then there is a sigma predicate $S(x)$ that interprets $\Box(x)$. Since $S(x)$ interprets DA's $\Box(x)$, the predicate $S(x)$ has the properties of at least a KD modality. But, by Theorem 5.4.2, DA $\vdash S[F]] \rightarrow S[S[F]]$, and therefore $S(x)$ would correspond to a KD4 modality. By Lemma 5.2.1, this is impossible since PA $\nvdash \bot$.

Where does this leave us in the choice between CoPA and DA? We can ask: is human reason the sort of thing that *could* be characterized using a simple arithmetical predicate? While a full answer to this question goes beyond the scope of this paper, it is *prima facie* implausible that so mysterious a notion might be given so simple a voice.

Chapter 6

Brouwerian Arithmetic

6.1 Mannoury's Challenge

> *Man does not know a sun and earth, but only an eye that sees the sun and a hand that feels the earth.* - Schopenhauer, (Schopenhauer, 1969, p. 3).

Beginning with his critiques of the use of logic in mathematics in his 1907 dissertation, L.E.J. Brouwer was consistently hostile to the use of formal methods in mathematics. Brouwer writes (Brouwer, 1907, p. 92):

> [Logistics] can teach us nothing about the foundations of mathematics, because it remains irrevocably separated from mathematics; on the contrary, in order to maintain an existence on its own account, i.e., to safeguard itself against contradictions, it must reject all its own special principles and acquiesce to be a faithful, automatic, stenographic copy of *the language of mathematics*, which itself *is not mathematics*, but no more than a defective expedient for men to communicate mathematics to each other and to aid their memory for mathematics.

This thought is distilled in Brouwer's *first act of intuitionism*, which is to "[separate] mathematics from mathematical language and hence from the phenomena of language described by theoretical logic, recognizing that intuitionistic mathematics is an essentially languageless activity of the mind" (Brouwer, 1981, p. 6). Intuitionistic mathematics is *completely separated from mathematical language* and *essentially languageless*, it is by its very nature independent of formal methods.

Nonetheless, we would be mistaken to say that Brouwer found *no* use for axiomatics. In his dissertation and in 1912's "Intuitionism and Formalism," Brouwer allows that the axiomatic method be used to aid memory and communicate with others (see (Brouwer, 1907, p. 73, 92, 97) and (Brouwer, 1912, p. 128)). We should not understand this characterization pejoratively. Indeed, Brouwer encouraged his student, Arend Heyting, to think axiomatically about intuitionism for his dissertation (see, van Atten (2017) and (Moschovakis, 2009, p. 81)). He also valued Heyting's work on intuitionistic logic; Hao Wang reports that Brouwer found Heyting's formalization of intuitionistic reasoning more important than Gödel's incompleteness theorems (Wang, 1987, p. 88).

In his 1956 *Intuitionism: An Introduction*, Heyting provides an articulation of the intuitionistic view, "[E]ven in intuitionistic mathematics the finished part of a theory may be formalized. [Though] . . . we can never be mathematically sure that the formal system expresses correctly our mathematical thoughts" (Heyting, 1956, p. 4). Formalization is thus acceptable in some cases for the intuitionist, though any such acceptance is at best tentative. This is given further elucidation in Heyting's 1962 distinction between the intuitionistically acceptable *descriptive* and non-admissible *creative* functions of axioms (Heyting, 1962, p. 238).[1] The descriptive function uses axiomatics to characterize an already present mathematical subject matter, that is, a subject matter that is constructed or finished. The creative function, on the other hand, delivers its own mathematical subject matter by means of consistency. Hence, while even though intuitionistic arithmetic can never be fully captured in language, the axiomatics when used descriptively can be accepted, albeit

[1] A view like this is endorsed also by A.N. Kolmogorov (Kolmogorov, (1925, p. 41):

> The intuitionistic point of view is based on the assumption of a real significance of mathematical propositions. Axioms forming the basis of mathematics are regarded as *expressions of* facts that are given to us. This approach allows the formal method for studying mathematical constructions as one of the possible methods. . .

Again, we see that the role of axioms is to characterize in some sense that which is already present.

with a grain of salt (Heyting, 1956, p. 4).

Even if axiomatic reasoning is no substitute for real mathematical reasoning, the intuitionist might take a given system to better or worse fulfill the descriptive function of the axiomatic method. Contrast, for example, classical propositional logic and intuitionistic propositional logic. It is consistent with the claim that neither CPC nor IPC provides a *perfect* characterization of propositional intuitionistic reasoning that IPC *better* axiomatizes intuitionistic reasoning.

In 1927, twenty years after Brouwer's dissertation, Gerrit Mannoury posed the challenge of axiomatizing intuitionistic reasoning through the Dutch Mathematical Society.[2] For the purpose of this study, we focus on arithmetic. Bringing with us the above information about the intuitionistic role of the axiomatic method, we can make precise Mannoury's Challenge:

> **Mannoury's Challenge**: Provide a descriptively adequate axiomatization of intuitionistic arithmetic.

In 1928, Heyting's answer:

> **Heyting's Response**: Heyting Arithmetic (HA) is a descriptively adequate axiomatization of intuitionistic arithmetic.

won and was selected for the prize of the Dutch Mathematical Society van Atten (2017). Heyting's Response was viewed favorably by Brouwer, as the aforementioned report from Wang shows (see (Wang, 1987, p. 88) and van Atten (2017)). The tradition has since made it clear that Heyting's is the accepted response to Mannoury's Challenge. Indeed, one could not be faulted for simply *identifying* intuitionistic arithmetic with Heyting Arithmetic.[3]

We argue that Heyting Arithmetic should not enjoy the hegemony it currently does. To clarify the reasons for which we should not simply accept Heyting's as the solution to Mannoury's Challenge, we should examine more closely the relation of Heyting Arithmetic to Peano Arithmetic.

[2] See van Atten's "The Development of Intuitionistic Logic," for discussion van Atten (2017). For a quote of the challenge explicitly, which was phrased in terms of set theory, see Troelstra's (Troelstra, 1988, p. 2).

[3] We focus on arithmetic because of its importance in the concept of twoity of the first act of intuitionism. A natural next step would be to extend our study to analysis.

When intuitionistic arithmetic is formalized as HA and classical or platonistic arithmetic is taken as Peano Arithmetic, (PA), the following will hold:

$$\textit{Classical Arithmetic proves } F \not\Rightarrow \textit{ Intuitionistic Arithmetic proves } F \tag{6.1}$$

The most famous examples of course include the intuitionistic failure of *tertium non datur* and inter-definability of the quantifiers. Indeed, this is philosophically well-motivated. Here we have an axiomatic marker of the difference between constructed and platonic objects. For example, in the context of *tertitum non datur*, while platonic objects are already determined, constructive objects are in a dynamic process of generation.

On the other hand, it is obvious that the following holds given we accept Heyting's Response to Mannoury's Challenge:[4]

$$\textit{Intuitionistic Arithmetic proves } F \Rightarrow \textit{Classical Arithmetic proves } F \tag{6.2}$$

The above expresses a connection between constructive arithmetic and platonistic objects. If we take intuitionistic arithmetic to be *simply* the theory of constructed arithmetical objects, then, insofar as constructed and platonistic objects share some properties, there is philosophical justification for the above. Indeed, this seems correct *as an account of the relation between constructive and platonistic objects*; while constructed and platonic objects are radically different sorts of objects, they nonetheless share in *some* properties insofar as both are sorts of objects.

The tacit assumption above, however, seems to be that intuitionistic arithmetic is *simply* the theory of constructed arithmetical objects. This assumption is incorrect as a characterization of intuitionistic arithmetic and inuitionistic mathematics more generally. Heyting himself writes in *Intuitionism: An Introduction*, (Heyting, 1956, p. 10):

> In fact, mathematics, from the intuitionistic point of view, is a study of certain functions

[4] Here the classical logician would say that the intuitionist has merely given the connectives a different meaning. From the intuitionistic perspective, the classicist accepts falsehoods. The question we are concerned with is whether, from the intuitionist's perspective, they have built enough into their formal system.

> of the human mind, and as such it is akin to [philosophy, history, and the social sciences].

Put another way, to think that the *only* difference between platonic and intuitionistic arithmetic has to do with the properties of arithmetical objects minimizes the role of *process* in intuitionistic arithmetic. We need to account for, in Heyting's terms, *the arithmetical function of the mind.*

In section 6.2, we discuss just what is required to account for the arithmetical process in intuitionistic reasoning. The key feature here is the subject's basic intuition of time, from which they generate arithmetic. In section 6.3, we expand upon the interpretation of Heyting Arithmetic as the arithmetic of constructed objects. We argue that while it is essential for formal systems describing the mental process of constructing such objects to contain something like HA, such a system does not exhaust an axiomatic description of the process of mental construction, and therefore that we must somehow go beyond HA.

In section 6.4, we discuss one method of doing this, namely, the option of extending HA with an operator to characterize the intuitionistic mental process. We argue that this method falls short of expressibility desiderata. There are two we focus on. The first is the inability to straightforwardly emulate second-order features. These sorts of worries have already been discussed in the context of motivating predicate-style treatments of modality (see, for example Halbach et al. (2003)). The second has to do with Brouwer's claim that the set of mathematical theorems is *denumerably unfinished.* Because of arithmetic's central position in intuitionistic thinking, a formal presentation of intuitionistic arithmetic should account for this. The operator approach makes good on neither of these counts.

In section 6.5, we discuss an alternative method of extending HA. There are many ways of doing so, but what they all have in common is the addition of a unary *predicate* to HA. Results of Gödel and Montague limit the ways in which we can do this, but nonetheless, two main contenders emerge. We argue that our preferred extension, Doxastic Heyting Arithmetic or DHA, best characterizes the process of intuitionistic creation. DHA is the intuitionistic counterpart of the classical system, Doxastic Arithmetic DA introduced in Peluce (2018) and Peluce (2020). In section 6.6, we provide

Kripke models for DHA. We then show that DHA is a conservative extension of HA and discuss some philosophical consequences of our proposal.

6.2 Brouwer's Basic Intuition of Mathematics

It is not hard to find passages in Brouwer where the subject's activity is of central importance (see (Brouwer, 1912, p. 85-6), and (Brouwer, 1907, p. 53), for example).[5] We find in his 1907 dissertation the following (Brouwer, 1907, p. 61):

> [W]e can call *a priori* only that one thing which is common to all mathematics and is on the other hand sufficient to build up all mathematics, namely the intuition of the many-oneness, the basic intuition of mathematics.

and (Brouwer, 1907, p. 70):

> Mathematics develops out of its basic intuition in a self-multiplication guided by an entirely free choice. The only synthetic judgements a priori generally are those obtained as possibilities of mathematical constructions by virtue of the basic intuition of time, or of many-one-ness

This basic intuition of time, here called *many-one-ness*, also known as *twoity* or *bare-two-oneness*, is of central importance to arithmetic. Brouwer tells us more in his 1913 "Intuitionism and Formalism" (Brouwer, 1912, p. 85):

> [I]ntuitionism considers the falling apart of moments of life into qualitatively different parts, to be reuinited only while remaining separated by time as the fundamental phenomenon of the human intellect, passing by abstracting from its emotional content into the fundamental phenomenon of mathematical thinking, the intuition of bare-two-oneness.

[5] See, for example, the discussion of the subject in Brouwer and Griss in Miriam Franchella's "Philosophies of Intuitionism: Why We Need Them," (Franchella, 2007, p. 74-5).

One notices the particularities of a given moment, and remarks upon its passing that it is different from the next moment. When all emotional content and other qualities are stripped away from the moments themselves, one is left with a bare-two-oneness. This process is iterated, while moments are held in memory, to generate natural numbers. Addition of x and y is explained as counting up to x, and then putting ordinals after x in a one-to-one correspondence with those invoked in counting to y (Brouwer, 1907, p. 15). He understands multiplication as the repetition of the above process and exponentiation as iterated multiplication (Brouwer, 1907, pp. 15-16).

While aspects of Brouwer's view did change over time, the role of twoity reains central. In his 1948 "Consciousness, Philosophy, and Mathematics," he writes (Brouwer, 1948, p. 480):

> *Consciousness* in its deepest home seems to oscillate slowly, will-lessly, and reversibly between stillness and sensation. And it seems that only the status of sensation allows the initial phenomenon of the said transition. This initial phenomenon is a *move of time*. By a move of time a present sensation gives way to another present sensation in such a way that consciousness retains the former one as a past sensation, and moreover, through this distinction between present and past, recedes from both and from stillness, and becomes *mind*.
>
> As mind it takes the function of a subject experiencing the present as well as the past sensation as object. And by reiteration of this two-ity-phenomenon, the object can extend to a world of sensations of motley plurality.

The role of twoity here goes beyond that of grounding arithmetic. Indeed, it is essential in the Brouwerian development of mind and the *world of sensations of motley plurality*.

These passages demonstrate a long-lasting commitment to the central philosophical importance of the intuition of twoity, and therefore also to that of the subject, in Brouwer's thought. We have already sketched arguments for the claim that HA does characterize the arithmetical process. The above shows that not only is it essential to intuitionistic arithemtical reasoning that experience and process be involved, but a very specific sort of experience is required, namely, that of the intuition of twoity.

We can put our task in another way. First, however, we need to supply a verb to characterize the mental activity of the agent. While many verbs are used in such context (perceiving, thinking, intuiting, for example), we suggest that we simply use *thinking*. Clearly what we are doing has idealized away from the epistemic activities of empirical agents, indeed, van Atten writes, "Intuitionism is a theory not about any thinking subject but about a correctly thinking one. This means that the object of study is what is intrinsic to the self-unfolding of the basic intuition, and therefore only essential and no accidental features of the subject are studied. Limitations of time, memory, attention and so on are abstracted from" (van Atten, 2004, p. 9).

What sort of formal feature could be used to represent a commitment to this sort of intuition? Consider the propositional Brouwer-Heyting-Kolomogorov BHK clauses:

Definition 6.2.1 *(BHK Semantics)*

- *A proof of $F \wedge G$ consists of a proof of F and a proof of G;*
- *A proof of $F \vee G$ consists of a proof of F or a proof of G;*
- *A proof of $F \rightarrow G$ is a construction that, given a proof of F yields a proof of G;*
- *There is no proof of $\bot$.*

The above conditions provide a straightforward way for thinking about intuitionistic connectives. While one might disagree with specific articulations of the above clauses, the general framework has application in the intuitionistic context.

A question arises. Are the above conditions themselves (or something like them) enough to express a commitment to Brouwerian intuition? We might indeed think that provided the correct interpretation is given of the formalism, we do not need anything beyond the connectives themselves. We can find an argument to the effect that something like the above is sufficient, given the correct interpretation, in Göran Sundholm's "Implicit Epistemic Aspects of Constructive Logic" (Sundholm, 1997, p. 194):

> The systems of doxastic and epistemic logic are metamathematical formalisms designed to express valid principles of reasoning concerning epistemic notions. The formalisms

> chosen, however, as simple extensions of propositional logic and the use of the propositional connectives K and B treats claims to knowledge and belief as if they were propositions. ...
>
> In the Frege-like, contentual paradigm, on the other hand, knowledge claims are part and parcel of the *use* of the system, but not in the form of propositional operators. Owing to the presence of the assertion sign, one is able to express not just (interrelations between) propositions, but actual assertions ("judgements") with the propositions as contents. An assertion, however, made through the utterance of a declarative sentence, contains (implicit) claims, as to knowledge and truth, with respect to the content expressed by the sentence in question.

It would be, for Sundholm, a mistake to add an epistemic feature for *assertions*; assertions are not the sort of thing that could even be a content of an epistemic feature. Therefore, the argument might go, the unmodified language is itself sufficient to express a commitment to the Brouwerian process of *thought*.

But the word "epistemic," is said in many ways. We require that our system describe constructed—in our case, arithmetical—objects, and in this sense, any theory that is correct with respect to those constructed objects will provide faithful epistemic content. But such a theory alone will not be able to express something like:

The Brouwerian agent constructs that F and thinks that they construct that F

Claims involving this sort of reflection upon construction seem essential to intuitoinistic arithmetical reasoning. Indeed, we saw that iteration of the intuition of twoity involves constructing elements by subtracting away extraneous details, while simultaneously holding others in memory. The entire process is fundamentally one in which the subject reflects upon the contents of their mind. For this reason, something to the effect of the following seems required to account for a Brouwerian arithmetic:

- A proof of $\mathbf{M}F$ requires a prior construction of F.

An intuitionistic formal system ought to reflect that if there is a construction of F, then the agent also *thinks*, again, in the idealized sense used by Brouwer, that F. The admission of such an epistemic feature sets the groundwork for expressing the subject's place in intuitionistic arithmetic, and, more specifically, the intuition of twoity. Note that as of yet we have not said anything about what that feature should look like, whether it should be a defined feature, a connective, or a predicate.

6.3 The Arithmetic of Constructed Objects

Since its acceptance as a response to Mannoury's Challenge, Heyting Arithmetic has held the claim to being *the* formal intuitionistic arithmetic. We argued that, while HA characterizes the properties of a constructed arithmetical objects, it fails to sufficiently differentiate formalized intuitionistic *reasoning* from the theory of classical objects. In section 6.2, it became clear that what was required was an epistemic feature to account for the Brouwerian subject to ultimately formally account for the intuition of twoity. We provisionally called this feature **M**. One might argue that we *can* express **M** within HA as a defined predicate. This is different from the approach of Sundholm that we examined insofar as a defined predicate would attach to codes and therefore allow for the possibility of iteration. In this section, we show that a defined epistemic feature cannot do the conceptual work required of **M**.

Heyting Arithmetic (HA) is the system presented in Definition 2.5.4.

Of course one might try to define the subject as an arithmetical predicate. To do so is to *build up* the subject in terms of arithmetical portions of our system. There is a clear philosophical problem with proceeding in this manner. Though a given defined predicate may encode attractive modal properties, by hypothesis it would be a *defined* predicate. By not including a primitive symbol for thought or intuition, one would suggest the priority of constructed objects over constructive process. But, to allow that constructed objects be prior to the constructing subject simply gets intuitionistic reasoning *wrong*. For this reason, the feature formalizing intuition or thought cannot be a defined one; any such predicate features in too late to the picture of intuitionistic arithmetic.

6.4 Extending HA with an Operator

We argued that Heyting's Response to Mannoury's Challenge was unsatisfactory insofar as it failed to axiomatically account for intuitionistic process. More specifically, the Brouwerian subject was conspicuously absent. We did not argue that HA goes *wrong*—indeed, it provides a valuable characterization of constructed arithmetical objects—but only that it omits a key feature of intuitionistic arithmetic. For this reason, we should preserve HA for its virtues and extend it where it is lacking. A natural way to do so is with axioms governing an added epistemic feature.

We immediately face a choice. We can treat our epistemic feature as an operator or as a predicate. To treat the epistemic feature as an *operator* is to add something to arithmetic that functions like a traditional modal connective, in accordance with the received practice of epistemic logic. Alternatively, if we treat our epistemic feature as a *predicate*, it will share more with the arithmetical predicates $Pr(x)$ and $Form(x)$. In this section, we consider the first approach. We argue that while it has the strengths that one would expect, it falls short in the intuitionistic, specifically, *Brouwerian*, context.

A natural way of articulating the operator approach is by making use of an intuitionistic version of Stewart Shapiro's Epistemic Arithmetic EA Shapiro (1985). Shapiro's EA is Peano Arithmetic extended with an S4 operator. The version we will discuss we call *Epistemic Heyting Arithmetic*, or EHA. Note that Shapiro's EA was not itself created for this purpose. Instead, what we examine here is the possibility of a repurposing of a system that conveniently exemplifies many desirable properties in our context. EHA is presented in Definition 2.6.3.

Reading $\Box F$ in our suggested interpretation as "our agent thinks that F," EHA proves both that intuitionistic thought iterates and that a contradiction is not thought. These correspond to modal axioms 4 (4 above) and D (as an instance of 3). In (3) we even have that if the intuitionistic agent thinks that F, then F holds (constructively). (2) and (5) are the K axiom and necessitation rule, respectively.

The following sort of objection can be raised to the operator approach generally. Halbach, Leitgeb and Welch point out that the operator approach to modality more generally limits the ease with which we can express connections between that operator and predicates (Halbach et al., 2003, p. 3). For example, while we can easily express the relation between an arithmetical predicate $Form(x)$ and the provability predicate $Pr(x)$ as, say:

$$\forall x(Pr(x) \rightarrow Form(x)) \tag{6.3}$$

we forsake the opportunity to emulate second-order features in our language with $\Box$ treated as a propositional operator. This is because, of course, the formula F in $\Box F$ is not the sort of object that could fall under the scope of a first-order quantifier. In this vein, we also cannot express that *something*—understood in terms of $\exists x$—is thought, or that the agent thinks that they think *something*, and so on.

There is a more pressing objection to the idea of taking some such modal extension of arithmetic as an account of Brouwerian arithmetic, however. In his notebooks, Brouwer claims that the totality of mathematical theorems are a *denumerably unfinished* set van Atten (2017) (see also (van Atten, 2004, p. 7-8), for discussion). By this he means that the set of mathematical theorems is such that (Brouwer, 1907, p. 82):

> . . . we can never construct in a well-defined way more than a denumerable subset of [that set], but when we have constructed such a subset, we can immediately deduce from it, following some previously defined mathematical process, new elements which are counted in the original set. But from a strictly mathematical point of view this set does not exist as a whole . . .

Each time we try to delineate more than a denumerable subset of mathematical theorems our circumscription leaves something out. For Brouwer the concept of mathematical theorem is inexhaustible in a deep sense; the set of mathematical theorems is ever-resistant to to formal codification.

How do we characterize this feature of Brouwerian thought within a formal system of intuitionistic arithmetic? If our response to Mannoury's challenge is to take seriously the aim of descriptive adequacy to Brouwerian thought, this seems to be a natural desiderata for such an account.

It is not uncommon to associate Gödel's theorems, and thereby the Gödel sentence, with the classical counterpart of Brouwer's denumerably unfinishedness of mathematical theorems (that is, *the inexhaustibility of mathematics*).[6] Now, what meaning does Gödel sentence:

$$G \leftrightarrow \neg Pr(\ulcorner G \urcorner) \tag{6.4}$$

have in the intuitionistic context? This says that if the agent has a construction of G then they can get one of $\neg Pr(\ulcorner G \urcorner)$, and *vice versa*. Hence the agent cannot proceed to either a construction of G or $\neg G$ within the system.

Above we argued that, in the intuitionistic context, (6.4) expresses a state of the reasoner in which they can decide neither G nor $\neg G$ within the system. This seems a good candidate to express the denumerably unfinishedness of *formal* theorems. In order to represent Brouwer's remarks on the denumerably unfinshedness of the set of *mathematical theorems*, however, we want a Gödel sentence that is not tied to the notion of Gödelian proof or to any specific interpretation, for that matter. Indeed, we want to be able to prove a version of Gödel's Theorem in our system that is not limited to some specific defined predicate. We want, for example, that it be provable that:

$$D \leftrightarrow \neg \Box D \tag{6.5}$$

The above is not provable using the usual argument for the diagonal lemma as here $\Box$ is an operator. Indeed, it is easy to see that adding (6.5) as an axiom to any system extending iT (intuitionistic modal logic T) or $\mathsf{iKD4}$ would render that system inconsistent. While investigating Gödel phenomena is worthwhile in itself, it is especially pressing in the context of an intuitionistic formal system for the reasons we have suggested. Because of this, then, the operator approach is unsuited for a formal characterization of Brouwerian arithmetic.

[6]See Gödel (Gödel, 1951, p. 305), (Gödel, 1931, p. 492), Hao Wang (Wang, 1996, p. 4), and Carlo Cellucci (Cellucci, 1992, p. 116).

6.5 Doxastic Heyting Arithmetic

To extend HA in such a way that our system provides a descriptively adequate characterization of intuitionistic arithmetic, we are immediately faced with two options: we can treat the epistemic feature as an operator or treat it as a predicate. In Section 6.4, we saw that the operator approach failed importantly since a descriptively adequate characterization of intuitionistic arithmetic requires an epistemic feature nuanced enough to allow for representation of the basic intuition of twoity. We now present an answer to Mannoury's Challenge. Our answer is Doxastic Heyting Arithmetic or DHA, which we argue is a descriptively adequate axiomatic characterization of intuitoinistic arithmetic. We present and motivate the system, and then discuss some of its properties.

Before turning to DHA, we need to mention some of the history of predicate treatments of modality. It is well known that the predicate approach has its own inherent limitations. Richard Montague (Montague (1962), 1962), showed that no such treatment of a T modality (a normal modal logic satisfying $\Box F \rightarrow F$) is possible *as a predicate*.[7] Since T is a sublogic of S4, there is no predicate treatment of EHA in this sense.

It is a consequence of the second incompleteness theorem and the three derivability conditions that there is no predicate treatment of a KD4 modality. This is straightforward when one observes the connection between D and the internalized consistency statement for the Gödelian provability predicate.

Given that the principle K and rule necessitation are both natural desiderata in an epistemic feature for an idealized agent, we know that our epistemic feature cannot have both 4 and D. Because of this, we face a choice between the two. We suggest now that the intuitionistically preferable choice is D. We motivated this project generally by a desire that formal intuitionistic arithmetic characterize the process of thought that is essential to intuitionistic arithmetic. We argued that at the heart of a Brouwerian account of the process of intuitionistic arithmetical thinking is the base intuition of twoity. This was the intuition that there are two moments, and they are *really* different.

[7] This result holds intuitionistically as well as an easy Corollary of Lemma 7, (Peluce, 2018, p. 277).

We consider now a new language, extending the language of arithmetic with the predicate $\Box(x)$, as presented in Definition 2.6.1.

The question of this section is whether or not we should allow our $\Box(x)$ to be governed by a version of the D axiom. A positive argument that it should be governed by D is an argument against allowing 4, then, insofar as the two are inconsistent given normality assumptions (again, by reasoning parallel to the second incompleteness theorem).

Let us return to the intuition of twoity. Take two moments, 0 and 1. What would it be to say that they are *really* different? Abbreviating **s0** as **1**, one might venture:

$$\Box(\ulcorner \mathbf{0} \neq \mathbf{1} \urcorner)^{8} \tag{6.6}$$

In our interpretation, this reads "the agent thinks that 0 is not 1." While this is a start, (6.6) is not enough. The above leaves open the possibility that the agent *also* think that 0 is 1. To see that this is so, one can observe that $\mathsf{PA} + \mathsf{K4}^{\mathsf{pred}}$—where this is a base of PA augmented with K4 rules and axioms governing an added predicate, $\Box(x)$—has models where this holds. We use PA here because $\mathsf{PA} + X$ consistency entails $\mathsf{HA} + X$ consistency.

Theorem 6.5.1 $\mathsf{PA} + \mathsf{K4}^{\mathsf{pred}}$ *is consistent with* $\Box(\ulcorner 0 = 1 \urcorner)$.

PA proves the axioms and rules of $\mathsf{K4}^{pred}$ when $\Box F$ is interpreted as $Pr\ulcorner F \urcorner$, hence $\mathsf{K4}^{pred}$ holds in each model of PA. Since $\mathsf{PA} \nvdash \neg Pr\ulcorner \bot \urcorner$, by Gödel's completeness theorem $\mathsf{PA} + Pr\ulcorner \bot \urcorner$ has a model. This, then, is a PA model of $\mathsf{K4}^{\mathsf{pred}} + \Box(\ulcorner 0 = 1 \urcorner)$.

The force of the intuition of twoity, however, does not seem to leave open the possibility that the first moment be identical with the second. In the context of Brouwerian philosophy, this is akin to saying that while most of the time when we take two moments and subtract all qualities from them, we arrive at a bare twoity; *sometimes* we subtract all qualities from two moments and arrive at equality. For this reason, we want a system that *does not* leave open the possibility of models

[8] This is a necessitated case of an instantiation with $x = 0$ of the arithmetical axiom $\forall x(s(x) \neq 0)$, and therefore clearly provable in such systems.

in which $\Box(\ulcorner \mathbf{0} = \mathbf{1} \urcorner)$ hold. While (6.6) is important, we need to augment it as follows:

$$\Box(\ulcorner \mathbf{0} \neq \mathbf{1} \urcorner) \wedge \neg \Box (\ulcorner \mathbf{0} = \mathbf{1} \urcorner) \tag{6.7}$$

This—a genuine difference between two elements—is what seems required to provide a descriptively adequate characterization of intuitionistic arithmetic. Note that the second conjunct is just another way of writing D with the predicate $\Box(x)$.

Given that we need $\neg \Box (\ulcorner \mathbf{0} = \mathbf{1} \urcorner)$ to characterize the Brouwerian twoity, we cannot allow the axiom $\mathsf{4}$ for reasons we have discussed. Therefore, we suggest the following answer to Mannoury's Challenge. Our system, Doxastic Heyting Arithmetic, or DHA, is presented in Definition 2.6.7. This is the intuitionistic version of Doxastic Arithmetic, introduced in Peluce (2018) and Peluce (2020). (1) is a base of HA.[9] (2) and (4) qualify $\Box(x)$ as a normal *predicate-style* modality. Lastly, (3) allows DHA to express Brouwerian twoity.

Observe now that the following can be proven of DHA:

Theorem 6.5.2 $\mathsf{DHA} \nvdash \bot$

It is already known that DA is consistent as it has an interpretation in PA (see (Peluce, 2018, p. 285)). Since DHA is a subsystem of DA, it too is consistent.

Next, we can observe that DHA proves in the usual way that:

Lemma 6.5.3 *(Generalized Diagonal Lemma)*

There is a D for which

$$\mathsf{DHA} \vdash D \leftrightarrow \neg \Box (\ulcorner D \urcorner) \tag{6.8}$$

By Gödelian reasoning we can prove a generalized version of Gödel's first incompleteness theorem. The following is more general than Gödel's proof because here we appeal to a predicate without needing to specify some particular definition of that predicate.

[9] As was mentioned, this assumes that HA adequately describes the temporal part of the Brouwerian intuition, that it captures the part that is *traditionally* considered arithmetical. Provided a *better* account of this, the system can be modified of course. We begin with HA now as it seems to best characterize the temporal part of intuitionistic arithmetic.

Theorem 6.5.4 *Generalized Incompleteness*

$\mathsf{DHA} \nvdash D$

$\mathsf{DHA} \nvdash \neg D$

Assume $\mathsf{DHA} \vdash D$. Then, by necessitation, $\mathsf{DHA} \vdash \Box(\ulcorner D \urcorner)$. By Lemma 6.5.3, $\mathsf{DHA} \vdash \neg \Box (\ulcorner D \urcorner)$, but then $\mathsf{DHA} \vdash \mathbf{0} = \mathbf{1}$. But by Theorem 6.5.2, $\mathsf{DHA} \nvdash \bot$.

We consider the case of DA (the intuitive classical version of DHA) and show that $\neg D$ is not derivable. The non-derivability result transfers to DHA since DHA is a subtheory of DA. Assume $\mathsf{DA} \vdash \neg D$. By Lemma 6.5.3, $\mathsf{DA} \vdash \Box(\ulcorner D \urcorner)$. By necessitation, $\mathsf{DA} \vdash \Box(\ulcorner \neg D \urcorner)$. Therefore, $\mathsf{DA} \vdash \Box(\ulcorner D \wedge \neg D \urcorner)$. By DA's D axiom—which here plays the role of Rosser's trick Rosser (1936)—we have that $\mathsf{DA} \vdash \neg \Box (\ulcorner D \wedge \neg D \urcorner)$, and therefore, if we assume $\mathsf{DA} \vdash \neg D$, then $\mathsf{DA} \vdash \mathbf{0} = \mathbf{1}$.

Gödel's second incompleteness theorem is often interpreted as saying that if a given theory can express a sufficient amount of arithmetic and is consistent, then it cannot prove its own consistency statement. We can observe that DA and DHA avoid Gödel's second incompleteness theorem with respect to $\Box(x)$, insofar as DHA and DA prove their consistency statements in their doxastic form.

Corollary 6.5.5 *We have both*

$$\mathsf{DHA} \vdash \neg \Box (\ulcorner \boldsymbol{0} = \boldsymbol{1} \urcorner) \textit{ and } \mathsf{DHA} \textit{ is consistent}$$

and

$$\mathsf{DA} \vdash \neg \Box (\ulcorner \boldsymbol{0} = \boldsymbol{1} \urcorner) \textit{ and } \mathsf{DA} \textit{ is consistent}$$

This follows by the fact that DA is consistent (by Theorem 6.5.2) and that DHA and DA contain their own consistency statements.

From the above we can see that there is a sense in which DHA (and DA) sidestep Gödel's second incompleteness theorem. The DA and DHA-consistency statements are provable. This is of course not the Gödelian consistency statement. Indeed, it would be strange for the intuitionist to desire that a marker of consistency be tied to a specific formal system. While in the general foundational context, DHA's consistency statement provides a method of working within the limitations of Gödel's second incompleteness theorem, in the intuitionistic context it has a philosophically important meaning. Namely, as we have argued, it expresses that the first and second moments—and more generally, *all distinct moments*—are thought by the agent to be genuinely different. We find consistency of reasoning as a consequence of twoity and an intuitionistic expression of consistency *just as* the intuition of twoity. Formal consistency follows because we have provided descriptively adequate characterizations of twoity itself; Brouwerian twoity thus remains conceptually privileged in our picture.

6.6 Kripke Models for DHA

In this section, we present Kripke models for DHA. A *heuristic* motivation for Kripke models of intuitionistic logic is to think of them as representing stages of a Brouwerian subject's development, thereby presenting a representation of what the intuitionist has in mind to the classicist.[10] The nodes of a Kripke DHA model (and HA model for that matter) are rather classical objects arranged in such a way to provide a model-theoretic picture of DHA. As is well known, Kripke models for HA make use of structures at nodes that are models of PA.[11]

We begin by introducing DHA frames:

Definition 6.6.1 *A* DHA *Frame* $\langle W, \leq, \{D_w\}, \{B_w\}, 0, s, +, \cdot, =\rangle$.

1. *W is a countable set of worlds;*

2. *$\leq$ is a partial order on W;*

3. *$\{D_w\}$ is a collection of sets. Each D_w is associated with $w \in W$ where:*

[10] See van Dalen (van Dalen, 2004a, p. 164).

[11] The Kreisel-Troelstra Theory of the Creating Subject (CS) brings a time parameter into the language itself while, of course, DHA does not. See van Atten's (van Atten, 2018, pp. 1588-1591) for an extended overview of CS. The time parameter in CS provides a nice characterization of the process of construction. DHA can characterize this process as well, not inside the language itself, but in the Kripke models for DHA. One might, though, object that intuitionistic Kripke models are not characteristic of the step-by-step construction of intuitionistic objects (see, for example (Shapiro, 2014, pp. 34-36)). The intuitionist sympathizer, however, can concede this pointing out that these models are not intended to convince the classicist of anything but rather are meant as an invitation for discussion of the *philosophy* at issue.

   (a) *D_w is the domain of w;*

   (b) *Each domain is denumerable;*

   (c) *If $w \leq v$, then $D_w \subseteq D_v$.*

4. *$\{B_w\}$ is a collection of sets, where each B_w is associated with a $w \in W$ where:*

   (a) *$B_w \subseteq D_w$;*

   (b) *If $w \leq v$, then $B_w \subseteq B_v$;*

   (c) *For sentences $F \in \mathcal{L}_w$, if* DHA $\vdash F$ *then* $\ulcorner F \urcorner \in B_w$;

   (d) $\ulcorner 0 = 1 \urcorner \notin B_w$;

   (e) *Where $F, G \in \mathcal{L}_w$, if $\ulcorner F \to G \urcorner, \ulcorner F \urcorner \in B_w$ then $\ulcorner G \urcorner \in B_w$.*

5. *0 is a constant which may be regarded as a zero-place function, s is a one-place function, $+$ and $\cdot$ are a two-place functions on each D_w. Whenever one such function has a given value in D_w, and $w \leq v$, then it has that same value in D_v.*

6. *$=$ is an identity relation on each D_w. For each identity $t = t'$ in the language of $\mathcal{L}_w$ that holds in D_w, if $w \leq v$, then $t = t'$ holds in D_v.*

The domain of the frame, $\mathcal{D}$ is defined as the union of all D_w for $w \in W$.

Definition 6.6.2 *At each world w, we consider* DHA *in the language that includes all elements of D_w as individual constants. Call this extended language $\mathcal{L}_w$. Let St_w be the set of sentences in $\mathcal{L}_w$.* DHA *truth is defined for closed formulas of* DHA *as follows for $w \in W$:*

1. *For closed terms t_1 and t_2 in the language of* DHA*,* DHA $\vdash t_1 = t_2$ *iff $w \Vdash t_1 = t_2$. In particular, $w \nVdash 0 = 1$.*[12]

2. *Standard conditions for intuitionistic first-order Kripke models:*

   (a) *$w \Vdash F \land G$ iff $w \Vdash F$ and $w \Vdash G$;*

[12] As we have shown in the previous Section, DHA is consistent.

(b) $w \Vdash F \vee G$ *iff* $w \Vdash F$ *or* $w \Vdash G$;

(c) $w \Vdash F \to G$ *iff for all* v *where* $w \leq v$, *either* $v \nVdash F$ *or* $w \Vdash G$;

(d) $w \Vdash \neg F$ *iff* $w \Vdash F \to 0 = 1$;

(e) $w \Vdash \forall x F$ *iff, for each* v *where* $w \leq v$ *and each* $c \in D_v$ *we have that* $v \Vdash F(c)$;

(f) $w \Vdash \exists x F(x)$ *iff for some* $c \in D_w$, $w \Vdash F(c)$.

3. $w \Vdash \Box(t)$ *iff* $t \in B_w$.

Definition 6.6.3 DHA *Model*

A DHA *Model* $\langle W, \leq, \{D_w\}, 0, s, +, \cdot, =, \{B_w\}, \Vdash \rangle$ *is a* DHA *frame augmented with* $\Vdash$ *such that for each* $w \in W$, *if* DHA $\vdash F$ *and* $F \in St_w$, *then* $w \Vdash F$.

Definition 6.6.4 DHA *Validity*

A sentence F *is* DHA *true in a model* $\mathcal{M}$ *if, for every* $w \in W^{\mathcal{M}}$ *it holds that* $w \Vdash F$, *which we write* $\mathcal{M} \vDash F$. *We write* $\mathcal{M} \vDash \Gamma$ *to mean that, where* Γ *is a set of sentences, each* $F \in \Gamma$ *is true in* $\mathcal{M}$. F *is* valid *if it is true in every* DHA *model. We write this* DHA $\vDash F$ *and abbreviate it as* $\vDash F$.

Lemma 6.6.5 *Hereditary Property: If* $w \Vdash F$ *and* $w \leq v$, *then* $v \Vdash F$.

We prove this by induction on complexity of sentences. If F is the atom $t_1 = t_2$, then, if $w \Vdash F$ and $w \leq v$, then $v \Vdash F$ by Definition 6.6.1 clause 6.

For $\wedge$, $\vee$, $\to$, $\forall$, $\exists$ sentences, the argument is the standard argument in first-order intuitionistic Kripke models.

Let $F = \Box(c)$. If $w \Vdash \Box(c)$, then $c \in B_w$. Since $c \in B_w$ and $w \leq v$, $c \in B_v$, by Definition 6.6.1 clause 4b. By Definition 6.6.2, it follows that $v \Vdash \Box(c)$.

Theorem 6.6.6 *Soundness*

If DHA $\vdash \varphi$ *then* DHA $\vDash \varphi$

We prove this by induction on proof length. It is easy to see that the cases of HA axioms will be covered by our definitions.

The DHA axioms $\Box[F \to G] \to (\Box[F] \to \Box[G])$ and $\neg \Box (\ulcorner \mathbf{0} = \mathbf{1} \urcorner)$ are special cases of the conditional. In the first instance, a failure of DHA's K axiom at w would require a case where fore some v, where $w \leq v$, $v \Vdash \Box[F \to G]$ and $v \nVdash \Box[F] \to \Box[G]$. Since $v \Vdash \Box[F \to G]$, by Definition 6.6.2, we have that $\ulcorner F \to G \urcorner \in B_v$. If $v \nVdash \Box[F] \to \Box[G]$, then for some x where $v \leq x$, it must hold that $x \Vdash \Box[F]$ and $x \nVdash \Box[G]$. Then, by Definition 6.6.2 again, we see that $F \in B_x$ and $G \notin B_x$. By Definition 6.6.1, since $v \leq x$, $\ulcorner F \to G \urcorner \in B_x$. But then $\ulcorner F \to G \urcorner, \ulcorner F \urcorner \in B_x$ but $\ulcorner G \urcorner \notin B_x$, which is impossible by Definition 6.6.1 clause 4e. The argument for DHA's D axiom should be obvious when we note that Definition 4d secures that for every world w, $\ulcorner 0 = 1 \urcorner \notin B_w$.

We now cover the case where φ follows by one of our rules. Recall that we have three rules: *modus ponens*, universal generalization, and necessitation. What if F follows by *modus ponens*? By induction hypothesis, $\vDash G \to \varphi$ and $\vDash G$, we show that $\vDash \varphi$. If $\vDash G \to \varphi$ and $\vDash G$, then every w will be such that $w \Vdash G \to \varphi$ and $w \Vdash G$. Take an arbitrary v in such a model. By the above, every u where $v \leq u$ is such that $u \nVdash G$ or $u \Vdash \varphi$, by Definition 6.6.2, clause 5. Since v is one such u, and $v \Vdash G$, it follows that $v \Vdash \varphi$.

If $\varphi = \forall x G(x)$ and follows by universal generalization, then it is provable that $G(a)$ for arbitrary $a \in D$, the domain of the model. By induction hypothesis, we assume that $\vDash G(a)$ for arbitrary a. Therefore, by Definition 6.6.2, any arbitrary $w \Vdash \forall x G(x)$, hence in every DHA world w our formula will hold.

If φ follows by necessitation, then $\varphi = \Box \ulcorner G \urcorner$ and $\vDash G$, by induction hypothesis. But then $\ulcorner G \urcorner \in B_w$, for each world w, due to Definition 6.6.1. Hence, for each $w \in W$ we have $w \Vdash \Box \ulcorner G \urcorner$, by Definition 6.6.2.

We now prove Completeness.

Theorem 6.6.7 *Completeness*

If $\mathsf{DHA} \vDash \varphi$ *then* $\mathsf{DHA} \vdash \varphi$

In order to simplify our argument, we take DHA along with all instances of K for any F and G

and any constants therein in any possible models. Observe that while we do account for new constants—we have any future instance of K already—this extension of DHA remains within the original language of DHA because K makes use only of Gödel numbers of formulas (as opposed to the formulas themselves). Indeed, this is possible because $\Box(x)$ is a predicate. This will not affect our argument because, despite this additional assumption, the model we create in which $w \nVdash \varphi$ will still be a DHA model.

Definition 6.6.8 DHA *Prime Theory*

Γ *is a* DHA *Prime Theory iff:*

1. $F \in \Gamma$ *for all sentences* F *such that* $\Gamma \vdash F$ *in* DHA*;*

2. $F \vee G \in \Gamma$ *iff* $F \in \Gamma$ *or* $G \in \Gamma$*;*

3. $\exists x F(x) \in \Gamma$ *iff for some constant* c*,* $F(c) \in \Gamma$*.*

Note that we do not posit necessitation for Γ; we only have necessitation on DHA theorems. Clauses 2 and 3 secure the correct properties of disjunction and the existential quantifier.

Lemma 6.6.9 *Deduction Theorem*

DHA *and* DA *enjoy the standard Deduction Theorem:*

$$\mathsf{DHA} \vdash F \to G \Leftrightarrow \mathsf{DHA}, F \vdash G$$

This follows by the same argument used in Theorem 2.1.3.

Now, set of sentences S is consistent if it $S \nvdash \bot$. Any consistent set of sentences extends to a consistent DHA prime theory. Here we reproduce the proof of van Dalen (van Dalen, 2004a, p. 170) in our setting.

Lemma 6.6.10 *Let* φ *be closed and in the language* $\mathcal{L}$*, of* DHA*. If* $\Gamma \nvdash \varphi$*, then there is a prime theory,* Γ'*, that extends* Γ *where* $\Gamma' \nvdash \varphi$*.*

We begin by extending the language $\mathcal{L}$ of DHA with a denumerable set of witness constants to $\mathcal{L}'$. We build up Γ' by creating a series of extensions of Γ_0 where $\Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \ldots$ and taking Γ' to be the union of all Γ_n. We do not need to add new axioms $\Box \ulcorner F \to G \urcorner \to (\Box \ulcorner F \urcorner \to \Box \ulcorner G \urcorner)$ for all new formulas because, as we stipulated above, DHA already contains the necessary instances of K. We ensure that Γ' extends Γ by setting that Γ_0 is Γ. The standard argument shows that φ remains unprovable in each Γ_k. We prove this by induction on Γ_k. In the base case, we know that $\Gamma_0 \nvdash \varphi$ by hypothesis.

Following van Dalen (van Dalen, 2004a, p. 170), we consider the two cases in which k is odd or k is even. If k is odd, then we find the first sentence $\psi_1 \vee \psi_2$ where $\Gamma_k \vdash \psi_1 \vee \psi_2$ that we have not treated. Since it could not be that *both* $\Gamma_k, \psi_1 \vdash \varphi$ and $\Gamma_k, \psi_2 \vdash \varphi$, we set that Γ_{k+1} is $\Gamma_k \cup \{\psi_1\}$ if $\Gamma_k, \psi_1 \nvdash \varphi$, or $\Gamma_k \cup \{\psi_2\}$ otherwise.

If k is even, we find the first existential sentence $\exists x \psi(x)$ where $\Gamma_k \vdash \exists x \psi(x)$ in $\mathcal{L}'$ that we have not treated. Take the first new constant that is not in Γ_k, c, and set that Γ_{k+1} is $\Gamma_k \cup \{\psi(c)\}$. Why does $\Gamma_{k+1} \nvdash \varphi$? If $\Gamma_{k+1} \vdash \varphi$, then $\Gamma_k \vdash \psi(c) \to \varphi$, by the Deduction Theorem. Note that c is a fresh constant. Therefore, the same derivation of $\psi(c) \to \varphi$ works when c is replaced by a fresh variable y. That is, $\Gamma_k \vdash \psi(y) \to \varphi$. Universal generalization yields that $\Gamma_k \vdash \forall y(\psi(y) \to \varphi)$. By first-order intuitionistic reasoning, we get $\Gamma_k \vdash \exists y \psi(y) \to \varphi$. We rename the bound variables from $\Gamma_k \vdash \exists x \psi(x)$ to get $\Gamma_k \vdash \exists y \psi(y)$, and then derive φ in Γ_k. This contradicts our assumption that $\Gamma_k \nvdash \varphi$.

We have seen that each successively generated Γ_k is such that $\Gamma_k \nvdash \varphi$. We now take the union of all such Γ_k and define this as Γ'. We now show that $\Gamma' \nvdash \varphi$, that Γ' is a prime theory, and that Γ' is closed under deduction.

First, we know that $\Gamma' \nvdash \varphi$, since if it did, there would a some $\Gamma_k \vdash \varphi$ which we showed above was impossible. Second, we know that Γ' is a prime theory. Consider first a $\psi_1 \vee \psi_2 \in \Gamma'$. Either we have that either $\psi_1 \in \Gamma_0$ or $\psi_2 \in \Gamma_0$, or not. In the first case, it follows that $\psi_1 \in \Gamma'$ or $\psi_2 \in \Gamma'$. In the second, there is a least number k where $\psi_1 \vee \psi_2 \in \Gamma_k$. The disjunction will be treated at some higher stage $k \leq h$, so for some h, we have that $\psi_1 \in \Gamma_{h+1}$ or $\psi_2 \in \Gamma_{h+1}$, and therefore, $\psi_1 \in \Gamma'$

or $\psi_2 \in \Gamma'$. Next, take a given $\exists x\psi(x) \in \Gamma'$. By similar reasoning, for some h and constant c, we have that $\psi(c) \in \Gamma_{h+1}$ and therefore $\psi(c) \in \Gamma'$. Finally, we see that if $\Gamma' \vdash \psi$ then $\psi \in \Gamma'$. This is because if $\Gamma' \vdash \psi$ then $\Gamma' \vdash \psi \vee \psi$, and therefore, at some stage h, Γ_{h+1} was defined as $\Gamma_h \cup \{\psi\}$.

Now we construct a model where the root node $r \Vdash \Gamma_0$ but $r \nVdash \varphi$. Without loss of generality, we assume that all functional symbols in DHA $(s, +, \cdot)$ are represented in predicate form $(sp, +p, \cdot p)$[13], assuming necessary postulates to ensure the functional behavior of our new predicates. For the purposes of this proof, we consider DHA without functional symbols. We closely follow van Dalen (van Dalen, 2004a, p. 170-1).

Lemma 6.6.11 *Model Existence Lemma*

If $\nvdash \varphi$, then there is a DHA *Kripke model $\mathcal{M}$ with a root node r where $r \nVdash \varphi$.*

We begin with our Γ_0 as DHA and extend it to a prime theory Γ' such that $\Gamma' \nvdash \varphi$, following Lemma 6.6.10. Where $\mathcal{L}'$ is the language of Γ', take the set of constants of $\mathcal{L}'$ and call that set D'. We define a new denumerable family of sets of denumerable constants disjoint from D' as $\{c^i_m | 0 \le i, 0 \le m\}$, from which we will draw to form new domains D^i as $\{c^i_m | 0 \le m\}$.

We take the finite sequences of natural numbers, including also the empty one which we write as $\langle\rangle$. We are going to exploit the fact that the relation "initial segment of" is a partial order on sequences of natural numbers, and that $\langle\rangle$ is in the root position.

We set that $D(\langle\rangle)$ is D', the set of constants in $\mathcal{L}'$. Then, for a k-long sequence of natural numbers $\vec{n}$, the associated domain $D(\vec{n})$ will be $D' \cup D^0 \cup \ldots \cup D^{k-1}$.

Let $\mathcal{L}(\vec{n})$ be the language resulting from extending $\mathcal{L}$ with the set of atoms $At(\vec{n})$ built up from constants in $D(\vec{n})$. Observe that $At(\vec{n})$ includes formulas $\Box(c)$ for constants of $\mathcal{L}(\vec{n})$.

From the above, we now need to rebuild prime theories as follows, which will be the truth sets associated with worlds in our construction. First, we define the prime theory $\Gamma(\langle\rangle)$ as Γ'. At $\Gamma(\langle\rangle)$ the language is $\mathcal{L}'$ and the domain is D'.

Recall, $\mathcal{L}(\vec{n})$ is the language at $\vec{n}$. Indeed, the relevant part of $\mathcal{L}(\vec{n})$ is the length, written $|\vec{n}|$, of $\vec{n}$. The language resulting from increasing the length of $\vec{n}$ by 1 we write $\mathcal{L}(|\vec{n}| + 1)$.

[13]For example, $s(t)$ yielding the value t' becomes the predicate $sp(t, t')$. See Smorynski Smorynski (1973) for an example.

Assume that we have $\Gamma(\vec{n})$. We enumerate all pairs of sentences in $\mathcal{L}(|\vec{n}| + 1)$ as $\langle \sigma_0, \tau_0 \rangle$, $\langle \sigma_1, \tau_1 \rangle$, ... where the theory $\Gamma(\vec{n})$ and σ_i does not prove τ_i in the extended language $\mathcal{L}(|\vec{n}| + 1)$, for each i.

We now use Lemma 6.6.10 to extend each $\Gamma(\vec{n}) \cup \{\sigma_i\}$ where $\Gamma(\vec{n}), \sigma_i \nvdash \tau_i$, to a prime theory. This gives us $\Gamma(\vec{n}, i)$ with $\Gamma(\vec{n}, i) \nvdash \tau_i$, where both $\sigma_i \in \Gamma(\vec{n}, i)$ and $\tau_i \notin \Gamma(\vec{n}, i)$.

Lastly, we define $B(\vec{n})$ where $B(\vec{n}) \subseteq D(\vec{n})$. Since $\Gamma(\vec{n}, i)$ is consistent and contains DHA, we set that $B(\vec{n}, i)$ consists of the Gödel numbers of each sentence F where $\Gamma(\vec{n}, i) \vdash \Box(F)$.

We call the set of all $\vec{n}$ our worlds W. We now check that, for a given $\vec{n}$ that,

$$\vec{n} \Vdash \psi \leftrightarrow \Gamma(\vec{n}) \vdash \psi$$

If ψ is an equality, then the equivalence holds by Definition 6.6.1. Consider now the case where ψ is $\Box(c)$. By construction, $\Gamma(\vec{n}) \vdash \Box(c)$ iff $c \in B(\vec{n})$ iff $\vec{n} \Vdash \Box(c)$.

The other cases follow by standard reasoning on intuitionistic Kripke models.

Therefore, the root world Γ' is such that $\Gamma' \Vdash \Gamma$ but $\Gamma' \nVdash \varphi$. It is clear that this is a DHA model insofar as we have our set of worlds; the indexes of our worlds are organized by the initial segment relation which gives us $\leq$; we have our sets of domains associated with worlds and subsets of those domains for $\{B_w\}$; and we have the standard arithmetical portion of the model.
From Lemma 6.6.11, we see that for $\Gamma \nvdash \varphi$, we can construct a Γ' where $\Gamma' \Vdash \Gamma$ but $\Gamma' \nVdash \varphi$.

We can observe that DHA does not prove the Barcan formula,

Lemma 6.6.12 $\mathsf{DHA} \nvdash \forall x \Box F(x) \to \Box[\forall x F(x)]$

We show that $\mathsf{DA} \nvdash$ the Barcan formula, and so DHA does not either. Note that the above is schematic, we only aim to show that some instances are not provable. Consider the Barcan formula where $F(x)$ is interpreted as the Gödelian proof predicate $Pr(x)$. Next, we let $\Box(x)$ be interpreted as a normalized Rosser proof predicate R. Our predicate $R[F]$ encodes that "there is an x that is a proof of F and for all $y \leq x$, that y is not a proof of F." We know there is an interpretation of $\Box(x)$ as a normalized Rosser proof predicate, which we call the *Rosser Interpretation*, and that under that interpretation all DA principles are provable in PA by (Peluce, 2018, pp. 284-285) and (Peluce, 2020, p. 11). Since under the Rosser interpretation, every formula provable in DA becomes provable in PA, it now suffices to check that the Rosser translation of the Barcan formula is not provable in DA. Consider the Rosser interpretation of the Barcan formula:

$$\forall x R[\neg Proof(x, \bot)] \to R[\forall x \neg Proof(x, \bot)]]$$

We observe that this fails in the standard classical model of arithmetic $\mathcal{N}$. The antecedent is thus easily satisfied; for it says that for each x, we have an independent proof that that specific x is not a proof of $\bot$. The consequent is not satisfied, however, since the formula inside the brackets is the Gödelian consistency statement and there is no Rosser proof of the Gödelian consistency statement.

What happens in the case of DHA's Gödel sentence $D \leftrightarrow \neg \Box [D]$? Since DHA $\nvdash D$, we first take a DHA-model where a root world v has $v \nVdash D$. Call this model $\mathcal{M}_1$, where $\mathcal{M}_1 = \langle W_1, \leq_1, \{D_w\}_1, 0_1, s_1, +_1, \cdot_1, =_1, \{B_w\}_1, \Vdash_1\rangle$. Then, since DHA $\nvdash \neg D$, we take another model with a set of worlds disjoint from $\mathcal{M}_1$, in which at a root world w we have that $w \nVdash \neg D$. Let this be $\mathcal{M}_2$, where $\mathcal{M}_2 = \langle W_2, \leq_2, \{D_w\}_2, 0_2, s_2, +_2, \cdot_2, =_2, \{B_w\}_2, \Vdash_2\rangle$.

We then build a new model $\mathcal{M}$. We make use of a new world r as our root world below v and w. We set r to be the standard model of PA. We build this new model $\mathcal{M}$ and set our worlds to $W = W_1 \cup W_2 \cup \{r\}$. We define our intuitionistic accessibility relation $\leq$ as the transitive closure of $\leq_1 \cup \leq_2 \cup \{r \leq v\} \cup \{r \leq w\}$. The domain of D_r is the set of natural numbers. The domains of our worlds, $\{D_w\}$, is defined by the following cases:

1. When $w = r$, D_r is the set of natural numbers.

2. When $w \in W_1$, D_w is as in $\mathcal{M}_1$.

3. When $w \in W_2$, D_w is as in $\mathcal{M}_2$.

Each of D_w contains D_r as its initial segment. Interpretations of $0, s, +, \cdot$, on those fragments are standard (which is determined by axioms of HA). This secures the required monotonicity conditions for $\mathcal{M}$. We define B_w for $w \in W_i$ as in M_i. We then define B_r as the set of Gödel numbers of formulas provable in DHA. We see, then, that $\mathcal{M}$ as defined is a DHA model.

Observe that $r \not\Vdash D$ since $r \leq v$ and $v \not\Vdash D$ and $r \not\Vdash \neg D$ since $r \leq w$ and $w \not\Vdash \neg D$, though $r \Vdash D \leftrightarrow \neg \Box (\ulcorner D \urcorner)$.

Lemma 6.6.13 DHA *Disjunction Property*

$\mathsf{DHA} \vdash F \vee G \;\Rightarrow\; \mathsf{DHA} \vdash F$ *or* $\mathsf{DHA} \vdash G$

We prove this by contrapositive, we show that if $\mathsf{DHA} \nvdash F$ and $\mathsf{DHA} \nvdash G$ then $\mathsf{DHA} \nvdash F \vee G$. By Lemma 6.6.11, there is a model $\mathcal{M}_1$ with root node r where $r \not\Vdash F$ and a model $\mathcal{M}_2$ with root node s where $s \not\Vdash G$. We need to just show that there is a model $\mathcal{M}_3$ with root node t where $t \not\Vdash F \vee G$.

We use the usual construction to build $\mathcal{M}_3$ from disjoint $\mathcal{M}_1$, $\mathcal{M}_2$, and the root world t. Notably, since t will be our root world, we preserve the partial order accessibility relations of the original two models but add t before every other world in our new accessibility relation. Suppose that $t \Vdash F \vee G$. Then, by Definition 6.6.2, $t \Vdash F$ or $t \Vdash G$. But, by Lemma 6.6.5, if $t \Vdash F$, then $r \Vdash F$ and if $t \Vdash G$ then $s \Vdash G$, contradicting our assumption.

Lemma 6.6.14 *Existence Property for* DHA

$\mathsf{DHA} \vdash \exists x F(x) \;\Rightarrow\; \mathsf{DHA} \vdash F(t),$ *for some term t*

We prove this by contrapositive. That is, if $\mathsf{DHA} \nvdash F(t)$ for any term t, then $\mathsf{DHA} \nvdash \exists x F(x)$. If $\mathsf{DHA} \nvdash F(t_i)$, then there is a model $\mathcal{M}_i$, for each t_i, where $r_i \not\Vdash F(t_i)$.

We now build a new model as we did in the proof of the Disjunction property. Take as our base node the standard model of arithmetic N. Therein, suppose for every term t, $N \not\Vdash F(t)$. We can suppose this because if for some numeral n, $N \Vdash F(n)$, then $\mathsf{DHA} \vdash F(n)$ and $\mathsf{DHA} \vdash \exists x F(x)$. We then take the disjoint models for each $\mathcal{M}_i$ where $r_i \not\Vdash F(t_i)$, and generate the new model in the usual way by setting N below (in terms of the accessibility relation) every other world in the new model but preserving their original accessibility relations. Suppose that $N \Vdash \exists x F(x)$. Then, by Lemma 6.6.5, each x is such that $x \Vdash \exists x F(x)$. But, then, by Definition 6.6.2, for each x there is a y where $y \Vdash F(t)$ for some t. But this cannot be because for every new term we added to extended models, we made sure that it was such that $\mathcal{M}_i \nVdash F(t_i)$.

6.7 Conclusions

In this Chapter, we introduced and motivated Doxastic Heyting Arithmetic or DHA as an answer to Mannoury's challenge and a formal characterization of Brouwerian arithmetic. We have aimed to axiomatically express the emphasis on mental *process* at the heart of the intuitonistic project. We suggest that DHA takes seriously this view of intuitionistic arithmetic.

Recall that we argued that HA was best construed as the theory of constructed objects. A natural question is: does DHA ascribe exactly the same properties to constructed objects as HA? That is, is DHA a conservative extension of HA? We answer this in the affirmative:

Theorem 6.7.1 DHA *is a conservative extension of* HA

For $F \in \mathcal{L}_{HA}$, *if* $\mathsf{DHA} \vdash F$ *then* $\mathsf{HA} \vdash F$.

By completeness of HA with respect to HA-models and DHA soundness with respect to DHA-models, it suffices to show that if there is an $\mathcal{M}_{\mathsf{HA}}$ with $r \in W$ and $r \nVdash F$, we can convert it into a DHA-model $\mathcal{M}$ with $r \in W$ where $r \nVdash F$.

Let $\mathcal{M}$ be an HA-model in which HA-formula F fails. We define now a new model $\mathcal{M}'$ by adding a family of relations $\{B_w\}$ for each world w in $\mathcal{M}$. We begin by proving some auxiliary lemmas about $\mathcal{M}'$ and then show that $\mathcal{M}'$ is a DHA-model in which the original HA-formula F fails. We use the same worlds from $\mathcal{M}$ as our new domain in $\mathcal{M}'$. We also assume that our extended language $\mathcal{L}_{\mathsf{DHA}}$ contains all domain elements from $\mathcal{M}$ as constants.

For any DHA-formula X, by its Rosser translation, X^R we understand the result of substituting all occurances of $\Box[.]$ with $R[.]$, where R is the normalized Rosser provability predicate. We define Rosser translations by induction on X with $(\Box[X])^R = R[X^R]$. Note that this definition works for formulas with Boolos brackets $[.]$ and the corresponding X's are not necessarily closed formulas. It is immediate that X^R is an HA-formula.

For each world w, define B_w as the set of Gödel numbers of closed DHA formulas X such that in $\mathcal{M}$, $w \Vdash R[X^R]$.

Therefore, we define the truth value of a closed formula $\Box[X]$ at w in $\mathcal{M}'$ as follows:

$$w \Vdash \Box[X] \Leftrightarrow w \Vdash R[X^R]$$

We note that our B_w's are subsets of corresponding domains, are monotone up $\leq$, do not contain $\ulcorner 0 = 1 \urcorner$ because HA refutes $R\ulcorner 0 = 1 \urcorner$, and that if $\ulcorner F \to G \urcorner \in B_w$ and $\ulcorner F \urcorner \in B_w$ then $\ulcorner G \urcorner \in B_w$. Note that for any HA formulas G, the truth values of G in $\mathcal{M}$ and $\mathcal{M}'$ coincide.

Lemma 6.7.2 *In $\mathcal{M}'$, $w \Vdash X \leftrightarrow X^R$ for each* DHA *formula X (where X is not necessarily closed).*

We prove this by induction on complexity of X.

If X is an HA-formula, then the Rosser translation of X just is X.

For the induction step corresponding to $\Box(x)$, let X be a DHA-atom then $X = \Box[Y]$ for some Y. We pick an interpretation of all free variables in Y. With this interpretation, we can regard Y as a closed formula. We defined that for all w and all Y, $w \Vdash \Box[Y] \Leftrightarrow w \Vdash R[Y^R]$. So therefore, this holds at our specific w with our Y and at every world accessible from w; hence, $w \Vdash X \leftrightarrow X^R$.

The cases corresponding to HA-connectives and quantifiers are straightforward.

Lemma 6.7.3 *$\mathcal{M}'$ is a* DHA*-model.*

This is immediate from Lemma 6.7.2. We note just that the induction axiom holds for DHA-formula X since it is equivalent to X^R, which is an HA-formula for which induction already holds in $\mathcal{M}$.

Since for some world w in $\mathcal{M}$, $w \nVdash F$ and F is an HA-formula, $w \nvdash F$ in $\mathcal{M}'$ as well. Hence we have shown that $\mathsf{HA} \nVdash F \Rightarrow \mathsf{DHA} \nVdash F$, for any HA formula F.

DHA embodies the aforementioned thesis that HA does not fall short as an answer to Mannoury's challenge in that it *goes wrong*. Indeed, Theorem 6.7.1 tells us that DHA does not ascribe any properties to constructed objects that HA does not already. Despite this, DHA does allow us to disrupt the traditional connection between intuitionistic arithmetic—when it is identified with HA—and classical arithmetic. Of course, every theorem of intuitionistic arithmetic as HA is also a theorem of classical arithmetic. On the other hand, when we understand intuitionistic arithmetic as DHA, a chasm between the axiomatization of the intuitionistic theory of mental construction and the classical theory of platonic objects forms:

$$\textit{Intuitionistic Arithmetic proves } F \not\Rightarrow \textit{Classical Arithmetic proves } F$$

We do not get the above because we have ascribed some new property to constructed objects, but instead, the above suggests that DHA accounts for the essentially epistemic character of Brouwerian arithmetic. This epistemic character—manifested through our doxastic predicate $\Box(x)$—is not represented in HA (or PA, for that matter). And after all, how strange it would be if formalized Brouwerian mental arithmetic could simply be extended so that its subject matter ended up matching with that of classical arithmetic, which is in many ways the antithesis of an epistemic ideology!

A classicist might object that Theorem 6.7.1 shows that we have just said with more theory what we could have said with less. To this, the intuitionist might respond first that HA oversimplifies and therefore obfuscates arithmetical construction by omitting the subject. They might continue, however, that to include the subject only insofar as it is expressed as an arithmetical predicate in HA is be to commit to a philosophical error; it is to grant priority to the constructed objects over the subject and in effect pull the cart out of the horse.

We have argued that an axiomatic characterization of intuitionistic arithmetic ought thus to disrupt the connection to classical arithmetic implied by Heyting's response to Mannoury's Challenge. We have posed one such system that we argue characterizes Brouwerian arithmetic better than HA is able to. We, of course, leave open the possibility that some future system characterize Brouwerian better than DHA; perhaps even such a possibility is to be expected given Brouwer's views on language. If nothing else, we still will have clarified the desiderata implicit in Mannoury's Challenge and contributed to the study of predicate-style modalities. While the latter is self-explanatory, the former contribution can be summed up as the claim that while epistemic classical arithmetic presents an interesting and fruitful extension of classical arithmetic, *epistemic intuitionistic arithmetic* is redundant in the sense that intuitionistic arithmetic already is and has been deeply epistemic.

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