Fermat's last theorem proved in Hilbert arithmetic.
II. Its proof in Hilbert arithmetic by the Kochen-Specker theorem with or without induction

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Abstract. The paper is a continuation of another paper (https://philpapers.org/rec/PENFLT-2) published as *Part I*. Now, the case of "n=3" is inferred as a corollary from the Kochen and Specker theorem (1967): the eventual solutions of Fermat's equation for "n=3" would correspond to an admissible disjunctive division of a qubit into two absolutely independent parts therefore versus the contextuality of any qubit, implied by the Kochen – Specker theorem. Incommensurability (implied by the absence of hidden variables) is considered as dual to quantum contextuality. The relevant mathematical structure is Hilbert arithmetic in a wide sense (https://dx.doi.org/10.2139/ssrn.3656179), in the framework of which Hilbert arithmetic in a narrow sense and the qubit Hilbert space are dual to each other. A few cases involving set theory are possible: (1) only within the case "n=3" and implicitly, within any next level of "n" in Fermat's equation; (2) the identification of the case "n=3" and the general case utilizing the axiom of choice rather than the axiom of induction. If the former is the case, the application of set theory and arithmetic can remain disjunctively divided: set theory, "locally", within any level; and arithmetic, "globally", to all levels. If the latter is the case, the proof is thoroughly within set theory. Thus, the relevance of Yablo's paradox to the statement of Fermat's last theorem is avoided in both cases. The idea of "arithmetic mechanics" is sketched: it might deduce the basic physical dimensions of mechanics (mass, time, distance) from the axioms of arithmetic after a relevant generalization, Furthermore, a future *Part III* of the paper is suggested: FLT by mediation of Hilbert arithmetic in a wide sense can be considered as another expression of Gleason's theorem in quantum mechanics: the exclusions about (n = 1, 2) in both theorems as well as the validity for all the rest values of "n" can be unified after the theory of quantum information. The availability (respectively, non-availability) of solutions of Fermat's equation can be proved as equivalent to the non-availability (respectively, availability) of a single probabilistic measure as to Gleason's theorem.
Keywords: arithmetic mechanics, Gleason's theorem, Fermat's last theorem (FLT), Hilbert arithmetic, Kochen and Specker's theorem, Peano arithmetic, quantum information

IX. INSTEAD OF INTRODUCTION: A SET-THEORETICAL "GESTALT CHANGE"[1]

This paper is the continuation (respectively, *Part II* to *Part I*) dedicated to the eventual arithmetical and inductive proof of Fermat's Last Theorem (further, notated as FLT) if one has proved that it is an insoluble statement after the Gödel (1931) incompleteness theorems valid to the triple of (Peano) arithmetic, (ZFC) set theory, and propositional logic. The idea of that proof (in the previous *Part I*) consists in the exclusion of set theory (and more precisely "actual infinity" after the axiom of infinity) from that triple, thus being reduced to the pair of "Fermat arithmetic" and propositional logic after an "epoché to infinity" (analogical to Husserl's "epoché to reality") furthermore being natural in Fermat's age, inexperienced to infinity (a concept entered

[1] The enumeration of the sections continues from *Part I*, so the first section in *Part II* of the paper turns out to be "IX".

mathematics a few centuries later after Cantor's set theory). Nonetheless, set theory in the framework of "Hilbert mathematics" is anyway used, but only as a "Wittgenstein ladder" being removable in the ultimate syllogism of the proof (which might claim to be the "lost proof of Fermat" heralded by Fermat himself, but without demonstrating it).

One can imagine a symmetric approach to the proof of FLT in the same paradigm if (now) arithmetic is utilized to be the "Wittgenstein ladder" to an only set-theoretical proof since the Gödel incompleteness does not admit the simultaneous use of arithmetic and set theory, but it does not predetermine which exactly to serve as a "Wittgenstein ladder" to the other one, in terms of which alone to be written the ultimate text of the proof.

A fundamental prejudice ostensibly rejected still by Lobachevski's geometry as the historically first non-Euclidean geometry, but persisting even nowadays, particularly as to FLT, can be subjected. That prejudice consists in the implicit and unarticulated postulate that any theorem (particularly FLT) is valid in any axiomatic system or respectively that there exists a "privileged" (somehow) axiomatic system, in which it has to be valid. For example, the theorem that the sum of the angles of any triangle is 2π is valid in Euclidean geometry, but it is false in general in any non-Euclidean geometry. So, one should expressly refer to which axiomatic system a theorem claims to be true.

For example, and as to FLT, the axiomatic system by default nowadays is meant to be that of arithmetic and set theory (and propositional logic to which both are first-order logics) since they are the foundations of the "standard mathematics" for almost all mathematicians or at least the "privileged reference frame" to which any other axiomatic system is to be reducible by a relevant model in it.

Nevertheless, Yablo's paradox can easily demonstrate that FLT obeys it and consequently, FLT is a Gödel insoluble statement (traced in detail in *Part I*). So, it is not a theorem in the standard mathematics at issue. Anyway, this does not reject that FLT is a theorem in some other axiomatic systems, e.g.: in Fermat arithmetic (i.e. Peano arithmetic after the "epoché to infinity") or in Hilbert arithmetic (borrowed the pattern of two dual Peano arithmetics from the qubit Hilbert space in turn originating from the usual separable complex Hilbert space of quantum mechanics).

As far as the last statement is already proved statement in the previous *Part I*, the present *Part II* is concentrated on the alternative, but not less consistent conjecture that set theory or its minimal extension, if need be, can also be a relevant axiomatic framework, in which FLT is a true theorem. The main obstacle still at first glance consists in the fact that FLT is an arithmetical statement rather than a set-theoretical one, thus needing at least a few arithmetic axioms even to be only formulated.

Different approaches can resolve that problem under the condition *sine qua non* for the Gödel dichotomy to be suspended somehow. The core of all those approaches consists in avoiding the direct contradiction of the axiom of induction (in Peano arithmetic) and the axiom of infinity (in ZFC set theory or equivalent to it in other axiomatic systems of set theory[2]), for example by means of: (1) Peano arithmetic ("PA" further) without induction (i.e. partial Peano arithmetic, or PPA

[2] Keyser (1903) suggests one from the first reflections on the relation of the two axioms.

further), by the way, being absolutely sufficient for the formulation of FLT, for which the axiom of induction is redundant; (2) a special infinite set of all natural numbers only within it the axiom of induction to be valid furthermore without any restriction of the bijection of any other set on that special set and guaranteed by the axiom of choice; (3) the axiom of induction can be kept even to all sets (unlike the first option where it is removed) under the necessary condition to be "doubled"[3] (i.e. as a model of transfinite induction by two finite inductions in two dual Peano arithmetics, thus being independent of each other).

A comment to the option (1) is to be the following. The axiom of induction is a strong and successful tool for mathematics and its eventual absence would make many proofs impossible, difficult or very complicated and sophisticated. However, it can be substituted by alternative or equivalent instruments based on the axiom of choice. Indeed, it, as in the option (2), allows for any infinite set (or even finite set, but under additional conditions or conventions) to be mapped bijectively into PPA. If one utilizes Hilbert arithmetic in a narrow sense, being dual to the qubit Hilbert space (thus both constituting Hilbert arithmetic in a wide sense), the property of unitarity (in turn being what is able to conserve energy conservation in quantum mechanics: *Penchev 2020 October 5*) together with the function successor verified to be valid as to an investigated set are sufficient to restore the axiom of induction in a modified form consistent to axiom of infinity (Penchev 2021 August 24).

Indeed, one can test the case generating the contradiction of Peano arithmetic and set theory, namely, the immediate corollary from the axiom of induction that all natural numbers are finite, but simultaneously the *set* of all natural numbers is infinite in virtue of the axiom of infinity (in ZFC set theory). If that is the case, the unitarity of Hilbert arithmetic is to be valid to the *set* at issue, and the axiom of induction, only to its elements. Then, one can refer to Gentzen's cut-elimination (1935) in order to verify the investigated property of the set at issue conserving both unitarity and function successor remaining consistent to each other unlike the converse case of the Gödel dichotomy implying either incompleteness or inconsistency as to the investigated property (e.g.: as in *Horská 2014*).

In other words, the difference from the Gödel dichotomy consists in the fact that the Gentzen cut-elimination can act "from an infinite set to its elements", all of which are finite, but the converse statement is false as the Gödel dichotomy can be interpreted: that is "from all natural numbers to their set". That asymmetry of the two opposite directions (namely: (1) "from an infinite set to its finite elements" after Gentzen; (2) "from all finite and enumerable elements to a certain infinite set" after Gödel) can be visualized also extensionally (as Gödel did) by the set of all insoluble statements (and meant by the second and dual Peano arithmetics in Hilbert arithmetic), but this is not necessary, i.e.: it is optional.

If the axiom of induction is substituted by unitarity underline by the axiom of choice[4] as this is described above, a much shorter pathway from the FLT(3) to FLT is thus pioneered: FLT(3)

[3] The papers of Maliaukiené (1997; 2000) share the context of an analogical idea.

[4] Another pathway (possibly equivalent) for the substitution of the axiom of induction by the axiom of choice can be outlined involving nonstandard models in relation to the axiom of inductions as Rabin (1961)

doubled dually in the dual Peano arithmetic implies immediately FLT without the "crutch" of induction, and rather directly by virtue of the completeness of Hilbert arithmetic. Anyway, this pathway is to be traced in mathematical detail, e.g as in *Section XIV*. Furthermore, that doubling can be interpreted also as a self-referential application of FLT(3) proved by the Kochen-Specker theorem to itself, getting discussable again there.

A comment to the option (2) is to be the following. If that is the case, FLT is valid only to that special set, only within which the axiom of induction is valid, but furthermore, to all bijections on it guaranteed by the axiom of choice. This means that FLT(3) should be proved to that set of all natural numbers, then it to be transferred within it as a statement to all natural numbers, to which the arithmetical variables x, y, z refer, and finally: the axiom of induction can be applied within it (as in the previous *Part I*) therefore proving FLT absolutely, but only to all natural numbers.

A comment to the option (3) is the following. This is the solution of Hilbert arithmetic. It establishes implicitly a new form of transfinite induction consisting of two *independent* or *dual* usual finite induction which can be also and rather loose interpreted as a "Hamilton version" of the usual finite inductions where its finite and transfinite "parts" are not successive, but *in parallel* and the transfinite part is homomorphic to a second finite induction dual to the initial finite induction[5].

Thus, the structure of the two inductions to which transfinite reduction has been reduced repeats that of the two dual Peano arithmetics[6] of Hilbert arithmetic and within which they can be exhaustively and unambiguously embodied. Furthermore, that scheme allows for transfinite induction to be interpreted as a single finite induction after the concept of natural numbers has been generalized to well-ordered series of information (strings or binary "messages") as well as if the extraordinary "2:1" ("transcendental") bijection is involved: $(P^{-} \otimes P^{+} \rightarrow P^{0}) \rightarrow P$ where the notations are the following: "P^{-}, P^{+}" for the two dual Peano arithmetics of Hilbert arithmetic; "$\otimes$" for Cartesian product; "$\rightarrow P^{0}$" for the mapping into "Fermat arithmetic" (defined in *Part I* as Peano arithmetic after a newly Husserl-like "epoché to infinity" rather than to reality); "$\rightarrow P$" for the mapping into the standard Peano arithmetic (i.e. granting implicitly the eventual context of set theory and thus, the Gödel incompleteness unlike Fermat arithmetic yet naively inexperienced to it or to the distinction of finiteness and infinity).

Indeed, that nonstandard bijection can be also called "informational" since it substitutes any arithmetic unit by a bit of information, any natural number "n" by a binary string of length "n" bits, and the single finite induction of Peano arithmetic, by two finite inductions independent of

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as well as (and absolutely independently) as Germansky (1961), the latter furthermore containing a wider reflection on the relation of those axioms. The relation of transfinite induction and recursive well-orderings (respectively, the axiom of choice) is mean in detail, for example, in: *Friedman, Ščedrov (1985)*.

[5] If the second "dual" induction is successively substituted by corresponding well-ordered qubits of the qubit Hilbert space (only in virtue of the axiom of choice), and then by, by probability distributions, that model of transfinite induction can be represented also by "epsilon substitution" as Towsner (2005) suggests. The approach of Rose (1972) can be interpreted as analogical.

[6] The approach of Hirst (1999) implies an analogical context.

each other and thus equivalent to transfinite induction[7]. This can convince us that the former option of the informational interpretation of "natural numbers" and the latter option utilizing the nonstandard bijection are equivalent to each other as well as to transfinite induction reduced to two finite and independent inductions.

An additional, but important notice is that the implicit connotation of *nonstandard* bijection to *nonstandard* interpretation (e.g. as in Robinson's "*Nonstandard*[8] analysis" in 1966) is intentional. In other words, *nonstandard* bijection can be considered as a *nonstandard* interpretation of the usual bijection as well as the unity of both standard and *nonstandard* analysis as the same. Indeed, if one delivers two interpretations (such as standard and nonstandard interpretations) of the same mathematical structure (e.g. that of infinitesimal analysis whether standard or nonstandard) it is simultaneously an interpretation of nonstandard bijection in turn: since two identical copies of the same can absolutely equivalently represent a single unit of the same and furthermore, that representation as whole, a bit of information.

The three options (1), (2), and (3) share the same first stage consisting in the set-theoretical proof of FLT(3). Indeed, that by the Kochen-Specker theorem eventually involves set theory necessarily since the separable complex Hilbert space needs some continuous topology (and thus "actual infinity" to be justified), though that topology is not determined unambiguously. Consequently, it should avoid arithmetic (more precisely the axiom of induction) and thus, should prevent FLT to be a Gödel insoluble statement once both arithmetic and set theory have been utilized.

So, one has to prove that either arithmetic (the axiom of induction) is not involved along with set theory for the proof of FLT(3) or, if it is involved anyway, the Gödel incompleteness is not relevant to the case: in other words, the latter option suggests that FLT (is) to be proved in Hilbert mathematics rather than in Gödel mathematics (only where it is an insoluble statement). Then, one can notice that the Kochen-Specker theorem (just for its statement) implies the latter case:

Indeed, the following rather philosophical observation can verify the above conclusion. The Kochen-Specker theorem states the absence of hidden variables in quantum mechanics: that is a rejection of the alleged "incompleteness of quantum mechanics", advocated in Einstein, Podolsky, and Rosen (1935) and also in many other papers before and after that article. The concept of Hilbert arithmetic (available in two dual forms: Hilbert arithmetic in a narrow meaning, and the qubit Hilbert space inferable from the separable complex Hilbert space of quantum mechanics) allows for transferring the statement about the completeness of quantum mechanics into the completeness

[7] Consistent models of transfinite induction within a single Peano arithmetic can be also considered (e.g. Sommer 1995). Then the gap between the two Peano arithmetic is able to be represented by an indefinite finite leap within a single Peano arithmetic therefore admitting also an exact estimation of the "provability transfinite induction in the initial segments of arithmetic" (Mints 1973). Another option for the implicit introduction of that gap is in the way of intuitionism, e.g.: as Scarpellini (1972). The classical paper of Kuratowski and Neumann (1937) can be also relevant to the same context.

[8] Italic mine, in order to emphasize the essential unity of "nonstandard bijection", "nonstandard interpretation" and "nonstandard analysis" sharing the same literal connotation of *nonstandard*.

of mathematics therefore avoiding or preventing any Gödel incompleteness at all, and for FLT to be an insoluble statement in particular.

Further, one can research the explicit way for the separable complex Hilbert space to be complete (as the Kochen-Specker theorem can be interpreted), but now: in parallel with Hilbert arithmetic (in its narrow meaning). The reason is that its dual Hilbert space is anti-isometric and thus, it is isomorphic to the class able to equate isometry and anti-isometry. Properly, this is the sense of rejecting the conjecture of "hidden variables" (granted to be true by Einstein and many other physicists even nowadays).

As von Neumann (1932) explained it, its essence consists in the "mysterious disappearance" of the half variables in quantum mechanics in comparison with the description of a physical system in classical mechanics. Then, the natural hypothesis might be that the description of quantum mechanics is incomplete and a future physical theory which will replace quantum mechanics will be able to include that vanishing half of variables missing in today's quantum mechanics. However, that special property of the separable complex Hilbert space (namely, to be isomorphic to the class able to equate isometry and anti-isometry of both dual spaces of it) allows for the half of variables to vanish, nonetheless remaining complete.

The pathway of Kochen and Speaker to prove their fundamental theorem is not less important to be traced and followed for deducing FLT(3). They demonstrated the *incommensurability* of those two dual spaces as they commute to each other (that is the case of entanglement) as they do not (the latter case was proved by von *Neumann* yet, in *1932*).

The expression of "commuting or non-commuting dual spaces" means more precisely the following: the Hermitean operators defined on them commute or not correspondingly. The same incommensurability translated in terms of Fermat's equation (to which FLT refers) means just FLT(3): the relation of any solution of Fermat's equation for the case $n = 3$ cannot be a rational number, but an irrational number implied by that kind of incommensurability, relative to that utilized by Kochen and Specker for inferring their theorem.

Furthermore, one can link immediately the Kochen-Specker (1967) and Gleason (1957) theorems as two counterparts[9], at least as to FLT(3). Indeed, Gleason's theorem states that a unique probabilistic measure exists for Hilbert space of dimension $n \geq 3$. Properly, Kochen and Specker's way to prove their theorem demonstrates the same unique probabilistic measure as an equivalent of the absence of hidden variables in quantum mechanics since that single measure means the incommensurability of quantum quantities also even in the case where their underlying Hermitian operators commute: after von Neuman (1932: 167-173) has proved the absence of hidden variables only about noncommuting operators.

On the contrary, the eventual availability of hidden variables would correspond to the commensurability, on the one hand, of natural numbers (what any rational number represents in fact), and on the other hand, many (i.e.: more than one) probabilistic measures: just this is the case

[9] For example, both theorems admit more or less direct experimental confirmations (e.g. Peres 1992; Campos, Gerry 2002; @)

for Hilbert space (whether real or complex) for dimensions $n = 1,2$, but false for dimensions $n \geq 3$ according to Gleason's theorem.

One is to emphasize that an "empty" qubit represented according to its standard and initial definition means two dimensions in the separable complex Hilbert space of Hilbert space and thus admitting many probabilistic measures according to Gleason's theorem. However, still one additional necessary condition for a qubit to be defined is that $|\alpha|^2 + |\beta|^2 = 1$ after $1\ qubit \equiv \alpha \mid 0\rangle + \beta \mid 1\rangle$ where $\mid 0\rangle, \mid 1\rangle$ are two orthogonal subspaces of the separable complex Hilbert space (as this will be repeated in detail in *Section X*). In fact, that additional equality restricts relevant probabilistic measures to single one unlike an "empty" qubit just obeying the case of two complex dimensions of Gleason's theorem and thus admitting any α, β satisfying that condition.

The same opposition of an empty qubit (uncertain) versus a certain value chosen (or "recorded") on it is conserved after the equivalent transformation of a qubit (defined as above) into a unit usual ball in Euclidean space; an empty qubit is represented by an empty ball (thus admitting many probabilistic measures); the establishment of a single value in it or that of a unique probabilistic measure are equivalent.

Hilbert arithmetic introducing an arithmetical unit of Peano arithmetic as the class of equivalence of all possible values restricts itself to the case of an empty qubit (respectively, an empty ball) consequently meaning the case "$n = 2$" of Gleason's theorem and many admissible probabilistic measures correspondingly. However, FLT(3) needs the case of incommensurability or a unique probabilistic measure respectively if it has been defined in Hilbert arithmetic where all units are classes of equivalence of qubits.

Fortunately, just that is the case of FLT(3) relating at least the two arithmetic variables "y^3" and "z^3" in Fermat's equation for "$n = 3$: $x^3 = y^3 + z^3$". Thus, the two qubits corresponding to the arithmetical units of the arithmetic variables "y^3" and "z^3" constitute a four-dimensional complex Hilbert space though satisfying still one condition, namely: Fermat's equation itself, therefore equivalently reducing the degrees of freedom to a three-dimensional complex Hilbert space, anyway absolutely sufficient to be immediately inferred the availability of a unique probabilistic measure according to Gleason's theorem.

That will be the proper subject of the next, *Part III* of the paper, which is only a horizon of the present *Part II* being concentrated exceptionally on the Kochen-Specker theorem as a relevant tool for proving FLT(3) (and even FLT in general). The last four paragraphs only vaguely marked the conceptual and logical pathway from the Kochen - Specker theorem to Gleason's theorem in the context of FLT, and especially FLT(3): it will be clarified in detail in the next part and will not be discussed any more now.

So, one has to trace rigorously and mathematically the "translation" (or more precisely, syllogism) from the incommensurability of Kochen-Specker theorem to that meant by FLT(3). The idea of that logical "translation" will be described in the next section.

X. AN IDEA FOR THE PROOF OF FLT FOR "N=3" BY THE KOCHEN-SPECKER THEOREM

One can utilize the conception of "natural number" as the class of equivalence of all sets consisting of the same numbers of elements. For example, the natural number "x" means the class of the equivalence of all sets consisting of "x" elements such as "x" pears, "x" apples, or better "x" qubits. Obviously, only the interpretation by qubits is relevant if one intends to use the Kochen-Specker theorem, and Fermat's equation can be interpreted particularly as: $x^n\, qubits = y^n\, qubits + z^n\, qubits$. Then, any solution of the general Fermat equation is a solution of it written as to qubits. Then, if one proves FLT(3) in qubits, this immediately implies FLT in virtue of *modus tollens*.

Thus, the new objective is that "$x^3\, qubits = y^3\, qubits + z^3\, qubits$" does not have any solution in natural numbers (and consequently, in rational numbers). Properly, FLT does not claim any statement about solutions in irrational numbers, however if the Kochen-Specker theorem has been already utilized for proving FLT(3), it implies the necessary existence of solutions in irrational numbers, which will be discussed in the next section: and just that is the case.

The standard definition of "qubit" in quantum mechanics is:

$1\, qubit \equiv \alpha \mid 0\rangle + \beta \mid 1\rangle$ where $\mid 0\rangle, \mid 1\rangle$ are two orthogonal subspaces of the separable complex Hilbert space (usually disjunctive to each other in default, but this is not necessary), and α, β are two complex numbers such that: $\mid \alpha \mid^2 + \mid \beta \mid^2 = 1$. That definition is isomorphic to an usual unit ball, within which two points on two orthogonal great circles are chosen to correspond unambiguously to those α, β.

Then, one can introduce the usual units of Peano arithmetic as "empty qubits", i.e. as the classes of equivalence to all possible values α, β which a qubit can accept (or which can be recorded on a cell of the quantum computer tape if "quantum computer" is granted to be a relevant generalization of "Turing machine: *Penchev 2020 August 5*). Just this definition is utilized as the definition of "unit" in Hilbert arithmetic[10]. Then, FLT(3) can be additionally converted in the form:

$$x^3\, empty\ qubits = y^3\, empty\ qubits + z^3\, empty\ qubits$$

One is to notice that the set consisting of empty qubits is unique in the following sense. On the one hand, its elements are specific such as "apples", "pears", etc. However on the other hand and simultaneously, they are "arithmetic units at all" in Hilbert arithmetic, and can be transferred identically as such in Peano arithmetic. Meaning the latter circumstance, one can state also that if "unit radius" is to notate "qubit radius" (as an three-dimensional unit ball isomorphic to a qubit):

$$(x^3\, unit\ radiuses = y^3\, unit\ radiuses + z^3\, unit\ radiuses) \equiv (x^3 = y^3 + z^3)$$

In other words, FLT(3) in the general case and it in the particular case of "empty qubits" are isomorphic after the mediation of Hilbert arithmetic. This property is not necessary right now, but it can be utilized where need be.

The following important observation is that the conceptions of "empty qubit" and "natural volume", the latter allegeable to be possible in Fermat's age, are relative. Indeed, any empty qubit

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[10] One can notice a fruitful ambiguity consisting in the uncertainty whether the "empty qubit" admits any internal structure or not: the concept of "unit" means rather the latter alternative of inseparable whonesess.

being a unit ball possesses an unambiguously determined three-dimensional geometrical volume: "$\frac{4}{3}\pi$" since its radius is a unit. Or said otherwise, Fermat's equation is identical to its interpretation by natural numbers, at that: the latter and the former are equally general statements though the last circumstance is also not necessary right now.

The identification of the triple of "natural number", "empty qubits", and "natural volumes" allows for the Kochen-Specker theorem, once it has been reformulated to "empty qubits" as a relevant corollary, to be directly transferred to both "natural numbers" and "natural volumes" (thus getting intuitively accessible even in Fermat's age). So, the next stage is to be that inference from the Kochen-Specker theorem just to "empty qubits":

Since "empty qubit" is defined as the *class of equivalence* (rather than as the *set*) of all possible values, its properties are shared by all values of a qubit. Any value is an additive member of a qubit wave function (respectively, being a component of qubit state vector) and thus: it can be considered in turn as an elementary wave function, in virtue of which the Kochen-Specker theorem can refer as to the single qubit of the arithmetical variable "x" on the left side of Fermat's equation, on the hand, as the sum of the two qubits "y" and "z" on the right side, on the other hand, as well as to the algebraic sum of all three qubits: thus rejecting any hidden variables in relation to them.

A problem is how to reinterpret the absence of hidden variables passing from qubits to usual arithmetic units. First, one can notice which meaning of them fits to prove FLT(3), and then, the same meaning is to be inferred rigorously and logically from the Kochen-Specker theorem, optionally utilizing relevant ideas of their proof as well.

The following meaning of the absence of hidden variables as to natural numbers in Hilbert arithmetic (rather than to qubits or wave functions in the separable complex Hilbert space which are literally meant by theorem) would verify FLT(3): the absence of any *solution in rational numbers*. Indeed, if one discovers a solution of that kind, it can be transformed into another solution in natural numbers. So, proving that any solution of FLT(3) is necessarily an irrational number, this is also a proof of FLT(3) immediately.

If that is the "useful" meaning of the absence of hidden variable for FLT(3) and what is to be proved, one can already define "hidden variable" in a proper arithmetic sense as the *commensurability* of two natural numbers since each of both consists of a finite number of that shared and thus common measure greater or equal to one arithmetical unit (i.e "1"); and vice versa, the absence of hidden variables right means their *incommensurability*: i.e. any finite measure of the one natural number is infinite to the other one, their ratio is an irrational number (rather than a rational number as in the former case).

If one has revealed that useful meaning of "hidden variable" in an arithmetic sense, it has to be then linked to *Theorem 1* (Kochen-Specker 1967: 70) stating that there does not exist any homomorphism of finite partial Boolean algebra (relevant to empty qubits as units) into "Z_2"[11] Hilbert space (relevant to bits). First, a few elements of the pathway to *Theorem 1* can be "paraphrased" about irrational and rational numbers as follows:

[11] This is the notation of Kochen and Specker for the field (in an algebraic sense) consisting of two elements.

The 'Z_2' Hilbert space of rational numbers is identical to all positive rational numbers however represented not as an enumerable well-ordered set, but as the Cartesian product of all natural numbers with themselves[12]: thus, being finite again in virtue of the axiom of induction since the Cartesian product of any two finite sets is finite as well.

On the contrary, the finite partial Boolean algebra of irrational numbers is an infinite Cartesian product of the set of all natural numbers with itself, therefore again touching the Gödel incompleteness after no homomorphism of any infinite (i.e.: proper set-theoretical) Cartesian product into any finite (i.e. proper arithmetic) Cartesian product. Anyway, the substitution of "finite partial Boolean algebra of irrational numbers" by the "infinite Cartesian product of the set of all natural numbers with itself" needs an additional justification not being obvious, after one defines rigorously the former concept following relevant ideas applied in the proof of the Kochen-Specker theorem by themselves. Furthermore, a few preliminary notices are necessary:

The Cartesian product of the two sets of all natural numbers can generate only algebraic irrational numbers being also a countable set just as the initial set of all natural numbers. So, all transcendental irrational numbers are not meant[13]. Nonetheless, the incommensurability of algebraic irrational numbers to natural or rational numbers is sufficient for forcing the set of all natural numbers (or more precisely, a relevant infinite subset of it for any certain algebraic rational number) to be involved necessarily. In other words, the incommensurability of algebraic irrational numbers allows for the set of all natural numbers to be used for proving FLT only in set theory without any reference to arithmetic (though its use is granted in advance for FLT to be able to be formulated at all).

A more general or philosophical reflection relates to the link of numerical incommensurability and Pythagoreanism. As this is commonly granted to be true, the discovery of (algebraic) irrational numbers by the ancient Pythagoreans generated a crisis in their doctrine since natural numbers turned out not to be enough for all mathematical (also geometrical or ontological) entities to be described as Pythagoreanism suggested and postulated: there existed entities which are not numbers (if they were restricted to natural and irrational numbers) and this can be inferred rigorously and deductively from the existence of numbers.

On the contrary (and thus, as if contradictory at first glance), the contemporary "quantum" neo-Pythagoreanism utilizes an analogical kind of incommensurability by the mediation of the Kochen - Specker theorem[14] to establish quantum information merging the mathematical and physical (and

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[12] Indeed, any positive rational number is a ratio of two natural numbers though the operation "division" is not commutative. Then, the pairs of the Cartesian product have to be granted to be non-commutative conventionally, i.e. being ordered: for example declaring the ordinate for the axis of all numerators and the abscissa, for that of all denominators.

[13] For example, following the classical paper of Gelfond (1960).

[14] The literature on the Kochen - Specker theorem is huge. Even the restriction for its relevance to the present context is not sufficient to be representative enough; an example list could include: Nagata, Nakamura, Farouk, Diep 2019; Nagata, Patro. Nakamura 2019; Nagata, Nakamura 2016; de Ronde, Freytes, Domenech 2014; Abbott, Calude, Conder, Svozil 2012; Waegell, Aravind 2012; Garola 2009; Dowker, Ghazi-Tabatabai 2008; Brunet 2007; Cabello, Estebaranz; García-Alcaine 2005; Döring 2005; Nagata 2005; Barrett, Kent 2004; Hrushovski, Pitowsky 2004; Huang, Li, Zhang, Pan, Guo 2003; Campos,

even ontological) bases as the ultimate substance of the world. So, one may be curious about how the same tenet of incommensurability can be utilized in both "pro" and "contra" Pythagoreanism: the ancient Pythagoreanism proved that there exist entities which cannot be described only by arithmetic (understanding in default that arithmetic is available in a single and unique copy); the contemporary quantum neo-Pythagoreanism utilizing Hilbert arithmetic does not need any other entity than those describable by two copies of arithmetic. Even more, one can use the mapping of the nonstandard bijection of both copies into a single one therefore being able to reduce today's Pythagoreanism to the original one as this is demonstrated as to FLT in particular in *Part I* of the present paper.

The intended proof of FLT(3) is based on a reformulation of its sense in terms of incommensurability, since only irrational numbers can be solutions of Fermat's equation for the case of "$n = 3$" (the absence of solutions which are natural numbers implies for them not to be rational numbers). Then, if one proves that the arithmetical variables y, z (those inin Fermat's equation "$x^n = y^n + z^n$" or any other pair among all the three arithmetic variables x, y, z) are incommensurable to each other, or in other words, that their ratio is necessarily an irrational number, Fermat's equation does not have any solution for the corresponding exponent "n".

Then, incommensurability as to FLT(3) is intended to be inferred following a corollary from the Kochen - Specker theorem establishing the inseparability of a qubit into halves in virtue of quantum contextuality, on the one hand, or the equivalent incommensurability of two qubits, on the other hand. The objective requires for the original proofs of the two authors to be mulled in order to be reinterpreted into relevant terms of the incommensurability of irrational numbers (as to to the proved statement about the absence of hidden variables in quantum mechanics) versus the commensurability of rational numbers (as to the rejected negation about hidden variables in quantum mechanics). The following statement is quite relevant as a "hint":

"*A necessary condition for the existence of hidden variables for quantum mechanics is the existence of an imbedding of the partial algebra Q of quantum mechanical observables into a commutative algebra*" (Kochen, Specker 1967: 66).

If one passes from qubits to arithmetical units, following the main idea of Hilbert arithmetic, the commutative algebra meant in the cited sentence is the field of rational numbers, which is a stronger structure (i.e.: satisfying additional axioms) than a commutative algebra. Then, *Theorem 0* (*ibid*.: 67) states that a necessary and sufficient condition that a partial Boolean algebra is imbeddable in a Boolean algebra is that every pair of distinct elements of the partial Boolean algebra can be unambiguously interpreted by a homomorphism as the field of two values, which any variable in a Boolean algebra can possess.

The pairs (namely ratios) of natural numbers (what rational numbers are, therefore satisfying the property of idempotency) constitute a partial Boolean algebra. "What makes a partial Boolean algebra important for our purposes is that the set of idempotent elements of a partial algebra forms a partial Boolean algebra" (*ibid*.: 65). So, FLT(3) allows for restricting only to partial Boolean

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Gerry 2002; Cabello 2000; Hamilton, Isham, Butterfield 2000; Isham, Butterfield 1998; 1999; 2002; Gill, Keane 1996; Peres 1996; Cereceda 1995; Mermin 1995; Lenard 1974.

algebra meaning that the Kochen - Specker theorem is intended to be utilized in order to prove the irrepresentability of the partial Boolean algebra of (quantum) hidden variables onto "Z_2".

One is to demonstrate furthermore that the condition at issue is not only necessary, but also a *sufficient* condition for the existence of hidden variables for quantum mechanics at least as to the case where those hidden variables constitute a partial *Boolean* algebra rather than a partial algebra at all. In other words, one needs the concept of hidden variables and that of the "partial algebra Q of quantum mechanical observables" after translating from the language of qubits into that of arithmetical units if one admits that the translation will reduce the general case of partial algebra to that of partial Boolean algebra: since just the Boolean case is sufficient for the need of FLT(3) to be proved[15]. The scheme of reinterpreting Kochen and Specker's proof in order to be relevant to FLT(3) is to be discussed in detail:

The idea of their proof is to infer a statement from the suggestion that the observables in quantum mechanics are commensurable and then, to demonstrate that the statement at issue is false and consequently, the suggestion that the observables in quantum mechanics are commensurable is not valid. Thus, the availability of hidden variables, which "is satisfied in the statistical mechanical description of thermodynamics" (Kochen, Specker 1967: 64), is reduced to commensurability. Then, the relation of commensurability is formally reduced, being notated by the authors as "♀", and utilized for the definition of a "partial algebra over field K" (*ibid.*) with two cases of interest: "The first is the field R of real numbers and the second is the the field Z_2 of two elements" (*ibid.*: 65).

For proving FLT(3), one needs a *relevant degenerative* realization (and reinterpretation) of all "the values of the polynomials in a_1, a_2, a_3 form a commutative algebra over the field K" (*ibid.*: 64) if the elements of set "A", on the Cartesian product of which with itself the "binary relation ♀ (commensurability) on A (i.e. $♀ \subseteq \boldsymbol{A} \times \boldsymbol{A}$)" (*ibid.*) are defined, are transponed from the qubit Hilbert space ("A") into Hilbert arithmetic in a narrow sense as the class of equivalence of all possible values.

In other words, if the Kochen - Specker theorem means the qubit Hilbert space, which is to be denoted as "A", and "♀" refer to physical quantities, being Hermitian operators, and thus defined just on "$\boldsymbol{A} \times \boldsymbol{A}$", one is to transfer the same description into its dual counterpart of Hilbert arithmetic in a *narrow sense* in the shared framework of Hilbert arithmetic in a *wide sense*.

The result of transporting can be immediately deduced meaning only that the set A (which is the qubit Hilbert space initially since it makes sense to be related to quantum mechanics and to the problem of hidden variables) degenerates now to the set A of all natural numbers, and the "commutative algebra over field K" has in turn degenerated to the field of all rational numbers if

[15] That is: the pair of two natural numbers, to which any rational number can be reduced in the final analysis, constitutes a member of a partial *finite* Boolean algebra. For example, if both natural numbers are represented equivalently in the binary notation system (i.e. only by "0" and "1" as their digits): the set of all rational numbers and the set of all members of partial finite Boolean algebras is the same. As a corollary, if any structure cannot be homomorphic to any partial finite Boolean algebra, it cannot be homomorphic to any structure of rational numbers needing necessarily finiteness, and thus, excludes necessarily any solution of FLT(3) meaning just that finiteness.

the interesting case of the field K is only Z_2: the necessary axioms of that commutative algebra are satisfied of the field of all rational numbers along with a few others, specific for a field, but not necessary for a commutative algebra though consistent with those necessary for it.

Then, as far as a unit in Hilbert arithmetic in a narrow sense originates from a qubit, the Kochen - Specker theorem, now interpreted in the defined above case of the degeneration of qubits into arithmetic units, implies that commutative algebra as to the pair of arithmetic variables y^3, z^3 (in Fermat's equation) that it does not exist: or FLT(3) is true. In other words, the solutions x, y, z of Fermat's equation for $n = 3$ can be only irrational numbers after they cannot constitute the commutative algebra in question.

What remains to be discussed is the meaning of hidden variables if they result into a partial Boolean algebra (rather than to a partial algebra) in order to be proved that the special case is not only a necessary, but also *sufficien*t condition of hidden variables (which can be alleged to be "hidden Boolean variables" if they induce right a partial *Boolean* algebra): then, one can utilize theorem being given (unlike Kochen and Specker, who should deduce it) as to that special case to infer the necessary irrepresentability onto Z_2.

Still one facilitation can be the consideration of the particular case of an arbitrary pair of qubits, belonging to the same qubit Hilbert space, for sharing hidden variables if they constitute a partial Boolean algebra according to the exact meaning of Kochen and Specker (1967: 64). All values of both qubits are naturally idempotent to each other, in virtue of which their partial algebra should be Boolean (*ibid*.: 65). Then, one grants that they generate a partial Boolean algebra in order to infer for them to share hidden variables therefore satisfying the condition, enumerated as "(4)" in their paper, just as "in the statistical mechanical description of thermodynamics" (*ibid*.)

After closely mulling why statistical thermodynamics fulfills it, one understands that it is due to the fact that any hidden variables are to specify or determine additionally a *finite* description complementing it to a wider, but again and necessarily *finite* description. So, if the description is infinite in definition, it might not be complemented in any way since the new description alleged to be wider is again the same as initially in fact. In other words, any infinite description does not admit any *hidden* variables since they have *already meant in advance* according to the definition of what an infinite description is.

Therefore, the way for the condition "(4)" not to be satisfied in quantum mechanics (and unlike thermodynamics) is rather extraordinary just as the general case of an identity which is idempotent trivially and thus it is not able to be idempotent properly, i.e.: nontrivially. Said otherwise, idempotency in a proper sense needs two discernable distinguishable states (for example, as the two alternatives of a bit of information *after choice* and eventually notable as "0" and "1"); on the contrary, if the relation of them is an identity (e.g. as the "coherent state" of those two alternatives *before choice*), one is not able to define idempotency in that proper sense (but it is trivially valid in virtue of the reflexivity of identity, applied twice).

Returning to the particular case meant in the condition "(4)", the extraordinarily of the way for quantum mechanics not to satisfy it consists in the indistinguishability (which can be also and not worse interpreted as a kind of "functional identity") of the Borel function of the "A" observable, denoted by the authors Kochen and Specker as "$g = g(A)$" from "$f_A \equiv \Omega \to R$", (i.e. the real-valued function of all pure states "Ω" of the investigated system). In other words, the same mapping is only represented in two (ostensibly different) ways: by the observable "A" as an argument of the function "$g = g(A)$" in the former case, but as the parameter "A" distinguishing among a family of mappings "$f_A \equiv \Omega \to R$"[16].

On the contrary, if statistical thermodynamics is the case where both operators "g, f" are usual functions quite non-identifiable with each other because they offer two finite descriptions (unlike the case of quantum mechanics where both mean infinite descriptions therefore inherently identifiable with each other), and the condition "(4)" makes clear sense, namely analogical to that of idempotency.

Now, one is already weaponed enough to trace how the relation of commensurability "♀" implies idempotency and therefore hidden variables at least in the case of a partial *Boolean* algebra initiated by the qubit Hilbert space. Though the property "1" postulated in the definition of the relation (*ibid*.) yet admits for it to be a relation of equality (such as that of "identity"), the next property "2" absolutely excludes that forcing "$\forall a, b$: a♀$\boldsymbol{b}$" for "a, b" to be distinguishable from each other and also idempotent in virtue of the definitive property "1".

So, one sees that the initial quantum (or rather, arithmetic, or "quantum and arithmetic") system to which the Kochen - Specker theorem is to be related is sufficient to be much simpler if the objective is defined by the imferability of hidden variables from any granted partial Boolean algebra. The distinguishable states of that system can be generalized to so wider classes that they would be different pairs of natural numbers, i.e.: excluding any *infinite set* of natural numbers and thus all irrational numbers.

Then, the proof that the partial Boolean algebra is also a *sufficient condition* (along with being necessary as Kochen and Specker's proof demonstrates literally in its text) for hidden variables as to the discussed extremely simplified case turns out to be almost obvious: of course, if set "A"[17], on which the definitive relation of "commensurability" is defined, is arithmetically reduced to all natural numbers (please notice: but not to the "*set* of all natural numbers"), it is necessarily *finite*[18]. In other words, the partial Boolean algebra defined on variables within Peano arithmetic is also finite according to the closeness of its operations (in the case, in relation to finiteness). Just the last circumstance necessarily forces the necessity of hidden variables in any initial quantum and arithmetic system under the condition for it to generate the partial Boolean algebra at issue.

Why the violation of the equation "(4)" (being valid in statistical thermodynamics, but not in quantum mechanics), meaning right "hidden variables" implies an "infinite description" of each

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[16] One can immediately notice the close and unambiguous link of that family of mappings with a corresponding functional defined on the set of all "f_A", which is the standard way for the *dual* Hilbert space to be introduced.

[17] One can suggest that the exact kind of the field "K" necessary to define the concept of "partial algebra" is not essential.

[18] That is in virtue of the axiom of induction: (a) "1" is finite; (b) if any natural number "n" is finite, the next natural number by the function successor "n+1" is finite; (c) then, the axiom of induction implies that all natural numbers are finite.

state in general, is explained in detail above. Thus, if that description is finite (as in the case of a partial Boolean algebra over natural numbers), the condition "(4)" is fulfilled, and hidden variables exist. This means that the partial Boolean algebra in the particular case in question is also sufficient for hidden variables.

The introduction of an infinite description rejecting hidden variables in a way maybe extraordinary for common sense therefore breaking and merging both separate mappings "g, f" simultaneously destroys the fundamental organization of any admissible cognition in Modernity after Descrates. Indeed, the interpretation of "g" as belonging to "mind" (and suitable only within it) unlike and opposed to "f" referring to the *objective* states of the system, i.e. to "body" seems to be obvious and natural, at least as to contemporary people.

Thus, involving "*infinite* description" to be a necessary condition in the case of quantum mechanics, this concept serves right for overcoming the opposition of "subject" and "object" being dominating the episteme of Modernity: the cognition of quantum mechanics cannot fit within it. So, a very sophisticated theory is invented to get rid of the modern episteme at least as to quantum mechanics since the kind of its cognition is irrelevant to those frameworks.

The proof of FLT needs an analogical release from the "shackles" of the same restrictions: then the simplest approach is to utilize the "key for those shackles" already created by quantum mechanics and perfected by quantum information: namely Hilbert arithmetic in a *wide sense* including the qubit Hilbert space.

Still a few notices can make clearer the following statements referring to the application of the Kochen - Specker theorem after degenerating it to Hilbert arithmetic. The problem of hidden variables is to be related even to a single quantum quantity (respectively a single Hermitian operator: this means, to the Cartesian product "$\boldsymbol{A} \times \boldsymbol{A}$" if "A" is the qubit Hilbert space). One may test that by the case of statistical thermodynamics justifying hidden variables. Indeed, if one considers any single phenomenological thermodynamic variable (such as pressure, temperature, volume, etc.), it admits the existence of relevant hidden variables (e.g.: such as masses, positions, velocities, accelerations of Boltzmann's "atoms") and their probability (density) distributions resulting into thermodynamic phenomenological observables by means of the average quantities (integrals) of those probability (density) distributions.

If one researches the formal and mathematical reason for the disappearance of any option of hidden variables passing from a phenomenological quantity in statistical thermodynamics to that in quantum mechanics (rather or too loosely speaking, because this is far from the subject of the present article), it consists in involving the Cartesian product "$\boldsymbol{A} \times \boldsymbol{A}$" for a Hermitian operator as any quantum quantity is defined.

A phenomenological thermodynamic quantity means a set "A" therefore allowing for some "hidden variable(s) B" such that the relevant Cartesian product can be "$\boldsymbol{A} \times \boldsymbol{B}$" in a consistent way. This means that statistical thermodynamics implies for both tuples of variables, though those in the former list "A" are alleged to be explicit unlike all physical quantities in "B", which are framed to be "hidden", are *finite*.

On the contrary, quantum mechanics is so “cunning” to postulate for “A” to be infinite and then, also for “$\boldsymbol{A} \times \boldsymbol{A}$” to be a sufficient condition for whatever claims to be a “physical quantity” (excluding maybe that of time). Then, any finite tuple “B” of Einstein’s “hidden variables” can be interpreted to have been in advance written in “A” once it had been granted to be an *infinite* description.

Somebody might call the solution of quantum mechanics in relation to the absence of hidden variables “fraudulent” (at least to common sense). Independently of its “moral” estimation, all experiments about the phenomena of entanglement confirm it again and again: thus and particularly, being consistent with the Kochen - Specker theorem.

One might say that the intention of utilizing the theorem for proving FLT(3) borrows the same kind of “trickery” from quantum mechanics in order to apply it in arithmetic, for such a “doubtful” purpose generalized to Hilbert arithmetic before that. Without the irrelevant ethic evaluation, that is the case in fact. Just the Kochen - Specker theorem as well as the logical pathway to be proved are the tool able to merge both contexts: that of FLT, though initially for FLT(3), with the absence of hidden variables in quantum mechanics interpreting the latter as that incommensurability embodied in the concept of irrational number and thus, relatable to FLT.

One can conclude that both conjecture of hidden variables and FLT to be proved during Modernity share the same kind of misunderstanding or said more exactly, “misframing”. The irrelevant cognitive reference frame pays attention only to the misleading distinction of explicit versus hidden variables rather than to the essential one: infinity versus finiteness. Analogically, FLT tried to be either proved or rejected: that is again a misleading opposition as Yablo’s paradox interpreting FLT demonstrates immediately. The correct framework of its solution turns out to be similar to that in the former case: infinity versus finiteness.

All physicists (in the former case) as well as all mathematicians (in the latter case) “shut up and calculate”. However, what they should do for resolving both problems is just the opposite: “stop calculating: to think”. The episteme of Modernity suggests a quite irrelevant reference mental frame for both granting for it to be imperative (or said philosophically, the condition of possibility of any thought in Modernity) for whoever, calculating and not wishing to think. On the contrary, the solution of both problems needs re-Gestalting, i.e. thinking rather than calculating.

Summarizing, the tool for utilizing the Kochen - Specker theorem in order to prove FLT(3) is absolutely ready: the absence of hidden variables in quantum mechanics implies that the corresponding partial Boolean algebra is not able to be embedded in “Z_2” under the meant particular additional conditions. The most fragments are literally available in their paper; what is necessary to be added is only that the partial Boolean algebra is also *sufficient* for the availability of hidden variables if one means Hilbert arithmetic in a narrow sense.

One can repeat quite concisely the idea of the proof of FLT(3) by the Kochen - Specker theorem (suggested above) once the necessary tool is already elaborated thoroughly. One writes Fermat’s equation for the case $n = 1$ possessing an unambiguous solution “x” for any pair of natural numbers “y, z”, now in Hilbert arithmetic, where all units are defined as “empty qubits”, and thus as three-dimensional unit balls. Then, it can be interpreted as a sum of three-dimensional volumes in relation to those unit radiuses by an exponent of "3" corresponding to the volume of a three-dimensional unit ball in turn relative to an "empty qubit", to which FLT(3) is right to be proved.

In other words, if FLT(1) stating solutions of Fermat's equation (sometimes notated as "FE' further) in natural numbers and unlike FLT(3) rejecting those solutions is reinterpreted in qubits and thus, as FLT(3), just the reinterpretation can be investigated as the reason causing the opposite meaning of the two propositions to each other, namely the Kochen - Specker theorem is relevant to FLT(3) rather than to FLT(1) in virtue of that "tool" being a corollary from it.

Speaking quite precisely, FE(1) where all units are empty qubits is what does no solution if one means radiuses of the corresponding unit balls (rewritable as FLT(3) in relation to the radiuses), but nonetheless the same FE(1) possesses solution if each qubit is considered "globally" (rather than "locally" as in the former case), i.e. as an *inseparable wholeness*, that is as if observed "outside" (rather than "inside" as in the former case).

Thus reflecting philosophically, what can be also proved formally and logically on the basis of the Kochen - Specker (therefore interpreting it indirectly as well), FLT(3) demonstrates that the transition from external, i.e. meant by FLT(1), and internal viewpoint, by FLT(3) is necessarily discrete or the corresponding units (just as a radius of a ball and the volume of the ball itself for the coefficient "π") cannot satisfy the relation of commensurability "♀", and thus in particular, to constitute a partial algebra or to share any hidden variables.

This means that they are inherently incommensurable to each other, which is already meant if the coefficient "π" is imperative to be involved. Properly, the tenet in the present paragraph is philosophically sufficient to justify FLT(3) as a corollary from the Kochen - Specker theorem. Anyway, one can added a few more formal and logical steps for who (if any) need "shut up and calculate" rather than reflect:

The "tool" elaborated above by means of the Kochen - Specker theorem states in the final analysis that the description of any system by Hilbert arithmetic in a narrow sense (i.e. where each empty qubit is interpreted as an inseparable wholeness or as a usual arithmetic unit) is incommensurable with the description of the same system in the qubit Hilbert space, i.e. constituting Hilbert arithmetic in a wide sense together with Hilbert arithmetic in a narrow sense (where each empty qubit is supplied with the internal structure of an "empty" unit three-dimensional ball).

Then, an analogical incommensurability can be also inferred from the former, meant in the above sentence, now in relation to any pair "y^3, z^3" of variables of FE(3) since they both can be considered as "Z_2" and thus satisfying literally the statement of the "tool" that the absence of hidden variables also in the particular case of Hilbert arithmetic in a narrow sense implies not to be embeddable in "Z_2" and particularly in any pair of "y^3, z^3"

What deserves to be noted instead of a conclusion of the present section is the two opposite interpretations of an empty qubit, namely external and internal as well as the transition between them, furthermore necessary for realizing the FLT(3) by the Kochen - Specker theorem as above, clarifies the kind of incommensurability relating the arithmetic world, as in FLT(1), with the geometrical or physical world, as in FLT(3), in the shared quantum neo-Pythagorean framework. This can clarify the philosophical significance of proving FLT(3) by involving a relevant correspondence of the physical and mathematical worlds, therefore transcending the gap being fundamental in our organization of cognition, or Foucalt's "episteme" (1966).

Furthermore, the two-directional transition "external - internal" implemented in the description of a system whether mathematical or physical or philosophical (discussed e.g. in Penchev 2021 June 8) underlies transcendentalism both as philosophical and as scientific (Penchev 2020 October 20). The Kochen - Specker can be directly related to that transition once it has been in advance utilized in the context of FLT (as in the present paper). The introduction of the separable complex Hilbert space of quantum mechanics and especially, that of the qubit Hilbert space of quantum information supplies the researcher with a general tool for investigating it since the main problem of quantumness at all belongs to the same class.

XI. FLT(3) BY THE IMPOSSIBLE DISJUNCTIVE DIVISION OF A QUBIT INTO TWO ABSOLUTELY INDEPENDENT PARTS

The Kochen - Specker theorem proves the absence of hidden variables in both cases of non-commuting (also proved yet by von Neumann in 1932, though in another way) and commuting Hermitian operators (respectively quantities in quantum mechanics). Just the latter (being the proper contribution of the authors in 1967[19]) is the case of entanglement, and particularly, the violation of Bell's inequalities (1964) is a sufficient condition for it.

As yet Einstein, Podolsky, and Rosen (1935) or Schrödinger (1935[20]) paid attention, even complementary quantities (linked by entanglement and corresponding non-commuting Hermitian operators[21]) can be simultaneously measured for a unique physical action (called by Einstein a "spooky action at a distance"), corroborated experimentally many, many times by contemporary physics and named "entanglement".

Its essence consists in the absence of space-time localization within any quantum system if the exact momentum-energy localization is granted, e.g. by virtue of the conservation laws of energy or momentum. Then, arbitrarily remote physical entities can be considered as a single physical system, the probability distributions of their namesake quantities can influence each other, that influence can be predicted exactly, e.g as the violation of Bell's inequalities (a sufficient, but not

[19] The relation of statements and proofs of the two theorems are not investigated enough though Kochen and Specker commented Neumann's paper in theirs; maybe undeservedly neglecting Neumman's proof (one of a few recent articles dedicated to it is: Dmitriev 2005).

[20] This is the paper of "Schrödinger's cat", the famous metaphor, became the emblem of quantum mechanics. Unlike the negative attitude of Einstein, Podolsky, and Rosen to that "spooky action at a distance", Schrödinger referred to those "verschränkten Zustände" as he called them, as to a still one corollary challenging common sense, but quite reasonable. Anyway, almost all authors citing Einstein, Podolsky, and Rosen's paper, with its polemical and even militant tone towards quantum mechanics, neglect the absolutely independent contribution of the Austrian physicist.

[21] The thought experiment suggested by Einstein, Podolsky, and Rosen (1935) can immediately illustrate how two complementary quantities mediated by entanglement and a relevant conservation law can share a certain part of their probability distribution: then the eventual change of that share probability distribution being the same in both cases is described by the same operator commuting with itself necessarily.

necessary condition for entanglement), and then confirmed experimentally. It is due to the overlapping (respectively sharing) of the same possible and thus measurable values of some namesake quantity independently of any spatial distance between the physical entities, to which it refers: indeed a "spooky action at a distance" in Einstein's words and contradicting common sense, classical physics or special and general relativity prohibiting any interaction claiming to be physical to propagate with a speed exceeding that of light in a vacuum.

Einstein, Podolsky, and Rosen (1935), after inferring that "spooky action" action from the mathematical formalism of quantum mechanics (also deducible even only from the separable complex Hilbert space underlying quantum mechanics), used it to prove the alleged (by them) incompleteness of quantum mechanics. Niels Bohr (1935) immediately answered to them that, on the contrary, quantum mechanics is complete nonetheless[22].

However, the precise proof of Bohr's statement is only the Kochen - Specker theorem justifying the absence of hidden variables also in relation to quantities, which are not complementary to each other, and accordingly, their corresponding Hermitian operators commute[23]. That circumstance refers directly to the proof of FLT(3) by the Kochen - Specker theorem, discussed in the previous *Section X*, in the following way:

The proof meant only the case of incommensurability, and thus, that of non-commuting Hermitian operators. Consequently, one can consider also the application to FLT(3) after "entanglement", i.e. if the corresponding operators commute: then the case is used to be notated as "quantum contextuality" consisting in the fundamental inseparability of any quantum system into non-interacting (in general substems). Particularly, a qubit cannot be cut into two not-linked parts, or respectively qubits, as a corollary from the Kochen - Specker theorem. The same observation can be visualized directly and rather elementarily by the definition of a qubit as $1\ qubit \equiv \alpha \mid 0\rangle + \beta \mid 1\rangle$ where α, β are two complex numbers obeying the condition: $\mid \alpha \mid^2 + \mid \beta \mid^2 = 1.$

Then a qubit can be constituted by the normalization of any two subspaces of Hilbert space and particularly (as an example sufficient for visualizing), that of two successive "axes" of the separable complex Hilbert space, to which a certain wave function establishes arbitrary complex coefficients C_n, C_{n+1}. Indeed:

$$\alpha = \frac{C_n}{\sqrt{|C_n|^2 + |C_{n+1}|^2}};\ \beta = \frac{C_{n+1}}{\sqrt{|C_n|^2 + |C_{n+1}|^2}}$$

Then, the qubit cannot be divided into two absolutely separated "parts" such as $C_n \mid 0\rangle$, on the one hand, and $C_{n+1} \mid 1\rangle$, on the other hand, otherwise than as two unambiguously determined additive members of a qubit (i.e. $\alpha \mid 0\rangle + \beta \mid 1\rangle$), therefore correlating necessarily just in virtue

[22] Plotnitsky (2010) discusses the quantum-theoretical shift of Bohr, Geisenberg, and Schrödinger's thinking involving probability in epistemology. Longair (2013) considers the probabilistic and indeterminist Gestalt change forced by quantum mechanics as the only alternative to be established as both objective and experimental science in the first decades of its existence.

[23] Maybe, the term of "partial commutability" would be more precious in a sense since commutativity relates only to the shared sub-operator due to entanglement (that sub-operator refers only to the correlating values).

of the mutual normalization involving the counterpart coefficient (i.e. C_n for C_{n+1} or C_{n+1} for C_n. The lesson is:

The introduction of probabilities as $|\ \alpha\ |^2$ or $|\ \beta\ |^2$ can be interpreted under the condition $|\ \alpha\ |^2\ +|\ \beta\ |^2 = 1$, which needs the normalization and thus unambiguous mutual correlation of any two "parts" of a qubit such as two qubits particularly. In other words, quantum contextuality (rigorously established by the Kochen - Specker theorem) can be visualized well enough by the necessary correlation of any two qubits, into which another one given in advance is suggested to be divided.

By the help of that visualization, one can further trace the sense in which Kochen and Specker's absence of hidden variables also means quantum contextuality in comparison to the alternative case of incommensurability: rather counterintuitively, they turn out to be equivalent. The difference consists only in the opposite interpretations of the same fact observed from the following two viewpoints: either *external* (resulting into incommensurability) or *internal* (resulting into quantum contextuality)[24]. The visualization can be still perfected as follows:

The division of a qubit into two parts, particularly, into two qubits, can be considered as a bit of information as whole, where the "two parts" are two alternative states of a "bit" meaning the initial qubit, which is to be halved. Then, the impossibility of that (which quantum contextuality implies) suggests only the statement that "no qubit is a bit", a rather trivial and obvious since a qubit is a choice among an infinite set of alternatives, and a bit, among two ones, which are a finite set. In other words, quantum contextuality visualized by the relation of "qubit" and "bit" as the statement "no qubit is a bit" can be reduced to be a corollary from the statement "no infinite set is a finite set"[25].

The general case meant by that visualization can be described also in terms of the nonstandard bijection as it is applied to Peano arithmetic: $(P^{-} \otimes P^{+} \rightarrow P^{0}) \rightarrow P$ above. Indeed, if it is granted to be a bijection, the converse statement is to be also valid, i.e. $P \rightarrow (P^{0} \rightarrow P^{+} \otimes P^{-})$[26]. Meaning also the visualization by a bit, the two directions of the nonstandard bijection can be interpreted as two directions of a bit of information (e.g. recording either "0" or "1" in a binary sell versus erasing either "0" or "1" in the same sell) or to be dual to each other. The last observation can serve as a formal non-contradictory definition of "nonstandard bijection" therefore rigorously distinguishing it from the usual "standard bijection": its two directions are *dual* to each other unlike those of "standard bijection" being *simultaneously* valid.

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[24] So, the relation of externality and internality, fundamental for scientific transcendentalism, is involved to verify the unity of quantum incommensurability and quantum contextuality, as the proper philosophical contribution of the Kochen - Specker theorem can be also interpreted.

[25] In fact, Skolem's "relativity of set" (1922) considers as relative also finite and infinite sets, which can be granted to be representable as dual in the context of Hilbert arithmetic: so the statement "No infinite set can be finite" can be complemented further by the statement that "they can be dual to each other". Just this precision allows for the Schrödinger equation not to be contradictory if it is interpreted to equate classical information (after measurement) to quantum information (before measurement) in the final analysis. The "duality of finiteness and infinity" will be also introduced in the present context a little below.

[26] One can pay attention to the "reverse" Cartesian product, which formally represents the duality of both directions of the bijection.

Once those perfections have been involved, both incommensurability and contextuality in virtue of the Kochen - Specker theorem can be unambiguously assigned to the two dual directions of the nonstandard bijections therefore suggesting for them to be dual to each other in turn. This means that incommensurability (for conjugate quantities or non-commuting Hermitian operators) and contextuality (for entanglement or commuting Hermitian operators) are to be considered as dual to each other in the virtue of the Kochen - Specker theorem. Then, the unambiguous correspondences at issue are:

"Incommensurability": "$(\{P\}^{-} \otimes \{P\}^{+} \rightarrow \{P\}^{0}) \rightarrow \{P\}$" refers to the one direction of the nonstandard bijection "$(P^{-} \otimes P^{+} \rightarrow P^{0}) \rightarrow P$".

"Contextuality": $\{P\} \rightarrow (\{P\}^{0} \rightarrow \{P\}^{+} \otimes \{P^{-}\})$ refers to the other direction of the nonstandard bijection $P \rightarrow (P^{0} \rightarrow P^{+} \otimes P^{-})$

The bracketing by "{}" notates that corresponding Peano arithmetics and natural numbers meant by them are interpreted as sets by the corresponding natural numbers therefore necessarily involving the Gödel incompleteness of set theory to arithmetic, in turn implying both incommensurability and contextuality, otherwise proved in the framework of th Kochen - Specker theorem.

An amazing symmetry appears after proving FLT by the Kochen - Specker theorem involving: one needs FLT(3) by virtue of whether incommensurability or contextuality since both follow from the Kochen - Specker theorem, on the one hand, and after that, by induction in a relevant form of induction for the general case of FLT($n \geq 3$), on the other hand. That circumstance has been already discussed in *Part I*, coining the term of the "paradox of proving FLT", meaning that the proof should involve both P and {P} and thus Yablo's paradox or the Gödel incompleteness by virtue of which FLT turns out to be an insoluble statement in Gödel mathematics therefore forcing necessarily reflection e.g. by Hilbert arithmetic or by Fermat arithmetic if one wishes a rigorous proof of FLT, in the framework of either of which (or other relevant framework) it is possible.

Discussing the proof of FLT in Fermat arithmetic by both MMT and MFD in *Part I*, an exchange of the order of "P and {P}" into "{P} and P" was discussed as necessary for the proof. The same exchange can be observed also now: after proving it by means of the Kochen - Specker theorem. Both P and {P} are necessary for the proof but only the order "first {P}, then P" is relevant, though being counterintuitive, rather than the alternative "first P, then {P} right generating the aforementioned "paradox of proving FLT".

Both (external and internal) viewpoints to the same fact, namely the two "parts" of the a single qubit, observed correspondingly either as incommensurable or as contextual hints at a new invariancy linking the externality and internality of a system in a philosophical sense or their quantitative descriptions mathematically including those relevant to the most fundamental theory of contemporary physics: quantum mechanics and general relativity.

A previous paper (Penchev 2021 June 8) exploits just that new kind of invariance to explain and infer corollaries from it about the relation of quantum mechanics (and quantum information) and general relativity, partly contradicting each other, e.g. as to exceeding / non-exceeding the fundamental constant of light speed in a vacuum. Its thesis is that quantum mechanics and general relativity describe the same, but each of them, from the correspondingly external and internal viewpoints at issue to the same system being investigated.

Particularly, their descriptions are complementary to each other (respectively, dual in a rigorous mathematical meaning): so, that conjecture allow for all troubles about "quantum gravity" to be explained since its subject consists of two complementary essences describing absolutely different only seemingly because both mean the same, but from those two viewpoints ostensibly inconsistent to each other.

The concept and theory of quantum information allows for their unification, starting from the viewpoint of quantum mechanics. Then, entanglement is linked to gravitation as its quantum, but necessarily complementary (respectively, mathematically dual) counterpart. On the contrary, a new generalization of "reference frame" is necessary if one shares initially the viewpoint of general relativity: "external reference frame" (or "discrete reference frame") is coined for it meaning accordingly a new "still more general principle of relativity" ruling the transition between any two reference frames being external (or discrete) to each other.

Indeed, the *only* "general principle of relativity" underlying general relativity (unlike the still the more general principle of relativity necessary for establishing the new invariance of both externality and internality of a mechanical system) allows for uniformly describing *only* reference frames moving smoothly to each other. This means that the still more general principle of relativity needs the uniform description to any two discrete (external) reference frames including, therefore allowing for any quantum entities to be described equally well in terms of general relativity if it has been generalized in advance as to external "reference frames". Once this has been done, the result is a description coinciding with that by utilizing entanglement, quantum information: or starting from the "other end" of quantum mechanics.

If hypothesis that quantum mechanics and general relativity have been granted to be two complementary theories of the same, both quantum information as a theory of entanglement and that "still more general relativity" as a theory able to include external reference frame can be identified both to each other and to the eventual theory of quantum gravity (however particularly excluding the pattern of "secondary quantization" inherent for the rest three fundamental interactions: weak, strong and electromagnetic[27]).

Then the unity of both externality and internality of a system suggest that one may construct a bridge able to link quantum information and the proof of FLT by the mediation of the Kochen - Specker theorem relevant to both. One might transfer concepts from the one area into the other: for example, the Gödel dichotomy of the relation of arithmetic to set theory, "either incompleteness or contradiction" (already proved to be relevant to FLT by means of Yablo's paradox in *Part I*) to be interpreted also to the pair of general relativity and quantum mechanics:

[27] One can notice that electromagnetic interaction allows for a description following the model of "still more general relativity" (by virtue of the fact that Minkowski space is a particular case of pseudo-Riemannian space) along with that of the Standard model as a "quantum force". However, that approach is inapplicable to weak or strong interactions (at least, immediately and literally).

Oppositely to Einstein's suggestion about the alleged incompleteness of quantum mechanics, general relativity turns out to be either incomplete or contradictory to quantum mechanics. Only if one complements it by a dual counterpart able to describe mechanical motions even to "external reference frames" uniformly, it is able to become complete and complementary to quantum mechanics, being also necessarily accompanied by quantum information to be complete.

The discussed relation after division into two parts of a qubit in virtue of the Kochen - Specker theorem (also demonstrated to be relevant to the proof of FLT above) can serve for the relation of quantum mechanics to general relativity to be visualized. The corollary at issue will be related only to the area of "quantum gravitation"[28] studied by external (discrete) reference frames, on the one hand, and by quantum information as entanglement, on other hand. Both viewpoints can be unified by halving a qubit: then, the viewpoint of the two parts as incommensurability will correspond to the conceptual construction of general relativity relevant generalized, and the alternative viewpoint of contextuality, to quantum mechanics (respectively, quantum information).

XII. ABOUT THE IDEA OF "ARITHMETIC MECHANICS" AFTER PROVING *FLT* IN HILBERT ARITHMETIC

Common sense's prejudice to infinity is that: to be "more" than finiteness, therefore the former is to be ordered as "second", i.e. after the latter as "first". Anyway, just that sequence of them generates the Gödel incompleteness if arithmetic and set theory are correspondingly attached[29]. Their opposite ordering (as well as their ordering in parallel), on the contrary, forces to be discussed as complementary to each other suggesting rather the dual structure of Hilbert arithmetic even where it is "naively" or in an inexperienced way reduced to Fermat arithmetic. Just this is the approach for proving FLT by induction, discussed in *Part I*, for preventing it to be a Gödel insoluble statement.

Thus, "quantum gravitation" investigated after the eventual complementarity of general relativity and quantum mechanics shares the same "reverse" sequence of finiteness and infinity, resulting furthermore into the complementarity at issue. The syncretic consideration of mathematics and physics in their foundations (naturally inherent for "quantum neo-Pythagoreanism" advocated here) can be continued, for example, to an arithmetic or set-theoretical justification of fundamental physical constants such as the Planck, "light", and gravitational constants. Though it is to be a subject of a future study, its outlines might be also mentioned in the present context:

What can be researched is their mathematical origin or necessity to exist rather than their certain values well-known to us as an empirical fact. In other words, the reason why their values

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[28] The term is within quotation marks since it is meant only in the present context, after which quantum mechanics and general relativity are complementary to each other, on the one hand, and the area as a whole is considered to be dual to the complement of arithmetic to set theory, where all structures relevant to the Gödel dichotomy are to be situated, on the other hand.

[29] Anyway the Gödel incompleteness can be overcome also in the framework of the same "prejudice" by involving transfinite induction as, for example, Caporaso, Pani (1980) demonstrate.

are just those[30] cannot be revealed by the sketched approach. Only the class of equivalence of all possible values for each of them can be discussed. Furthermore, they are able to generate basic physical dimensions of the world by means of the unambiguously determined Planck length, mass, and time interval. Analogically, their certain, "Planck" values do not admit any relevant discussion in that framework, however the mathematical origin of the corresponding mathematical dimensions may be in turn traced:

Then, the relation of arithmetic needs just the following three mathematical constants able to be attached unambiguously to the three fundamental physical ones. Those can be called: the unit of actual infinity; the unit of arithmetical induction (or that of function successor[31]); their relation (being uncertain in virtue of the inconsistency of the axioms of induction and infinity): it can be also likened to be the "ratio" of a qubit (usually defined) to a measured qubit (thus restricted only to rational values, and to natural numbers in the final analysis).

Indeed, arithmetic grants for its least element and the constant of the function successor (utilized in the axiom of induction) to be the same ("1"), and the third one can be defined only in Hilbert arithmetic rather than in Peano arithmetic since it represents the relation of the corresponding units in the two dual counterparts of Peano arithmetic distinguishable from each other only in Hilbert arithmetic.

On the contrary, the concept of actual infinity implicitly establishes the class of all possible units and even determines a series of them such as that of enumerability, continuum, etc. as well as the famous "continuum hypothesis" (proved to be an independent axiom of set theory by *Gödel 1940*, consistency, and by *Cohen 1963*; *1964*, independence) in terms of them. However, the axiom of choice implies their "relativity in a Skolemian manner" (Skolem 1922) thus generalizing their "units" to a single one, that of actual infinity.

The discussed exchange in the order of finiteness and infinity implies for that unit of actual unit to be identified as the least unit which is to be a unit, "1", again and then identified to the aforementioned (coinciding) units of induction and function accessor. So, one tends to generalize arithmetic if it is considered as the particular case where the three "fundamental mathematical constants", namely the unit of the least element, the unit of induction (function successor), and the unit of their relation (or the ratio of the units of the two dual arithmetics) coincide with each other: all being conventionally notable as "1".

On the contrary, if one admits for them to be different from each other, each of them can be further identified with just one fundamental physical constant in turn implying the three basic

[30] A hypothesis to be just "ours" suggested Dirac (1937; 1038), and it is exploited (maybe too widely and closely) in "anthropic principle" mentioned for the first time by Dicke (1957), a rather philosophical or metaphysical conception.

[31] The addition of an unit, i.e. "n+1" for any natural number "n" is postulated in the axiom of function successor, and then used as a condition in the axiom of induction. The natural admission for them is to be identical; and just that is the case which will be researched as to the correspondence with the fundamental physical constants. However, one can allow for two different "additions of a unit" therefore noy coinciding with each other not less consistently. If that were the case, which is relevant to the fundamental mechanical constants, still one had to exist besides those three ones (Planck, "light", and gravitational). Anyway, the hypothetical admission that it exists, but is yet unknown for us cannot be rejected *a priori*.

physical dimensions of mass, distance, and time: they are enough for classical mechanics. Then, the unit of the least element is the Planck constant; the unit of induction (function successor) is that of the light speed in a vacuum, and their relation corresponds to the gravitational constant[32]. The last statement seems to be rather unobvious and it might be loosely and briefly justified as follows though its rigorous proof in detail is to be the subject of a future study:

A necessary preliminary notice is to clarify why and how the fundamental physical constants are units of c*ontinuous* physical quantities, but the arithmetical unit(s) even granted to be three different refer(s) to *discrete* quantities, what all arithmetic variables are. Their unification or identification is only from the viewpoint of quantum mechanics (and articulated especially discernibly by quantum information) since it is forced to describe uniformly the *macroscopic* apparatus, as it is in terms of the continuous motion of classical mechanics, and the *microscopic* quantum entities obeying discrete laws due to the Planck constant and its magnitude commensurable with their physical actions. Then, quantum mechanics introduces the separable complex Hilbert space as its basic mathematical structure, then evolving into the qubit Hilbert space of quantum information and into Hilbert arithmetic in a wide sense.

The concepts of general relativity need a parallel development, e.g. the "still more general principle of relativity" valid also to discrete (external) reference frames advocated in the cited previous paper (Penchev 2021 June 8) to be analogically applicable to both continuous and discrete physical variables. So, all those preliminary considerations are granted in advance to the idea of "arithmetic mechanics" investigating the roots of the three fundamental physical constants in a Pythagorean manner, but rigorously and mathematically. After that, one can justify the identifications of all three pairs of constants as follows:

The least element of arithmetic (due to its well-ordering) *as the Planck constant*: indeed, it must be a constant not admitting any values of physical action less than it in virtue to be the least element of arithmetic.

The unit of induction (function successor) *as the constant of light speed in a vacuum*: the ratio $\frac{n+1}{n}$ as a constant can be interpreted to be a paraphrase of the necessary condition for the validity of induction and translated in a continuous quantity such as velocity. As the axiom of induction implies for any natural number to be finite, as the "light" constant implies for any physical distance to be finite. As the axiom of induction contradicts the axiom of infinity in set theory, as the non-exceeding of the "light" constant contradicts all phenomena of entanglement happening "instantaneously" (i.e. exceeding the "light" constant) in definition.

The relation of the former two constants, i.e. that *of the least element of arithmetic and the unit of induction corresponds to the gravitational constant*. That statement is the most sophisticated and thus most difficult for justifying. It is again a unit, as the ratio of two units, in Peano arithmetic. However, those last two units are different from each other in the generalization of Peano arithmetic relevant to arithmetic mechanics, on the one hand, and much more important, the

[32] A very interesting observation is that three fundamental physical constants originate even from arithmetic alone, but only after the identification of the set-theoretical constant of actual infinity as the arithmetic instance of the unit of the least element.

contradiction of the axiom of induction to the axiom of infinity (if the latter is "pre-ordered" and involved indirectly by the Planck constant) implies for their ratio to be uncertain and needing an additional own specification as an absolutely independent, third fundamental physical constant, on the other hand.

It may be also recognized as the "Planck length" ($l_0 = \sqrt{\frac{\hbar G}{c^3}}$) depending on the gravitational constant after the other two constants (the Planck and "light" constants) participating in its definition have been already verified in advance as above: the Planck length can serve just for elucidating the sense and meaning of the gravitational constant.

Three circumstances need relevant clarification as to the cited formula of the Planck length, namely: "$\sqrt{\ }$"; the exponent of "3" for the "light constant"; and finally, whether the Planck length can be identified as a new constant corresponding to that of the function successor and thus absolutely independent of the other arithmetical constant of induction (with the physical counterpart of the "light" constant); only in a few words about each of those circumstances:

A shared preliminary notice may be that all the three circumstances need also Hilbert arithmetic in a wide sense to be made clear. Anyway, it can be abandoned similarly to a "Wittgenstein ladder" if one grants to distinguish the constants of induction and function successor as different from each other after elucidating the last, third problem:

About "$\sqrt{\ }$": the formulas of all Planck units, and particularly that of length, share the same peculiarity, to be determined by "$\sqrt{\ }$", and consequently mean two values: "$\pm\sqrt{\ }$". Each of them can be associated with the one of the two dual Peano arithmetic if they are connoted in a wide sense in virtue of their idempotent anti-isometry. So for example, $+l_0$ can be conventionally related to P^+, and $-l_0$, to P^+ accordingly.

About the exponent of "3" for the "light constant": if the reduced Plank constant, "$\hbar$" has been already associated with the least arithmetic element, it is meant as a qubit, i.e. as a unit ball in the usual three-dimensional Euclidean space. Then, it should correspond to an also three-dimensional unit of induction, i.e. "c^3" (for "c" in each of all the three dimensions). If the axiom of induction (to which the "unit of induction" refers) and the axiom of infinity (to which a qubit or respectively, the least arithmetic element refers) were consistent to each other (i.e. if the Gödel dichotomy were not inferable), the relation of a qubit to its measurable counterpart, and particularly, the ratio "$\frac{\hbar}{c^3}$" would be unambiguously determined without needing an additional constant (which is the gravitational constant) for being absolutely defined. In other words, just the Gödel incompleteness in fine, if one means it in arithmetic mechanics, implies the gravitational constant (but not its certain value in our universe) to overcome that incompleteness.

The ratio "$\frac{\hbar}{c^3}$" interpreted once as the ratio of two arithmetic units, the least element to the unit of induction (both granted to be "1" in Peano arithmetic), can be further interpreted also as corresponding to the relation of a qubit to its measurable value. Indeed, while the value of a qubit is a pair of irrational numbers in general, its measured value is always reduced to a rational number (which is a ratio of natural numbers). Then, "c^3" corresponding to the "three-dimensional unit of induction" refers just to the class of equivalence of all measurable values, i.e. to an "empty qubit being already measured".

If the Gödel incompleteness were not valid, the uncertainty of quantum measurement would not exist. It is overcome in any measurement by a fundamentally random choice corresponding to the axiom of choice which is able to consistently regulate the relation of set theory to arithmetic. Analogically. the gravitational constant (and consequently, gravitation as the interaction obeying it) determines additionally and unambiguously quantum uncertainty, transforming it into an absolutely certain values which is measured, by a certain state of entanglement of the measured quantum entity *with all the universe* as far as entanglement is the dual counterpart of gravitation. So. the physical "picture of the world" becomes perfectly consistent only in arithmetic mechanics.

About whether the Planck length *can be identified as a new constant corresponding to that of the function successor and thus absolutely independent of the other arithmetical constant of induction* (with the physical counterpart of the "light" constant): indeed, it can if one wish to avoid the mediation of Hilbert arithmetic to Peano arithmetic though generalized by arbitrary values of the "arithmetic constants" (all granted to be "1" in Peano arithmetic).

In other words, one can introduce the Planck length as the unit of the function successor and therefore postulating for it to be distinguishable from the unit of induction since their corresponding axioms are independent of each other in the framework of the axiomatics of Peano arithmetic. Anyway, this is rather a formal permission or option, but the motivation is due to the Gödel incompleteness properly implying an additional "degree of freedom" right realizable or embodible into the independence of the units of function successor and induction of each other. Nonetheless, that motivation can remain hidden or only "hinted", and Hilbert arithmetic, from which it originates, to be abandoned as a "Wittgenstein ladder".

All the three meant theorems, namely: FLT, the Kochen - Specker theorem, and Gleason's theorem, would be fundamental to the intended arithmetic mechanics because they make clear why a qubit is a natural unit of arithmetic mechanics after it has been rather conventionally postulated to be that in the basis of Hilbert arithmetic in a wide sense by defining of an arithmetic unit as an "empty qubit". As far as the concept of "qubit" is irrelevant to the traditional or standard context of FLT, one can speak of the uniqueness of the case of three dimensions, articulated differently in each theorem as follows:

FLT means the uniqueness of the exponent "3" in Fermat's equation, which is the least one for which it is not soluble. Gleason's theorem determines the dimension "3" of Hilbert space as the least one for which the probabilistic measure is unique. As to the Kochen-Specker theorem, the considered corollary about "halving a qubit" demonstrates directly the qubit as the structure of least dimensions. One can speak of the absence of hidden variables in the sense of the theorem.

A more philosophical justification of the special case "3" is suggested in the introduction of the paper in *Part I*. If one relates "idempotency" and "hierarchy", "3" is the least element able to distinguish them for belongs to the latter, but does not to the former. The same observation can be interpreted even ontologically and transcendentally as far as transcendentality means implicitly idempotency by the identification of the externality and internality of the totality. Then, the case of "3" meant fundamentally what the least, which does not refer to idempotency, is, however partly paradoxically: it is able to feature and distinguish the totality.

Furthermore, one can speak of "idempotency" and "hierarchy" abstractly and philosophically as the two properties (or relations), which are necessary and sufficient to represent the meta-relation of "completeness" meant by "the totality", on the one hand, and the transcendental method, on the other hand. The approach of "scientific transcendentalism" (e.g. as in: *Penchev 2020 October 20*) tends to represent the traditional philosophical transcendentalism in a rigorous, mathematical and logical, falsifiable and non-metaphysical way, after which to interpret the completeness of quantum mechanics in virtue of the theorems about the absence of hidden variables as an example of that scientific transcendentalism, on the one hand, and then, to extrapolate it in relation to the problem of completeness already as to the foundations of mathematics, on the other hand.

Following the same impetus originating from scientific transcendentalism, Hilbert arithmetic in a narrow sense as well as in a wide sense is introduced in order to unify the problem of completeness and its solution in both quantum mechanics and mathematics forcing a kind of quantum neo-Pythagoreanism, to which Husserl's phenomenology can be considered to the "closest relative" within the philosophical tradition.

The idea of arithmetic mechanics (though only loosely sketched here) is a next step for implementing scientific transcendentalism as a fruitful methodology able to unify the foundations of mathematics (in the case: arithmetic and set theory) and physics (the origin of the fundamental physical constants). Common sense's prejudice enumerates the former to the area of "mind", or the mental, and the latter to that of "body", or the physical: following the general organization of cognition in Modernity (and which is able to be traced at least to Descartes).

Thus, the idea of arithmetic mechanics contradicts that contemporary "episteme" (utilizing the term coined by Michel Foucault), which determines what is allowed to be studied and what is not in its framework. The eventual links between "body" and "mind", able to occur independently of the "figure of human being" (again by Michel Foucault's expression or metaphor) "on the sand of the ocean" of cognition, are prohibited in Modernity (at least as to science), in the scope of which "arithmetic mechanics" should be situated. On the contrary, FLT, and first of all: its proof, generates a philosophical context and attitude for that area of prohibited cognition to be investigated in not less detail than its traditional counterpart of allowed cognition.

XIII. INAPPLICABILITY OF THAT PROOF IN THE FRAMEWORK OF GÖDEL MATHEMATICS COMPARED WITH ITS REASONABILITY IN HILBERT MATHEMATICS

The inapplicability of that proof of FLT(3) by means of incommensurability due to the Kochen - Specker theorem in Gödel mathematics can be demonstrated quite simply. The axiom of induction implies for all numbers to be rational and thus the inexistence of both irrational numbers and incommensurability. Indeed, any natural number is finite, as a corollary from it, and then, the ratio of any two finite natural numbers is a certain rational number: so neither irrational numbers nor incommensurability can exist.

On the contrary, set theory (just for the axiom of infinity) implies for both to exist and consequently, the incommensurability necessary for FLT(3) to be proved, is a Gödel insoluble statement. The way out is traced in Hilbert mathematics by Hilbert arithmetics and projectable into Gödel mathematics based on both (Peano) and (ZFC) set theory either as an only arithmetical proof of FLT (in *Part I*) or as only set-theoretical proof of FLT(3) and then, that of FLT in general (in the present *Part II*).

A rather philosophical reflection on incommensurability is relevant since it both refuted the ancient Pythagoreanism, but confirms the contemporary quantum neo-Pythagoreanism by means of the Kochen - Specker: as this was made clear already in *Section X*. The transition from the former, rejecting incommensurability to the latter, corroborating incommensurability consisting in the complementation by a second and dual Peano arithmetic following the approach by which quantum mechanics is able to justify its own completeness. Thus, the incommensurability is embodied in that counterpart, therefore its homomorphism (or "almost" homomorphism due to its anti-isometry) to the other "twin" or copy of Peano arithmetic being provable.

The pair of two dual, but homomorphic Peano arithmetics changes the concept of natural number as an implicit or self-obvious concept indirectly defined by Peano axioms. However, the result of that change is well known for a long time ago, but as an absolutely independent entity in a science with a subject quite different from arithmetic in first glance or tradition: theory of information or the concept of information accordingly.

Indeed, the pair of two namesake (or better, "numbersake") numbers in the two dual Peano arithmetics constitutes a bit of information, and both Peano arithmetics constitute a well-ordered series, respectively "string" of binary information. It can be also considered as a language, the symbols of which are digits, and any subseries, respectively substring, can be a word, sentence, or text of that language: a text whether meaningless or meaningful in a certain context. A bit can be introduced furthermore as a unit, i.e. as an elementary bijection of the nonstandard bijection (as this is already demonstrated in detail in *Section IX*): "$(P^{-} \otimes P^{+} \rightarrow P^{0}) \rightarrow P$".

So the new viewpoint to "natural number" imposed by Hilbert arithmetic understands it as a certain bit of information of a well-ordered sequence of bits being able to represent a message of information and thus interpretable not less or worse in terms of the theory of communication: speaking loosely all natural numbers constitutes the languages, by which all entities can be linked to each other. If one adds the concept of quantum information, those entities themselves can be considered as informational, and thus numerical in the final analysis.

Properly, the incommensurability at issue and in the same context is interpretable as the relation of binary information and quantum information: that of all entities in relation to their communication. This is obviously a worldview quite relevant to a kind of Pythagoreanism: and the provability of FLT needing the completeness of mathematics and the world for its proof (as in Hibert mathematics) can serve as a symbol for that transition to Pythagoreanism.

Whatever way (e.g., each of those three options enumerated in *Section IX*) for avoiding the Gödel incompleteness, whether in the framework of only set theory or that of Hilbert arithmetic can be relevant and consistent to the proof of FLT(3) by the Kochen - Specker theorem.

Furthermore, one can reflect even the exact reason for that proof to be applicable to Hilbert mathematics (arithmetic) rather than to Gödel mathematics. This is the explicit reference to the dual qubit Hilbert space, thus to both dual ones (respectively, both dual Peano arithmetics) in the former one unlike the only single one Peano arithmetics in the latter case. The Kochen-Specker theorem suggesting the separable complex Hilbert space as its necessary condition (for being inferable) is consistent only with Hilbert mathematics, but not with Gödel mathematics directly and in the final analysis.

The same statement is not literally valid as to FLT in general as *Part I* has already demonstrated. Hilbert arithmetic can serve also as an only heuristic "Wittgenstein ladder" to find an only arithmetic proof localizable thoroughly and absolutely in Peano arithmetic alone in fine. Anyway, one may mull the conjecture whether statements provable only in Hilbert arithmetic (mathematics), but fundamentally irreducible to Peano arithmetic (Gödel mathematics) can exist in principle.

Two corollaries from that hypothesis seem to reject it immediately in virtue of *modus tollens* being absurd, but maybe only at first glance and only in relation to common sense (which science has been often forced to ignore).

The *first* one is that there exist statements (namely those valid only in Hilbert arithmetic or mathematics) able to change reality directly, i.e.: without the mediation of human activity. Of course, the contemporary Cartesian episteme excludes them as even mystic and anti-scientific. However, once a relevant paradigmatic change, i.e.: the transition to quantum neo-Pythagoreanism has been postulated, their eventual existence, though not necessary, anyway is not more inconsistent. Furthermore, the mediation of human mind-brain seems to imply somewhere on their boundary of the *separated* mind and brain their real availability as extraordinary "Centaur-like"[33] *inseparable* thought-actions irreducible as to only thoughts as to only actions.

Quantum mechanics and especially the theory of quantum information introduce the physical phenomena of entanglement as a subject right studied by them. The Kochen-Specker theorem is fundamental for those particularly implying "contextuality" as to a mental statement about any quantum fact and the fact at issue (otherwise interpreted in classical physics or epistemology as a fact "by itself" and independent of any statement about it), consequently inseparable in two "texts" with a zero intersection. Nonetheless, the same inseparable contextual unity as a whole can be considered consistently and equally well as an only mental statement or as a state of only external reality.

Thus, the concept of entanglement is consistent with the discussed conjecture, but does not imply it necessarily. If one is to paraphrase the hypothesis in terms of quantum information and contextuality established by it, this would sound like that: there exists contextuality fundamentally irreducible to two more absolutely separable "texts". The same can be expressed in a still one

[33] The term was coined by the Georgean (Soviet) philosopher Merab Mamamrdashvili. Vladiv-Glover (2010) suggests a study of Mamardashvili's concept and idea. Furthermore, they are frequently encountered in Marxist philosophy due to the idea of "objective contradiction", borrowed from Hegel's or Hegelian "dialectics".

essential way: there exists entities which cannot be fundamentally any system or wholeness, since the latter divides disjunctively into its internality belonging to the system and its externality not belonging to it.

In fact, the last observation is justified by "scientific transcendentalism" (coined e.g. in: *Penchev 2020 October 20*) as a falsifiable version of philosophical transcendentalism by and after Kant (especially in Husserl's phenomenology: *Penchev 2020 June 29* or *Penchev 2021 November 18*). So, as far as the totality of scientific transcendentalism as an ultimate premise implies any wholeness or system in its framework, those entities, which cannot be fundamentally a system or wholeness to be granted, suggest logically (in virtue of *modus tollens*) rejection of the totality itself (at least that of scientific transcendentalism).

Then, one can trace the ambiguous or dual role of transcendentalism to Cartesian (i.e. contemporary episteme): on the hand, it generalizes the inherent "mind - body" (respectively, "subject - object") opposition right by transcendentality; however on the other hand, it guarantees for the same opposition to be universally possible after reducing transcendentality either to mind or to body; or said otherwise, the binarity of the totality in the final analysis (e.g. as the dichotomy of "God and the devil" or that of "good and evil", etc.)

The *second* seemingly ridiculous corollary is that there exist elements of Peano arithmetic, i.e. natural numbers, to which the nonstandard bijection "$(P^{-} \otimes P^{+} \to P^{0}) \to P$" is not applicable, or otherwise said: there exists natural numbers which cannot be represented by bits fundamentally. If one utilizes the axiom of choice[34] and thus set theory, this means that there exist sets, not to all elements of which the axiom is valid. As far as the restricted validity of the axiom of choice is painlessly grantable to (ZF) set theory, this demonstrates after tracking back, that those natural numbers irrepresentable as bits of information are also a matter of convention just as the limited applicability of the axiom of choice.

If one passes to quantum information, that second corollary from the considered conjecture implies that there exist qubits which cannot be represented by bits. If the interpretation of the Schrödinger equation as a universal equivalence of bits and qubits is granted (e.g. as in: *Penchev 2020 July 16*), the hypothesis implies the existence of quantum entities not obeying that equation[35].

Thus, one can conclude that physical objects in the scope of the conjecture are not confirmed by any experiments and even that nobody has searched for them though relevant axiomatic systems (e.g., restricting the applicability of the axiom of choice as above) might be consistently suggested.

Nonetheless, the concept, theoretical model, and partial implementation of "quantum computer" can be developed in both alternative frameworks: either postulating the totality (as scientific transcendentalism does) or rejecting it. The former viewpoint is rather that by default. According to it, any quantum computer cannot resolve any problem which a Turing machine cannot. Anyway, it may do it eventually faster and even much faster than a Turing machine for

[34] For example, as in the formulation of Russell (1906) by the corresponding Cartesian product not to be empty.

[35] For example, as in: *Sokolovski 2007*.

certain classes of problems, for which quantum algorithms[36] have been elaborated already. Furthermore, it is able to simulate a Turing machine as Feynman (1982; 1986) has yet demonstrated.

Meaning that the universe itself or any subarea of physical reality can be considered as a quantum computer (e.g. Penchev 2020 July 24), the interpretation of it even in the former framework is also possible though very rarely met (e.g. Penchev 2020 July 20). If that is the case, a quantum computer can resolve, not worse, problems being fundamentally insoluble for any Turing machine; in other words, that kind of algorithm is not only faster, but inherently better and irreducible to any algorithm of any finite length and applicable to a Turing machine.

A quantum computer interpreted in this way is able to manage that by embodying an attitude to reality absolutely or oppositely different from that of "Turing machine", which is strictly restricted to "mind" in Cartesian episteme and this means: no calculation accomplishable by any Turing machine computer can influence on reality besides in a neglectable and exactly predictable degree due to the corresponding physical carrier of information, which the computer at issue uses. Accordingly, any Turing machine processing any algorithm (both being theoretical models) cannot influence reality at all.

On the contrary, a quantum computer (being simultaneously a "fabric of reality" by Deutsch's metaphor coined in 1997) can resolve problems inaccessible to any Turing machine, because it involves all the universe in the calculation in question, as Lloyd (1997) explains for example, however, therefore, changing reality as a "side effect" of its work. A quantum computer postulated to be beyond the framework of scientific transcendentalism can be figuratively likened to "God's Intellect" ("GI") rather than to AI (presumably comparable with human intellect).

GI should not be interpreted only as loose metaphor, but as a scientific idea in the following sense. Quantum computers interpreted so may change reality by the quite physical mechanism of entanglement, due to which they might change the probabilities of certain processes in their environment to happen or not. In other words, their environment would be involved in their calculations therefore adding unlimited resources for them to be accomplished, but not with impunity. Reality turns out to be changed as a result, too. That is: if one distinguishes discernibly that quantum computer and its environment as the Cartesian dogma needs, the former calculates a result, but its environment does a new reality so that the result is valid in it. It sounds fantastical, but any natural law prohibiting that is not yet known.

[36] The literature about quantum algorithms is huge. Only a non-representative sample can be the following, for example: Dowling 2021; Kommadi 2021; Kurgalin, Borzunov 2021; Lipton, Regan 2021; Grumbling, Horowitz 2019; Johnston, Harrigan, Gimeno-Segovia 2019; Dowling 2013; Portugal 2013; Ykhlef 2011; Young, Knysh, Smelyanskiy 2011; Liu, Koehler 2010; Cheng. Tao 2007; León, Pozo 2007; Romanelli, Auyuanet, Donangelo 2007; Chen, Hsueh 2006; Love, Boghosian 2006; Kreinovich, Longpré 2004; Wei, Yang, Luo, Sun, Zeng 2004; Karafyllidis 2003; Steane 2003; Mahler, Gemmer, Stollsteimer 2002; Ortiz, Knill, Gubernatis 2002; Yu 2002; Ettinger, Høyer 2000; de Raedt, Hams, Michielsen, de Raedt 2000; Pittenger 2000; Chau, Lo 1998; Cleve, Ekert, Henderson, Macchiavello, Mosca 1998; Lloyd 1997; Liberman, Minina 1995; Vieira, Sacramento 1994.

Anyway, the interpretation of the proof of FLT either only arithmetically as in the previous *Part I* or only set-theoretically as in the present *Part II* is thoroughly in the framework of scientific transcendentalism problematized by Wiles's proof being out of Gödel mathematics as this demonstrated by mediation of Yablo's paradox in *Section IV* (of *Part I*). Once the question whether FLT does not belong to that hypothetical class of problems, the solution of which is able to change reality, is actual after Wiles's kind of proof, this paper answers negatively. The proof of FLT (though FLT is a Gödel insoluble statement) is reducible to Gödel mathematics whether in the framework only of arithmetic or in that of only set theory. Its proof does not change reality as the conjecture of "quantum computer as GI" admits.

XIV. A SHORT PROOF OF FLT FOR ANY EXPONENT "n" BASED ON ITS "KOCHEN - SPECKER" PROOF FOR "n=3"

Once FLT(3) has been proved by the Kochen - Specker theorem, one can use a relevant form of induction in an admissible way consistent with the context of set theory, and more especially, with the axiom of infinity. The enumerated above approaches are: (1) the proof of FLT to a special set of all natural numbers, only within which one can use the axiom of induction consistently and only to which FLT refers or can be proved; (2) a proper set-theoretical proof of FLT (that is to all sets rather than to a special one, that of all natural numbers after doubling the axiom of induction by its dual counterpart accordingly valid in the dual counterpart of Peano arithmetic in the framework of Hilbert arithmetic). Furthermore, still one approach can be added: (3) utilizing the axiom of choice rather than the axiom of induction: it can be considered in detail in the next *Section XV*.

The option (1) is a direct repetition of the proof of FLT by induction according to that in *Part I*. The differences are only two and external to the proof of induction itself. The one relates to the way of FLT(3) to be proved: citing the Kummer (1847) proof or following the pathway based on the Kochen - Specker theorem and traced in the previous *Section XII*. The other peculiarity restricts the proof only to a special set, that of all natural numbers in the present case. So, both are not essential and do not need any additional attention.

The option (2) involving two "dual" inductions is much more interesting due to its immediate relation to the proof of FLT(3) by the Kochen - Specker theorem, on the one hand, and to the nonstandard bijection "$(P^{-} \otimes P^{+} \to P^{0}) \to P$" which can be now limited only to corresponding inductions, symbolically "$(I^{-} \otimes I^{+} \to I^{0}) \to I$", on the other hand. Then, "$I^{-} \otimes I^{+}$" can be interpreted as transfinite induction "decomposed" into two absolutely independent standard, "finite" inductions (according to the Peano axiom of induction) in a rather "Hamiltonian" manner.

Particularly or as an illustration, "diagonalization" associable weather with "$P^{-} \otimes P^{+}$" or respectively with "$I^{-} \otimes I^{+}$" would correspond to transfinite induction therefore incommensurable with each of both finite inductions participating in the Cartesian product at issue. So, the incommensurability involved by virtue of the Kochen - Specker theorem for FLT(3) is repeated once again for the general case of FLT by means of induction. One can even trace that the repetition of incommensurability in both cases may be interpreted to be literal or "complementary" in a sense:

Indeed, since y^3 and z^3 are incommensurable, Fermat's equation does not possess any solution.Then, their incommensurability can be embedded in "$(P^- \otimes P^+ \to P^0) \to P$" so that they satisfy: "$(y \in P^- \otimes z \in P^+ \to x \in P^0) \to x, y, z \in P$", and thus, into the complementarity of "$y \in P^- \otimes z \in P^+$" whether for the mutual anti-isometry or as the two "axes" of the Cartesian product. Consequently, both cases of proving FLT(3) by incommensurability (mediated by the Kochen - Specker theorem), on the one hand, and then proving FLT for any exponent greater or equal than 3 by two gapped inductions (able to model transfinite induction), each of which is attached to one of both gapped dual Peano arithmetics, on the other hand, are unified. That observation about option (2) is already suitable to be further developed in detail as the option (3) in the next *Section XV*.

One can notice, that the unity of incommensurability and quantum contextuality revealed by the Kochen - Specker theorem after generalizing the absence of hidden variables from the non-commutativity of two quantum quantities or their Hermitian operators, proved yet by von Neumann (1932) and relevant to incommensurability, also to the commutativity of them, therefore including the violation of Bell's inequalities (1964) or the phenomena of entanglement at all and corresponding to contextuality, is now repeated in Hilbert arithmetic in relation to its two dual Peano arithmetic, but only "in half": to incommensurability alone.

However, that logical pathway, by means of Hilbert arithmetic in a wide sense, implies also the "other half" just in virtue of the Kochen - Specker theorem: namely, quantum consistently applicable not worse to the dual Peano arithmetic but rather indirect, or "complementary" in the following sense. Each "normal": the finite Peano arithmetic of both can be continued "over the gap", i.e. "transfinitely", and the identified with the there found other finite Peano arithmetic. Thus, the pair of two gapped finite Peano arithmetics can be equivalently exchanged by their transfinite continuation yet also gapped.

Then, both dual transfinite Peano arithmetic can be identified as both dual qubit Hilbert spaces (Penchev 2021 August 24), to which mutual quantum contextuality as entanglement can be already defined painlessly. Incommensurability and quantum contextuality may be distributed correspondingly to the pair of two dual *finite* Peano arithmetics and to the pair of two dual *transfinite* Peano arithmetics, therefore the two pairs dual to each other embodying the analogical kind of duality: incommensurability and quantum contextuality.

The same observation suggests that Kochen - Specker theorem supplying the completeness of quantum mechanics is also inferable from the completeness of Hilbert arithmetic if the latter is proved or postulated in advance. In other words, the completeness of quantum mechanics and that of Hilbert arithmetic are equivalent, which is not surprising after just this serves as the motivation for Hilbert arithmetic to be introduced.

A much more extended perspective is open so that the binary relations of incommensurability (particularly embedded in the concept of irrational number) and quantum contextuality (involving complex as a necessary condition for it) can be complementary to each rather than those of pairs of Peano arithmetics whether finite or transfinite. Then, they can be linked also to the properties or relations of hierarchy, idempotency and completeness discussed in more detail in *Part I*.

XV. THE OPTION OF AN ONLY SET-THEORETICAL PROOF OF *FLT*

That option can be defined by excluding the axiom of induction in any form since it is not available in the list of axioms of set theory. This a stronger requirement than only overcoming the Gödel incompleteness, implicit in FLT after Yablo's paradox since also any perfections or modifications of the axiom of induction consistent with the axiom of infinity (or relative to it) in set theory are not allowed, too. Reflecting philosophically, one can question whether hierarchy can be removed, absolutely, on the one hand, or partially, on the other hand, i.e. restricting it only in the framework of what is not total: a whole or a system (which is the equivalent of conserving the function successor of Peano arithmetic simultaneously rejecting the axiom of induction).

The function successor seems to be definable even only within set theory and the axiom of infinity, which can be also formulated explicitly by the function successor: now, from an element to the same element as a set. Then, all axioms of Peano arithmetic excluding only the contradictory axiom of induction are admitted, and FLT can be also formulated thoroughly in set theory. Finally, the problem is: can FLT be proved in that framework alone?

Anyway, the proof of FLT(3) by the Kochen-Specker theorem according to the subject of the present part of the study will be included conventionally among the conditions. All ways enumerated in *Section XIII* do not belong to the class of only set-theoretical proofs as it is defined above since they need the axiom of induction or its modifications substantially. On the contrary, the present section is concentrated on the option FLT to be proved without them, after the axioms literally necessary for the formulation of the theorem have been included or deduced in the *framework of set theory*.

That approach can be interpreted to be symmetric or "antisymmetric" to that exploited in *Part I* by "Fermat arithmetic" defined by a Husserlian kind of "epoché" but to infinity rather than to reality. Of course, set theory by and after Cantor has appeared as a counterpart of arithmetic since its fundamental concept of actual infinity contradicts any arithmetic recursive process remaining always finite as actual though continuable unlimitedly or as if "potentially infinite"[37].

However logically, one may disregard that impetus of actual infinity pushed off finiteness inherent to arithmetic therefore defining a set theory (conventionally further notable as "phenomenological set theory") sharing with Fermat arithmetic the same epoché to infinity (though one can now articulate it rather stylistically as an "epoché to finiteness" conserving the same Fregean "Bedeutung" as the "epoché to infinity"). Indeed, that new sense of phenomenological set theory would not allow for the Cantorian hierarchies of infinities to be established or Fraenkel's scheme of axioms (1921; 1922; 1922a; 1922b; 1922c; 1924; 1925) to be introduced, being irrelevant to FLT and its eventual only set-theoretical proof. Then, the way for

[37] The concept of "potential infinity" attached to arithmetic and opposed to the "actual infinity" of set theory is not correct or at least misleading once "potential infinity" is finite in fact and rigorous speaking by virtue of the axiom of induction. It rather hides and obscures the direct contradiction of arithmetic and set theory, and thus prevents mulling the Gödel incompleteness in a fruitful way. Nonetheless, it can be met in many scientific papers. For example, Linnebo and Shapiro (2019) discuss the problem: "Particular attention is paid to the question of whether potential infinity is compatible with classical logic or requires a weaker logic, perhaps intuitionistic" (p. 160).

the Gödel incompleteness to be overcome is also shared with Fermat arithmetic: the Gödel incompleteness can appear only between areas of finiteness and infinity[38]. Once both areas are not distinguished from each other, no incompleteness is generable.

The main tool able to substitute the axiom of induction successfully and remaining exceptionally in the framework of set theory is the axiom of choice. Indeed, it can create a corresponding well-ordering homomorphic to all natural numbers as an equivalent of any infinite virtue of the well-ordering "theorem". The "relativity of the concept of set" inferred by Skolem (1922) can be immediately related to phenomenological set theory and its epoché to infinity / finiteness. It suspends the entire Cantorian hierarchy of infinities reducing all of them to a single enumerable infinity, but not only: the boundary between infinity and finiteness is also erased, e.g. in virtue of the Dedekind set-theoretical finiteness defined by the absence of any bijection between an infinite set and another set at issue, which turns out to be finite in that sense[39].

This implies that the area of all Gödel insoluble statements between the arithmetic finiteness and the set-theoretical infinity also vanishes after the axiom of choice and just because of Skolem's tenet (often called Skolem's paradox). One may pay attention that Gödel's incompleteness and Skolem's relativity of "set" mean the same, but from two opposite viewpoints: correspondingly, from that of Peano arithmetic versus from that of set theory in relation to the transfinite area between them. Then, the former generates right the Gödel dichotomy (either incompleteness or contradiction between them), but the latter by means of the axiom of choice avoids it, therefore implicitly involving the epoché to infinity / finiteness shared with Fermat arithmetic.

Thus, the introduction of the axiom of choice in set theory (granted to be commonly accepted) is sufficient for FLT to be a soluble statement in set theory and an only set-theoretical proof of FLT to be possible in principle. Anyway, the constructive mechanism of how the axiom of choice (together with the well-ordering "theorem") is able to substitute the successive, and "unit by unit" series of induction is to be discussed in detail:

The axiom of induction starts from the least element of a well-ordering (said in terms of the axiom of choice and well ordering "theorem") and advances *forwards uniformly*. That uniformity absolutely necessary for applying induction is concentrated into the condition as to the statement "S" at issue: "$\forall n: S(n) \rightarrow S(n+1)$", The same requirement translated in terms of the axiom of choice would be rather the corresponding equivalence than only that implication at issue, namely:

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[38] That area between finiteness and infinity can be described, for example, by the following scheme starting from the countable cardinal number (usually notated as "a") "backwards": a_1 is the cardinal number of a set such that the cardinal number of the set of all subsets of it is countable. Then, one can construct a recursive series a, a_1, a_2, a_3, … a_n, … of cardinal numbers such that any set of that cardinal number possesses a number of elements more but not equal to any set with a finite cardinal number, and simultaneously, rigorously less from that of any enumerable set. That sequence of cardinal numbers is a mirror or reciprocal image of the Cantorian hierarchy of infinities introducible by virtue of the axiom that the class of all subsets of any set is a set. However, this does not imply that a set such that the set of all subsets is equally powerful to any given set exists necessarily. Anyway, this can be postulated therefore initially avoiding the problem of how the new axiom relates to the rest of (ZFC e.g.) of set theory only to investigate "what would be up". The idea was suggested for the first time in *Penchev 2005*.

[39] The idea is demonstrated in more detail in: *Penchev 2020 August 5*.

"$\forall n$: $S(n) \leftrightarrow S(n + 1)$". Obviously, what is added in the latter case is the converse implication: "$\forall n$: $S(n + 1) \rightarrow S(n)$": it can be immediately recognized as the dual condition as to the dual counterpart of Peano arithmetic in the framework of Hilbert arithmetic and what is able to supply completeness furthermore necessary for FLT to be proved set-theoretically.

The interpretation of the condition "$\forall n$: $S(n) \leftrightarrow S(n + 1)$" as to FLT means that FLT(3), FLT(4), … FLT(n), … are to be considered uniformly as a single statement just by virtue of the axiom of choice. Meaning the way for FLT(3) to be proved by the incommensurability of two qubits (in turn inferable from the Kochen-Specker theorem) interpreted as two usual three-dimensional unit balls, those balls are to be generalized as four-dimensional, five-dimensional, …, "n"-dimensional, …, etc.: because all of them are the same in virtue of the axiom of choice. The last statement may be traced in detail as follows:

One may constitute a series of *geometrical* units, all of which to be discussed as the same *arithmetical* unit (usually denoted as "1") after that: those are: a 1-dimensional unit (a unit segment); a 2-dimensional unit (a unit circle); a 3-dimensional unit (a unit ball); a 4-dimensional unit; a 5-dimensional unit, a generalized n-dimensional unit (all those units for dimensions more than three are beyond our sensual experience, but nonetheless accessible mathematically). All of them (1, 2, 3, 4, 5, n-dimensional, etc.) are constituted uniformly as in relation to the immediately previous dimension, i.e. "$n - 1$", as in relation to the immediately next dimension, i.e. "$n + 1$". Only the latter is meant by the mechanism of induction, but both former and latter, if one involves the axiom of choice in order to substitute the axiom of induction in a certain proof (which is that of FLT in the case). Those are accordingly:

"$n + 1$" (shared by both approaches: by the axiom of induction and by the axiom of choice): an infinite set of elements which are identical in the first "n" dimensions, but different as to the "$n + 1$" dimension at issue. This construction generates the next, "$n + 1$" geometrical unit. For example (in the scope of our sensual experience), a 1-dimensional unit segment is varied ("rotated") arbitrarily in the next dimension, "2", therefore constituting a unit circle (i.e. two-dimensional); a 2-dimensional unit circle is varied ("rotated") in the next dimension, "3", therefore constituting a unit ball (i.e. three-dimensional).

The same construction can be continued uniformly out of the scope of our experience to four, five, "n" dimensions, etc., constituting new and new geometrical units. Furthermore, they cannot be interpreted as identical without the necessary condition "$\forall n$: $S(n) \rightarrow S(n + 1)$" involved by the axiom of induction. So, one is to interpret the sense of that condition as the verification of the identity of all members of the successive series in relation to the property "S" at issue. If the axiom of choice is used for the same, it confirms the equivalence which implies immediately:
"$\forall n$: $S(n) \rightarrow S(n + 1)$" (as well as both "$\forall n$: $S(n + 1) \rightarrow S(n)$", for the axiom of induction, and "$\forall n$: $S(n) \leftrightarrow S(n + 1)$", for the axiom of choice).

"$n - 1$" (being specific only for utilizing the axiom of choice): one considers the class of equivalence of the set of all elements of "$n - 1$" elements in relation to the dimension "n", therefore reducing the latter one. For example (in the scope of our sensual experience), one may reduce all great circles of a single unit ball to as a single unit great circle as a class of equivalence;

then, all unit diameters of a single unit circle to the class of equivalence of a single unit segment (repeated uniformly in any diameter of that circle). The operations can be continued "before that" beyond the scope of our sensual experience successively to four, five, "n" dimensions, etc. The axiom of choice means the equivalence and thus both cases, "$n + 1$" and "$n - 1$" unlike induction needing to validate expressively the direction "$n + 1$" to be able to involve the equivalence at issue.

On the one hand, the axiom of induction can be discussed to be more general than the axiom of choice since it admits it not to be valid in the reverse direction of "$n - 1$". On the other nand, it can be considered to be incomplete to the more general case including the direction "$n + 1$" as the direction "$n - 1$".

Returning to the only-set theoretical proof of FLT by the axiom of choice (rather than by the axiom of induction), one is to investigate the unique event happening between the cases "$n = 2''$ and "$n = 3''$ simultaneously in both directions: from "$n = 2''$ to "$n = 3''$, on the one hand, and from "$n = 3''$ to "$n = 2$", on the other hand; that is: why the corollary from the Kochen-Specker theorem is valid for two qubits, respectively for the case "$n = 3''$ but not for the previous case "$n = 2''$ (in other words, two unit circles just as two unit segments can be commensurable rather than two unit balls or units of any higher dimension, which is the proper sense rather of Gleason's theorem).

The Kochen - Specker theorem can imply the case only for two qubits (i.e. for $n = 3$), but not for two unit circles or two unit segments (accordingly, the cases $n = 2$ and $n = 1$), just by virtue of which FLT(3) is true, and the same statement as to $n = 2$ and $n = 1$ is not, as it is demonstrated in *Section 10*. The axiom of choice allows for FLT(3) to be generalized for any exponent greater than two since two unit balls generalized to any of those dimensions are incommensurable just those of three dimensions.

The same observation grounded rigorously only to the Kochen - Specjer theorem or Gleason's theorem (as this will be demonstrated in detail in the next *Part III* of the paper) can be at least visualized anyway as follows. The axiom of choice establishes for any choice of a point among a unit ball of dimension 3, 4, 5, …, n, …, and what any qubit is, *to be the same*: and more precisely, the choice of a single element among a countable set (after the axiom of choice), to which all those unit balls of different dimensions can be equated. Furthermore, any pair of two choices from those is incommensurable under the meaning of being absolutely independent of each other, therefore allowing for FLT to be proved right by virtue of the axiom of choice rather than by the axiom of induction.

On the contrary, the cases of two unit segments or two unit circles are commensurable or dependent on each other in the framework of a single qubit and thus relevant to FLT(1) and FLT(2). Obviously, the suggested visualization is based on the privileged position of a qubit rather than of any other unit "ball" of any dimension. However, that privilege is not justifiable by the axiom of choice, but rather by the separable complex Hilbert as the mathematical and formal ground of quantum mechanics, from where it enters Hilbert arithmetic for being postulated. Anyway, the privilege of a qubit (rather than other ball of any dimension) is proved in quantum mechanics: on the one hand, by Gleason's theorem, after which just the pair of two qubits is the "atom", or the least unit of a unique probabilistic measure; on the other hand, by the Kochen - Specker theorem, after which the same pair is also (and due to similar reasons) the "atom", or the least unit of incommensurability or respectively, the absence of hidden variables.

One can question why just the separable complex Hilbert space among many other similar vector spaces is chosen to be the fundamental mathematical formalism of quantum mechanics. The aforementioned two theorems justifying the qubit as the relevant unit of Hilbert arithmetic to be postulated can be one relevant answer. Another answer clarifies any wave function as the characteristic function of a corresponding probability (eventually, density) distribution, after all quantum quantities (unlike all in classical physics) are probabilistic and supplied by probability (density) distributions. Then if one dare link both answers to each other, what is observed is the following:

The "natural unit" embodied in the unit of a qubit (and then in the unit of Hilbert arithmetic) is the relation of the changes of two probabilities. The sense of just that to be the natural unit consists of the identification of a qubit of entanglement with a qubit of a single quantum quantity represented in the qubit Hilbert space. In other words, a qubit is the universal unit being the natural unity of the totality in the following sense: the unit of a qubit can be related equally well (and thus, indistinguishably) to both cases: a single quantity (quality) and two quantities (qualities). Consequently, "anything is connected to any other thing" just as the concept of the totality needs after all is expressed in qubits: which is the fact in turn able to verify the qubit as the universal unit being the least amount, "atom" of that omnipresent unity embodied in the totality.

Interpreted in that way, FLT is a corollary from the totality and its universal unity of a qubit, therefore able to symbolize that fundamental statement from which originates: and both theorems (Kochen and Specker's and Gleason's) can serve for that "noble" origin to be certified.

In fact, the axiom of choice involved for the only set-theoretical proof of FLT is only consistent with the above consideration, but its immediate objective is different compared with the analogical tool of the axiom of induction. First of all, the axiom of choice is able to remove the Gödel incompleteness as the main obstacle for FLT to be proved set-theoretically (or in other words, to avoid Yablo's paradox inherent for it). The Gödel incompleteness appears between the finiteness of arithmetic for the axiom of induction and the actual infinity of set theory for the axiom of infinity:

Hilbert arithmetic resolves that problem fundamentally by complementing with a second and dual Peano arithmetic, within which the Gödel numbers of all insoluble statements can be situated in a consistent way. The same dual Peano arithmetic featured by its anti-isometry (just as the dual separable complex Hilbert space of quantum mechanics) is represented relevantly enough by condition for the "reverse induction" "$\forall n: S(n+1) \to S(n)$" along with the "normal" "$\forall n: S(n) \to S(n+1)$" and resulting into the ultimate circumstance establishing that "$\forall n: S(n) \to S(n+1)$" being inherent to the axiom of choice properly (as this is explained above).

Thus, the two areas, that of Peano arithmetic and its complement to the actual infinity of set theory being successive, that is "the one after the other" (as in the case transfinite induction "after" finite induction), are re-organized to be “in parallel” in virtue of the anti-isometry at issue. And resulting into the equivalency featuring the axiom of choice (unlike the axiom of induction) in the final analysis. For example, considering FLT in terms of Yablo’s paradox, any insoluble statement is doubled by a soluble statement, furthermore dual or complement to the former, and thus its solubility is the logical disjunction of both, and consequently it is always valid: insolubility does not exist just as in Hilbert mathematics based on Hilbert arithmetic.

One can notice that all mathematical proofs in the paper are rather unusual or “extraordinary” and quite different in comparison with those in a mathematical journal. Common sense’s idea about a mathematical proof consists in a very extended syllogism, too artificial and technically sophisticated, accessible in detail only to a close circle of very professional mathematicians, furthermore specialized in a few quite narrow mathematical areas (and often even only a single one). Wiles’s proof of FLT is a typical example of that kind of proof.

On the contrary, the proofs in the present paper are absolutely different as this is obvious even according to the use of natural language rather than the artificial specialized mathematical and logical symbolism (which can be called “language’ only metaphorically). So, the proofs here seem to be “proofs” (in quotation marks): that is “unserious”, “intuitive”, “non-rigorous”, “illustrative”, “doubtful”, “false”, etc. compared with an extended syllogism consisting presumably of thousand elementary links (traceable theoretically by anyone).

However, that impression is wrong, The idea itself of mathematical proof is changed in the present paper fundamentally, for the merging of philosophy and mathematics (as well as physics, but this is not so important according to the subject of the present paper) in the advocated framework of quantum neo-Pythagoreanism. What is elaborated is a much wider context, in which FLT is to be situated in order to discover the shortest pathway for its proof.

Building that context (or any relevant context) is the philosophical approach for any problem to be resolved since the crucial change for it consists in the appropriate Gestalt switch, so that the solution is almost obvious from the newly introduced viewpoint. A top mathematician dare not swap the Gestalt because he or she does not even suspect whether it is possible or the mathematical education learns how to do this, or the relevant proficiency or background are available. Common sense “knows” that FLT is a (very difficult) mathematical problem so it is to be resolved or not by mathematicians sharing the above prejudice rather than by philosophers (which as even only an intention seems to be “nonsense” and “ridiculous”, in fact).

However, if one interrogates why FLT should be qualified as a mathematical problem (rather than a philosophical one), the answer is only wordless bewilderment at how such a childish question can be asked. The admission that it has not been proved so long just because it is a rather philosophical (than mathematical) problem: as if it is harder to break through than to break through the wall with a growing syllogism. Anyway, the difficulty of Gestalt change (by the way, featuring usually the deficit of enough intellect) can be overcome as the present paper demonstrates:

So, the main effort consists in making clear the Cartesian episteme of Modernity and the position of mathematics within it. Further, this reflects on the foundations of mathematics, on the relation of arithmetic and set theory, on the realization of the Gödel incompleteness theorems.

Once that opening has been accomplished, the idea of Hilbert mathematics based on Hilbert arithmetic can be suggested by the verified completeness of quantum mechanics by the theorems of the absence of hidden variables. Then, FLT can be reframed in Hilbert mathematics, and its solution to be astonishingly simple just for a philosophical problem (what FLT is) has been researched wrongly in virtue of its appearance to be only mathematical.

So, even that exceptionally complicated and sophisticated solution revealed by Wiles more than four centuries later obeys the same proper philosophical issues, and this can be demonstrated elementarily even in mathematical language by Yablo's paradox. In other words, the approach of the present paper originates from FLT itself and this is unavoidable in the proper framework of any attempt for proving only mathematically.

Thus, that aforementioned impression about a doubtful approach is wrong and it is due to the very deep prejudice that FLT is a mathematical problem (even rather a many centuries old curio), but quite not a fundamental philosophical problem implying changes of our understanding of what cognition is.

XVI. THE "TRANSLATION" OF THE PROOF FROM HILBERT ARITHMETIC INTO PEANO ARITHMETIC

The previous *Part I* shows that the relevant proof created in Hilbert arithmetic rather heuristically can be translated in Peano arithmetic painlessly and even its origin may be absolutely hidden as a "Wittgenstein ladder" at the cost only of some artificiality. The same question can be also asked about that set-theoretical proof exploited in the present *Part II* and based on the Kochen-Specker theorem.

One may notice that the representation of the proof of FLT from Hilbert arithmetic into Peano arithmetic in *Part I* and consisting of two discernible elements called "MFD" (modified Fermat descent) and "MMT" (modified *modus tollens*) can be also related as to the interpretation of the only set-theoretical proof of FLT suggested in the previous *Section XV*. Then, the correspondence is: the axiom of choice is depicted as MFD, and the Kochen - Specker theorem applied to be proved FLT(3), as MMT, accordingly.

This is not a loose interpretation, but rather a homomorphism grounded on the mapping of the qubit Hilbert space in Hilbert arithmetic in a narrow sense (after the two ones together can be meant as "Hilbert arithmetic in a wide sense"). Then the axiom of choice applied to the former (i.e. to a *certain* wave function for example) remains the same after the homomorphism; this means to be applied to the latter and furthermore corresponds to the description of how it is to relate to the axiom of induction in the previous section: namely, as a doubling by adding the "reverse" direction "$\forall n: S(n+1) \rightarrow S(n)$" to the "straight" direction "$\forall n: S(n) \rightarrow S(n+1)$". Both "reverse" and "straight" are in quotation marks because they are chosen conventionally, being idempotent to each other. The two directions of induction are conserved in the pair of isometry and anti-isometry featuring as the two dual qubit Hilbert spaces as the two dual Peano arithmetics.

Those two directions of induction (being absolutely necessary for the proof of FLT by induction for preventing of the Gödel incompleteness relevant in a single direction of induction) are represent also in MFD in the left and right part linked by the logical equivalence:

$[(x^{n+1} = y^{n+1} + z^{n+1}) \to (x^n = y^n + z^n)] \Leftrightarrow [\neg(x^n = y^n + z^n) \to \neg(x^{n+1} = y^{n+1} + z^{n+1})]$

This means that the one direction of induction is represented in the axiom of choice, discussed only in relation or in terms of the axiom of induction as "$S(n+1) \to S(n)$", and in MFD, as "$(x^{n+1} = y^{n+1} + z^{n+1}) \to (x^n = y^n + z^n)$"; accordingly, the other direction, both:

"$S(n) \to S(n+1)$" and "$\neg(x^n = y^n + z^n) \to \neg(x^{n+1} = y^{n+1} + z^{n+1})$".

Both directions of induction originate from the duality of two dual structures (whether qubit Hilbert spaces or Peano arithmetics) providing the property of completeness necessary for proving FLT in the approach advocated in the present paper.

The other homomorphism for discussing is that resulting in MMT as to Peano arithmetic. Indeed, if its initiate element belongs to the qubit Hilbert space, the corresponding argument is just the equality of a single qubit to two ones: a statement inferable from the Kochen - Specker theorem and implying as its necessary condition for the absence of hidden variables as to the qubit Hilbert space, on the one hand, and the mutual incommensurability of y^3 and z^3 as to Hilbert arithmetic or Peano arithmetic, on the other hand.

However, the way of mapping of the relevant corollary of the Kochen - Specker theorem into MMT can be likened to be rather "anti-homomorphic" in the following sense. MMT means the statement that x^3 can substitute $y^3 + z^3$ where x, y, z are defined to be arithmetical variables on all natural numbers. This implies that FLT(3) is false being a direct negation of FLT(3) proved by the Kochen - Specker theorem. Even more: the same direct negation links the general case of MMT and the idea of an only set-theoretical proof of FLT based thoroughly on the Kochen - Specker theorem and suggested in the previous *Section XV*.

One can reflect on the deep reason of that "anti-homomorphism". The Kochen - Specker theorem as well as its eventual utilization for proving FLT(3) or FLT at all is rooted in set theory and in actual infinity in the final analysis. On the contrary, MMT is an arithmetical statement (i.e. in the framework of the first-order logic of arithmetic), thus obeying the axiom of induction implying in turn the finiteness of all natural numbers. In terms of the Gödel dichotomy, that kind of anti-homomorphism embodies its alternative of contradiction (rather than that of incompleteness) of arithmetic to set theory. Indeed, the alternative of incompleteness is to be rejected since FLT needs completeness to be proved (as Yablo's paradox makes obvious). Nonetheless, MFD is just homomorphic rather than anti-holomorphic similar to MMT because each of both represents different (or properly, dual aspects) of the nonstandard (2:1) bijection, that is: "$(P^- \otimes P^+ \to P^0) \to P$" as follows:

The "anti-homomorphism" of the relevant (for FLT) corollary of the Kochen-Specker theorem, on the one hand, onto the MMT means rather the partial resultative mapping "$P^- \to P$", and its dual counterpart "$P^+ \to P$" featuring the straight homomorphism of the axiom of choice into the axiom of induction though both kinds of homomorphism are forced to involve the same Carstesian product "$P^- \otimes P^+$". One might say that MMT interprets that Cartesian product from the viewpoint of its "ordinate" (e.g. "P^-"), and MFD by its "abscisa" (respectively, e.g. "P^+"), being conventionally chosen as idempotent to each other.

One can notice an asymmetry after attempting to map the set-theoretical into the arithmetical proof. The proof of FLT(3) by the Kochen - Specker theorem does not correspond in any way to FLT(3) proved arithmetically, e.g. by Kummer (1847) as cited in *Part 1*. Even more, the set-theoretical proof of FLT(3) is absolutely absorbed in the mechanism of induction fitted to be applicable to FLT.

In fact, both theorems, as Kochen and Specker's as Gleason's, distinguish the dimensions "1" and "2" from those greater than "2": being explicit in the formulation of the latter, but implicit in the proof of the former. The same observation as to Peano arithmetic (rather than to the qubit Hilbert space or Hilbert arithmetic) is embedded in FLT itself. The opposition of "$n = 1,2$" versus "$n \geq 3$" valid as to the former two theorems as to FLT can be reflected by the relation of idempotency (for "$n = 1,2$") to hierarchy (for "$n = 1,2$, 3, 4, …"), and FLT to be restricted to the complement of idempotency to hierarchy (that "hierarchy which is not idempotent", or the values of an "arithmetical variable which is not a Boolean variable"; i.e.: speaking even more loosely, to "arithmetic which is not logic").

This can explain why just the relation of idempotency and hierarchy is the subject of the introduction *Section I* in *Part I* of the paper. It is able to unite FLT (and its proof by induction justified in *Part I*) with the Kochen - Specker theorem (and its utilization for FLT(3) and FLT in *Part II*) and with Gleason's theorem (analogically used in *Part II*I). Hilbert arithmetic (as dual to the qubit Hilbert space) is the relevant tool for the intended unification.

Meaning the above consideration, one can interpret the necessity of FLT(3) to be proved absolutely separately and independently as a problem of two alternatives. The one suggests that the concept of actual infinity utilized implicitly by the Kochen - Specker theorem as incommensurability (as far as commensurability means the availability of hidden variables) cannot be at all and fundamentally translated into (Peano) arithmetic language being inherently finite. Anyway, that proof should possess some relevant dual counterpart in Hilbert arithmetic in virtue of its duality to the qubit Hilbert space, to which the Kochen - specker theorem can be immediately related. Obviously, Kummer's proof is properly arithmetical and cannot be interpreted in this way.

The one suggests alternatively that the counterpart at issue exists, but it is not discovered yet. If that is the case, Peano arithmetic seems that it should somehow involve irrational numbers (in the framework of the finite Peano arithmetic): that is an obvious paradox. Indeed, one can refer to the ancient Pythagorean discovery of geometric *incommensurability* right by two geometrically orthogonal segments, but arithmetically *commensurable* to each other. Thus, the yet ancient Pythagorean incommensurability hints at two orthogonal Peano arithmetics (for the two orthogonal segments) to be inferred, and Hilbert arithmetic has only said over the same, but already explicitly and linked to the absence of hidden variables[40], which is that form of incommensurability relevant and important for contemporary physics and science.

In other words, one touches again the "extraordinary" bijection "$(P^{-} \otimes P^{+} \rightarrow P^{0}) \rightarrow P$" as the only possible way to prove FLT(3) by an exact arithmetical copy of the utilization of the the Kochen - Specker theorem for the same objective since "$P^{-} \otimes P^{+}$" (or "diagonalization") is the

[40] This idea is developed in detail in another paper (Penchev 2020 August 5).

only site within that kind of bijection where whether incommensurability or irrational numbers can be revealed. Anyway, that "clever" tool able to concentrate that property just for FLT(3) is not yet known (at least to me[41]).

XVII. INTERIM CONCLUSION: FROM THE KOCHEN-SPECKER THEOREM TO GLEASON'S THEOREM BY MEANS OF THE CONTEXT OF *FLT*

The next *Part III* intends to clear up all links between the formulations and proofs of FLT and Gleason's theorem following the pathway between FLT and the Kochen - Specker theorem established in *Part II*. So, one relevant interim conclusion is to describe the shared context of all three theorems, which is some relevant substructure of Hilbert arithmetic in a wide sense (i.e, including the qubit Hilbert space as a dual structure to Hilbert arithmetic in a narrow sense), since the former theorem refers to a statement valid in Peano arithmetic, and the latter two theorems can be easily "retold" in the qubit Hilbert space though originally "written" for the separable complex Hilbert space of quantum mechanics.

In fact, that context has been already reflected but rather only philosophically as the relation of hierarchy and idempotency: thus one has to reveal that link both sufficient and necessary to embody the relation at issue, but already as a proper mathematical substructure of Hilbert arithmetic in a wide sense. One would straightforwardly suspect just "qubit" whether "empty" or specified by a certain, e.g. "recorded" value to be that: or speaking loosely, the atom of incommensurability, and thus, the relevant "inseparable" (just as an "atom") substructure of the absence of hidden variables meant literally by the Kochen - Specker theorem, on the one hand, and rather mediately by Gleason's theorem, on the other hand: as that unique mathematical entity admitting many probabilistic measures granted to be two-dimensional complex (i.e. an "empty qubit"), but only a single one granted to be three-dimensional real (i.e. a qubit within which a unique value has been chosen).

Then one can consistently conclude that the *record* of a value in a qubit means in the final analysis that a unique measure has been unambiguously determined. Indeed, the condition $|\alpha|^2 + |\beta|^2 = 1$ allows for both coefficients to be interpreted as the counterparts of two certain probabilities already mapped in the characteristic function, what the natural interpretation of "wave function" (particularly, that of "qubit") is. Then, a pair of probabilities obviously determine a single probabilistic measure (e.g. by their ratio or any other unambiguous relation of them)[42].

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[41]Though one might mention that the Schrödinger equation links classical information (in its left side) and quantum information (in the right side) therefore being a possible applicant able to link FLT(3) proved in each of both "languages" dual to each other in the framework of Hilbert arithmetic in a wide sense.

[42] "Probabilistic measure" is defined by the mapping of an "event space" into the real (or empirically and experimentally: rational) numerical interval [0, 1]. The statement of a single probabilistic measure in Gleason's theorem means for that mapping to be single as well. Then, if the case refers to two probabilities they have to be linked as an ambiguous quantitative relation, e.g. such as that in a qubit. Furthermore, the concept of probabilistic measure is fundamental for quantum neo-Pythagoreanism since it links the world at all (where any event space is to be situated) with that of numbers: if the latter is additionally supplied by that of "infinity", it may consistently claim to be isomorphic to the former, and thus the same in a sense (at least mathematical).

Accordingly and on the contrary, any true substructure of an “recorded” qubit (e.g. that of an “empty” qubit) admits more than a single probabilistic measure in virtue of the incomplete certainty of the pair of two probabilities (therefore implicitly suggesting some class of equivalence).

Thus, the context of the unity of all three theorems is the mapping of a qubit into an arithmetic unit after their complementarity in Hilbert arithmetic in a wide sense. It is worth to notice the complementarity of Hilbert arithmetic in a narrow sense and the qubit Hilbert space is different from that of two dual Hilbert spaces, which is due to the complementarity of “function” and “functional”. A qubit and an arithmetic unit in Hilbert arithmetic are complementary by virtue of the correspondence of an element of a certain class of equivalence and that class at issue. In other words, the case of a unit and a qubit is the generalization of the case of two dual Hilbert spaces.

Then, the approach already established in the present *Part II* as the correspondence of a qubit and a bit by means of the relation of the Kochen - Specker theorem to FLT should be modified after substituting the Kochen - Specker theorem by Gleason’s theorem as to the forthcoming *Part III*. The core of that relevant reinterpretation consists in the replacement of proving the incommensurability of two qubits (featuring the corollary from the Kochen - Specker theorem, necessary as to FLT) with specifying a certain unique probabilistic measure after recording a given value in an empty qubit (in turn featuring the corollary from Gleason’s theorem as to FLT).

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