From the four-color theorem to a generalizing "four-letter theorem": A sketch for "human proof" and the philosophical interpretation

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Abstract. *The "four-color" theorem seems to be generalizable as follows. The four-letter alphabet is sufficient to encode unambiguously any set of well-orderings including a geographical map or the "map" of any logic and thus that of all logics or the DNA (RNA) plan(s) of any (all) alive being(s). Then the corresponding maximally generalizing conjecture would state: anything in the universe or mind can be encoded unambiguously by four letters.*

That admits to be formulated as a "four-letter theorem", and thus one can search for a properly mathematical proof of the statement.

It would imply the "four colour theorem", the proof of which many philosophers and mathematicians believe not to be entirely satisfactory for it is not a "human proof", but intermediated by computers unavoidably since the necessary calculations exceed the human capabilities fundamentally. It is furthermore rather unsatisfactory because it consists in enumerating and proving all cases one by one.

Sometimes, a more general theorem turns out to be much easier for proving including a general "human" method, and the particular and too difficult for proving theorem to be implied as a corollary in certain simple conditions.

The same approach will be followed as to the four colour theorem, i.e. to be deduced more or less trivially from the "four-letter theorem" if the latter is proved. References are only classical and thus very well-known papers: their complete bibliographic description is omitted.

Key words: *alphabet of nature, four-color theorem, four-letter theorem, human proof versus "machine" proof, two-letter vs four-letter alhabet*

INTRODUCTION

How many "letters" does the alphabet of nature need? Nature is maximally economical, so that that number would be the minimally possible one?

One can approach the problem by the next question. What is the common in the following facts?

(1) The square of opposition

(2) The "letters" of DNA

(3) The number of colors enough for any geographical map

(4) The minimal number of points, which allows of them not be always well-ordered[1]

The number of entities in each of the above cases is four though the nature of each entity seems to be quite different in each one.

The first three facts share that to be great problems and thus generating scientific traditions correspondingly in logic, genetics, and mathematical topology. However, the fourth one (4) is almost obvious: triangle do not possess any diagonals, quadrangle is just what allows of its vertices not to be well-ordered in general just for its diagonals.

Thus the limit of three as well as its transcendence by four seems to be privileged philosophically, ontologically, and even theologically: It is sufficient to mention Hegel's triad, Peirce's or Saussure's sign, Trinity in Christianity, or Carl Gustav Jung's discussion about the transition from Three to Four in the archetypes in "the collective unconscious" in our age.

One suggestion might be: The base of all cited absolutely different problems and scientific traditions is just (4). Thus the square of opposition can be related to those problems and interpreted both ontologically and differently in terms of each one of the cited scientific areas as well as in a few others.

Here are a few arguments:

(A1) Logic can be discussed as a formal doctrine about correct conclusion, which is necessarily a well-ordering from premise(s) to conclusion(s). To be meaningful, that, to which logic is applied, should not be initially well-ordered just for being able to be well-ordered as a result of the application of logical tools.

(A2) Consequently, the initial "map" (to which logic is applied) should be "colored" at least by four different types of propositions, e.g. those kinds in the square of opposition. They are generated by two absolutely independent binary oppositions: "are – are not" and "all – some", thus resulting exactly in the four types of the "square".

(A3) Five or more types of propositions would be redundant from the discussed viewpoint since they would necessary iff the set of four entities would be always well-orderable, which is not true in general.

(A4) Logic can be discussed as a special kind of encoding namely that by a single "word" thus representing a well-ordered sequence of its elementary symbols, i.e. the letters in its alphabet. The absence of well-ordering needs at least four letters to be relevantly encoded just as many (namely four) as the "letters" in DNA[2] or the minimal number of colors necessary for a geographical map.

[1] Here and bellow, the term of well-ordering as to cyclic orderings means the option any point in those to be able to be chosen as the "beginning", i.e. as the least element in well-ordering. This corresponds to the prohibition of „vicious circle" in logic.

[2] They are: adenine (A), guanine (G), cytosine (C), and thymine (T).

(A5) The alphabet of four letters is able to encode any set, which is neither well-ordered nor even well-orderable in general, just to be well-ordered as a result eventually involving the axiom of choice in the form of the well-ordering principle (theorem). It can encode the absence of well-ordering as the gap between two bits, i.e. the independence of two fundamental binary oppositions (such as both "are – are not" and "all – some" in the square of opposition).

A more rigorous, logical and mathematical introduction into the problem might be the following

All logics seem to be unifiable as different kinds of rules for conclusion. Thus any logic is a set of correct well-orderings (i.e. sequences from the premise to the conclusion). The axiomatic description of logic consists in explicating the characteristic property of that set so that one can decide for any well-ordering whether it belongs or not to that set. To be a well-ordering 'correct' means just that it belongs to the set defined by its characteristic property as a certain kind of logic.

Then, the characteristic property of the set of all logics seems to be the set of all sets of well-orderings in a class identifiable as language as a whole. The advantage of that definition is that one can "bracket" (in a Husserlian manner) the latter class being too fussy, unclear, and uncertain. It is substituted by the set of all natural numbers perfectly sufficient for representing all well-orderings. Indeed, this is the sense of the well-ordering principle equivalent to the axiom of choice. The initial class of language can be interpreted as what is enumerated, then "bracketed" and "forgotten". This follows the essence (though not the "letter") of Gödel's approach for the arithmetical "encoding" of

If all logics as that set of all sets of well-orderings of natural numbers are granted, one can define the concept of the 'map' of any given logic as the graph of all correct conclusions in the logic at issue. The vertices of the graph are natural numbers. Just four colors are enough to be colored that graph so that any two neighboring vertices to be colored differently according to the direct corollary from the "four-color" theorem. Then the maps of all logics share the same property. One can choice any four certain and disjunctive "colors" for all maps, e.g. those of the square of opposition according to the tradition, or the "A-C-G-T" alphabet of DNA. Nature always simplifying maximally has also "proved" the "four-color" theorem as to DNA.

The "four-color" theorem seems to be generalizable as follows. The four-letter alphabet is sufficient to encode unambiguously any set of well-orderings including a geographical map or the "map" of any logic and thus that of all logics or the DNA (RNA) plan(s) of any (all) alive being(s).

Then the corresponding maximally generalizing conjecture would state: *anything in the universe or mind can be encoded unambiguously by four letters*.

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Sometimes, a more general theorem turns out to be much easier for proving including a general "human" method, and the particular and too difficult for proving theorem to be implied as a corollary in certain simple conditions.

The same approach will be followed as to the four colors theorem, i.e. to be deduced more or less trivially from the "four-letter theorem" if the latter is proved.

THE GENERAL IDEA, PLAN AND METHOD FOR THE FOUR LETTERS THEOREM TO BE PROVED

The general plan for proving the theorem rests on a certain mathematical structure, its properties and interpretations. It is the separable complex Hilbert space, which is a universal mathematical structure in the following sense. It can unify nature by the its interpretation in the scientific doctrine of quantum mechanics, on the one hand, cognition and thus mind as both rest on some kind of logic in a broad sense, on the other hand. One can add a third universality, that of the self-foundation of mathematics even in the Pythagorean framework including reality within itself.

Hilbert space is a vector space closed to a scalar product, infinitely dimensional in general. If it is separable, this means that the number of its "axes" is countable without utilizing the axiom of choice. If it is complex, this means that it is defined on the field of complex numbers, and its axes are topologically compact internally, e.g. as a usual 3D ball. It consists of two identical dual spaces mutually disjunctive as any two dual spaces. This involves the concept of choice in its base for a choice between two disjunctive and equally probable alternatives is necessary to determine which of both dual spaces is meant. That kind of choice is an elementary choice, a bit, the unit, in which the quantity of information is measured. Thus both concept and quantity of information are inherently involved in the structure of Hilbert space and in the separable complex Hilbert space particularly.

Identity to the choice of each one dual Hilbert space implies still one extraordinary property: the invariance to the axiom of choice and thus, to well-ordering. The former is widely utilized in quantum mechanics as the identity of a coherent state to the corresponding statistical ensemble and interpretable ontologically as a kind of invariance to the possible and actual. The latter, the equivalent invariance to well-ordering is not widely used yet, but can be utilized to unify all logics, their relation to the non-logical in a broad sense as the non-ordered including both any unordered mental contains and any natural staff such as matter and energy and what may consist of them.

The separable complex Hilbert space admits a few mathematical interpretations essential for the applications in physics, which will be considered in the next section in detail. However, one of them, that in terms of quantum information refers directly to the plan how the theorem may be proved as well as to the foundation of mathematics, and more especially to the relation of finiteness and infinity.

In fact, the common framework of set theory (including the axiom of choice) and Peano arithmetic (including the axiom of induction) is not absolutely consistent if one admits arithmetizability in the sense of one-to-one mappings between infinite sets and sets of natural number. Indeed, any set of natural number should be finite for the following argument: 1

is finite; adding 1 to any finite number, another finite number is obtained; consequently, all natural numbers are finite according to the axiom of induction.

On the contrary, set theory postulates the existence of infinite sets (e.g. the “axiom of infinity” in ZFC even utilized the same construction as the axiom of induction), however any contradiction does not appear for the axiom of induction is not an axiom in set theory. If one unifies set theory and Peano arithmetic, a contradiction appears as to infinity for the former postulates it, and the latter excludes it by the axiom of induction. The so-called Gödel inconsistency argument (1930) elucidates that discrepancy in the common framework of set theory and Peano arithemetic. It is not too difficult to be removed, e.g. by replacing induction by transfinite induction as Gentzen (1936, 1938) or by Heyting arithmetic in an intuitionist manner.

Furthermore, one third way may be outlined by Skolem’s conception about the “relativeness of the notion of ‘set’” (1922) conserving the framework of Peano arithmetic (with the axiom of induction) and set theory (with the axiom of choice). That pathway has the advantage to be both intuitively clear and linking the concepts of probability distribution, naturally representable in the separable complex Hilbert space as its equivalent characteristic function, and those of finite natural numbers and infinite sets. Once that pathway is marked, both above ones, by transfinite induction or by Heyting arithmetic, may be described as equivalent to it. However, one can emphasize expressively that neither transfinite induction nor Heyting arithmetic, nor any equivalents of them will be utilized. The proof is intended rigorously in the framework of Peano arithmetic and set theory.

The concept of information, and particularly, of quantum information is directly connected to that of probability. Indeed, the concept of an elementary choice between two equally probable alternatives is shared by both concepts of information and probability correspondingly as a bit of information and as an elementary event in the space of events, in which probability can be defined by Kolmogorov’s axioms. Indeed, the choice between any alternatives, which are a finite number, can be always represented as a series of binary choices.

The essential generalization of ‘choice’ is to be admitted for an infinite set of alternatives therefore involving the axiom of choice. That generalization of a bit of information is a qubit of quantum information. The way for ‘qubit’ to be introduced in quantum mechanics as the normed superposition of two orthogonal subspaces of the separable complex Hilbert space is equivalent to the above definition. Then, the separable complex Hilbert space can be represented as a series of “empty” qubits, and any element of as a record of a certain value in each qubit. The separable complex Hilbert space might be likened to the free variable of quantum information, any record of a “wave function” corresponds to its bound variable. The opposition and transition from ‘free variable’ to ‘bound variable’ features furthermore the concept of information.

Further, one needs Skolem’s *relativeness* to represent any qubit as finite sequences of bits though purely and mathematically by virtue of the axiom of choice:

The axiom of choice in the common framework of set theory and Peano arithmetic implies that relativeness of finiteness and infinity as follows. Any set can be well-ordered and thus enumerated and represented by an initial segment of natural numbers. However, all natural numbers are finite in virtue of the trivial deduction demonstrated above as well as all initial segments of natural numbers. Consequently, the axiom of choice implies a one-to-one mapping of any infinite set to some finite set. However, one can state only a pure existence of that correspondence therefore excluding any constructive way in a narrow sense to be demonstrable.

That fact can be interpreted as a probability distribution for all initial segments of natural numbers to be attached to any infinite set even unambiguously. That unambiguous attachment of a probability distribution rather than a certain finite set can be anyway considered as a constructive way in a broad sense as to the above pure existence in virtue of the axiom of choice. However, that assumption is not necessary for the proof though it is consistent to it. Skolem’s relativeness is absolutely sufficient.

Then any qubit can be represented as a finite though unknown series of bits. Any element of the separable complex Hilbert space consisting of some finite number of qubits, each of which in turn representable as a finite series of bits, is some finite though unknown series of bits.

Any element of those series of bits consists of a pair of complementary bits, each one for each one of the two dual Hilbert spaces. Both bits being disjunctive (or “complementary” in terms of quantum mechanics) can share one and the same two “letters” e.g. such as “0” and “1”. Still two signs are necessary in the meta-level for the two bits of the pair to be able to be distinguished from each other: totally four. Consequently, four different letters are necessary for the alphabet of the separable complex Hilbert space

Any state of any physical system might be exhaustedly described by a wave function, i.e. by an element of the separable complex Hilbert space, which in turn needs four “letters” to be recorded as an unambiguous string. Consequently, any physical entity needs alphabet of four letters to be written down and distinguished from any other physical entity.

One can admit that any mental entity also needs not more than four letters for its alphabet as far as mind reflects the physical world. Anyway that is a speculative philosophical hypothesis, and a direct proof would be preferable.

That direct proof can rest on the assumption that any mental entity or phenomenon is also cognizable by science and human knowledge as those in the physical world. Being cognizable, they should be representable as a certain *consistent* text. That text being right consistent should share certain rules of consistency, or a certain logic of some kind. Any logic generates a class of true proving sequences which are well-ordered from the premises to the conclusions. All other sequences including those belonging to different logics are interpreted as false in the framework of the logic at issue.

What is well-ordered by that logic, as it refers to that, a *text* to be represented consistently, are some units of meaning such as propositions, words, etc. belonging to language and even to a certain language of some kind. Those units may always be enumerated in virtue of the axiom of choice even where the contextual links between them are arbitrarily many and arbitrarily strong. Then all logics will be different sets of series of natural numbers well-ordered in a way different from the natural one in general. Any logic is the characteristic property, explicating the rules of conclusion and logical implying, of the corresponding set of well-ordered series of natural numbers.

Furthermore, one needs to demonstrate that any logic can be described as an element of the separable complex Hilbert space and as that describable by four letters.

Indeed, any logic being represented as a set of well-orderings (eventually of natural numbers) corresponds though not unambiguously to a single wave function, e.g. as the statistical ensemble of natural numbers after all well-orderings of the logic at issue are mixed. That corresponding wave function can be exhaustedly represented by four letters according to the previous consideration. Consequently, the set of all logics corresponding to that wave function can be represented also by four letters. Furthermore, those logics are disjunctive to each other as different ways one and the same text (sharing one and the same natural numbers) to be represented consistently i.e. as a set of well orderings. If they would not be disjunctive, the absence of contradiction in their intersection (being a nonempty set) cannot be guaranteed, and the text would not be represented consistently.

Even any physical or mental entity to be representable thoroughly by four letters as this is already sketched above, the theorem is not yet a universal statement for some entity neither physical nor mental might anyway exist. Indeed, the theorem is valid under the condition of Peano arithmetic and set theory, but Gödel proved (1931) the existence of undecidable statements under the same condition.

In other words, one needs a proof of completeness somehow distinguishing the case from Gödel's incompleteness argument. This would be realized in a few ways.

The one of them is to be demonstrated that Gödel's theorem ("Satz VI", 1931) obeys its own condition and thus it is self-applicable (Penchev 2010). This means that its validity implies its undecidability: its status should be that of an axiom, a meta-mathematical axiom about the existence of reality external to mathematics, which can be as accepted as rejected. Then, Gödel's proof would transform its meaning into a proof of independence to the axioms of Peano arithmetic and set theory.

That mathematics accepting the axiom of existence of reality external to mathematics may be called Gödel mathematics. That mathematics rejecting the axiom as a mathematical one may be called Hilbert mathematics in honour of Hilbert and his program about the self-foundation of mathematics. Consequently, Gödel's incompleteness argument demonstrates only that Hilbert's program cannot be accomplished if any reality external to mathematics exists. However, that condition is implicitly presupposed in Gödel's argument (1931). If it is removed, "Satz VI" is unprovable in the sense that its provability implies its non-provability.

Another way for distinguishing from Gödel's incompleteness argument is by elucidating that the pair of Peano arithmetic (including the axiom of induction) and set theory (including the axiom of choice) are inconsistent to each other as to infinity. Indeed, the axiom of induction implies that all natural numbers are finite, set theory (e.g. the "axiom of infinity") states the existence of (actual) infinity, and the axiom of choice generates a one-to-one mapping between any infinite set and some subset of the set of all natural numbers, which is necessarily finite because of the axiom of induction.

Then Gödel's incompleteness argument means the following. Any system containing Peano arithmetic is incomplete to set theory for the former does not admit infinite natural numbers while the latter admits infinite sets, and the axiom of choice guarantees for them to be enumerable. If Peano arithmetic is merely complemented by infinite natural numbers, it turns out to be inconsistent for any infinite natural number contradicts to the axiom of induction.

The practice and tradition of mathematics demonstrates that infinity is necessary for its complete exhibition. This supposes the axiom of induction in Peano arithmetic to be somehow generalized, e.g. directly, by transfinite induction, or indirectly, by Heyting arithmetic. In each of both frameworks, Gödel's incompleteness argument is invalid.

One can easily show the way of the "axiom of reality" is accepted or rejected correspondingly in the pair of Peano arithmetic and set theory vs. the pair of the modified Peano arithmetic (either as Heyting arithmetic or by transfinite induction) and set theory:

In the former, the concept of infinity is external to Peano arithmetic, and the set theory itself, including it, represents reality within mathematics. Any mathematical model is "Gödelized" as some subset of the set of all natural numbers. A fundamental gap between it and reality in mathematics represented in set theory exist always right for the concept of infinity.

In the latter, either the infinity itself (by means of transfinite induction) or the gap between finiteness and infinity (by Heyting arithmetic) is internal for the mathematical model also arithmetical in the final analysis. So, any difference rather than only the gap between model and reality is removed and therefore reality is not more external to mathematical model.

The separable complex Hilbert space can be also considered as a relevant generalization of Peano arithmetic so that any inconsistency between it and set theory is removed, and mathematical model and reality coincide in a Pythagorean manner.

Indeed, the separable complex Hilbert space can be considered as that generalization of Peano arithmetic where natural numbers are substituted by qubits. Furthermore, the proofs about the absence of hidden variables in quantum mechanics resting only on a few relevant properties of the separable complex Hilbert space can be considered as a proof of completeness as to that space. Any hidden variable would mean a certain nonzero difference between model and reality.

The completeness of the separable complex Hilbert space can be also demonstrated immediately in terms of the above consideration by Skolem's "relativity" (1922) representing the difference between any infinite set and the finite set of all natural as a certain probability distribution. That probability distribution implies a wave function as its characteristic function. Thus any difference between the separable complex Hilbert space and set theory turns out to be an element of the former and therefore included in it and removed as in the cases of transfinite induction or Heyting arithmetic.

One can question which mathematics, either Hilbert or Gödel one, is the relevant as to our being. The philosophical term of 'being' should be preferred rather than of 'reality' for the two mathematics differ from each other right by its relation to reality.

As far as quantum mechanics is a universal physical theory exception of which are not yet observed, Hilbert mathematics is that of our being.

One can conclude that the four letters theorem as well as the four colours theorem as far as and if the former implies the latter are valid in Hilbert mathematics. As to Gödel mathematics, the problem needs an additional discussion, which follows.

The separable complex Hilbert space is shared by Hilbert mathematics and by Gödel mathematics. However, the former accepts it as the most fundamental mathematical structure, that arithmetic, which is able to generalize the Peano one in a way to underlie both mathematics and reality in the framework of mathematical being in a Pythagorean manner.

The latter considers it as one among the many mathematical structure, which cannot be featured in any way. Thus, it can serve as a model of Hilbert mathematics in the framework of Gödel mathematics. The existence of that kind of model can serve as the relative consistency of Hilbert mathematics: it is not less consistent than Gödel mathematics.

What distinguishes the two uses of the separable complex Hilbert space in each one mathematics is one condition which can be called the "axiom of completeness", respectively the "axiom of incompleteness" as to Gödel mathematics. That axiom admits at least a few more or less equivalent formulations, some of which have been already mentioned above: transfinite induction, Heyting arithmetic, and the unambiguous representation of any infinite set as a certain probability distribution whether of all initial segments of natural numbers or of all finite sets. The latter rests on Skolem's "relativeness of set" after the axiom of choice.

That relativeness is essentially used in the proof of the four letters theorem for reducing any qubit to a finite series of bits. However, the additional condition, which is properly equivalent to the "axiom of completeness" is not used. Furthermore, the Skolem relativeness is valid in Gödel mathematics as far as the axiom of choice is valid. Consequently, the four letters theorem is both valid and proved in Gödel mathematics.

Two questions are to be discussed about the four colours theorem:

1. Does the four letters theorem imply it as in Hilbert mathematics as in Gödel mathematics?
2. Can it be directly proved in Gödel mathematics even if need be not?

As to Hilbert mathematics, (1) the four colours theorem is implied trivially for the middle different from the physical world and mind, both representable absolutely in the separable complex Hilbert space does not exist fundamentally. Furthermore, (2) one can directly represent any "geographical map" as a well-ordering of all areas in it. Then, the same well-ordering is valid also if any area is substituted by the group of all areas having a common border with the corresponding area at issue including itself. Then any group of those, whether a physical quantum system or an imaginary map in mind, can be represented by four letters, which might be interpreted as four colours.

As to Gödel mathematics, the argument (1) above is invalid. One would need an additional proof that the four colours theorem is not an undecidable statement. Any direct proof of the four colour theorem can be considered as that proof of decidability. As the "computer proof" of the four colour theorem is a direct one, it implies also that the above argument is a valid proof in Gödel mathematics. Then the "computer proof" can be even removed similar to Wittgenstein's "stairs" for the above argument itself have already been a direct proof implying itself. Anyway that seems to be rather doubtful being too similar to a "vicious circle".

The argument (2) seems to be valid as far as the "map" exists in the world or in mind. However, the being of the "map" otherwise is both unprovable and irrefutable in Gödel mathematics.

Those considerations impose rather the necessity of a direct deduction of the four colours theorem from the four letters theorem without any referring to the axiom of completeness or any relative condition.

One rather elegant and instructive way for that direct deduction is the above "method of Wittgenstein's stairs" to be perfected by demonstrating, figuratively speaking, the fact that after removing the "stairs", one is still able to "leap down" and that the proper statement of the four colours theorem consists just only in that "jump down" independently of the way, in which that one has turn out to "be up". That construction follows:

The proof rests on the difference between induction and transfinite induction (bar induction). However, transfinite induction implies completeness for mathematics and thus it is inherent in Hilbert mathematics while the task now is the theorem to be proved also in Gödel mathematics. So, transfinite induction will be used only as a "Wittgenstein stairs" to be removed in the ultimate construction grounded only on the axiom of choice, which is admissible in Gödel mathematics.

Constructing a "Wittgenstein stairs" consists in decolourizing the map coloured by four colours under conditions of the four colour theorem to a single colour (among the initial four ones). That construction needs the transfinite induction. The intention is to be utilized the corresponding reverse process for colouring the one-colour map into four colours under the conditions of the four colours theorem. In that reverse direction, the axiom of choice is what necessary rather than transfinite induction. The axiom of choice corresponding to transfinite induction in that reverse direction is available in Gödel mathematics unlike transfinite induction generating completeness and therefore involving Hilbert mathematics.

Transfinite induction is applied between two domains of different colour. All domains are enumerated by the axiom of choice. Because Peano arithmetic is finite, any enumeration remains within each one corresponding domain. So one obtains a set of well-orderings, thus a logic and a certain wave function as above. Wave function can be represented by 4 letters. Consequently, all domain is designable by four letters. (The topological structure is describable by a wave function, the wave function by four letters, thus the topological structure is describable by four letters.)

One can remove the meditation of wave function for it represents the relation between the inductions. The necessary algorithm is for example the following:

If two neighbouring areas are versicolour, they can be unified in a single domain with either of the two colours. One uses any of both. The process continues.

The process stops if it colours two neighbouring areas in one and the same colour. Then, it returns to the previous level and continues again.

The map can be called by any natural number of different colours initially. Nevertheless, the described algorithm will decolorize the map to a single colour always.

Indeed, if the process reach two neighbouring one-color domains, this is reduction ad absurdum as to the given way of colouring.

If the process met the same problem in the last case, this means that induction and transfinite induction can be identified at least in one case of transfinite transition, which is false[3]. This is reduction ad absurdum about the premise that the process can stop before it to colour all domain in one colour.

The process is also well describable by bar induction: It states that exists at least one branch which leads from any area to the first, initial area from which the process starts.

The above description about checking whether the process can anyway continue in a certain other branch of the tree of the process is necessary only to express clearly relativeness or equivalence of the pair of transfinite induction and Peano induction (or merely transfinite induction if one accepts that transfinite induction includes Peano induction) to intuitionist bar induction. The intended algorithm will work without returning to any previous branch. However, it can be directly referred to bar induction because of the above equivalence.

The problem is why the minimal number of colours should be four. The intention is to be demonstrated that the algorithm of decolorizing needs at least four letters though the number of initial colours in the map is arbitrary under the condition that any two neighbouring domains are coloured differently. Then, one can utilize the fact that both (transfinite) induction and the (axiom of) choice identify the result and process though in "opposite directions":

Induction accepts the process as given and then postulates the result as equivalent. Choice accepts that result as given and restores a process (well-ordering) as equivalent to the set eventually resultative by induction.

Thus, if either of the process and result needs four "letters" for its "alphabet", and both are identified, then the other of the process and result will need also four "letters" for its "alphabet".

One can utilize the property of Hilbert space to be compounded by two identical mutually dual spaces. One orders the map in one string by two colours. If two neighbouring domains turn out to be one-color, one can utilize a second string (this exists as far as the map is not well-ordered), so that if the two neighbouring domains turn out to be one-color, their corresponding domains in the second string are versicolor. Obviously less than four colours cannot realize that structure: two colours for each of the disjunctive two binary strings as being disjunctive they can share the same two colours; still two "meta-colours" to be distinguished the two disjunctive binary strings from each other; i.e. totally four colours.

If any string is representable in thus, the process cannot stop before to colour the domain one-colour because it continues in either of the two branches above.

Then the answer of why four colours are sufficient can be sketched as follows:

The sense of removing the bound between two versicolor areas is transfinite induction. Indeed, the points of the one area are enumerated by one induction, those of the other one by another induction. Both inductions divided by the border between the two areas can be unified only by transfinite induction, which is defined as able to accomplish any transfinite transition, e.g. such as between two different inductions.

Its dual operation in the context of the problem is the opposite case: where a bound between two one-color areas exist ceasing the process of decolorizing of the map area by area. The corresponding correcting operation is the change of the colour of the second area so that the barrier between it and the former one to make sense. Once that latter operation is accomplished, the former one can be accomplished in turn, too.

So, the process can be algorithmically described by two disjunctive operations and a meta-operation switching between both. The first disjunctive operation is: "write the value of colour of the n^{th} area in the $(n+1)^{th}$ one if these values are different initially". The second disjunctive operation is; "change the value of colour $(n+1)^{th}$ one if these values are equal initially". The meta-operation is: "if any of both disjunctive operations strops, switch to the other operation.

These three operations can be described by three Turing machines, each one of which corresponds to exactly one operation of the above three ones. They cannot be unified by a less number of machines because a Turing machine cannot know whether or where it will stop. Any stop needs a meta-level for its description or processing. That kind of a pair Turing machines mutually activating each other by means of the stop of the other can be called complementary.

These three Turing machines needs four different "digitals" or "colours" for them to work: The first pair of "colours" for both disjunctive machines for they work never together in definition and thus can share economically one and the same pair of colours. The meta-machine is what needs the second pair of colours.

The second stage consists in removing the "Wittgenstein stairs": the above four-letter algorithm for decolourizing any map to a single colour needs right just four letters and it is able to decolourize any map. Furthermore, the essence of both transfinite induction and axiom of choice is the identification the process and result including as to any infinite set. The difference between them consists in the opposite directions of identifying: transfinite induction accepts the process as given and then identifies the result with it; the axiom of choice starts from what is the result of transfinite induction and identifies a process to it. The axiom of induction and the obvious option for any finite set to be well-ordered by an equivalent series of elementary choices are the corresponding finite analogies of transfinite induction and the axiom of choice.

One can utilize that identification of the process and result after the axiom of choice right available in Gödel mathematics unlike transfinite induction for removing the "Wittgenstein stairs" as follows. If the process can be described by any alphabet of four letters, and the process and result are identified

[3] The finite and transfinite might be identified in virtue of Skolem's relativity of the finite and infinite, but that identification can exist only "purely" while the meant case would be concrete referring to a finite and thus constructive process generating an equivalence of finite and transfinite transition.

for the axiom of choice, then the result can be also described by four letters or "colours".

One might object that the process of colouring a map starting from its unicolor version needs the axiom of induction or transfinite induction rather than both option and axiom of choice. This objection is not correct for the axiom of choice is applied to the transition between the points of a unicolor area to divide it into two neighbouring areas rather than to the set of all areas. This is so as transfinite induction above is applied in the same way, i.e. between the points of two neighbouring areas, just to unify them into a single unicolor one.

Now, the axiom of choice is necessary for the opposite task: to divide points into two areas therefore generating a border between them similarly to a "Maxwell demon" distributing atoms or molecules in two areas according their velocities or energies, or momenta, etc. Indeed, two complementary "Maxwell demons" therefore working alternatively can reconstruct any state of an ideal gas from an initial state of maximal entropy and absolutely equal energies of all atoms or molecules. That interpretation of the four colours (or four letters) theorem states that those two complementary Maxwell demons are sufficient.

The axiom of choice having available four alternatives or two levels of choice generalizes the pair of complementary Maxwell demons including the case of colouring any map according to the condition of the four colours theorem.

The four alternatives necessary for the axiom of choice to "colour the map" might be elucidated as follows. If any level of choice needs two colours to distinguish its element, still two "meta-colours" are necessary to distinguish any two neighbouring levels, e.g. any two colours for the odd levels, and any other two colours for the even levels. Then no neighbouring elements (alternatives) as in the one and the same level as in two neighbouring levels will share one and the same colour. In other words, the other pair of colours are necessary for the next level of choice.

More colours are not necessary because the n^{th} level make impossible any neighbourhood between levels $(n-1)^{th}$ and $(n+1)^{th}$. The topology of any map can be exhaustedly represented by the alphabet of four alternatives for the axiom of choice in virtue of the identification of the process and result after it: any map is then the process of dividing the points into different areas if the result, from which the axiom of choice starts, is the homogenous unicolor (or "grey") ensemble of all points of the map.

To be avoided any logically suspicious teleology of the process of successive choices for dividing the points, one can reason so. The axiom of choice accomplishes any possible division of the points of the initial "grey" map. It is sufficient for proving the theorem in Gödel mathematics that one division among all possible ones represents right colouring the map in four colours according to the condition of the theorem for the theorem states only that the option exists. Then the foresaid process will happen randomly among all possible processes rather than intentionally, but it is enough for proving the theorem

The main difficulty for understanding is the prejudice that the map is not a process, but a result. If one realizes the map just as a process, right the process of dividing the points, the four colours theorem is almost obvious after the axiom of choice.

That kind of proof of the four colours theorem seeming to be a topological one is rather set-theoretical than topological, though. The theorem is considered as a topological interpretation or an even only topological expression of much deeper fundamental dependencies referring to the foundation of mathematics, relation of finiteness and infinity, and subordination of mathematics and reality. These dependencies being philosophical admit anyway a rigorous and properly mathematical discussion in the generalizing four letters theorem stating that anything can be unambiguously "written down" by an alphabet having only four letters. It identifies infinity with the non-well-ordered finiteness and elucidates that the "atom" of the latter consists of four elements.

Then and particularly, the "geographical map" can be considered as a topological expression of the non-well-ordered finiteness and as such, representable by the "letters" of four colours.

Comparing to the properly topological proof, the calculating complexity of which needs unconditionally computers, this "human" proof resolves the problem by its reinterpretation from a much more general viewpoint, from which its calculating complexity is minimal and even almost zero. Therefor the necessity of computers for the proof is substituted by the human capability to change the framework, "gestalt" of the problem choosing just that, in which its resolving is as simple as possible though challenging prejudices.

The next paragraphs consider the steps of that general plan one by one formally and in detail.

INSTEAD OF CONCLUSION: WHAT A MORE DETAILED PAPER WILL CONTAIN:

Discussing the separable complex Hilbert space and its interpretation in quantum mechanics and in theory of (quantum) information.

Demonstrating the correspondence between classical information and quantum information as the correspondence between the standard and nonstandard interpretation (in the sense of Robinson's analysis) of one and the same structure.

Elucidating the link between that last structure and Skolem's "relativity of 'set'" (1922) as the one-to-one mappings of infinite sets into finite sets under the condition of the axiom of choice.

Deducing the four letters theorem and interprets the theorem as to the physical world after any entity in it might be considered as a quantum system.

Interpreting the theorem as to mind seen as the set of all logics by means of representing the well-orderings in the separable complex Hilbert space.

Discussing the unification of the physical world and mind under the denominator of the four-letter theorem.

Deducing the four colors theorems from the four letters theorem including the case of an infinite number of domains by attaching ambiguously a wave function to any map (the axiom of choice may be excluded for any finite number of domains).

References in detail are not necessary for only classical, well-known papers are meant, for which the author and publication year (available in brackets in the text) are absolutely sufficient.