

DUMMETT AND WITTGENSTEIN'S PHILOSOPHY OF MATHEMATICS

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(original draft of the paper: although a bit naïf, it may still have some suggestions CP 2015)

“Explanations come to an end somewhere” (Dummett LBM, p. 311)

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INTRODUCTION

Summary: in §§1, 2, 3 I will give a reconstruction of Dummett's analysis of Wittgenstein's philosophy of mathematics. Then, in §§4 and 5 I will analyse the development of Wittgenstein's ideas to sustain two theses: (a) some aspects of Dummett's interpretation (especially the idea of decision as fundamental in philosophy of mathematics) are a reconstruction of ideas held by Wittgenstein in his early period, and later discarded. (b) Wittgenstein somehow entertained a position similar to Dummett's pattern-recognition strategy and abandoned it, or at least gave to it a different focus (somehow linked it with the distinction between the context of discovery and the context of justification). Relying then mostly on Wittgenstein's latest remarks, in §6 I will give a slightly different assessment of rule following consideration. In §§7 and 8 I will try some criticism of Dummett's interpretation and of his general views on philosophy of mathematics.

1. FULL BLOWN CONVENTIONALISM

In his review of Wittgenstein's *Remarks on the Foundations of Mathematics* M. Dummett states a claim, which has been generally accepted later by other philosophers, against standard conventionalism (all necessary truths derive directly or indirectly from linguistic conventions). This claim, –whose argument reminds us of Carroll's in “What the tortoise said to Achilles” – says that standard conventionalism, held by logical positivists, is unable to explain logical necessity: it leaves unexplained why certain conventions (expressed in axioms and rules of inference) necessarily have certain consequences. If you want to recognise the necessity of the consequences through another conventional rule, you are bound to fall into an infinite regress¹.

¹ This is one of the most accepted results of Dummett's analysis: to extract from Wittgenstein's remarks the confutation of standard

Apparently the only way to retain conventionalism, and at the same time to escape from this undesired situation, is to accept a radical form of conventionalism, a full blown conventionalism: all necessary truths (axioms, rules and theorems) are a direct expression of a linguistic convention. This is supposed to be the position held by Wittgenstein as a means to solve the debate between Platonism and Constructivism in mathematics. All of Dummett's papers on Wittgenstein's philosophy of mathematics² seem to be devoted to displaying his natural reaction of resistance against such a "radically faulty vision", a conception "extremely hard to swallow, even though it is not clear what one wishes to oppose to it" .

Dummett gives his first reconstruction of Wittgenstein's philosophy of mathematics through some fundamental topics which will also be basic for his later discussions:

1. *no fact (of the matter or of the mind) compels us* - Once given the axioms, in order to follow a proof, we have to recognise each step as an application of the rules of inference. Dummett places much emphasis on the possibility envisaged by Wittgenstein: somebody may apparently accept the formulation of the axioms and rules while refusing to accept the proof. In fact "there is nothing in our formulation of the axioms and of the rules of inference, and nothing in our minds when we have accepted them before the proof was given, which of itself shows whether we shall accept the proof or not; and hence there is nothing which forces us to accept the proof."³

2. *Necessity is given by free decision* - Then, this would be Wittgenstein's conclusion, we are free at each step to choose whether to accept or reject the proof: if we accept it we confer necessity on the theorem and from here onwards we will consider it unassailable, we will put it in the archives. We are not making explicit a previously implicit decision given through the axioms and the rules of inference: we are making a new decision.

3. *Proofs will yield new concepts* - How then to explain a training in which an explicit formulation of the rules is given, such that a machine could follow it? "Where does a human being gain a freedom of choice in this matter which a machine does not possess?"⁴ Wittgenstein could answer that a formal system must somehow correspond with the intuitive conception (a point made later also by Kreisel); but the most

conventionalism. The argument from infinite regress is explicitly stated, in a slightly different manner, in Dummett 1978. Here Dummett says that we cannot explain or give foundation to logical necessity calling in question a general formulation of a rule because every formulation could still be misapplied, and we cannot, without circularity or infinite regress, suppose that someone's knowledge of how to apply a formula consists in his knowing another formula. An explicit connection between Wittgenstein's arguments and Carroll's paper has been given by Robinson 1965. The strategy given by Geach 1972 in his paper on assertion in order to overcome the difficulty of Carroll's argument (imposing a distinction between semantic and pragmatic justification of deduction) was unsuitable for Wittgenstein given his rejection of the assertion sign, and we will not pursue the argument here.

² Besides Dummett 1965, 1978 and 1994, it would be useful to keep in the background the papers "Truth" and "The Philosophical basis of Intuitionistic Logic" (in Dummett 1980). The quotations which follow are from Dummett 1978: 68, and 1959: 173.

³ See Dummett 1959: 171.

⁴ Ibid., p. 172

convincing answer is that we cannot previously circumscribe all possible forms of proof, and the proofs we have now could arrive at some limitations, beyond which it would be necessary to devise new methods.⁵

4. *Foundation is agreement in practice* - The most general conclusion of Wittgenstein's argument may be expressed by saying that "the existence of a rule ... rests ultimately upon the fact of agreement in practice, amongst human beings who have been taught the rule, over its application, a fact not susceptible of further explanation."⁶

2. INTERNALISM AND THE MISTAKE OF RULE FOLLOWING

In reaction to one of his main critics, Barry Stroud,⁷ Dummett restates his analysis of Wittgenstein. Some aspects of his previous interpretations seem to be partially superseded by his new position,⁸ and in what

⁵ See Dummett 1978 wherefrom I quote, gives the example of the method for finding prime numbers. On the same example see Wright 1980 (I, vi, 4). Perhaps Wittgenstein's hints on new concepts given by new proofs are more perspicuous if considered under the topic of changes of numerical systems: in these cases you have genuine change of concepts (the concept of natural number differs from the concept of real number, as the corresponding proofs on properties of numbers in the two different systems). The interpretation Dummett gives (also elsewhere; e.g. Dummett 1973a: pp. 300-301) appears to be too strong; and Wittgenstein seems to be better represented by the alternative given at pages 302-303). But see below, note 8 (3).

⁶ See Dummett 1978: 65.

⁷ See Stroud 1965

⁸ Among some of the changes in Dummett's attitude we find the following points:

1. the role of (counter) examples: after having criticised Wittgenstein for having given only "thin and unconvincing examples" Dummett (1978: 67-68) contrasts Wittgenstein's examples with the detailed analysis of mathematical proofs given by Lakatos. The claim that necessity is reduced to what is taken as necessary contrasts with the historical development of mathematics: we have many examples of proofs, once accepted as necessary by the scientific community later discarded for the discover of an error. Disregarding the role counterexamples had in history of mathematics in changing the status of certainty held by some proofs, Wittgenstein runs the risk of conflating necessity with certainty (necessary is what is held for certain). Dummett 1994, however sustains that Wittgenstein's thesis is not bound to this conclusion and can be formulated so as to allow revision in what we take as having been proved: we may admit proofs "to be only provisionally compelling (...) What is necessary is what we treat as such and will continue to do so".

2. The danger of a break down in communication: the idea given in Dummett 1959 (p.195), that - if a single mathematical statement is rendered necessary by a particular decision to treat it in a certain way - "communication will be in constant danger". To this it has been replied that communication *is* in constant danger (Stroud 1965). Dummett (1991: 311) however identifies Wittgenstein's viewpoint as the idea that communication is in no danger at all; the only problem is that our certainty of that does not rest on anything.

3. The role of mistakes: Dummett 1959 has devoted a lot of discussion to the problem: does the acceptance of a mathematical statement as a rule gives a radically new criterion? If a criterion given by a new proof is radically new, how is it related to previous criteria? How can a mistake be detected in this case? Is it possible to speak of a mistake we are not able to detect with previous criteria? Or is it possible to speak of a mistake we are not at all in principle able to discover? The discussion aims at building up an argument against the plausibility of Wittgenstein's position. The detailed discussion is more or less restated in *Wittgenstein on necessity*: some reflections show that all the discussion is based on a misunderstanding of Wittgenstein's internalist stance. The

follows we rely on the solid core which remains unchanged in Dummett's interpretation.

Stroud stresses the fact that, for Wittgenstein, mathematical necessity is contingent: it depends on the peculiar form of life in which mathematics has grown;⁹ but if "human nature, or our *form of life*, determines the consequences of the basic necessary truths, or of the conventions that directly confer necessity upon them"¹⁰ Stroud's position seems to be only a refined version of moderate conventionalism. Stroud's interpretation misses Wittgenstein's primary contention: obviously it is a fact of our existence that we agree on the consequences of our basic conventions. "But, on Wittgenstein's view it is a *brute* fact: nothing explains it." If our nature determined what would constitute a consequence of our conventions, there would be necessary truths (consequences of our necessary conventions) not yet recognised by us, and may be such that we will never be able to recognise them. On the contrary, Wittgenstein maintains that nothing determines a necessary truth in advance: "only when we have accepted the proof and put the theorems in the archives does it *become* consequence of the initial conventions."¹¹

A good label for this extreme position is the recently coined *internalism*: Wittgenstein wants us to do without the idea that we can imagine the circumstances from an external, superhuman point of view; he refuses to talk "about what would have been true even though we had not recognised it as true". Due probably to the fact that Wittgenstein refuses *any* reference to an external point of view, his position is labelled 'full blown internalism' (hereafter FBI)¹², of which Dummett says:

1. FBI is *unsayable* - Following *Tractatus*' terminology, the FBI attitude cannot be said, but only shown; in fact any attempt to state the general thesis will contrast with the actual practice which it claims to be the source of necessity and truth and will be self-refuting: it would be an attempt to view our language from the external vantage point which it declared to be inaccessible.

2. FBI and extreme externalism coincide - On the other hand, also the externalist, who wants to say how things are in themselves, attempts to say the unsayable. In a very Wittgensteinian move Dummett says that "it is at this point that Realism and Idealism seem to coincide" : they cannot be established because that would require us to say the unsayable. They are both said to be incoherent: extreme externalism through an argument given by Putnam on quantum mechanics and paradoxes in logic¹³ the FBI for discrediting the

recognition of the internalist point of view seems to deprive the original argument of all its strength, and Dummett seems to abandon it.

Even if very interesting in themselves, I will not discuss these topics here, for the problem would then become a more general one: is it possible to accept a radical internalist point of view which holds that 'truth can attach to a statement only in virtue of its being treated as true'? But I still doubt that this conclusion is very much dependent on the argument 3 which has been much criticized as an interpretation of Wittgenstein (see: [174] I,v.pp.6-8; [178] in [155], pages 22-27;[144] pp. 294-7).

⁹ Putnam on Stroud quoted in [56].

¹⁰ See [56, pp. 6-7]

¹¹ Ibid., p. 8.

¹² This name reminds one of previous 'full blown conventionalism'. The change of label doesn't mean the change of the frame of discussion; as in his previous papers, Dummett finds a way to restate the old debate between Platonism and Constructivism.

¹³ Putnam quoted in M. Dummett, Ibid. It seems that the two philosophers are going in a strange symmetric movement: Putnam

practices considered the sources of necessity (see 4).

3. *The apparent advantages of externalism* - Frege and Wittgenstein also believed that Realism and Idealism coincided; but, while Frege and Wittgenstein wanted to find a way out of the dichotomy, Dummett searches for a compromise: he says that “a reasonable position must lie between the two extremes”.¹⁴ While Putnam accepted a moderate internalism (or an internal realism) Dummett reminds us that externalism is apparently “more congenial to common sense” because it uses the distinction between how things are in themselves and how they appear to us; and from where. do we get such a distinction? The answer runs: from our own linguistic practices, especially from the practice of deductive inference, where we are forced “to form a conception of how things are independently of any observation we make” .

4. *FBI does not explain actual practice* - FBI forgets that we maintain our practice of rule following not only because “it is what society expects us to do” ; we maintain it because it appears to have a motivation, a “rationale” , “a method of ascertaining how things in fact are” . If you do not take this fact into account, you are bound to think that “only inertia makes us go on reasoning as before”.¹⁵

5. *FBI is question begging* - The actual argument of FBI derives, from the right premiss that “there is nothing *by which* we judge something to be a correct application of the rules” to the conclusion that “there is no specific thing that would have been a correct application”. But in order to reach such a conclusion you need a hidden premiss: that “there is nothing to truth beyond our acknowledgement of truth”.¹⁶ Then the “huge mistake” embodied in the rule following considerations seems to be that the argument of FBI begs the question, because FBI is required as a premiss “to take us from the epistemology of rules to their metaphysical standing”. This hidden premiss is “totally implausible” and it (or its conclusion) induces “a scepticism so profound that few can swallow it”.¹⁷

6. *A practical (weak) incoherence* - Actually the last conclusion seems to be more a matter of taste than an argument; indeed, after having said that FBI is incoherent because it “makes our linguistic practices the whole source of necessity and of truth only by discrediting those practices” , Dummett recognises that may be FBI is not actually incoherent, but it “would render uninteresting and pointless a great deal that we regard as interesting and full of point”: how? By dismantling the distinctions that the externalist underscores, such as the distinction between how things are in themselves and how they appear to us; but a

from realism towards anti-realism; Dummett from anti-realism towards something more and more similar to realism, as if all his efforts to defeat realism were efforts against something deeply embedded in himself - as in the most part of us.

¹⁴ M. Dummett, *Ibid.*, pages 25 and 31.

¹⁵ *Ibid.*

¹⁶ *Ibid.*, p. 31 - The major claim made by Dummett is that the internal thesis about necessity must imply the internalist thesis about truth. The assumption that “necessity attaches to a statement only in virtue of its being treated as necessary” brings - Dummett argues - to the recognition that “truth is to be equated with being recognised, or, better, with being treated, as true”. But exactly this claim is defined at the beginning of Dummett's paper, as such that nobody would accept it. The paper's problem is then the validity of this implication from 'internalist' necessity to 'internalist' truth.

¹⁷ *Ibid.*, p. 32.

lot of scientific research (e.g. the description of the physical world with the idea to attain the ‘completely objective’) are based on these distinctions, and there is much in our form of life that supports it and it would be ‘nihilist’ to treat it with scorn.¹⁸

3. KOENISBERG’S BRIDGES AND PATTERNS

FBI is criticised for not taking into account the idea of a ‘rationale’ behind our rule following; Dummett has to explain then what he means by this idea. In order to explain his point Dummett gives an example: what does Euler’s proof concerning the bridges of Königsberg show? Euler’s proof is a very good example of a deductive practice which induces us to form a conception of how things are independently of the observations we are able to carry out. The proof shows that, even if we could not have directly verified that a person passed a bridge twice, the simple verification that he crossed every bridge entitles us to assert that he must have crossed some bridge twice. The proof gives us a new concept, a new way of looking at facts; moreover the proof is considered as a justification for the assertion given, even in the absence of direct observation. This proof is a very good example of ‘indirect verification’: we did not observe the person crossing the bridge twice; but if we had observed it, we should have seen that he did exactly that: we come to have a “method of ascertaining how things in fact are” independently of our observation.¹⁹

While Dummett had once spoken of “the same underlying principles underlying diff methods”, now, using the example of Königsberg bridges, he takes up the metaphor of patterns underlying a deductive consequence: “even the simplest judgment imposes a pattern upon reality, a pattern in common between the variegated circumstances which would verify it”.²⁰ On one hand a pattern seems to be something like the *Tractatus’* logical form, which is in common between reality and description; on the other hand a pattern seems to be some kind of similarity perceived through different thoughts or sentences. In *The logical basis of metaphysics* Dummett holds that “the fundamental idea is that inference involves the discernment of pattern. The pattern is not, in general, imposed: it is there to be discerned.”²¹

By a detailed description of the phenomenology of pattern recognition in inventing or recognising a structure of the proof, Dummett concludes his paper on Wittgenstein’s necessity in a way very similar to his 1959 paper: “The ability to discern patterns depends upon the stock of concepts available to us. But we do not impose patterns, but discern them: the capacity of a pattern to be transformed into another is intrinsic, is not created by our ability to perform the transformation”. As already attempted in previous papers, here again Dummett seeks a compromise between two extreme positions: FBI and extreme external realism

¹⁸ Ibid

¹⁹ Ibid. pp. 24-25. On Euler’s proof see also [42] in [50], pp. 307-308.

²⁰ See Dummett 1978 and 1994.

²¹ See Dummett 1991: 198.

(both of which he contributed to create). Is there a moderate externalism? And is there a moderate internalism? And do they coincide? Frege had both components which can be reconciled with the Tractarian metaphor of a grid: “we need a grid, and might use one or another; but, given the grid, what constitutes a correct description is wholly independent of us”.²²

4. WITTGENSTEIN’S DECISION

First of all I want to discuss a central aspect of Dummett’s interpretation: the idea that we are free at each step to decide whether to accept or reject a proof and the idea that any new theorem is given the character of necessity by an act of decision. This picture of Wittgenstein’s philosophy of mathematics is widespread and shared by many authors, among them Putnam.

The argument appears to be like this: *if there is nothing which compels us to accept a proof, then any acceptance of a proof is grounded in an act of decision*. We will see that this idea has been entertained by Wittgenstein for some time, but, later, Wittgenstein’s argument was eventually developed the other way round: *if accepting a proof is not grounded in any act of intuition or decision, then there is nothing which compels us to accept it*. How was this change possible? It was possible because of the ambiguous relation Wittgenstein held with intuitionism; following the development of Wittgenstein’s attitude towards Brouwer will help to obtain a certain amount of clarification on the matter.

Certainly Wittgenstein was fascinated by Brouwer’s lectures and writings, and in his remarks we find many references to intuitionistic ideas. Sometimes these ideas are accepted and developed,²³ while at other times they are entertained with doubt and eventually sharply rejected. Among one of the latter cases we find Brouwer’s idea of ‘basic intuition’.

This may have been a stimulus and an inspiration for Wittgenstein, while he was trying to reject logicism as held by Russell. We cannot forget that in Wittgenstein’s time the discussion of intuition and the synthetic a priori was one of the most widespread, not only in the writings of Brouwer and Hilbert, but also in the debate between Couturat and Poincaré, a debate to which Russell dedicated some of his attention. Since his return to philosophy after the *Tractatus* Wittgenstein insisted in recognising mathematics as based on “the intuition of symbols” (WWK, §219), or on the “insight into the number calculus” (e.g. “no investigation of

²² Ibid., pp. 26-27.

²³ In the first case we find the idea of a real number as a law; this idea is almost certainly taken from Brouwer and became an example of the ‘intensional point of view’ in mathematics against the extensional point of view. Here Wittgenstein found a good ally for one of the fundamental ideas on mathematics developed in the *Tractatus*: in mathematics process and result are the same (Tractatus, §6.1261); after meeting with Brouwer’s ideas Wittgenstein writes that in mathematics there is no dualism between description and reality, between description and object (*Philosophical Remarks*, §180; *Philosophical Grammar*, §457; *Remarks on the Foundations of Mathematics*, I, §81 ff V, §19). God does not know the infinite expansion of a real number because there is no infinite expansion to be discovered: real numbers are techniques, laws of construction (see *Philosophical Remarks* §128; *Philosophical Grammar*, §481; *Remarks on the Foundations of Mathematics*, VII, §41).

concepts, only insight into the number-calculus can tell us that $3 + 2 = 5$ ", *Philosophical Grammar*, §347). These passages are directed against Russell's idea of the complete replacement of arithmetical equations by tautologies; in contrast with the idea that an arithmetical computation could be reduced to an analytic statement, Wittgenstein refers to Kant's synthetic a priori (PR, §108) possibly aiming to contrast Russell with positions similar to the ones held by Brouwer.²⁴ It is easy to think that the complex of discussions about the role of intuition in mathematics may have affected Wittgenstein very much in his fight against both Logicism and Platonism.

In fact, born in antilogistic context, the stress that was placed on intuition in mathematics was later found to be useful in the critique against Frege's Platonism. Wittgenstein reacts mostly against the platonistic idea that every result of a calculation is already there, and we could find it, as if an act of foresight could permit us to see the development of a proof or of a calculation in advance.

At the end of the twenties, still strongly influenced by Brouwer, Wittgenstein expressed this reaction very clearly:

Is it like this: I need a new insight at each step in the proof? This is connected with the individuality of each number. Something of the following sort: Supposing there to be a certain general rule (therefore one containing a variable), I must recognise each time afresh that this rule may be applied here. No act of foresight can absolve me from this act of insight. Since the form to which the rule is applied is in fact different at every step. (PR, §149). [A note written by Wittgenstein beside this passage says: "Act of *decision*, not *insight*", and this change is considered again in PG, §301].

Between the twenties and the thirties Wittgenstein seems satisfied with this idea, but soon afterwards he begins to reject it sharply. From that time on, the idea of an act of decision in rule following is presented as belonging to an argument aiming at destroying this very idea. In *The Brown Book* the idea is already clearly stated:

²⁴ Perhaps Kant's influence on the later Wittgenstein is deeper than normally thought (see Penco 1979). Some ideas of Wittgenstein's can be taken as a part of the process of reconstruction of the Kantian a priori which engaged many philosophers from the twenties onwards, starting from the neokantian Cassirer and the neopositivist Reichenbach. Reacting both to Platonism and Empiricism, Wittgenstein seems to reach some kind of 'liberalised' transcendentalism, where a priori becomes a feature of our constructive practice given in mathematics: necessity a priori is not attached to some kind of sentences, but to the use of them: an empirical proposition can be used as a rule if made stable and independent of empirical facts. Using Crispin Wright's quasi transcendental argument, we may assert that for Wittgenstein the practice of mathematical reasoning "requires that there is such a thing as necessity" (See Wright 1980: 415). It would be interesting to study how much Wittgenstein has been considered 'conventionalist' in the literature, starting at least from Waismann 1966, who seems to assimilate his positions to the conventionalist ideas held by Russell (against this interpretation Penco 1991). Other philosophers, such as e.g. Specht or Bensch 1973 have worked on Wittgenstein's discussion on 'synthetic a priori' in mathematics, which perhaps deserves more attention than normally given.

It is no act of insight, intuition, which makes us use the rule as we do at the particular point of the series. It would be less confusing to call it an act of decision, though this too is misleading, for nothing like an act of decision must take place, but possibly just an act of writing or speaking. (BB, §143; see e.g. BB, §141-143, RFM, 1, 3; LFM, 28-30, 237; PI, 214).

Here Wittgenstein fights against two ideas: (1) the platonistic Fregean idea that the expression of a rule ‘foreshadows’ all the transitions which are to be made according to it; (2) the idea that there must be ‘some-thing’ which makes us do what we do. “We need have no reason to follow the rule as we do. The chain of reasons has an end” (Ibid.).²⁵

At the end of the thirties, in *LFM*, the process of revising the idea of decision is completed: Wittgenstein explicitly rejects the idea that we “have an intuition . . . at each step of a rule” (the idea is labeled ‘intuitionistic’ because of its origin in Brouwer’s writings):

We might as well say that we need, not an intuition at each step, but a decision. Actually there is neither. You don’t make a decision: you simply do a certain thing. It is a question of a certain practice. Intuitionism is all bosh - entirely. Unless it means an inspiration. (LFM 237)

What was wrong with Brouwer’s ideas? Brouwer wanted to provide a basic reconstruction of mathematics passing from proofs which simply convinced us (as e.g. classical proofs) to proofs ‘with indubitable steps’. For these steps you need a special intuition or decision. But there are no indubitable steps; there are only steps which persuade us and are part of a technique; the correct application of a rule finds its justification within the practice of the technique, and has no justification outside it, in any special act of decision. The extremely rude way in which Wittgenstein speaks of ‘intuitionism’ is probably due to the delusion of the hopes which certainly the early encounter with Brouwer had provoked in the young

²⁵ Compare this passage with Lectures given in ‘34-’35: here Wittgenstein, discussing intuitionistic ideas, was still saying that in following a rule “if any mental process is involved, it is one of decision, not of intuition” (p. 134). Wittgenstein then discusses the idea of an infinite regress in justification: somehow you can say that the chain of reasons has no end: you *may* always give a new answer to any question of justification of a rule; but you *need* not: you may stop asking. At this point you can make a decision (see [170], lect. VI, pp. 132-133). In *The Blue Book* we find a more perspicuous criticism of the idea of an infinite regress, and the acceptance of the idea that you may have no reason at all for accepting a rule. In other remarks of the period Wittgenstein eliminates the idea of decision: you do not take any decision; the chain of reasons has an end. Looking for a solid ground you cannot pay attention to psychological processes, but to the practice of rule following.

Wittgenstein: he had really found a good source of inspiration. Dummett's interpretation has the merit of picking up one stage in Wittgenstein's development and making a coherent "Wittgensteinian" theory from it;²⁶ but the internal development of Wittgenstein's remarks compels us to look more carefully at this question of interpretation. I have suggested that Wittgenstein rejected the idea of 'decision at each step'. I would like to suggest now that some aspects of Dummett's attempt to overcome the difficulties of 'decision at each step' have been somehow already considered and later discarded by Wittgenstein. What happened then after the rejection of the idea of 'decision at each step' in Wittgenstein's thought?

Actually the idea of an intuition or a decision at each step of the proof was early assumed by Wittgenstein with a definite purpose: to give an alternative justification which could take the place of the platonistic justification given by Frege, that is the idea that the result of a calculation - or a step in a proof - is not determined in advance. This strategy did not work and Wittgenstein abandoned the idea of a 'decision at each step' once expressly held in *Philosophical Remarks*. But he never abandoned the idea that the result in a calculation is not determined in advance. Given the total failure to justify or give foundation to mathematics through intuition or decision Wittgenstein finds himself left only with that: as Dummett puts it, the acceptance of a proof, or the agreement in calculation, is a brute fact: nothing explains it.

But Wittgenstein was partly dissatisfied with this immediate outcome of the failure of the 'decisionistic' or 'intuitionistic' strategy for the foundations of mathematics, and the remarks on mathematics that began after the end of the thirties can be considered an attempt to face this new theoretical situation, an attempt to express this dissatisfaction.

5. PATTERN RECOGNITION IN WITTGENSTEIN AND DUMMETT

The point can be summarised as follows: when faced with the contrast between decision and recognition we seem to be left without any way out. In Wittgenstein's view mathematical necessity is not based on recognition of some kind of mathematical reality, nor on some kind of decision. We are then left with the brute fact of our agreement on rule following.

How can we explain or clarify this fact? And which kind of clarification can be given? It seems that Wittgenstein mainly tried two kinds of answers: (i) one dealing with the theme of perspicuity of the proof, (ii) the other dealing with rule following considerations. We will discuss here the first attempt, which sounds somewhat similar to Dummett's strategy of pattern recognition, to see the different conclusions Wittgenstein drew from it. In the next section we will give some hints as to the answer which followed the failure of the first attempt.

²⁶ Possibly Dummett's interpretation has been influenced by his first source: Bosanquet's notes of the end of thirties (see Dummett 1978). As we have seen, in the thirties Wittgenstein is still fighting an internal battle on the idea of decision at each step. The main changes of the theme are subsequent to this period.

It is worth noting that after the '39 lectures the main preoccupation in Wittgenstein's remarks on mathematics (e.g. part III of RFM was on perspicuity of proof. It is as if, having reached such a hard conclusion (inference has no justification) he tried to find some solid ground for mathematical necessity in the geometry of proof. The writings at the end of the thirties and the beginning of the forties when they are considered as an attempt to face the problems created with the breakdown of the transition from the stress on 'intuition' to the stress on the 'perspicuity' of a proof, on the "new way of looking at the old system of signs".²⁷ The phenomenology of proof formation is in many respects similar to the one given recently by Dummett. Wittgenstein says that "the proof makes one structure generate another. It exhibits the generation of one from the other" (RFM, III, 29). Many psychological and physical facts are considered to be part of the feeling of constriction that the proof gives us. The properties of shapes (e.g. of numbers) show to be "physical and psychic possibilities of decomposition, of arrangements, etc." (IV, 11). But Wittgenstein distinguishes between the physical and psychological facts which convince or persuade us of the correctness of a proof, and the anthropological fact of the way in which we give a certain role to mathematical propositions.

In *The Logical Basis of Metaphysics* Dummett too speaks of physical and mental facts connected with the observations which establish a statement as true; and he suggests that beyond this fact we have also a discernment of patterns. It looks as if we are faced with another level, different from the physical and the psychological, I suppose a 'conceptual' level (which reminds one of Frege's three realms).²⁸

But what is doubtful is exactly this idea: does pattern-recognition belong to a conceptual level distinct from the physical and psychological? Certainly, to use a traditional wittgensteinian jargon, we may not only see something, but also see something as a certain pattern, a pattern previously given, there to be perceived, not invented by us, but lying there, where perceptual experience gives it to us. But recognising such a pattern seems to be only a moment of discovery, a psychical suggestion which can push us to accept it as pattern for a proof. But only when we take them as fundamental patterns of proof do we give them a determinate role: we give these patterns the status of basic features in our proofs. We accept them as a representation of the way in which we use a succession of inference steps as a representation of our 'logical must', that is, as necessary.

²⁷ I tried to give some textual evidence in Penco 1981.

²⁸ Actually Dummett (1991: 195-199), relies heavily on §9 of Frege's *Begriffsschrift* speaking of perception of a pattern as a basis of concept-formation. But, to state my point in this context, I would stress the distinction between (i) the perception of a pattern common to different thoughts as a process of concept formation; (ii) the use of the concept, formed in such a way (i.e. via extraction of predicates) in the context of a logical (linguistic) system. We have to distinguish the problem of concept formation from the problem of what a concept is and how it is used (how it works in a system). The excessive attention paid to problems of the source or origin of concepts threatens to bring us back to that centrality of epistemology in philosophy which Dummett criticised in his final chapter of *Frege: Philosophy of Language*.

If we call the recognition of these patterns a ‘rationale’, we have more than a ‘rationale’ to explain the discovery or invention of proofs; but the role attributed to a proof, the role of criterion, even if it depends on this “rationale”, does not derive from it. Even if we could discern patterns in the brain which strictly correspond to steps in our proofs, this would not justify the role we give to these patterns. The pattern is not the proof (RFM, III, 11) and the attribution of necessity to the use of a certain pattern of signs is given a priori. Here we must rely on the peculiar way in which Wittgenstein uses the term ‘a priori’, not as a mark of certain kinds of sentences, but as a certain use of sentences. Certainly you need a perceptual clarity in recognising the structure of a proof (the proof must be surveyable); but the proof is not given through perception; the ability to recognise some perceptual patterns or regularity is a physiological and psychological ability of human beings, and may be considered among the causes of concept formation. But our interest in philosophy is not in the possible causes of concept formation (*Philosophical Investigations* II, XII).

Wittgenstein seems to follow strictly Frege’s advice to distinguish carefully between *justification* and *explanations*, between reasons which justify our judgments and causes which give origin to our judgments. He sustains that the *motivations* for accepting the proof are not to be considered as the source of necessity; may be they can be considered ‘causes’ for our accepting the proof (RFM, VI, 47). They are grounds for inventing and accepting proofs, grounds which we could study under the label ‘context of discovery’; but they cannot be considered under the label ‘context of justification’.

The problem posed by Wittgenstein is the following: in which context must we consider the recognition of patterns in our practice of deductive inference? The answer is that our pattern-recognition may perform the role of explanation of our rule following, but cannot take the role of justification which was erroneously supposed to be performed by intuition or decision. Certainly we recognise certain patterns; but we could recognise other patterns, or other kinds of patterns. And we are still left with a question about the ground on which we give such a strong status of necessity just to the patterns we recognise.

6. THE FOUNDATION OF OUR LANGUAGE GAME

The phenomenology of proof has given us a certain amount of ideas on the natural, psychological and physical grounds on which we form our concept; we have a clear explanation of the discovery of our concepts; certainly there is something ‘there to be discovered’; but once we have extracted patterns from our perceptual experience, there is nothing which compels us to use them as necessary; actually we use some of them as tools to form necessary constraints on the description of our future experience. But nothing is said of this step, from the extraction of such patterns to the status of necessity we give to them: it really seems that necessity is a brute fact which seems to be a matter of arbitrary decision. But if it is not a matter of arbitrary decision, then it appears to be an arbitrary chance which befalls us human beings, like a mere habit to which we conform by inertia, following our natural inclination to perceive certain kinds of patterns.

On the contrary Wittgenstein's attitude seems to point to a different direction: in remarks from 1939 and 1940 he begins to speak of agreement in calculation as essential to the technique of calculation: the technique of calculation, or more generally the technique of rule following, is based first of all in training and shared reaction. The gap between pattern recognition and necessity seems to be placed on training and shared reaction. But it took some time to arrive at this point; I will try to sketch the stages which brought Wittgenstein to this conclusion.

In the thirties Wittgenstein was very deeply involved in the idea of a calculus as an autonomous system, a "free floating calculus" (PG, 314), where the meanings of the constituent symbols are given by the rules of the game. This idea, developed at length in the *Philosophical Grammar*, appears to be linked to an holistic view of language: you can't grasp the 'meaning' of a move of chess unless you are aware of the entire set of rules. In this context Wittgenstein develops the idea that the meaning of an assertion can be somehow identified with its justification (PG, 81). The place of justification is placed in the context of a discussion on which kinds of proofs, or inferential steps, are used to give reasons for an assertion.

It is certainly true, as Wittgenstein stressed many times, that the meaning of a mathematical proposition receives life from the context of the calculus in which it is embedded, as the meaning of a move of chess receives light from the rules of the game; but, in a sort of reaction to this view, Wittgenstein later placed a greater emphasis on the connections with nonlinguistic attitudes: the calculus (or the game) is not separated from, but strictly embedded in human practice: it receives light from the actions connected with it: the training, the teaching of it, and of its role in language. Starting from these ideas Wittgenstein begins to describe language games, on one hand with a strong holistic attitude (each game represents an entire language), but on the other hand with some kind of reaction to the holistic position implicit in the *Philosophical Grammar*: different language games are presented in a succession such that a language game can be learned only on the ground of a preceding one; the most exemplary evidence for that is given in *The Brown Book*, but also the *Investigations* give many hints in this direction which is made more radical in part VI of *Remarks on the Foundations of Mathematics*.

Here Wittgenstein tells us a story. Let us imagine different language games, more and more complex: a first one is a situation in which people play regular rhythm or look at simple ornaments, just for the pleasure of simple regularity. A second is a language game where somebody is taught to obey an order like 'beat regularly'. He is exposed to a certain rudimentary technique and he is taught to react in certain ways to what is called 'mistake'. In the teaching of the technique there is an element of deixis, like "do *that way*". We may then imagine an other language game in which the person becomes conscious of applying a technique, of following a rule: he expresses himself saying "If I obey, I do that". He can also foresee what he will do later, starting with simple statements as "I conceive the rule (or the command) *so*, then I have to go on *so*". But, in order to do anything like that, he needs to have a distinction between *taking it so* and *continuing so* (VI, 29). He has to distinguish between his *interpretation* or *expression* of the rule and his *following* the rule.

On this ground we are driven to accept the idea that the expression of the rule convinces us, but only given the fact that we have been trained to react in certain ways: I cannot firmly possess the rule without an expression. But, on the other hand, I possess the rule if I react as I have been taught; the reaction is “our guarantee of understanding” (VII, 47). In this way we may witness that training another person to follow a rule is integral to rule following. We need to start with orders and examples. “As we employ the word ‘order’ and ‘obey’, gestures no less than words are interwoven in a net of multifarious relationships” (VI, 48). In the case of training, the criterion of understanding is the agreement of individual actions.²⁹

How is it possible to follow a rule?³⁰ It seems that, in answering this question we have reached something like a robust ground: we don’t follow rules in certain ways because society wants us to do so; on the contrary it is essential to a construction of society, or human beings entertaining language games, to reach a level in which people don’t dispute about the use of some basic assertions. In order to have human society it is necessary to have necessary statements. These statements are somehow physically and physiologically based on sense perception (pattern recognition), but they are given the role of necessary statements by means of a training in agreement with interpersonal reactions. Thus the activity of rule following becomes “the FOUNDATION of our language game”, without which we cannot reach what we call description (VI, 28); it is constitutive of the grid through which we see the world.

These ideas are taken again in *Philosophical Investigations* §240, where Wittgenstein remarks that normally no dispute arises among mathematicians to decide if one has followed a rule. And this fact belongs to the structure (Gerüst) on whose ground language works, e.g. in order to give descriptions. In the next cryptic remark and similar passages (e.g. RFM, VI, 39; [170], 183-184) Wittgenstein tries to make

²⁹ Here Wittgenstein remarks that agreement in action is different from “instruction in receptivity” (in RFM, VII, 53); this is another evidence of the contrast Wittgenstein makes with his previous strategy of pattern recognition in the phenomenology of proof as we have described it.

³⁰ A framework which helps us to find a place for Wittgenstein’s philosophical aims is given by Dummett’s “Justification of Deduction” (in Dummett 1980). Here Dummett distinguishes three levels of the question of justification: (i) the use of inference forms simpler than the one used in the argument being justified; (ii) the justification of a set of inference forms given in some area of logic; (iii) the discussion of how deductive argument is possible at all (page 297). The main interest for Dummett resides in the intermediate problem, to which a great part of his book *The Logical Basis of Metaphysics* is devoted. Wittgenstein is more interested in the third level of discussion: “how can one follow a rule?” (RFM, VI, 38). But his latest remarks are somehow very far from the style of *Philosophical Grammar* and can be better understood against the background of his concern with ‘certainty’; we could translate the general question about justification in a slighter different form: how is it possible for human beings to hold something for certain? Wittgenstein does not reduce necessity to certainty, but attaches much more importance to the discussion of the second, not as a psychological analysis, but as a search of the anthropological conditions in which logical necessity can be constituted. In this he seems to make a very Aristotelian move, and his discussions of the mathematician who does not recognise our rule following has been compared to the Aristotelian idea that one who does not follow our logical laws is given a status more similar to a plant than to a human being (see Robinson 1965). But, besides his repeated stress on the fact that “interpretation comes to an end” (RFM VI, 38), as distinct from Aristotle, Wittgenstein also recognises that “anything can be justified somehow”. But, just after this assertion, he goes on to say that “the phenomenon of language is based on regularity, on agreement in action” (RFM VI, 39).

clear his stance: the agreement in action (in form of life, or shared reactions) is a presupposition of logic. He insists that he is not speaking of agreement of opinions (which would directly lead to conventionalism). The logical ‘must’ which expresses the need of convention and which has been discussed in the phenomenology of proof is ‘a component part of propositions of logic’. Propositions of logic are grounded in techniques developed in the teaching of rule following - and “also in physical and psychological facts that make the technique possible” (RFM, VIII, 1) .

7. SCEPTICISM ? DOUBTS ON DUMMETT’S INTERPRETATION .

Dummett says that rule following considerations are to be totally rejected for the skepticism they compel us to assume (see here §4); and elsewhere he remarks that the holism held by Wittgenstein “removes all desire to ask for a justification”. But he does not say what kind of questions and answers Wittgenstein still pushes us to desire. It seems to more than one author that the later Wittgenstein’s kind of analysis is by no means bound to bring about skepticism, even if perhaps it may remove some desire for justification.³¹

Skepticism would occur if there was a total reduction of our rule following to arbitrary decision, or to conventional agreement in opinion; but Wittgenstein’s stress on the learning of rule following in primitive language games as an anthropological ground of our techniques (among them the technique of counting) is bound to expel skepticism from that stage: our fundamental practices are rooted in our natural history, that is in our perceptions or pattern recognition; if our physical and psychological structure were different, things could have been different. But in any case they are more deeply rooted in shared reactions and constructions of the agreement that any rational being needs to build up society. Logical necessity springs from this need: what is necessary is not what is arbitrarily held necessary, but what is accepted as necessary in response to our basic constitution and our needs.

What surprises me is the contrasting attitude Dummett takes towards different aspects of Wittgenstein’s philosophy; in the Introduction to *Truth and Other Enigmas* he reminds us that “Wittgenstein’s argument against the conception of the private ostensive definition is *incontrovertible*” (page xxxiii); but the “private language argument” is strictly based on the rule following considerations. How is it possible to accept one application of anti-skeptical argument in philosophy of mind and to reject it in philosophy of mathematics? Perhaps there is a problem with respect to the charge brought against Wittgenstein of reducing truth to

³¹ Especially since the publication of Kripke 1982 (but the main lines of his argument had been already given in Fogelin 1976, pp. 142-147) critics have been discussing Wittgenstein’s stance in relation with the skeptical problem. But, leaving aside the details of Kripke’s interpretation (on whose limits see e.g. Wright 1984: 769-771) most of Wittgenstein’s interpreters see his philosophy as a demolition of skepticism, that is just the contrary of what Dummett sustains for FBI. In the following I will try to give some evidence in favour of this more standard interpretation of an anti-skeptical Wittgenstein.

recognition of truth, as a consequence of necessity understood as a free decision. But this is too strong a charge to bring against Wittgenstein, as if he rejected the idea of description of facts as they are. Something strange is going on here; Wittgenstein is very insistent about the risk of misunderstanding certain kinds of uses of statements as if they had the role of stating facts. That means that he sharply distinguishes mathematical discourse (which performs the role of rule description: it gives us the form of what we call ‘facts’) from fact-stating discourse.³² We must never forget the very sharp difference Wittgenstein holds between mathematical and empirical truth, maybe more radical than the one held by neopositivism. Instead of speaking of mathematics as “antecedent to truth”³³ we may also speak of truth of a mathematical statement as ‘overdetermined’ (RFM, VI). This ‘overdetermination’ can be considered as another way of saying that truth is reducible to recognition of truth, that in mathematics “the process and the result are the same”, as an old Wittgensteinian motto says. But empirical statements are not overdetermined; their truth is to be discovered, with the help of mathematics. In fact-stating discourse truth cannot be reduced to recognition of truth, just because mathematics teaches us rules according to which we judge empirical facts. Dummett gave the example of indirect proof as a way to build up the distinction between the world as it is and as it appears to us; it appears that this example is very much in agreement with the Wittgensteinian picture.

We follow rules in the way in which we do, not because society wants us to do so; on the contrary our rule following is what has determined the society in which we live. The acceptance of the distinction rejected by FBI does not seem to contrast with this picture; given the basic structure of our mathematical practice we cannot but have a distinction between how we describe things and how things are in themselves, that is how they might possibly have been described inside our grid.

We follow rules as we do, not because a basic instinct makes us do so: Wittgenstein distinguished carefully between rule following and a natural expression of regularity, on which rule following is based (RFM, VI, 44). Rule following is a fact strictly connected with shared reactions; and facts are to be described, not justified; but it is not simply a ‘brute fact’ as the natural expression of regularity might be; it is a basic fact with which we have to begin in order to define what we call ‘rational’. Rationality cannot be defined as performing some behaviour which depends on some objective external reality; nor with performing some behaviour which depends only on language; but with performing a behaviour which builds up agreement with other beings, till a point at which they are able to speak of right or wrong, of true and false. To think that our reason and what we call ‘rational’ springs from a complex pre-linguistic ground does not compel us to discredit the practice we call ‘rational argument’ or ‘proof’ ; even if we learn to follow them through some basic ‘shared reactions’, these basic reactions are just the precondition of the formation of rational activity. Once we have formed a grid in which to describe facts, we are no longer free

³² Wright 1984: 771 makes a similar remark.

³³ Wright 1980.

to decide every time how to go on.³⁴

8. CONCLUSIONS

In *The Logical Basis of Metaphysics* Dummett maintains that Wittgenstein “held that logical laws need no justification and fix the meaning of the logical constants without the need for further explanation or for the backing of a semantic theory” (p.337). I believe this claim is a bit overboard, and his work shows that he was somehow engaged in devising some systematic explanation of the meaning of the logical constants. Certainly he did not pursue the aim of constructing a detailed semantic theory, but he gave some hints that can be received in a more systematic frame.³⁵

On the other hand we cannot always be satisfied to consider Wittgenstein’s analysis as something which provokes interesting ideas but is too vague to be considered by itself. But my impression is that the interesting reconstruction given by Dummett gives too little space to some aspects of Wittgenstein’s philosophy. Once he suggested developing some ideas of Wittgenstein’s which are in contrast with the standard reconstruction given by himself, e.g. the idea that a proof has ‘a point’ (M. Dummett, *Reckonings: Wittgenstein on Mathematics*, Ibid., page 67). Unfortunately Dummett did not discuss any more explicitly this aspect of Wittgenstein’s ideas.

The resulting attitude of Dummett’s interpretation is to see Wittgenstein as an anarchist (M. Dummett, Ibid.) or as a *nihilist*. If by nihilism we mean the attitude of free, arbitrary will or power to decide what we want in making up our rules, this charge has to be rejected. On the other hand, if we mean the attitude of taking as ultimate ground of our practices only the agreement among human beings, then this definition is apt for the Austrian philosopher. But what could be said about the charge of ‘anarchist’? The picture would be something like a philosopher who, instead of intuitionistic ‘bolshevism’, might be supposed to bring anarchy into mathematics, as if the ‘variety of mathematics’ were only a chaos of different methods of proof without any connection with one another. Certainly Wittgenstein was not a fan of mathematical bolshevism as Ramsey would have thought. But neither does the label ‘anarchist’ suits his philosophy well. After all in his overall picture of mathematics we are not left only with arbitrary language games growing without any order at all. On the contrary Wittgenstein tried to

³⁴ The interpretation of Wittgenstein given by Dummett comes to the point of attributing to Wittgenstein the idea that there is no determinately correct result of a calculation (Dummett 1994: 31), as if until someone has done the calculation it is not determined by what would count as the result. Although this attitude has been sometime entertained by Wittgenstein, Wittgenstein’s examples are not to be taken always at face value and his claims have always a point, beyond which they become futile. Think of the idea of mathematics as ‘invention’: obviously a boy who performs a simple calculation does not ‘invent’ it anew, and “there is nothing wrong in saying that he found it out” (LFM: X); and if it is correct that he found it out, that mean that *the result was already determined by the number system and the technique of counting*. The point here is the contrast between calculation and experiment.

³⁵ See, for instance, the partial reconstruction of the ‘Wittgensteinian’ meaning of the quantifiers given by Cellucci 1995.

clarify which *kind* of order we find in mathematics: an order which is given, on one hand, by the anthropological ground of the most fundamental human need of agreement, on the other hand, given by application (use) and interest.³⁶

A better term than ‘anarchy’ would be ‘heterarchy’: different language games, systems of proof, are confronted without one being fundamental for all the others, but being connected in a net (or in different possible nets) with all the others. Part of the philosopher’s task is to analyse the relationships among the different language games, the different systems, showing the intermediate passages which lead from one to the other, in order to get a clear view of mathematics.³⁷ This work, based on the idea of language games, questions the charge of holism normally brought against Wittgenstein, giving a peculiar complexity of his kind of holism. For it is very difficult to disregard the construction by stages given in his presentation of language games in the *Philosophical Investigations* and even more clearly in *the Brown Book*. It seems that in Wittgenstein’s analysis of language games we are bound to individuate some more basic language games before entertaining more complex ones. Also the description given in the previous paragraph shows how far Wittgenstein went in this analysis of what is more basic in the construction of language, and in the learning of rule following. You may speak of rule following only after certain general conditions have been given in basic language games.

Probably Dummett is right in observing that there is an “ineradicable ambiguity in Wittgenstein’s attitude to whether or not it is possible to give a comprehensive and systematic meaning-theory for a natural language”.³⁸ He recognises also that Wittgenstein’s remarks are not as unsystematic as they appear to be. And he also recognises, in an answer to Crispin Wright, that “a minimum of holism is indispensable”: a sentence can be understood only in the context of a language, or in the context of a fragment of the language; but that fragment, Dummett says, in a very Wittgensteinian mood, “must be one that could be the whole of language”.³⁹ But, on the other hand, the strategy given by Dummett is sharply in contrast with some aspects of Wittgenstein’s. Dummett strives to reach a clear vision of the working of language through a definite hierarchical vision: the complexity of the different fragments of language is seen as a strictly hierarchical organization: “there must be a hierarchy of dependence, the understanding of any sentence never presupposing the understanding of sentences standing higher in the hierarchy. If there is not, if dependence runs in all directions, the significance of a linguistic utterance will become unsurveyable”.⁴⁰

³⁶ See Penco 1981a

³⁷ I would like to point out that the idea of heterarchical systems is well developed in Artificial Intelligence, at least starting with seminal works in natural language understanding such as SCHRDLU in Winograd 1972. This reference to Artificial Intelligence suggests a possible analogy with the Wittgensteinian strategy of language games and definition of meaning as use.

³⁸ See Dummett 1991: 306.

³⁹ See Dummett 1987: 233.

⁴⁰ Ibid., p. 234.

Is it possible to have an intermediate picture between total anarchy and strict hierarchy? Is it possible to have a set of dependencies which do not run in all directions, but at the same time do not run in a single direction? We may imagine some kind of passage from monarchy to democracy also with respect to mathematics. After all not every bit of mathematics is based on set theory or on category theory. The picture of Wittgenstein I have tried to give here tends to suggest this possibility.

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REFERENCES

- Bensch R. 1973, *L. Wittgenstein: Die apriorischen und mathematischen Sätze in seinen Spätwerk*, Bonn: Bouvier.
- Cellucci C. 1995, "On the Meaning of Logical Symbols" in R. Egidi (ed.) *Wittgenstein, Mind and Language*, Dordrecht: Kluwer.
- Dummett M. 1973, *Frege. Philosophy of language*, London: Duckworth.
- Dummett M. 1978, "Reckonings: Wittgenstein on mathematics", in J. V. Canfield (ed.) *The Philosophy of Wittgenstein*, New York: Garland.
- Dummett M. 1980, *Truth and Other Enigmas*, Cambridge (Mass): Harvard University Press.
- Dummett M. 1987, "Reply to Essays" in B.M.Taylor (ed.) *Michael Dummett, Contributions to Philosophy*, Dordrecht: Nijhoff.
- Dummett M. 1991, *The Logical Basis of Metaphysics*, Cambridge (Mass): Harvard University press.
- Dummett M. 1994, "Wittgenstein on Necessity: some reflections" in B.Hale, P. Clark, *Reading Putnam*, Oxford: Blackwell.
- Fogelin R. J. 1976 *Wittgenstein*, London: Routledge & Kegan Paul.
- Geach P.T. 1972, *Logic Matters*, Oxford: Blackwell
- Kripke S. 1982, *Wittgenstein on Rules and private language*, Oxford: Blackwell.
- Penco C. 1979 "Matematica e regole: Wittgenstein interprete di Kant", *Epistemologia*, 2: 123-154.
- Penco C. 1981, "Intuition in Mathematics", *Epistemologia*, 4: 77-94.
- Penco C. 1981a, "Mathematik und Interesse" in R. Haller (ed.) *Sprache und Erkenntnis als Soziale Tatsache*, Wien: Hölder-Pichler-Tempsky.
- Penco C. 1991, "Induzione matematica e convenzione: Wittgenstein tra Russell e Poincaré" in G.Cimino,U.Sanzo,G.Fava (eds) *Il nucleo filosofico della Scienza*, Lecce: Congedo.
- Robinson G. 1965, "Following and Formalization", *Mind* 73:46-63.
- Shanker S.G. 1986, *Wittgenstein and the Turning Point in the Philosophy of Mathematics*, London: Croom Helm.
- Stroud B. 1965, "Wittgenstein and Logical Necessity", *The Philosophical Review*, 74: 504-18.
- Waismann 1966, *Introduction to Mathematical Thinking*, New York: Harper Torch Books.
- Winograd T. 1972, *Understanding Natural Language*, New York: Academic Press.
- Wittgenstein L. 1953, *Philosophical Investigations*, Oxford: Blackwell.
- Wittgenstein L. 1964, *Philosophical Remarks*, Oxford: Blackwell.
- Wittgenstein L. 1969, *Philosophical Grammar*, Oxford: Blackwell.
- Wittgenstein L. 1975, *The Blue and Brown Books*, Oxford: Blackwell.
- Wittgenstein L. 1976, *Lectures on the Foundations of Mathematics, Cambridge 1939*, Ithaca: Cornell University Press.
- Wittgenstein L. 1978, *Remarks on the Foundations of Mathematics*, Oxford: Blackwell, 3rd. edition.
- Wright C. 1980, *Wittgenstein on the Foundations of Mathematics*, London, Duckworth.
- Wright C. 1984, "Kripke's account of the argument against private language", *The Journal of Philosophy*, 81: 759-778.
- Wright C. 1987, "Dummett and Revisionism" in B.M. Taylor (ed.) *Michael Dummett, Contributions to Philosophy*, Dordrecht: Nijhoff.

