Gödel mathematics versus Hilbert mathematics.
I The Gödel incompleteness (1931) statement: axiom or theorem?

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Abstract. The present first part about the eventual completeness of mathematics (called "Hilbert mathematics") is concentrated on the Gödel incompleteness (1931) statement: if it is an axiom rather than a theorem inferable from the axioms of (Peano) arithmetic, (ZFC) set theory, and propositional logic, this would pioneer the pathway to Hilbert mathematics. One of the main arguments that it is an axiom consists in the direct contradiction of the axiom of induction in arithmetic and the axiom of infinity in set theory. Thus, the pair of arithmetic and set are to be similar to Euclidean and non-Euclidean geometries distinguishably only by the Fifth postulate now, i.e. after replacing it and its negation correspondingly by the axiom of finiteness (induction) versus that of finiteness being idempotent negations to each other. Indeed, the axiom of choice, as far as it is equivalent to the well-ordering "theorem", transforms any set in a well-ordering either necessarily finite according to the axiom of induction or also optionally infinite according to the axiom of infinity. So, the Gödel incompleteness statement relies on the logical contradiction of the axiom of induction and the axiom of infinity in the final analysis. Nonetheless, both can be considered as two idempotent versions of the same axiom (analogically to the Fifth postulate) and then unified after logicism and its inherent intensionality since the opposition of finiteness and infinity can be only extensional (i.e., relevant to the elements of any set rather than to the set by itself or its characteristic property being a proposition). So, the pathway for interpreting the Gödel incompleteness statement as an axiom and the originating from that assumption for "Hilbert mathematics" accepting its negation is pioneered. A much wider context relevant to realizing the Gödel incompleteness statement as a metamathematical axiom is consistently built step by step. The horizon of Hilbert mathematics is the proper subject in the third part of the paper, and a reinterpretation of Gödel's papers (1930; 1931) as an apology of logicism as the only consistent foundations of mathematics is the topic of the next second part.
Keywords: Boolean algebra, completeness, dual axiomatics, Euclidean and non-Euclidean geometries, Gödel, the Fifth postulate, finitism, foundations of mathematics, Hilbert arithmetic, Hilbert program, Husserl, incompleteness, logicism, Peano arithmetic, phenomenology, *Principia mathematica*, propositional logic, Riemann's "space curvature", Russell, set theory

I. INSTEAD OF INTRODUCTION: EUCLIDEAN TO NON-EUCLIDEAN GEOMETRY

A visualization of the approach advocated in the present paper follows the first historical example of two alternative axiomatics distinguishing from each other in a single axiom, the famous Fifth postulate of Euclid: there exists only a single straight line in a point out of a straight line, parallel to the latter (in a contemporary formulation equivalent to Euclid's original one).

Lobachevsky (or eventually János Bolyai rather than the "King of mathematics" Carl Friedrich Gauss) was the first mathematician who published his result that no contradiction appears if one admits the negation of the Fifth postulate of Euclid. Riemann generalized the works of Lobachevsky introducing the real parameter (or respectively, tensor) of space curvature, by means of which Euclidean geometry can be considered to be a particular case of all non-Euclidean geometries: where the space curvature at issue is zero[1].

So, all non-Euclidean geometries share the negation of the Fifth postulate, on the one hand and following literally the approach of Lobachevski, but simultaneously, Euclid geometry is a particular case of all non-Euclidean geometries, on the other hand and following literally Riemann's approach. Obviously, both approaches mean the same subject thus representing two alternative pathways to it.

If one distinguishes them, the following description should be relevant. Lobachevsky's approach, which can be called logical, restricts itself to the usual tridimensional Euclidean space shared by both Euclidean geometry and any non-Euclidean geometry, after which the former accepts the Fifth postulates, and the latter, its negation.

On the contrary, Riemann's approach, which can be called mathematical, implicitly introduces a fourth dimension to which the parameter (or respectively, tensor) of space curvature may make a "visual" sense. Then, no logical inconsistency exists between Euclidean geometry and any non-Euclidean geometry (and unlike following literally the approach of Lobachevsky) since they are distributed as two alternative values according to the generalizing parameter (or tensor) offered by Riemann: "space curvature".

One may notice further, that Riemann has followed an unarticulated (and very fruitful) postulate in science, according to which any direct contradiction (such as the Fifth postulate and its negation in the case at issue) can be removed by the addition of a relevant parameter (a tensor in general such as space curvature in the case in question). Furthermore, the abstract general structure meant by that unarticulated postulate corresponds to Hegel's "triad" (or respectively, to his "dialectical logic") trying to describe the unarticulated postulate at issue only in terms of logic, but transcending the framework of classical logic (sometimes called "formal logic" by the Hegelians in the opposition to the new "dialectical logic"), or more precisely, the "excluded middle", since Hegel's "synthesis" seems to be that "middle" excluded in classical logic in virtue of a special axiom.

So, one may notice that the unarticulated postulate at issue, being granted in science, on the one hand and Hegel's dialectic logic, on the other hand, shares the same subject considered from two alternative viewpoints just as the initial example by Lobachevsky's approach versus that of Riemann, but only in relation to the particular case of the pair of Euclidean geometry and any non-Euclidean geometry. However, the analogical abstract pair of a logical contradiction and its

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[1] The Fifth postulate refers to a plane, in the framework of which the Riemann curvature tensor is reduced to a real number being identically zero for any plane in the case of Euclidean geometry properly. The distinction of whether the parameter is a constant or a tridimensional function (tensor) is inessential as to the meant analogy of the pairs of Euclidean and non-Euclidean geometry, on the one hand, and arithmetic and set theory, on the other hand. There are many papers about the history, philosophy and innovations which led to the class of non-Euclidean geometries (e.g. Redei 2014; Greenberg 2010; 1979; 1972; Ravindran 2007; Prékopa, Molnár 2006; Stamp 1991; Schein 1979; Torretti 1978; Wiredu 1970) as well as to Riemann's generalization by "space curvature" (Pesic, ed. 2007; Laugwitz 1999; Farwell, Knee 1990; Portnoy 1982; Zund 1983; Douglas 1938; Eisenhart, Veblen 1922). There exist papers (e.g. Hunter 1980) investigating the consistency proofs of non-Euclidean geometries by models in Euclidean geometry.

overcoming (whether "dialectically" or "scientifically" as this is distinguished above) admits one more description: now, in terms of the Gödel-like dichotomy: either *contradiction* or incompleteness (literally, about the relation of arithmetic to set theory).

Indeed, one can immediately interpret the relation of contradiction between Euclidean geometry and any non-Euclidean geometry meant particularly in Lobachevsky's approach as the contradiction in a new Gödel-like dichotomy and thus implying a corresponding *incompleteness*: now that of Euclidean geometry to the general case of all non-Euclidean geometries right meant by Riemann by his newly introduced parameter (or tensor) of space curvature. Obviously, Euclidean geometry restricted to the case of zero space curvature can be called incomplete to the general case depicted by Riemann.

Furthermore, if one means the pair of Euclidean geometry and all non-Euclidean geometries one can resume the literal Gödel dichotomy in a new form (called above "Gödel-like dichotomy"): either a direct contradiction (i.e. that of the Fifth postulate and its negation in Lobachevsky's approach) or a corresponding incompleteness (i.e. that of Euclidean geometry to all non-Euclidean geometries according to Riemann's idea).

So the intention of the present paper is to realize the literal Gödel dichotomy as a particular (though initial) case of the Gödel-like dichotomy (sketched above) due to the foundations of mathematics since it can be grounded only to arithmetic and set theory as long as the literal Gödel dichotomy did not exist.

II. THE PAIR OF ANY TWO AXIOMATICS DISTINGUISHABLE BY A SINGLE AXIOM: THE CASE OF THE ONE TO BE INCOMPLETE TO THE OTHER

The description in the previous section can be generalized to any two axiomatics distinguishable by a single axiom so that the one to be incomplete to the other meaning just the general case. Obviously, they will share a set of identical statements in the inference of which the axiom at issue does not participate, but they distinguish from each other by another set of statements, each of which is valid only in the one axiomatic system due to the different versions of the axiom at issue.

One can suggest further the case where the set of identical statements are not literally the same, but isomorphic or homomorphic to each other as to any pair of corresponding propositions, which can be confirmed by both models: the one axiomatics without the differing axiom modeled into the other one also without the differing axiom and vice versa. So, those parts of both axiomatics are mathematically the same though interpreted in two alternative tuples of axioms, relations, properties, and concepts.

As to the complement parts depending on a statement and its negation in the two corresponding versions of the axiom at issue (e.g. the Fifth postulate in Euclidean geometry versus its negation in non-Euclidean geometries), one can anyway notice a certain asymmetry though both versions are idempotent to each other (which can be also interpreted as a kind of logical symmetry valid for any pair of mutual negations in the framework of classical logic).

A metaphor for that asymmetry can suggest the famous first sentence of Lev Tolstoy's novel "Anna Karenina": "Happy families are all alike; every unhappy family is unhappy in its own way"[2]. Indeed, the happy families can be interpreted as an idempotent logical negation to the unhappy families according to the pair of statements "Those families are happy / unhappy" (respectively, "Those families are happy" versus "Those families are unhappy"), on the one hand. On the other hand, however, the sentence itself states the identity (literally "similarity") of the one case of "the happy families" versus the non-identity (literally "non-similarity") of its negation (i.e. all unhappy families).

Following the suggestion of that metaphor, one might call the one version of the axiom at issue to be "happy" unlike the other version (i.e. the logical negation of the former) which is to be called "unhappy". For example, the Fifth postulate means the "happy case" versus the "unhappy case" of all non-Euclidean geometries sharing its negation.

Then and meaning now the Gödel-like incompleteness, one may say that the "happy statement" is incomplete unlike its "unhappy negation" being "almost complete" as excluding a single case, namely that meant by the "happy statement". For example, "This flower is red" is a "happy statement" unlike its negation "This flower is not red" being "unhappy" since a yellow flower is not similar to a blue flower though both are not red. Another example can be a measured value of any quantum quantity being "happy" unlike all "unhappy" possible, but not actually measured values of the same quantity.

The above observation about *one* "happy" case versus *many* "unhappy" ones can be extended further to a set of arbitrarily many "happy" cases versus a complementing set (being furthermore a set-theoretical complement of the former to that all possible cases) of also arbitrarily many "unhappy" cases including where the probability of the "happy" cases is equal to that of the "unhappy cases" therefore constituting together a structure isomorphic to a bit of information.

Respectively, the ratio of any finite set of "happy" cases to any finite set of "unhappy" cases determines unambiguously a certain quantity of information as well as a certain probability: and thus any probability implies just one value of information and vice versa[3]. If the set of "unhappy" cases is infinite, but that of "happy" cases is finite, a qubit of quantum information is determined also unambiguously at least in virtue if the axiom of choice, but furthermore, a constructive convention can map by any bijection any natural number as an ordinal or a cardinal number of a finite set and an exact value of a qubit (Penchev 2020 July 10)[4].

Now, the Gödel-like incompleteness though originating from the literal Gödel incompleteness as the non-contradictory alternative of the relation of arithmetic to set theory can be thoroughly interpreted probabilistically or "informationally" (i.e. by the quantity of information), and in fact, rather trivially: the happy cases are always less than all cases, i.e. the unhappy cases are a nonempty set. Then, if one means all happy and unhappy cases together, they always are all cases generating

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[2] According to Volokhonsky's translation in English: **Tolstoy**, L. (2001) *Anna Karenina* (Original work published 1875-1877) (R. P. L. Volokhonsky, trans.). New York, NY: Viking Penguin.

[3] A statement also obvious after the usual definition of quantity of information (Shannon 1948), respectively entropy.

[4] The corresponding probability is always zero in the case of quantum information, but a probability *density* distribution is relevant to any continuum of qubits: both determining unambiguously the same wave function.

furthermore an obvious instruction for overcoming any Gödel-like incompleteness (including the literal Gödel incompleteness itself): namely the consideration has to include the "unhappy" cases along with the "happy" ones.

One can test the validity of that instruction about the very discussed completeness of quantum mechanics. Indeed, its "secret" consists in Bohr's "complementarity" concentrated mathematically in the dual Hilbert space inherently, available for any Hilbert space, able to represent just all "unhappy" (that is "conjunctive") quantities to the "happy" ones actualized by any certain measurement.

Then the lesson learnt from quantum mechanics can be generalized and utilized for overcoming the literal Gödel incompleteness ("poisoning" the foundations of mathematics) by only doubling Peano arithmetic as a dual (and anti-isometric) counterpart for the "unhappy" cases (properly, all Gödel insoluble statements) to be meant as dual to all "normal", i.e. soluble statements meant by propositional logic (e.g. Penchev 2021 August 24). So, one immediately notices that if a single Peano arithmetic is postulated, the literal Gödel incompleteness appears right away, but if, on the contrary, both Peano arithmetics dual and anti-isometric to each other are postulated, it vanishes "in thin air", and all troubles about the foundations of mathematics are due to an unarticulated hidden postulate (seeming to be so obvious that nobody is able to reflect it): namely that Peano arithmetic is a *single* one rather than *two dual* ones as Hilbert arithmetic postulates.

Then, one can suggest that literal Gödel incompleteness is equivalent to that postulate rather than a conclusion from it, or in other words, the Gödel incompleteness theorem[5] ("Satz VI" in his original paper) is an axiom, in fact, rather than a theorem as Gödel himself heralded his result. Meaning it as an axiom (even too fundamental not to be easily articulable, or said otherwise, being "metamathematical"), one can call that mathematics, to which it is valid, "Gödel mathematics" featured just by its incompleteness of mathematics to the world therefore being absolutely harmonious with the organization of scientific cognition, valid nowadays or after Descartes and his dualism of "body" and "mind". Indeed, mathematics originating only from that "mind" cannot but be incomplete to the "body" at issue, i.e the world.

The Gödel incompleteness axiom only exemplifies and repeats the Cartesian conviction of our age that the "body" (i.e., the world) cannot but be external to the "mind" after an abyss between them is postulated to be surmountable only by "God" or by "God's Vicar on earth", humankind therefore justifying human chauvinism, a self-flattering self-delusion about the ostensible human domination over the world. So, people willingly believe in the Gödel incompleteness statement for being harmonious with the self-esteem of humanity as "the master of the world".

Once the Gödel incompleteness statement has been realized to be an axiom, in fact metamathematical, but in the framework of mathematics itself, referring to the relation of

[5] One might state that the influence, interpretations or generalizations of Gödel incompleteness paper (1931) in the foundations of mathematics, philosophy and science is so huge that it generates an implicit *new discipline researching incompleteness* (e.g.: Smith 2013; Berto 2009; Fitting 2007; Goldstein 2006; Goldstern, Yehûdā 2005; Goldberg 2000; Smullyan 1992; Chihara 1972).

mathematics to the world and postulating for the latter to be thoroughly external to it, one can immediately introduce its negation, namely that mathematics is not external to the world, i.e. the world is mathematical though in one or other degree, particularly and eventually, absolutely mathematical.

That mathematics meaning the negation of the Gödel incompleteness axiom is called "Hilbert mathematics" in honor of David Hilbert, its program for the self-foundation of mathematics on arithmetic (properly, an arithmetic called again "Hilbert arithmetic" substituting by doubling the standard Peano arithmetic) and his mathematical achievements (such as Hilbert space being inherently dual, or doubling itself by itself). Furthermore, the term Hilbert mathematics can be used in another recognition connoting the particular and extremal case of an absolute coincidence of mathematics and the world though being dual to each other, from the viewpoint of mathematics, or complementary, from the viewpoint of physics or philosophy.

An analogue in essence between Euclidean and non-Euclidean geometries, on the one hand, and Gödel mathematics and Hilbert mathematics in a wide sense, on the other hand, can visualize the statement promoted above that Hilbert mathematics means for the world by itself to be mathematical in one or other sense. As Euclidean geometry, Gödel mathematics is a "happy" case, and one can introduce a parameter (or respectively, a relevant function, e.g. a tensor function) analogical to Riemann's "space curvature" which would determine the exact degree in which the world is mathematical in a certain "universe"[6] (or any point of a certain "universe") reaching its maximum "value" in Hilbert mathematics in that narrow sense determined above.

The analogue can be even continued by means of Einstein's general relativity involving non-Euclidean geometry to be the "real geometry of the world" due to gravitation. Respectively, physical observations or experiments to determine the "curvature of the universe" or those of certain space-time domains of it are possible once the advocated worldview has been granted. Then, one can suggest observation and experiments (properly in the area of quantum mechanics and information) intended to discover the exact "degree" in which our universe or its physical and cognitive areas by themselves are mathematical. Thus, the naive conviction of the age of Modernity that our universe obeys Gödel mathematics can be rejected in general though it might be corroborated particularly and only its parts might turn out to be "Hilbert" ones.

That viewpoint to Hilbert mathematics rejecting Gödel incompleteness axiom is rather external, i.e. described in terms of the world rather than in those of mathematics itself though absolutely relevant in virtue of the degree of coincidence of mathematics and the world once mathematics has been granted to be Hilbert one. The proper viewpoint of mathematics, on the contrary, relies on Hilbert mathematics, or more precisely, on the relation of the two dual and anti-isometric Peano arithmetics in the framework of Hilbert arithmetic.

For example, the degree at issue, in which a certain world is mathematical, can be described by the quantity of overlapping, respectively distancing, of the two "twin" Peano arithmetics.

[6] The quotation marks serve to denote that the concept of universe is meant as a totality, being cognitive as well in order to be simultaneously mathematical, unlike "universe" in physics where it is granted to be only material.

Intuitionistic mathematics[7] after postulating the rejection of the "excluded middle" to infinite sets would correspond (in the analogue above) to hyperbolic geometries (such as Lobachevsky geometry interpreting the negation of the Fifth postulate as an empty set of parallels). Then, the measure of distancing (respectively, the "measure" of the "admitted middle" in intuitionistic mathematics), if that measure would be relevantly introduced, would correspond to a "negative" degree, in which the world is mathematical and thus intuitionistic mathematics is harmonious with the episteme of Modernity and the Cartesian abyss between Mind and Body.

On the contrary, all cases of overlapping (i.e. those corresponding in the geometric analogue to all geometries of Riemann featured by positive space curvature") would mean a "positive" degree in which the world is mathematical. They would reject the law of noncontradiction as to infinite sets rather than that of the "excluded third" as intuitionistic mathematics does. Then, the term of "dialectic mathematics", an counterpart of intuitionistic mathematics, can comprise all cases of overlapping of the two twin Peano arithmetics of Hilbert arithmetic.

Accordingly, the special case meant by classical propositional logic, in which both laws are valid, would correspond to the special unique case of Gödel mathematics[8], i.e. definable by both "zero overlapping" and "zero distance" of the two twin Peano arithmetics. The term of "*dialectic* mathematics" is chosen due to Hegel's *dialectic* logic, which has admitted for the first time the rejection of the law of noncontradiction.

An impressing visualization of the distinction of intuitionistic versus dialectic mathematic by means of the proof of Fermat's last theorem (especially that demonstrated in Penchev 2022 May 11; Penchev 2022 June 30) since it relies explicitly on no overlapping of the two Peano arithmetics though eventually modeled into a single one, as in the first cited paper). It is not a theorem in dialectical mathematics. It is a theorem in Hilbert mathematics (or only arithmetics), but it is an insoluble statement in Gödel mathematics as in intuitionistic mathematics. Thus, Wiles's proof of Fermat's last theorem inferring it as corollary from the modularity theorem (therefore needing set theory for elliptic curves along with arithmetic for modular forms) cannot but involve Hilbert mathematics though implicitly and secretly, for the "Boeotians" (meant by Gauss as to his explanation not to publish his results in non-Euclidean geometry) not to be able to notice or pay attention.

Summarizing, one can suggest that both properties of incompleteness and completeness[9] can be exhaustively modeled by a single bit of information, respectively by idempotency able to generalize the idempotency of the two alternatives of a bit of information therefore allowing for

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[7] The contemporary context of the relation between logicism, intuitionism, and formalism is discussed in the following compendium: *Lindström, Palmgren, Segerberg, Stoltenberg-Hansen, eds. 2009*.

[8] That relevance of classical propositional logic to Gödel mathematics does not implies that the Gödel incompleteness theorems can be proved as such within the triple of (classical) propositional logic, (Peano) arithmetic, and (ZFC) set theory: an observation, which will be supported in detail by many arguments and tenets in *Part II* of this paper.

[9] For example, a series of papers (e.g. Franks 2014; Manzano, Alonso 2014; Avron 2011; Gaifman 2006; 2006a; Cintula 2005; Srivastava 2001) investigates various generalizations, interpretations or the origin of Gödel's completeness idea(1930),

the conjecture that a bit of *classical* information can be considered as a *unit of idempotency* just as a bit of *quantum* information, as unit of hierarchy (well-ordering), both naturally unified in Hilbert arithmetic (unlike Peano arithmetic, in which the former is not available). Thus, one can further suggest that Hilbert arithmetic is able to be complete due to including idempotency along with hierarchy (well-ordering) inherently featured even a single Peano arithmetic.

Indeed, any alternative of a bit of classical information is incomplete to both alternatives (a single binary opposition), moreover to a bit being two complementary elementary binary oppositions: the opposition of the two alternatives at issue, but furthermore, a frequently unarticulated meta-opposition, that before the choice of any alternative versus the state after the choice among them (in more detail in: *Penchev 2021 July 8; Penchev 2021 August 24*).

Thus, a bit of classical information involves two idempotencies in turn idempotent to each other (a rather complicated structure, often without paying enough attention to it). The most considerable property of that deep structure of the most elementary unit of a bit of information consists in the observation that the explicit (and seemingly a single) choice among the alternatives automatically involves both oppositions since the preliminary opposition "before choice" versus "after choice" is a necessary condition for the ostensibly single, explicit choice between the alternatives.

So, the ultimately most elementary unit of a bit (also interpretable as the most elementary distinction, i.e. as elementary as possible) needs not two "letters" (notated usually as "0" and "1"), but four ones since each of both oppositions needs two "letters" for its exhaustive description (therefore pioneering a fundamental generalization of the "four-color theorem" to a new "four-letter theorem" about how many letters the "alphabet of the universe" needs minimally and thus really: in detail in *Penchev 2020 July 17*). Those four letters can be provisionally distinguished from the standard two ones, "0" and "1" as follows: "potential 0", "potential 1" (both "before choice" and eve indistinguishable from each other in a sense by virtue of their shared "potentiality"), "actual 0", and "actual 1" (both "after choice").

Analogically, the property of idempotency doubles automatically itself by itself, or said otherwise: a single doubling is impossible since any doubling implies its counterpart therefore a pair of doubles as a necessary condition of any single doubling though as un ultimate result any doubling is a single one hiding its necessary condition (speaking philosophically and speculatively, for example as Heidegger about "truth" as "Aletheia": hiding itself by itself in its revealing). Then, the pair of properties "completeness" versus "incompleteness" can be attached directly and ultimately to idempotency and its extraordinary peculiarity to double itself by itself (i.e. as *causa sui* should do).

One can interpret that in relation to logical negation since both alternatives of it are idempotent to each other as in Hegel's triad (by the by, being identical in classical propositional logic), in which the idempotent pair of "thesis" and "antithesis" is in turn doubled by the newly introduced "synthesis" and relevant only to "dialectical logic" suggested by him. In fact, the only problem to logical negation is how the puzzle of the doubled operation of logical negation to be ordered in a consistent way, not contradicting classical logic, as far as Hegel “resolved” it too radically, canceling “formal logic” at all for his “dialectical logic”.

Bohr’s complementarity originating from quantum mechanics and inherently needing to be consistent with it and with classical logic in the final analysis, because quantum mechanics as a rigorous science implies classical logic as its necessary condition, is what manages to resolve that “puzzle” not in a Hegelian “revolutionary” way, but consistently with the scientific tradition.

For example, complementarity applied to the pair of Euclidean and non-Euclidean geometries (respectively to the pair of the Fifth postulate and its negation) implies their mutual complementarity as a necessary condition of completeness while each of them alone is incomplete. As above, their relation can be interpreted by the Gödel-like dichotomy: either contradiction (indeed, any geometry cannot be simultaneously both “flat”, i.e., Euclidean. and “curved”, i.e., non-Euclidean) or incompleteness of each of both (as far as it would describe a particular case thus inherently incomplete to the general one).

Then, the observation explicated in the last paragraph can be generalized to any pair axiomatics differing from each other by a single postulate and its formal negation in order to be further applied to the literal Gödel incompleteness statement about the relation of (Peano) arithmetic to (ZFC) set theory, both granted to be first-order logics to propositional logic, by the mediation of Bohr’s complementarity. That pair of axiomatics can be considered to be inherently complete unlike each of them, remaining incomplete without its dual counterpart. However, they are complementary to each other by virtue of the axiom in question in the one counterpart versus its negation in the other counterpart. The two axiomatics cannot be simultaneously applied to avoid the direct contradiction and inconsistency of their conclusions to the same subject.

One should always and in advance choose which of both means therefore rejecting the counterpart in any case of application. The two counterparts are idempotent to each other, thus and in virtue of the discussed above doubling of idempotency by itself. The necessary condition of the explicit choice between the axiomatics is a preliminary implicit choice (or rather “meta-choice” choosing the choice itself) between the state “before choice” (i.e. before the choice of any of both axiomatics) versus the state “after choice” (i.e. after the explicit choice of the one of them).

Then, the state “before choice” can be interpreted to be *complete* unlike that “after choice”, being inherently *incomplete* to both inconsistent alternatives of the choice as a set of them. So, the explicit choice of the one axiomatics (never mind which namely) implies incompleteness, which can be interpreted to be the Gödel-like incompleteness. This means that a *meta-axiom postulating incompleteness* has been preliminarily involved to make possible the explicit choice of the only one of both axiomatics.

Nonetheless and not worse, one can choose the negation of that meta-axiom requiring incompleteness: that is one to postulate explicitly completeness but not in any Hegelian way implying inconsistency to classical logic, but by the mediation of the Bohr’s complementarity. Following the example of the pair of axiomatics generated by the Fifth postulate, Euclidean geometry can be distinguished from non-Euclidean one, but simultaneously unified with it by means of the concept of “smooth manifold”:

Any infinitesimal neighborhood in any point of the non-Euclidean manifold obeys Euclidean geometry. In other words, Euclidean geometry is valid *locally*, and non-Euclidean geometry is valid *globally*. One is to choose either the investigation is to be local and thus Euclidean geometry, or, on the contrary, it is to be global therefore predetermining non-Euclidean geometry. So, the concept of smooth manifold is able to reconcile the two inconsistent alternatives as complementary to each other in the framework of the same description consequently choosing the meta-axiom of completeness.

Obviously, the sketched approach to the pair generated by the Fifth posture can be generalized to the pair of any two dual axiomatics sharing an axiom and its negation even literally, i.e. by introducing both "local" (infinitesimal or differential) and "global (finite or integral) considerations, for example, to any vector space featured by the length between any two points of it, thus being able to be either infinitesimally small (namely "local") or finite (namely "global").

In fact, just that approach is also shared by the two most fundamental physical theories nowadays: general relativity and the Standard model (though their mutual inconsistency generates the "problem of quantum gravitation"). Indeed, the global pseudo-Riemannian space of general relativity is locally Minkowski space in any point of it (featured to remain smooth rather than to collapse in any singularity). Analogically, the Standard model introduces both local and global spaces, but identifies them to be the same, namely the separable complex Hilbert space of quantum mechanics, distinguishable from each other by virtue of quantum discreteness imposed by the Planck constant (i.e. prohibiting the smooth infinitesimal transition inherent for both classical mechanics and general relativity).

In fact, the same approach of interpreting a pair of axiomatics sharing an axiom and its negation as reconcilable and together complete as two complementary, local and global aspects of the same vector space can be borrowed from the fundamental physical theories for the foundations of mathematics, especially for the pair of arithmetic and set theory as first order logics to propositional logic by the *mediation of Robinson's nonstandard analysis* (1966). Indeed, if the axiom of choice (as in ZFC set theory) is granted, the condition of the weaker ultrafilter lemma (necessary for the nonstandard and non-Archimedean infinitesimal quantities to be introduced) is satisfied.

One can notice that Cauchy's classical approach for the infinitesimal quantities to be introduced only "potentially" (i.e. by definitions such as "$\forall \varepsilon: \ldots, \exists n: \ldots$" or "$\forall \varepsilon: \ldots, \exists \delta: \ldots$" respectively[10]) suffers from the same Gödel-like incompleteness for the description of any

[10] "n" usually means a natural number as a number of an element among a discrete sequence, and "δ", an interval containing all values of a certain function. However, the distinction between them is not essential in relation to the sense of the "Cauchy-like definition" in the present context: it serves to link a *set* specified by the quantifier "for each" and any *arithmetically* enumerable mathematical entity specified by the quantifier "there exists", being obvious in the case of "there exists some natural number ***n*** so that …". However, the fact of arithmetical enumerability can be implicit and more or less hidden in the general expression for Cauchy-like definitions being concentrated in the non-constructiveness of the quantifier "there exists" meaning in fact "there exists *some uncertain* …": this means "there exists *some fundamentally indeterminable* ***x*** so that …".

infinitesimal transition, by the by, well-known at least still since Zeno's age and his aporia about the "Tortoise and Achilles". Those Cauchy-like definitions translated in terms of Zeno's paradox would state that there exist necessarily insoluble statements about intermediate states in the process corresponding to the precisely determined moment of how Achilles is outrunning the Tortoise:

In any of those insoluble statements, both "Achilles is behind the Tortoise" and "the Tortoise is behind Achilles" would be valid. Then, one can immediately continue the Gödel-like incompleteness from Zeno's aporia to the Cauchy-like definitions of infinitesimal quantities as the necessary existence of insoluble propositions stating simultaneously that both propositions, namely "$\forall\varepsilon: \ldots, \exists\delta: \ldots$" and "$\neg(\forall\varepsilon: \ldots, \exists\delta: \ldots)$", are simultaneously valid. In fact, the Gödel-like incompleteness as well as set theory at all (invented by Cantor later) had been yet unknown in Cauchy's age and when they appeared, nobody has checked that the Cauchy-like definitions suffer the same the Gödel-like incompleteness once arithmetic and set theory are used simultaneously[11].

So, the common statement that Robinson's and Cauchy's justifications of infinitesimal analysis are both valid is to be specified: the former overcomes the Gödel-like incompleteness therefore being valid in both arithmetic and set theory and thus in so-called standard mathematics grounded on both. On the contrary, the latter is valid only in arithmetic alone and thus complementing it by set theory immediately implies the Gödel-like incompleteness. In a not quite correct historical retrospect, only Leibniz's "differentials" can serve as the basis of infinitesimal calculation, consistent with both arithmetic and set theory though set theory had not yet appeared in Leibniz's epoch.

One can in detail trace the way of how Robinson's approach is able to overcome the Gödel-like incompleteness in relation to infinitesimal quantities. His paper (1966) introduces infinitesimally great quantities along with the infinitesimally small ones of classical analysis as reciprocal to the latter. Then, an infinitesimally great Gödel number (along with an arithmetical and thus finite one, properly meant by Gödel) can be assigned to any Gödel insoluble statement. Moreover, that assignment is unambiguous in virtue of the ultrafilter lemma being a corollary from the axiom of choice valid in ZFC set theory meant in Gödel's paper (1931). That infinitesimally great Gödel number assigned to any Gödel insoluble statement can be conventional called its "set-theoretical Gödel number" to be distinguished from its initial finite arithmetic Gödel number as only the latter is meant explicitly in Gödel's paper (1931).

So, the conceptual "pun" about "all natural numbers" (being *finite* according to the axiom of induction of arithmetic, but *infinite* according to set theory as the "set of all natural numbers") underlying the literal Gödel incompleteness can be reasonably resolved accordingly by distinguishing the *local* aspect to "all natural numbers" meant by arithmetic from the *global* one meant by set theory.

[11] Indeed, the Cauchy-like definitions can be interpreted in a probabilistic sense if the quantifier of "pure existence, $\exists$" is meant indefinitely, i.e., as a quite uncertain interval or natural number therefore generating a relevant probability distribution (respectively its characteristic function furthermore identifiable as a wave function being a "point" (vector) in the separable complex Hilbert space as well).

Then, the Gödel insoluble statement can be consistently reduced to two soluble statements though the one of them possesses an infinitesimal great Gödel number (however properly, only after Robinson's approach rather than after Gödel's one followed literally), and the other one is finite being the single one meant immediately in Gödel's paper (1931). So, the proper Gödel number of any insoluble statement of his can be interpreted to be local (arithmetical) and then doubled by its global (set-theoretical) counterpart therefore removing its insolubility; for example so: "The Gödel number of this statement is 'this'" possesses a local and finite Gödel number (which alone is meant in his paper), and its counterpart interpreted to be global, namely "The statement possessing 'this' Gödel number is false", i.e. an infinitesimally great (set theoretical) one[12]. Indeed, it is to be infinitesimally great since it needs implicitly all statements of the class "The Gödel number of this statement is 'this'" to make sense, i.e. their set.

Summarizing, if one opposes the local and inherently finite arithmetic aspect of mathematics versus its global, set-theoretical and thus actually infinite aspect, the Gödel dichotomy about their relation ("either incompleteness or contradiction") turns out to be postulated. However, it cannot be consistently inferred from (Peano) arithmetic, (ZFC) set theory and (classical) propositional logic (as Gödel's paper [1931] ostensibly did) since their opposition is an independent (eventually meta-mathematical) axiom, which can be demonstrated by the consistency of the unification of the arithmetical and set-theoretical aspects just correspondingly local and global aspect and following the model of vector space borrowed from fundamental physical theories as above.

In other words Gödel's proof cannot but fail, however, the logical mistake able to "deduce" an axiom, and thus the dependence of a *really* independent statement, will be explicitly demonstrated in the next several sections, furthermore building an extended context, in which that mistake to be obvious and inferable just logically in a few self-supporting ways though all of them are linked to the elucidation of the aforementioned confusion of all natural numbers being finite in arithmetic in virtue if the axiom of induction, but the set of all natural numbers is simultaneously infinite in set theory in virtue of the axiom of infinity.

Thus, the model of the relation of arithmetic to set theory is to be that of the pair of two axiomatics differing from each other in a single axiom; at that the two corresponding versions are a statement and its negation, namely: "All natural numbers are finite" (equivalent to the axiom of induction in the framework of Peano arithmetic as this will be shown in the next section) and "The set of all natural numbers is infinite" (equivalent to the axiom of infinity in ZFC set theory).

Formally, those two statements can be consistently reconciled if one admits that a set can possess a property exactly opposed to another property shared by all elements of it (as "infinite" and "finite"). This is a conjecture which will be also researched in the next section, first of all, in relation to its consistency with all ZFC axioms. Independently of that hypothesis, the confusion of

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[12] This means in detail the following. It should be greater than any natural number as far as it can refer to a proposition possessing an arbitrary Gödel number. For example, if Fermat's last theorem is proved to be a Gödel insoluble statement by means of Yablo's paradox (Penchev 2021 March 9), its Gödel number is an infinitesimal great one, which is simultaneously finite in the framework of Peano arithmetic alone due to the axiom of induction implying for all Gödel numbers being natural numbers to be finite in its scope (i.e., out of set theory).

"all elements of a set" and "their set" as to "all natural numbers" will be explicitly demonstrated in Gödel's proof.

Further, *Section IV* is intended to investigate the general structure of any syllogism, in the logical chain of which a true paradox is included, since Gödel (1931: 175) himself explained the idea of his proof by Richard' paradox or the "paradox of Liar" just in this way, still emphasized by a linked footnote (*ibid*.) stating that any true paradox can substitute it.

One notices immediately that definition of an insoluble statement at the same page of his original paper, that is "$n\epsilon K \equiv \neg\{Bew\,[R(n); n]\}$", links definitively the natural number "n" belonging to the set (though called "class" by Gödel himself) of all natural numbers "K" in the right side to the natural number "n" (i.e. notated to be the same), but already meant arithmetically as what Gödel called "sign" of a "class-sign".

In other words, "class-sign" is still one Centaurus-like concept introduced by Gödel and discussed in detail during the present paper so that it is able to be finite in the scope of (Peano) arithmetic, but infinite as to (ZFC) set theory. This means that Gödel built his famous "Liar-like statement" (therefore sharing the formal structure of any true paradox) in an ostensibly rigorous definition just by means of the ambiguity of "all natural numbers" since all of them are finite in arithmetic in virtue of the axiom of induction, but nonetheless, constitute an infinite set in set theory by virtue of the axiom of infinity.

Of course, one should also check whether the syllogism itself (rather than Gödel's loose idea in the introduction) really includes a statement equivalent to a true paradox as well as the way of the Gödel statement stating for itself to be false "helps" for the confusion of "all natural numbers" with the "set of all natural numbers" therefore, for involving a direct contradiction in the syllogism though implicitly.

Section V will consider whether the statement that Gödel's proof (whether true or false) is constructive is valid after distinguishing thoroughly "all natural numbers" from the "set of all natural numbers" and then, will make clear the problem of the Gödel number of the Gödel statement stating to be false as well as the eventual application of the Gödel incompleteness theorem ("Satz VI") to itself if it can be interpreted as a theory satisfying its own premises. The question why all enumerated deficiencies of the proof are to be related only to its independence / dependence from all axioms of arithmetic, set theory, and propositional logic rather than to its inconsistency to them will be also investigated. Particularly, why and how the proof of that *independence* at issue can be granted to be a proof under the condition of its "obvious" *dependence* (which is wrong) are discussed in the same section.

Section VI considers the result about the modelability of any mathematical structure in arithmetic being a sub-area of mathematics under the consideration that the axiom of infinity is the negation of the axiom of induction (equivalent to the finiteness of any arithmetical structure). Then, the pair of arithmetic and set theory can be interpreted similarly to that of Euclidean and non-Euclidean geometry, i.e., as a pair of axiomatics differing from each other only by a single axiom and its negation as this is elucidated in the present *Section II*.

The final *Section VIII* outlines the horizon of the next, *Part II* and *Part III* of the paper by the relation of Gödel mathematics and Hilbert mathematics. The former can be identified as the contemporary "standard mathematics" suffering inherently from a variety of the Gödel-like incompleteness and originating from the literal Gödel incompleteness in its foundations.

Many of its most fundamental unresolved problems (for example, those famous "Seven Millennium problems" suggested by Clay Mathematics Institute) can be interpreted to be due just to different embodiments of the same Gödel-like incompleteness, thus being less or more difficulty resolvable under the necessary condition for the literal Gödel incompleteness (as the only cause of all forms of Gödel-like incompleteness) to be removed from the foundations of mathematics. The ostensible obviousness of the Gödel incompleteness statement to be granted is due, in fact, to the organization of cognition in Modernity, originating from Cartesian dualism, after which mathematics restricted thoroughly within the framework of "mind" cannot but be incomplete for "body" is opposed as absolutely external to it. So, an incorrect syllogism (what Gödel's "proof" is) can be trusted uncritically for stating what is expected and normal in the contemporary "episteme".

On the contrary, Hilbert mathematics relies on the negation of the Gödel incompleteness statement being equally admissible as the statement itself being in fact an independent meta-axiom about the relation of mathematics and the world though thoroughly in the framework of mathematics alone. The discussion of the relation (or opposition) of Gödel mathematics and (versus) Hilbert mathematics as well as its consequences are the proper subject of *Part III* of the paper.

Part II reinterprets both Gödel papers (1930; 1931) in a positive way as long as they are an apology of logicism and its inherent intensionality bracketing the opposition of finiteness and infinity as relevant only to extensionality though being the proper subject of set theory[13]. Then *Part III* is able to "synthesize" (even in a Gödelian sense) the criticism of the first part with the adoption of the second part to those papers of Gödel.

Indeed, their interpretation after logicism can be also seen as an "apophatic" Hilbert mathematics thus inherently complete though implicitly, after a possible kind of "epoché to infinity" (similar to Husserl's "epoché to reality"). On the contrary, the explicit Hilbert mathematics relied on Hilbert arithmetic is "cataphatic" in the above sense as far as it is able to suggest an extensional model unifying the finiteness of arithmetic with the infinity of set theory by doubling the former by its dual anti-isometric counterpart.

III, THE GÖDEL INCOMPLETENESS PAPER (1931) BY THE RELATION OF ARITHMETIC TO SET THEORY

The problem about the foundations of mathematics appeared in the beginning of the 20th century after discovering many paradoxes, which can be related more or less directly to the concept of "actual infinity" inherent for any infinite sets and thus, for set theory at all. In fact, the concept of "*actual* infinity" itself is misleading since it admits two oppositions (also interpretable as logical

[13] That idea can be traced in many forms of logicism (e.g. Demopoulos 2013; Gandon, Halimi 2013).

negations): *actual* infinity versus *potential* infinity, on the one hand, and (actual) infinity versus (actual) finiteness, on the other hand.

In fact, the former opposition does not correspond to the axioms of neither arithmetic and set theory and even, it is inconsistent with either of them. The potential infinity is an absurd "invention" of so-called philosophy of mathematics for which one can ostensibly "admit" (being a philosophy though that of mathematics) to be inconsistent with mathematics itself (more exactly with arithmetic and set theory as far as it is a meta-conception about their relation).

Once "potential infinity" has been incorrectly allowed though being inconsistent with both arithmetic and set theory, the literal Gödel incompleteness can be interpreted as a proper mathematical theory corresponding to the philosophical doctrine of "potential infinity". However, after "potential infinity" enters as an only philosophical category, for which is permitted to be inconsistent with mathematics just being a *philosophical* one, the literal Gödel incompleteness as its rigorous counterpart and subject of a proper mathematical theory would be to remove that inconsistency in a correct logical way.

However, the removal of inconsistency at issue is more than strange (right said, ridiculous and absurd) since it consists in being postulated by involving of a true paradox such as that of "Liar" (at that Gödel himself emphasized in a footnote in his paper: any true paradox can serve for the same purpose, consequently meaning the abstract, formal and logical structure shared by any true paradox) and without any analogy in the entire history of mathematics and logics.

If one grants the propositional formula "$(A \to \neg A) \land (\neg A \to A)$", which is always "false", for that formal and logical structure of any true paradox, it is relevant to the inconsistency meant by "potential infinity", but in a too extraordinary way reducible to an example of the general tautology "any inconsistency is false" (thus the inconsistency at issue is also false). In other words and speaking more loosely, Gödel resorted to an unexpected logical tautology, namely "'What is false is false' is true" (but only on the next metalevel if "false" means a proposition of the proper level to that metalevel).

Indeed, "potential infinity" is inconsistent with the finiteness of arithmetic due to the axiom of induction equivalent to that finiteness. (1) The axiom of induction implies for all natural numbers to be finite. "1" is *finite*. "The natural number N is *finite*" implies that "The natural number N+1 is *finite*". Then, the axiom of induction implies for all natural numbers to be *finite*.

(2) The statement that all natural numbers are finite implies the axiom of induction. Let S(1) mean a proposition, which is true for the natural number "2", S(2) for "2", …, S(N) for "N". Furthermore, both necessary conditions for the axiom of induction are satisfied; those are: "S(1)" and "$S(N) \to S(N+1)$". Then, the *finite* transitivity of the logical operation of implication in propositional logic implies the axiom of induction.

In other words, the *finite* well-ordering of natural numbers and the finite well-ordering of implications are isomorphic (or homomorphic if the viewpoint is algebraic) therefore both originate from the same property of the finite well-ordering. Thus "potential infinity" is inconsistent with arithmetic for all natural numbers are finite, which is equivalent to the axiom of induction.

As to the axiom of infinity (for certainty, in ZFC), it also admits only a kind of infinity, which is commonly accepted to be called "actual infinity" meaning that the process of the constitution of any infinite set is granted to be *completed*. Nonetheless, that process is isomorphic to the axiom of induction in arithmetic, but stating just the opposite about the complete collection of all natural numbers: it is *infinite*. Furthermore, the collection determined in this way to be infinite is an *infinite set*. Literally, the axiom of infinity states the following:

"'$\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\{\{\varnothing\}\}\} \dots$ ' is an infinite set"

One can immediately see that it is only another notation of the sequence of all natural numbers though interpreted otherwise:

'1, 1 + 1, (1 + 1) + 1, (1 + 1 + 1) +1, …'

Obviously, the latter means all natural numbers in arithmetic, for which the axiom of induction implies to be finite. However, the former (properly the axiom of infinity in set theory) though only notated otherwise states for the same to be an infinite set therefore involving the literal Gödel dichotomy about the relation of the former notation to the latter one under the implicit and unarticulated condition for any finite set to be mapped bijectively into a true subset of any infinite set. That is: the lather notation (i.e. that meant by arithmetic) is either contradictory (if one grants that the "set of all natural numbers" being infinite and "all natural numbers", each of which is finite, mean the same) or incomplete (if they mean different entities) to the former notation.

Then, but only at first glance, one can declare either of the two notations is absolutely redundant and Occam's razor should cut it; moreover, this is necessary for they contradict each other, but meaning the same entity. As far as paradoxes have been found in set theory (but no one in arithmetic), the natural suggestion is that the former notation is redundant (since it is associated with so-called actual infinity). This is the initial idea of Hilbert's finitism[14] consisting in the absolute "translation" of set theory into the language of arithmetic (Stenlund 2012):

In fact, the literal Gödel incompleteness refutes it only under the *additional* suggestion that set theory is totally necessary for mathematics and its foundations, which is granted rather empirically, i.e. in virtue of the intuitive conviction of many mathematicians that it is really necessary. The Gödel (1931) incompleteness paper also shares the same conviction, and only under granting it, his study can really make sense.

However, the present research interprets the same intuitive conviction of many mathematicians (for example, the school "Bourbaki") about the necessity of set theory for the foundations of

[14] The relationship of Gödel's second incompleteness theorem (and thus, though indirectly, that of the first one) and Hilbert's program is discussed by (e.g.) Tait (2010; 1981), Akiyoshi (2009) or Feferman (2008); the paper of Zach (2003) means the relation of epsilon calculus and finitism. Finitism can be discussed in a wide sense, for example by Incurvati (2015), as Ebbs (2016) or Kemp (2017) in relation to Quine, by Marlon (1995) to Wittgenstein, or as Hämeen-Anttila (2019) generalizing finite ordinals differently from infinite ones; also, as "finitism in geometry" (Suppes 2001) or as the "geometric concept of number" (Sobczyk 2012): an approach followed by the present paper as well. The advocated neo-Pythagoreanism can be eventually interpreted as synthesizing finitism and Platonism (usually granted to Gödel's thought, e.g.: Taotao 2011; Tieszen 2011; Parsons 1995; Silvers 1966), the eventual link of which is researched by Haukioja \(2005) or after Martin (1997) and the "need for properties". One can speak even for a class of finitist conceptions as (e.g.) Ganea (2010) does.

mathematics absolutely formally, proving that any complete theory needs two axiomatics differing from each other by a single axiom and its negation. That kind of complete mathematics called "Hilbert mathematics" means set theory in relation to arithmetic analogically as non-Euclidean geometry to Euclidean geometry, expressly emphasizing that any complete theory (after mathematics needs that) is to be equivalent to a similar pair of axiomatics (mathematical theories).

Then, the (as if) absolutely different language of set theory (though quite analogical to that of arithmetic as a dual pair of axiomatics as above) is rather psychologically necessary, namely hiding the direct contradiction of the axiom of induction and the axiom of infinity, under the veal of an ostensible absolute distinction of arithmetic and set theory so as if the consumption of "induction" and "infinity" means thoroughly unconnected entities rather than two ones each of which is the direct logical negation of the other one (in the sense above of two dual axiomatics).

On the contrary, one of the main ideas of Hilbert arithmetic consists in the rigorous formulation of the similarity, relationship, and relation of arithmetic and set theory as two dual axiomatics therefore removing the misleading veal of their ostensible utter distinction. Meaning that, one can trace back the too sophisticated way how Gödel kept the veal at issue by means of a "proof" relying on the formal and logical structure of any true paradox. However, the formal and logical structure of any true paradox is ambiguous, therefore in turn admitting a few essentially different interpretations, which will be considered one by one:

(1) The most frequent formal and logical interpretation of a true paradox is the aforementioned one: "$(A \rightarrow \neg A) \land (\neg A \rightarrow A)$"; in fact, it is equivalent[15] to the logical inequality of any statement to its negation: "$A \neq \neg A$", which is obviously a tautology. So, if a compound implication, i.e. a syllogism, contains a true paradox *and it is interpreted to be a tautology*, it cannot change its true value and can be neglected. However, if an identical true statement is inserted in a compound implication between its initial premise and its initial conclusion, the compound implication only repeats the conclusion without proving it:

That is: given "$a \rightarrow b$", and "$a \rightarrow 1 \rightarrow b$" is not equivalent to it, but to "b" alone; then, one is to realize whether "$(a \rightarrow b) \rightarrow b$" is equivalent to "$b$" alone, i.e. whether "$b \rightarrow (a \rightarrow b)$" as well, which indeed is a tautology, however involving the "paradox of material implication" being interpreted in any first-order logic (such as arithmetic and set theory) since the proposition "a" can be irrelevant to the proposition "b" after interpreting them in the corresponding terms of that first-order logic. The relation of relevance or irrelevance of any two propositions make sense only in any first order logic rather than in propositional logic by itself.

So, if a true paradox is included into a syllogism already interpreted in terms of any first- order logic, a subtle logical fallacy might sneak: the identification of the case of a premise relevant (as to the first-order logic at issue) to the conclusion with that of irrelevance since both relevance and irrelevance do not refer to propositional logic by itself, i.e., before any interpretation.

[15] More precisely, that formal notation of a true paradox (being always false) and the idempotent logical inequality of any proposition to its negation (being always true) are equivalent in the sense in which the laws of noncontradiction and the excluded middle are "equivalent": formally, after the use of de Morgan's rules.

In other words, one can identify the case of irrelevance (formally, that of logical inequivalence) with that of relevance (formally, that of logical equivalence) after de Morgan's rules, but that identification does not hold after the interpretation as any first-order logic therefore distinguishing cases of relevance and irrelevance of any two propositions (including the compose proposition, what the implication is) already interpreted to be meaningful or meaningless as to the first-order logic at issue.

This means particularly as to any first-order logic: if one deduces a true paradox from any premise, so that the latter implies any conclusion, that conclusion is not proved, but only repeated: i.e. it is false if it is false, and it is true if it is true, consequently allowing for both statement (whatever it be) and negation of it to be equally well postulatable.

So, a practical conclusion or rather warning about any proof involving a true paradox should be that it might be invalid in terms of any first-order logic, and the special case meant in Gödel incompleteness paper (1931) is especially complicated since it considers the relation of dual axiomatics (Peano arithmetic and ZFC set theory, in the meaning above) just as two first-order logics (thus implicitly transforming their binary relation in a triple relation after meaning propositional logic itself as its identification with set theory as the same structure of Boolean algebra).

In the final analysis and only as to that special case, one can suggest that a proof valid in propositional logic is *illegally* transferred in terms of a pair of dual first-order axiomatics where it can be both valid and invalid (i.e., similar to an independent axiom in regards of accepting itself or its negation), but granting to be ostensibly necessarily true by virtue of the analogical proof valid only in the framework of propositional logic alone.

(2) Anyway, a true paradox can be interpreted formally and logically more or less otherwise, namely so that the initial premise of the compound syllogism (implying a true paradox) and the initial conclusion (being inferred form a true paradox) to be logically unequal to each other, or symbolically (where "T" means a true paradox: "$[(A \rightarrow B) \land (A \rightarrow T \rightarrow B)] \rightarrow (A \neq B)$". The difference in the interpretation (to that above) of a true paradox consists in the following: now "T" means logical variable obeying an additional condition rather than a tautology (i.e. the constant "true", formally):

The transitive two equivalences, "$[(A \rightarrow \neg A) \land (\neg A \rightarrow A)] \Leftrightarrow (A \neq \neg A) \Leftrightarrow (true)$" are reduced to "$[(A \rightarrow \neg A) \land (\neg A \rightarrow A)] \Leftrightarrow (true)$" (though the latter expression is false, however interpreting it at the next metalevel as "'What is false is false' is true"), but to the following expression in the latter case: "$[(A \rightarrow \neg A) \land (\neg A \rightarrow A)] \Leftrightarrow (A \neq \neg A)$", which means rather the mutual complementarity of any statement and its negation than the tautological logical equivalence of any statement to its negation,

The difference between the former interpretation and the latter can be still more emphasized by the following consideration: "$(A \rightarrow \neg A) \rightarrow \neg A$", on the one hand, versus "$(\neg A \rightarrow A) \rightarrow A$", on the other hand; that is: as two alternative (mutually excluding each other) proofs by *reductio ad absurdum*. Interpreted so, the insertion of a true paradox as an interim conclusion in the course of any total syllogism disconnect it into to complementary parts, namely "$A \rightarrow T$" versus "$T \rightarrow B$", therefore decaying the syllogism which is to be proved into a premise and a conclusion turning out to be complementary to each other. In other words, if one manages to insert a true paradox in the chain of implications of any syllogism, what is really proved is that the initial premise and conclusion are complementary to each other, or they (themselves) constitute a derivative true paradox in virtue of the inserted one.

In addition, one can investigate the formal and logical structure of the Gödel dichotomy "either incompleteness or contradiction" after the last consideration that any insertion of a true paradox into a syllogism transforms it in a derivative true paradox in turn. Indeed, it can be equivalently translated (or speaking metaphorically, "translated" into the newly involved language of Bohr's complementarity borrowed from quantum mechanics) "either complementarity or contradiction" meaning just that the insertion of a true paradox into any syllogism transforms the latter into a true paradox in turn. However, and after that is the case, the premise and the conclusion of the syllogism at issue can be interpreted as a statement and its negation, each of which can be equally well postulated as an axiom though idempotent to each other, inconsistent and inadmissible simultaneously in the same context.

So, whatever is the interpretation of the insertion of a true paradox into a syllogism (i.e. whether as a tautology or as a logical nonequivalence in terms of first-order logic though identifiable in the framework of propositional logic alone), the conclusion is the same: a new axiom is to be added though formally different in the former interpretation to the latter. If it is interpreted to be a tautology, the conclusion of the investigated syllogism and its negation can be added as two alternative versions of the same axiom. If it is interpreted to be nonequivalence, the premise and the conclusion are the alternative versions of the same axiom.

The syllogism which Gödel's paper (1931) investigated can be concisely reduced to that: (whether or not) "Peano arithmetic implies set theory under the condition that both are first-order logics to classical propositional logic". What is really proved is that a true paradox (namely the Gödel insoluble statement) can be correctly inserted into the syllogism at issue. Then: (1) the set theory and its negation to so-called actual infinity can be equally postulated as an axiom; or as well as (2) (Peano) arithmetic and (ZFC) set theory constitute a pair of dual axiomatics being opposed in virtue of the logical nonequivalence due to the insertable Gödel insoluble statement.

The present paper demonstrates that the Gödel incompleteness statement ("Satz VI") in fact means that it is an independent (eventually meta-mathematical) statement about the relation of arithmetic to set theory after propositional logic as their shared basis therefore admitting equally well its negation, which implies an alternative mathematics: Hilbert mathematics versus Gödel mathematics in turn granting the literal Gödel incompleteness statement (rather than its negation as Hilbert mathematics does).

The next *Section IV* will demonstrate in detail following the proof of Gödel himself step by step, that it shares thoroughly the logical scheme sketched above.

IV. WHAT A PARADOX INVOLVED IN A SYLLOGISM IMPLIES FOR THE LATTER

In fact, Gödel’s original proof should be equivalent to the scheme outlined in the end of the last section. That necessary equivalence contains a few main shared ideas which are to be commented:

The logical and formal structure of a true logical paradox is absolutely necessary to be inserted between the premises (which Gödel himself determines generally as Peano arithmetic, ZFC set theory, and propositional logic) and the conclusion (whether about any axiomatics able to include Peano arithmetic consistently, i.e. “Satz VI”, or Peano arithmetics itself particularly, i.e. “Satz X”) consisting in the dichotomy about the relation of arithmetic to set theory, namely: either incompleteness or contradiction. In other words, the enumerated premises have to imply the logical structure of any true paradox, from which in turn the Gödel dichotomy to be unambiguously inferred.

The Gödel dichotomy follows only if the premises are organized as an investigation about whether Peano arithmetic implies (or can imply) ZFC set theory under the additional condition that both are first-order logics to classical propositional logic. Then, he demonstrated that the Gödel insoluble statement sharing the structure of a true paradox can be inserted so that Peano arithmetic implies it, and it in turn implies ZFC set theory under the cited condition.

That insertion consists in the enumeration of all propositions relevant to ZFC set theory by assigning unambiguously a “Gödel number” (a natural number) following conventional constructive rules expressly formulated by him in the study. Then, the Gödel insoluble statement being a true paradox is one of all those unambiguously enumerated statements relevant to ZFC set theory. The availability of one single statement necessarily possessing the formal and logical structure of a true paradox forces all of them to acquire the same structure of a true paradox.

Then and summarizing, all statements relevant to ZFC set theory share the same structure of a true paradox, imply set theory (being all statements relevant to it), and follow from Peano arithmetic (being unambiguously enumerated by natural numbers).

One can distinguish two stages, in which the Gödel insoluble statement “infects” collections of propositions with its insolubility: (1) being one among all of them, it transforms all propositions relevant to ZFC set theory into a huge true paradox after they have been enumerated unambiguously by natural numbers; (2) necessarily mediating the syllogism from Peano arithmetic to ZFC set theory, it transforms that syllogism itself into a true paradox as this is described in detail at the end of the last section. Finally, the Gödel dichotomy about the relation of arithmetic to set theory “either incompleteness or contradiction” is involved to describe exhaustively the statement already proved that the syllogism from arithmetic to set theory possesses necessarily the same formal and logical structure of a true paradox.

Meaning the crucial influence of the Gödel insoluble statement mediating in the syllogism from arithmetic to set theory, one can “stare at” what exactly causes its insolubility being in fact a conjunction of two propositions such that the one refers to arithmetic, but the other one to set theory, accordingly: (1) “The Gödel number of this propositions is ‘this’”, and (2) “The proposition possessing 'this' Gödel number is false". The former meaning the enumeration by natural numbers is to be related to arithmetic unlike the latter being 'false' is one of those relevant to set theory.

One can "gaze at" what causes its insolubility even more deeply: (1) The Gödel number being a natural number is *finite* in virtue of the axiom of induction (as this is described in detail in the previous section); (2) The set of all propositions relevant to set theory is *infinit*e in virtue of the axiom of infinity (and thus as containing an infinite subset whether true or not). Then one can conclude that the ultimate nature of the way how the Gödel insoluble statement acquires its structure of a true paradox is quite trivial in fact: if it is *finite* in virtue of (1), it is *infinite* in virtue of (2); and simultaneously, if it is *infinite* in virtue (2), it is *finite* in virtue of (1).

The same observation about the opposition of finiteness and infinity however involved as a true paradox by means of the Gödel insoluble statement can be immediately transferred to the investigation of the total syllogism from arithmetic to set theory: what Gödel's proof is in the final analysis. Arithmetic is inherently finite just as set theory is inherently infinite: so, either a direct contradiction appears if they are simply unified; or arithmetic is incomplete to set theory if finiteness is subordinated as a true part of infinity just as any infinite set contains true finite subsets (and all of them are necessarily true subsets).

Then, what Gödel's paper postulates properly is just what "proves" ostensibly: the nonequivalence of finiteness and infinity since what is proved is a premise of the proof (though veiled and hidden) and reducible to trivial nonequivalence of the axiom induction and the axiom of infinity since each of them is the idempotent logical negation to the other.

Quite analogically, one can state that Euclidean geometry is incomplete to non-Euclidean geometry and considered in detail in the introductory section. Even more, one can build an exact analogue of the Gödel insoluble statement so that it will mediate after the investigation whether Euclidean geometry implies (or can imply) non-Euclidean geometry, i.e. in a Gödelian manner. This can be any surface locally flat, but globally curved under the additional condition the global and local aspects to be simultaneously valid and thus indistinguishable: for example, the upper limit of *extending flat neighborhoods*, but simultaneously the lower limit of *shrinking curved neighborhoods* of the same point, being *both flat and curved* so that its flatness implies its curvature and vice versa just as the formal and logical structure of a true paradox needs.

The Gödel "incompleteness paper" suggests another new concept, "ω-consistency" (Gödel 1931: 187) and the split of consistency into both "ω-consistent consistency" and "non-ω-consistent consistency" meaning accordingly the consistency of an infinite set of propositions versus its counterpart of a finite set of propositions. Thus, that new concept is possible only from the viewpoint of set theory rather than from that of arithmetic. So, and for example, a Gödel number in the framework of arithmetic can be unambiguously assigned to any proposition belonging to an infinite set of propositions claiming to be ω-consistent.

Thus, the notion at issue can serve to illustrate once again the fundamental difference between the reflection of Gödel himself in his proof (1931) and the approach of the present paper. Gödel believed that his incompleteness statement ("Satz VI") is a theorem inferable from (Peano) arithmetic, (ZFC) set theory, and propositional logic. Inspired by that intention, he introduced as a series of Centaurus-like concepts (such as "ω-consistency" in particular) being finite to arithmetic, but infinite to set theory, the class of which was already discussed by the general idea of "potential infinity", in fact inconsistent with both arithmetic and set theory. Thus, he hoped to deduce absolutely rigorously, thoroughly formally and logically, the dichotomy about the relation of arithmetic to set theory. His paper describes that hope, for which all those Centaurus-like concepts invented by him should serve.

On the contrary, the present paper suggests that the dichotomy, which Gödel intended to prove, is available implicitly and hiddenly in the premises enumerated expressly by him, and quite precisely, in the fact that the axiom of induction in arithmetic and the axiom of infinity in set theory are related as the logical negation of each other. So, all Centaurus-like tools crafted by Gödel for his hope to "prove" the relation "either incompleteness or contradiction" are irrelevant formally and logically since they are inconsistent with both arithmetic and set theory turning out to be inherently finite from the viewpoint of arithmetic, but not less inherently infinite from the alternative viewpoint of set theory.

So, each of them contains a logical contradiction in its definition or said otherwise, all of them are wrong in definition and thus inadmissible formally and logically. Their real function is only rhetorical: to give the appearance of a standard syllogism able to demonstrate the dichotomy at issue as necessarily originating from the enumerated premises. However, what it really does is to camouflage and veil (by means of those notions being wrong in definition) the fact that what is proved is assumed in advance in the premises consequently not worse substitutable by its negation after it is an independent axiom.

So, the paper of Gödel is rather philosophical or "ideological" in a wide sense than mathematical or logical being directed to persuade its readers in an assumed in advance thesis, which is consistent with the modern organization of cognition, in the frameworks of which mathematics creates (and can only create) mental models gapped from reality and thus needing humans (or humankind, after generalizing) as an arbiter who alone can decide about a given model whether it corresponds to reality or not. Gödel's pathos consisted in taking the prejudice at issue out of any possible criticism stating that it is a theorem supportable by all authority, experience and tradition of mathematics and logic rather than a convention as any postulate, thus not worse substitutable by its negation (just as after the famous precedent about the Fifth postulate of Euclid).

One can allow for even a much more general notice that science partly substitutes religion nowadays since the latter has lost its authority even thoroughly at least as to the most educated people, but nonetheless the functions which it performed in the past are absolutely necessary for the unity of society and its hierarchy. So and particularly, a postulated meta-mathematical statement about the relation of mathematics to the world can be inferred as an ostensible "theorem", therefore blocking the way to any alternative relied on its negation. So, the authority of science replacing that of religion has been involved in proving a "truth" which is not more than a prejudice originating from the organization of cognition in Modernity and from the spot assigned to mathematics only within it.

Thus, if one doubts about the Gödel incompleteness statement ("Satz VI") stating, as the present paper does, that it is an independent axiom therefore admitting equally well its negation being not less consistent with the axioms of (Peano) arithmetic, (ZFC) set theory, and classical propositional logic, the spot assigned to mathematics and thus all the organization of cognition, the modern knowledge itself would be questionable or dubious, as a result: and then, by virtue of that, even the unity of society and its hierarchy would be undermined.

However, if one abandons those considerations, too important for science as a huge institution similar to a church (rather a metaphor coined by Ernst Mach) and charged with the responsibility of watching over the order in knowledge and thus in society, in favor of its essence by itself: *to search for the truth*, the proof itself of Gödel can be immediately reinterpreted (particularly as above) therefore elucidating that the insertion of any true paradox within a syllogism transforms the whole syllogism into a true paradox, which implies for its premise (as a whole including a series of statements) and its conclusion to be linked (or rather divided) by logical non-equivalence in turn interpretable as the pair of a compound statement and its negation.

If Gödel meant to investigate an eventual deduction of (ZFC) set theory starting from (Peano) arithmetic under the condition of classical propositional logic so that both are first-order logics, one can immediately see, that the essence of his proof consisted in inferring their logical nonequivalence (i.e. as a Boolean function of two variables) due to the option of inserting a true paradox (namely, the Gödel insoluble statement) to mediate the deduction of the former from the latter. So, this might be interpreted as a "theorem" as Gödel himself reflected on it.

However, the present paper demonstrates that the logical nonequivalence of arithmetic to set theory, ostensibly proved by Gödel as a theorem, in fact is assumed in advance by virtue of the fact that the axiom of induction in arithmetic and the axiom of infinity are mutual idempotent logical negations therefore sharing the same structure as Euclidean and non-Euclidean geometry, discussed in the introduction section.

Then if one adds the observation also discussed in detail above that any bit consists really of two oppositions rather than a single one (as the common prejudice states), the opposition meant by the pair of arithmetic and set theory (respectively, by the pair of axioms of induction and infinity), a necessary condition (for it) is the implicitly preceding opposition of the state "before choice" being inherently *complete* versus the state "after choice" of either alternative in turn being inherently *incomplete*. The one alternative is arithmetic, but set theory is both alternatives, in fact, after the other alternative would be a sub-theory of set theory restricted only to infinite sets (as far as set theory in default studies both finite and infinite sets).

Thus, Gödel ostensibly proving the incompleteness of arithmetic to set theory (under the condition not to contradict each other) in fact had assumed it in advance implicitly, but necessarily in virtue of the consideration of arithmetic and set theory correspondingly as the discernable compound premise and conclusion of a very sophisticated deduction.

Meaning that observation (namely that the incompleteness at issue is presupposed in advance, i.e as an axiom), one can (not worse) admit its negation, i.e., the completeness of arithmetic to set theory even building a consistent model of the relation of incompleteness (meant by Gödel) to that of completeness (meant by an alternative mathematics, called “Hilbert mathematics” and investigating the negation of that meta-mathematical axiom about the relation of mathematics and the world, namely as “completeness”). Hilbert arithmetic corresponding and underlying Hilbert mathematics is in turn inspired by the way, in which quantum mechanics after utilizing the separable complex Hilbert space manages anyway to be complete (as the theorems about the absence of hidden variables in it demonstrate: *Neumann 1932*; *Kochen, Specker 1967*).

V. A FEW MORE DOUBTS ABOUT THE GÖDEL INCOMPLETENESS STATEMENT: WHETHER A THEOREM OR AN AXIOM?

The Gödel incompleteness statement can be loosely featured (as above) to be a “pun” or “wordplay” relying on the ambiguity of “all natural numbers”, which are *finite* according to arithmetic, but simultaneously their set is *infinite* according to set theory. The same “pun” can be embodied and thus hidden by a series of Centaurus-like notions, the class of which can be investigated as “potential infinity”. In fact, all of them contain the same logical fallacy yet in their definitions, namely the unification of two opposite properties: “finite” and “infinite”. However, if they are granted as notions (similar to all “normal” notions not containing any contradiction in their definitions), the Gödel incompleteness statement can be justified to be a theorem in an ostensibly consistent way analogical to that utilized by Gödel himself by inserting necessarily a true paradox in the syllogism inferring set theory from arithmetic according to the laws of propositional logic.

The observation sketched in the previous paragraph allows for “demystifying” a series of concepts relevant to the ostensible “proof” of the Gödel incompleteness statement (or a general method for that), which can be demonstrated by its application to a few statements of that kind: (1) the ostensible “constructiveness” of Gödel’s proof; (2) the Gödel number of the Gödel incompleteness statement itself; (3) the problem about the Gödel incompleteness statement itself to be applied to itself if it satisfies its own conditions.

(1) A prejudice shared by Gödel himself is that his proof of the incompleteness statement is constructive (unlike that in his “completeness paper” (1930), which will be discussed in detail in *Part II*). In fact, there is a unilateral and jug-handled viewpoint, according to which Gödel’s proof can be consistently demonstrated to be really constructive. However, there exists a not less consistent viewpoint, according to which it is non-constructive and it will be sketched a little below. The option for two contradictory theses about the same statement to be equally well advocatable relies on the aforementioned “pun” about the finiteness (constructiveness) of all natural numbers in arithmetic versus the infinity (and thus, implicit non-constructiveness) of the set of all natural numbers in set theory.

In other words, Gödel’s proof is really constructive, but only from the viewpoint of arithmetic just in virtue of the fact that all Gödel numbers are finite, namely because all natural numbers are finite according to the axiom of induction in arithmetic. However, Gödel’s proof refers to the problem about the eventual derivability of set theory from arithmetic therefore admitting the viewpoint of set theory as equally possible to that of arithmetic. Then and after interpreting all natural numbers as the set of all natural numbers according to set theory (thus being infinite), Gödel's proof turns out to be non-constructive, not worse.

Indeed, Gödel's proof involves all natural numbers in order to be able to enumerate all statements relevant to set theory. However, their set is infinite according to the axiom of infinity. Therefore, an infinite syllogism in the sense of Gentzen's cut rule is to be involved. Really, the relevant cut-elimination allows for that infinite syllogism to be equivalent to a finite one but only *non-constructively*, i.e. without being able to demonstrate the finite equivalent syllogism at issue.

However, the latter equivalent finite syllogism though uncertain can be reduced always to a single implication linking the (most) initial premise to the (most) ultimate conclusion, which are always the same: so meaning that observation, Gödel's proof even involving an infinite set of interim conclusions is anyway equivalently constructive from the viewpoint of set theory just as from that of arithmetic. Nonetheless, one is to consider also how the insertion of a true paradox (namely the Gödel insoluble statement) influences the total syllogism:

Then and again, the non-constructiveness can be shown even in a quite elementary way after utilizing the nonequivalence of the premise versus the conclusion of any syllogism which inserts a true paradox in its chain as Gödel did. The nonequivalence at issue consists of two possible options: (1) the premise is false, but the conclusion is true, and the implication is also true, however in virtue of the "paradox of material implication" therefore being fundamentally *non-constructive* (stating only the existence of some *unknown* premise implying the conclusion constructively); (2) the premise is true, but the conclusion is false, and the implication is also false, which is equivalent to the alternative of "contradiction" within the Gödel dichotomy of "either incompleteness or contradiction".

Indeed, the former option (1), being relevant to the alternative of "incompleteness", is fundamentally *non-constructive* stating indirectly that there exists some true premise of the explicit conclusion, but it cannot be determined explicitly unlike the explicit conclusion, which is absolutely determined. In other words, that proved by Gödel incompleteness itself implies the *non-constructiveness* of his proof since it makes sense only as a relation to set theory therefore involving it as well as its viewpoint to all natural numbers now to be the *infinite set* of all natural numbers.

However even more, one can immediately notice that the above conclusion about the non-constructiveness from the viewpoint of set theory is equally valid as that of arithmetic because it does not refer to the premise that an infinite set of interim conclusions mediates the ultimate result[16]. The *non-constructiveness* is due only to the insertion of a true paradox into a conclusion consisting of whether a finite set of elementary implications or infinite one implying the resultative logical non-equivalency of the "first" premise (speaking generally, it is "arithmetic") to the "last" conclusion (which is "set theory", speaking generally as well).

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[16] This statement can be reasonably disputed in the following sense: the non-constructiveness of the paradox of material implication in the case of a finite set of possible premises (as this is always in the framework of arithmetic) can be distinguished from that in the opposite case of an infinite set of those. Though the relevant premise is not known in both cases, it can be constructively demonstrated only in the former case, just by virtue of the *finiteness* of the set of all possible premises.

The seeming constructiveness relies on the fact that ostensibly the chain of the proof is finite or equivalent to finite and expressly determinable. However, this is an invalid argument since syllogism involves unavoidably the paradox of material implication, which is non-constructive even in the case of a single implication because the true premise which implies the true conclusion remains unknown fundamentally and definitively. The same circumstance is obvious even in the term and concept of "incompleteness" being inherently *non-constructive*, i.e. without any claim to indicate explicitly what in set theory misses in arithmetic to be complete.

(2) If the Gödel incompleteness statement is joined to set theory, which is natural since it is obviously relevant to set theory, an ambiguity appears about its Gödel number. On the one hand, it should be finite as any Gödel number is a natural number and thus finite as to arithmetic in virtue of the axiom of induction. On the other hand, one can generalize the procedure of assigning an unambiguous Gödel number to any proposition whether true or false (i.e. necessarily soluble) to be relevant in terms only of set theory as follows:

Any Gödel number assigned in arithmetic is interpreted to be an ordinal number of a set admissibly infinite rather than only finite. Then, any statement containing within itself a reference to any infinite subset of the set of all natural numbers will mean an *infinite ordinal*, but nonetheless: the counterpart of the Gödel number of the same statement only in the framework of arithmetic will be finite since all natural numbers are finite in arithmetic.

Thus, one can state that any proposition referring to an infinite set of the set of all natural numbers will acquire "unambiguously" two *different* Gödel numbers since the one is a finite ordinal (as the finite natural number assigned as a Gödel number in arithmetic and transformed into a finite ordinal), but the other is an infinite ordinal (as involving particularly an infinite set and corresponding to a derivative set being necessarily infinite).

Then, one can notice that the Gödel incompleteness statements (i.e. both "Satz VI" and "Satz X") belong to the afore-discussed class of statements containing within itself a reference to infinite sets and thus possessing two different Gödel numbers from the viewpoint of set theory (according to the suggested procedure for assigning Gödel numbers in the framework of set theory) since both indicate explicitly "all natural numbers", which is immediately transformed into the set of all natural numbers once the viewpoint of set theory is granted.

However, one can further consider the Gödel insoluble statement as belonging to the same class of statements needing the Gödel number to be doubled after the "bifurcation" between finite arithmetic and infinite set theory, though implicitly, but necessarily. It can be decomposed into two sub-statements: (1) "The Gödel number of this statement is 'this'" & (2) "The statement possessing 'this' Gödel number is false". Since "false" should be related to "(1) & (2)", it can be equivalently transformed into "$\neg(1) \vee \neg(2)$"; that is: (1^a) "The Gödel number of this statement is not 'this'" or (2^a) "The statement possessing 'this' Gödel number is true". So, if "The Gödel number of this statement is not 'this'" is a true statement, the composed statement (whether in the former or latter equivalent forms) is true regardless of whether the latter sub-statement is true or false; on the contrary, if "The Gödel number of this statement is 'this'" is a true statement, the composed one is true if and only if “The statement possessing ‘this’ Gödel number is false” is a true statement.

The conclusion relevant in the present context refers only to the fact that “The Gödel number of this statement [i.e. the Gödel insoluble statement itself] is not ‘this’” is an admissible option consistent with the Gödel insoluble statement. It involves all natural numbers implicitly or non-constructively since its real Gödel number is some other among them, which can make sense if all natural numbers are available in advance. Then, “all natural numbers” from the viewpoint of set theory are already the “set of all natural numbers” being both infinite and necessarily involved in the Gödel insoluble statement once it has been reinterpreted in set theory.

(3) Let one assume that the Gödel incompleteness statement (“Satz VI”) can be applied to itself because it satisfies its own conditions. This means that it is considered to be a theory in a mathematical sense (i.e., an axiomatics) including arithmetic speaking generally though it is a single statement. Then, the Gödel dichotomy would be also relevant to its conclusion, which would be: either (1) it contradicts set theory (which is a premise of it to be inferred) or (2) it is incomplete to set theory.

The option means that “(1)” after contradicting its own premise is an example of *reductio ad absurdum* therefore implying its own negation, namely that its direct logical negation; that is: “The Gödel incompleteness statement is not incomplete to set theory” though it contradicts set theory. This means that it implies the proposition that set theory is false if the Gödel incompleteness statement is true (there is no formal contradiction but only by virtue of the “paradox of material implication”).

Alternatively, the option (2) means the Gödel incompleteness statement is to be complemented by at least an additional statement (respectively, a nonempty tuple of axioms), after which the Gödel incompleteness statement will be equivalent to set theory. In other words, the eventual self-applicability of “Satz VI” means that “arithmetic” can be substituted in “Satz X” with “Satz VI” itself consistently and equivalently therefore just repeating the formal structure of “Satz X” after the only change due to the fact that “Satz VI” can replace “arithmetic” in it.

Then, one can demonstrate that the only possible complement in the sense of the option (2) furthermore consistent with the option (1) is just the direct logical negation of the Gödel incompleteness statement. Summarizing, the eventual self-applicability of the Gödel incompleteness statement (i.e. to itself) implies for it to be an independent axiom (rather than a theorem) since it implies its own negation as the only possible complement of itself to set theory.

Before that formal demonstration, one can visualize its sense by a bit of information to pioneer the pathway to it. The idea is that the Gödel incompleteness statement shares the same structure as a bit of information and it is quite sufficient to exhibit its self-referentiality after the discussion above that a bit of information means two oppositions complementary to each other rather than a single one (what is common sense’s prejudice) and substituting the one by its complementary opposition in the framework of the same bit:

This means: the explicit choice of either alternative of a bit requires as a necessary condition a preliminary choice of the state “after choice” versus that “before choice” and that preliminary choice is again an elementary opposition, which can be identified with that of the explicit choice without any contradiction: this is due to the mutual complementarity (or formally, idempotency) of the opposition of the explicit choice and that of the preliminary choice (for example, as any statement and its negation are not contradictory if each of them is considered separately as e.g. Bohr's complementarity needs).

Indeed, if one considers the preliminary choice as a meta-choice to the explicit choice, this would generate an hierarchy (or for example, a temporal series of choices) *ad lib*, which in turn can be equivalently interpreted as a single idempotent pair such as the opposition of any statement and its negation and thoroughly representable by a bit of information (in much more detail in: *Penchev 2022 June 30*; *Penchev 2022 May 11*; *Penchev 2021 March 9*).

Then and quite concisely, the Gödel incompleteness statement can be interpreted as the preliminary choice to the choice embodied by what itself states to the pair of arithmetic and set theory. Respectively, the preliminary choice between the state before the choice "either arithmetic or set theory" (reducible to "either the axiom of induction or the axiom of infinity") and the state after the same choice follows from the Gödel incompleteness statement since that preliminary choice is a necessary condition for the Gödel incompleteness statement.

Then, the state before the choice "either arithmetic or set theory" corresponds to the negation of the Gödel incompleteness statement and would be a corollary from the eventual self-applicability of the Gödel incompleteness statement. The state before choice can be visualized furthermore by the concept of "epoché" borrowed from Husserl's phenomenology.[17] He meant originally an "epoché to reality", which is to be now modified as an "epoché to infinity" as refraining from judging whether a set is finite or infinite, therefore as suspending the axiom of infinity in set theory (though suspending the axiom of induction in arithmetic would be formally not less relevant).

In fact, that "epoché to infinity" can be related to mathematics *before* Cantor's set theory, only in which the concept of actual infinity really contradicts that of finiteness, first of all in arithmetic. For example, Cauchy's (so-called "epsilon - delta")definition of infinitesimal quantity implies a Gödel-like insolubility (e.g. one can prove that there exists some "epsilon", for which the corresponding "delta" both exists and does not exist as an interpretation of the Gödel insoluble statement in terms of any infinitesimal transition to a limit), which can be illustrated by Zeno's aporia about "Achilles and the Tortoise" as the necessary existence of states, in which Achilles is simultaneously before and after the Tortoise (or in other words, the probability of being before the Tortoise increases smoothly, and that of being after the Tortoise decreases also smoothly in any infinitesimally small distance between them).

However, Cauchy (as well as many of his adherents) created his famous class of infinitesimal definitions in the age preceding Cantor's set theory or respectively that of "actual infinity". So, the above kind of objections originating from the Gödel-like insolubility can be granted to be

[17] A few authors (e.g. Atten 2015; Tieszen 2011; 2010; 1994; Føllesdal 2011; Liu 2010; Cassou-Noguès 2007) discussed the gradual accession of Gödel's self-reflection to Hisserl's viewpoints about philosophy of mathematics.

irrelevant historically as well as formally and mathematically if one ascribes that "epoché to infinity" at issue to mathematics belonging to his epoch. Furthermore, one can utilize the metaphor that Cauchy and mathematics in that age had been in "Eden" naively, speaking metaphorically, before the cognition of infinity and the "expulsion from the Paradise Garden".

Another impressive example can be "Fermat arithmetic" naively not distinguishing a finite arithmetic series consisting of any natural number of members from their set able to be infinite. One can demonstrate that Fermat's last theorem is a provable statement in that Fermat arithmetic, but it is unprovable in the contemporary standard mathematics underlain by both arithmetic and set theory since they imply a Gödel-like insolubility in relation to Fermat's last theorem, which can be demonstrated by means of its interpretation in terms of Yablo's paradox (Penchev 2021 March 9). Anyway, it can be proven in a more powerful mathematics able to overcome any Gödel-like insolubility and among which Wiles's proof of Fermat's last theorem is to be enumerated though implicitly, but necessarily just because of the relevance of Yablo's paradox to Fermat's last theorem[18].

However, after the "biblical fall" of set theory has happened (as that is the case of our age), the naive sojourn in Paradise (as that of Fermat or Cauchy) is already impossible. Anyway, that approach can be restored in virtue of having enough knowledge about infinity rather than by virtue of the naive absence of any cognition of it. What is necessary is the reinterpretation of the Gödel incompleteness statement as an axiom rather than as a theorem therefore allowing for its negation (in turn relevant to the naive approach of pre-Cantorian mathematics), which would be inadmissible if a proved theorem (as Gödel and his disciples stated) was the case.

What remains to be shown is the applicability of the Gödel incompleteness statement to itself, by means of which it is to be recategorized once again from a theorem into a meta-mathematical axiom relevant to the relation of arithmetic to set theory under the condition of propositional logic as their "zero-order" logic and more precisely said, predetermining it to obey the Gödel dichotomy "either incompleteness or contradiction". Its negation, i.e. postulating for their relation to be both "completeness and non-contradiction" is not less admissible just if the Gödel incompleteness statement can be considered as a theory sharing the same relation to set theory as arithmetic (i.e. "Satz X").

Discussed substantively, not formally, the Gödel (1931) "incompleteness paper" is a meta-theory (and thus a theory) about the relation of the two most fundamental mathematical theories, arithmetic and set theory, furthermore able to be the foundations of all branches of mathematics, if both are considered as first-order logics to classical propositional logic. Consequently, the "incompleteness paper" satisfies the conditions of its main statement ("Satz VI") therefore implying in turn its applicability to the paper as a whole and itself particularly. Then, all above conclusions in favor of the interpretation of the Gödel incompleteness paper as a meta-mathematical axiom rather than a theorem are valid since they are direct corollaries from that self-referentially at issue.

[18] Fermat's last theorem can be interpreted to be a true statement among those "true assertions about the natural numbers which do not follow from the Peano axioms" meant as a class by Weiermann (2009).

A purely formal consideration is also relevant elucidating the implicit necessary condition for a class of consistent syllogisms to be meant as a single theory according to the proof of both “Satz VI” and “Satz X”. It consists in the option for all statements relevant to that class of consistent syllogisms claiming to be a single theory to be enumerable by some subset of all natural numbers (particularly following the constructive rules for assigning a certain “Gödel number” suggested explicitly by himself). In other words, the necessary condition for a class of syllogisms to be considered as a single theory according to the proof itself consists in being consistent to each other and unambiguously certain Gödel numbers to be assignable to all statements relevant to those syllogisms.

As to a single statement (such as “Satz VI”), the requirement to be consistent is reduced to be non-contradictory. Furthermore, it is to possess a Gödel number, a problem discussed a little above and it is to be unambiguous, i.e a single one from the viewpoint of arithmetic, or more precisely said according to the Gödel rules for the enumeration of all relevant statements. Thus, it is self-applicable if it is not contradictory therefore repeating literally the Gödel dichotomy “either incompleteness or inconsistency” meant by it: now to itself.

Consequently, the Gödel incompleteness statement allows to be applied to itself self-referentially an an absolutely consistent way and implies the above conclusion that it is rather an (eventually metamathematical) independent axiom than a theorem, thus, alternatively admitting also its negation to be added to the corpus of all axioms of arithmetic, set theory, and propositional logic.

The statement often called “diagonal lemma” and proved a little later (Carnap 1934) suggests the following consideration to be investigated: whether the application of arithmetic to set theory by means of the Gödel enumeration implies or not for set theory to be interpretable as still one copy of arithmetic (only modified by the substitution of the axiom of induction with that of infinity) after Gödel himself demonstrated the “immovable point” of that mapping what the “diagonal lemma” implies as well[19]. That conjecture will be discussed in detail in the next section.

It can be also visualized by the analogy with the pair of Euclidean and non-Euclidean geometries differing from each other only in relation to the Fifth postulate (either it or its negation). Indeed, non-Euclidean geometry can be interpreted as a second copy of Euclidean geometry, after which the immovable point of that mapping (i.e. that of Euclidean geometry into its second copy embodied as non-Euclidean geometry) is truly paradoxical just as the Gödel insoluble statement itself: that is a unique “glocal” neighborhood of any point in Euclidean space (respectively, any point of non-Euclidean space) able to be both locally “flat”, but globally “curved” in virtue of the fact that both local and global interpretations are relevant to it. Obviously, Riemann’s idea of space curvature as a smooth (even being only continuous) quantity implies the necessary existence of that “glocal” neighborhood at issue.

[19] Adamowicz and Bigorajska (2001) suggest a proof of the “second incompleteness theorem” without the diagonal lemma also relevant to “what Gödel's second incompleteness theorem shows” according to Moore (1988) or to “another look” (Wisser 2030; 2012) to it.

VI. THE VIEWPOINT TO ARITHMETIC AND SET THEORY AS A PAIR OF AXIOMATICS DISTINGUISHABLE BY A SINGLE AXIOM

The conjecture meant in the last paragraph of the previous section can be formulated loosely as follows. Arithmetic and set theory seem to be absolutely different, but maybe this is only an appearance at first glance underlined rather by the tradition than by their substantial distinguishability otherwise than by a single axiom and its negation: either the axiom of induction for arithmetic versus its negation as the axiom of infinity for set theory. The parallelity with the pair of Euclidean and non-Euclicdean geometries often used in the paper visualizes and exactly corresponds to the same idea.

The arguments in favor of the conjecture that arithmetic and set theory are the same structure only doubled in two dual copies according to the two idempotent versions of the same axiom can be supported by a few groups of arguments:

(1) Any mathematical structure which is a first-order logic possesses an arithmetic model;

(2) Peano axiomatics for arithmetic originated from set theory and more precisely, from the well-ordering relevant to any set by virtue of the axiom of choice following after ideas Dedekind's "Was sind und was sollen die Zahlen?" (1888);

(3) arithmetic is not more than a doctrine about the structure shared by all well-orderings assignable to all sets to which the axiom of choice is applicable;

(4) Auxiliary algebraic considerations about the operations definiable arithmetically versus those as to set theory.

A few notices about (1):

The statement that any mathematical structure which is in the framework of a first-order logic possesses an arithmetic model can be considered as an interpretation of Lindström's theorem (1969)[20] defining a first-order logic by both compactness (due to the compactness theorem[21]: Gödel 1930: "Satz X"; Tychonoff 1930) and Löwenheim–Skolem properties (for the Löwenheim (1915)-Skolem (1920) theorem)[22]. The latter implies that a theory referring to sets of any cardinal number is isomorphic to a theory referring only to countable sets, and intuitively, it is due to the axiom of choice able to reduce any cardinal number to countability. The former requires that a theory has a model if and only if any *finite* set of its sentences has a model.

The restriction for mathematics to be thus limited only to all first-order logics is not too strong because of the following. Any mathematical structure featured by any higher order logic can be equivalently translated into the "language" of a certain relevant first-order logic by adding self-referential entities external to the mathematical theory at issue:

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[20] It is generalized in many ways (e.g. Enqvist 2016), and discussed by Lindström (2017) himself in relation to incompleteness.

[21] The "compactness theorem" can be discussed as a class of theorems relevant to various branches of mathematics, but sharing all finite representations of a structure as a necessary and sufficient condition for the literal statement relevant to the branch at issue (e.g., Danhof 1974; Kimber 1974). As to that of first-order logic (e.g. Rubin 1973, or in a series of proofs as in: e.g. Paseau 2010) or Dawson (1993) have traced it from Gödel to Lindström; and Kohlenbach (2012), to "weak compactness".

[22] A generalization is suggested by Bloom (1973).

Speaking loosely, any mathematical theory can remain a first-order logic only by complementing it with a relevant external set to be its object "holistically", i.e., including corresponding systems among the studied, or in other words, only projecting the self-referentiality of any eventual higher-order logic as an equivalent self-referentiality of what is studied by means of the concepts of "whole" or "system".

That approach can be traced back to quantum mechanics defined, for example by Niels Bohr, by its self-referentiality in fact: indeed, as a science investigating quantum *systems including the measuring apparatus itself* by the readings of the macroscopic apparatus (Bohr 1934)[23] (thus described by the smooth differential equations of quantum mechanics). Then, the theory of the separable complex Hilbert space introduced by it can be a "normal" first-order logic referring to quantum reality being "abnormal" for its inherent definitive self-referentiality.

In a sense, quantum mechanics forced by its subject (quite extraordinary in comparison with all ones accessible to classical science) generates a general approach and model for how any reality of higher order to be studied scientifically, objectively, and quantitatively, on the one hand immediately, but on the other hand, as Hilbert arithmetic in both narrow and wide meanings, can demonstrate, a not less relevant approach and model for any higher-order mathematics to be reduced to a corresponding first-order logic, by the by, able to borrow the proved completeness of quantum mechanics (by the theorems of the absence of hidden variables: *Neumann 1932*; *Kochen, Specker 1967*) for the completeness of mathematics itself[24].

In proper mathematical terms, the adoption from quantum mechanics consists in the equivalency (postulated to be a formal definition of completeness if need be) of duality (idempotency) and hierarchy (well-ordering) reflectable even by the equivalence of the well-ordering "theorem" and the axiom of choice[25] and discussed in detail in other papers (Penchev 2022 June 30; Penchev 2022 May 11; Penchev 2021 March 9).

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[23] For example, an exact quotation of Niels Bohr is the following: "... I advocated a point of view conveniently termed "complementarity," suited to embrace the characteristic features of individuality of quantum phenomena, and at the same time to clarify the peculiar aspects of the observational problem in this field of experience. For this purpose, it is decisive to recognize that, *however far the phenomena transcend the scope of classical physical explanation, the account of all evidence must be expressed in classical terms*. The argument is simply that by the word "experiment" we refer to a situation where we can tell others what we have done and what we have learned and that, therefore, the account of the experimental arrangement and of the results of the observations must be expressed in unambiguous language with suitable application of the terminology of classical physics" (Bohr 1949: 209).

[24] A few authors (e.g. Ben-Ya'acov 2019; Srikanth, Hebri 2008; Hon 2004; 1996) noticed and researched the relationship of incompleteness (respectively, completeness) in both foundations of mathematics and physics; as well as causality in Gödel's philosophy (Kovač 2020). As to incompleteness in quantum mechanics, their most relevant intersection can be situated in the concept of entanglement or nonlocality, after which any local description in quantum mechanics can be interpreted to be incomplete (e.g., Redhead 1987).

[25] "Institute for Advanced Study, Princeton, NJ, Univalent Foundations Program": for example, (2013) Homotopy type theory: univalent foundations of mathematics. Princeton, NJ: Lulu Press, Univalent Foundations Program.

Indeed speaking more intuitively, one can rid of the dichotomy of "either finiteness (respectively, the axiom of induction in arithmetic) or infinity (respectively, the axiom of infinity in set theory)", furthermore resulting in the Gödel dichotomy of the relation of arithmetic to set theory (which is the proper subject in the present part of the paper) by duality (idempotency) as far as any finite number of repeated idempotency seems to be equivalent to that of any infinite cardinal number since the options are only two (either of both). That is: an eventual infinite cardinal number of records of either "0" or "1" in a binary cell (for example, of a Turing machine[26]) cannot but be either "0" or "1" (at least out of intuitionistic mathematics, which also resolves an analogical problem though otherwise, but equivalently in the final analysis).

So, any first-order logic starting from classical propositional logic can effectively accomplish a Husserlian "epoché" to infinity (therefore interpreting Husserl's original one, being "to reality") only at the cost of the equivalent substitution of hierarchy by duality particularly getting rid of Gödel's insolubility situated between any finite hierarchy and any infinite hierarchy. One can interpret that "getting rid of" in the epistemological model pioneered by quantum mechanics for describing hierarchic reality (for its inherent and definitive self-referentiality) by means of a first-order logic, but necessarily to be inherently dual, such as the separable complex Hilbert space.

Summarizing both premises in the last paragraph, one may state that just the definitive duality of the separable complex Hilbert space allows for uniforming the descriptions of any quantum systems with its readings by the apparatus therefore authorizing the new generalized kind of objectivity possible within quantum mechanics. In other words, one could say that the duality at issue involves "quantum phenomenon" in a Husserlian manner, namely by an "epoché" to the apparatus: that is, "quantum phenomenon" relies on an "epoché" in relation to the alternative of "whether the readings of the apparatus or the studied quantum system by itself".

Finally, one can observe, that both compactness and Löwenheim-Skolem properties relevant to Lindström's definition of "first-order logic" can be in turn unified after Skolem's (1922) "relativity of the notion of 'set'" or by the correspondence of Hilbert arithmetic in a wide sense to that in a narrow sense.

Indeed, Skolem's paper (1922) expressly indicates that suggested by him idea about the "relativeness of the concept of "set"" can be extended from infinite sets of any arbitrary cardinal number following literally the Löwenheim-Skolem theorem to all finite sets after a Dedekind-like, purely set-theoretical definition of "finiteness" meaning the absence of any bijection into its own true subset (i.e., as the negation of Dedekind's definition of 'infinite set'). That extension of the "relativeness of the concept of 'set'" to all finite sets allows for a kind of indistinguishability of infinite sets from finite ones therefore recollecting Husserl's "epoché" now applied to the pair of infinity and finiteness. Furthermore, it can be related to the compactness theorem implying as a

[26] Gödel's "first incompleteness theorem" in relation to "Turing computability" is investigated (e.g.) by Hadley (2008), and his "second incompleteness theorem" to "Kolmogorov complexity": by Kikuchi (1997); both to computer science, by Murawski (1997) or Uspenski (1994). Accordingly, Gödel's completeness can be related to Markov's principle (McCarty 1994), or respectively the compactness of first-order Gödel logics to be researched (Pourmahdian, Tavana 2013).

corollary any model relevant to an infinite set to be equivalent to a set of models relevant to all finite sets of the infinite set at issue.

For example, one can "name" all true subsets (which are in turn infinite) of any infinite set by the cardinal number (whether finite or infinite) of its complement to the infinite set issue and then, consider the reciprocal value of that cardinal number (respectively, whether an infinitesimally small quantity equivalent to zero or a finite rational number less than a unit) therefore resulting effectively into a certain probability distribution in turn unambiguously corresponding eventually to a probability density distribution, and in the final analysis, to a wave function as its characteristic function.

Then, all infinite sets could be "named" by the set of all wave functions, i.e. by all points of the separable complex Hilbert space (before that utilizing the equivalence of cardinal numbers in virtue of the Löwenheim-Skolem theorem) and any of those names would mean a unique profile to all finite sets so that those finite sets relevant to an infinite set in virtue of the compactness theorem to be featured by any finite (i.e. nonzero, not being infinitesimally small quantities) in the corresponding "name" representing a wave function in the final analysis (or an equivalent probability density distribution eventually).

The same "name" relevant to all sets of all cardinal numbers[27] can be interpreted by the following procedure borrowing the Dedekind-like finiteness (elucidated above) after granting the absence of any bijection within it as the absence of any *permanent* bijection therefore allowing for some, fundamentally random bijection of any infinite set into a certain finite, but necessarily different in general after the repetition of the same bijection. Then, if one considers the probabilistic space of all those trials for an infinite set to be mapped by a bijection into any finite set, a probability (density) distribution would be relevant to it. In other words, the absence of any permanent bijection in virtue of a Dedekind-like definition of finiteness is interpreted as an eventual admissibility of a probabilistic bijection in the exact meaning above.

Then, the established name of any infinite set (as a certain infinite set) to be a wave function allows further for involving Hilbert arithmetic and more exactly the also idempotent duality of both Peano arithmetics of Hilbert arithmetic in a narrow sense (i.e. the pair of two dual, being anti-isometric to each other, Peano arithmetics[28]) and Hilbert arithmetic in a wide sense[29] (i.e. the pair of two dual and anti-isometric qubit Hilbert spaces relevant or equivalent to the two dual separable complex Hilbert spaces of quantum mechanics).

Particularly, that inclusion of Hilbert arithmetic is able to illustrate immediately the Löwenheim-Skolem theorem as obvious by the transformation of any hierarchy (respectively, the

[27] Though sets of different cardinal numbers can be "called" by the same "name" due to the Löwenheim-Skolem theorem.

[28] The "second" Peano arithmetic allows for infinity to be interpreted as a second finiteness as (e.g.) Fjelstad (2007) or Firby (1971) did, though rather implicitly or in the context of "more infinity for a better finitism" (Sanders 2010).

[29] One might trace back the idea of "Hilbert arithmetic in a wide sense" to Chaitin's "information incompleteness" (e.g., 1987; 1992) or philosophically, to probabilistic epistemology and ontology being linkable also to ideas of logicism (Galavotti 2003).

well-ordering of successive cardinal numbers) into an equivalent duality of only two members (for example interpretable as "finiteness and infinity"): however repeatable *ad lib*.

Furthermore, one can show even the famous "continuum hypothesis" of Cantor (proved to be and independent axiom of set theory by two separated papers of Gödel in 1940 and of Cohen in 1963 - 1964) again as an obvious corollary from the equivalence of "hierarchy" and "duality", at that, in a generalized form: a new one cardinal number can be inserted between any two cardinal numbers, not touching all the rest axioms of set theory, in virtue of which the equivalence at issue is relevant. In other words, the duality of two "alefs" (for example, "alef (1)" and "alef (2)" as in the "continuum hypothesis" itself) can correspond to any series of successive "alefs" consistently insertable between them just as the class of any pair of two different natural numbers admits an arbitrary series of consecutive natural numbers between them (for example, as rational numbers within the same interval).

Elucidations about (2): in fact, Peano's paper follows Dedekind's (1888) work (which is expressly cited in the preface namely as such: *Peano 1889*: V), and the essence of his axiomatics consists in determining all natural numbers as *finite* ordinal numbers corresponding to the classes of well-orderings (i.e. those, each of which shares a certain finite ordinal number). The axiom of induction is equivalent to the statement that all natural numbers are finite (as it is proved in an elementary way above).

Thus, Dedekind's idea accomplished explicitly by Peano can be reduced to the derivability of both operations definable in arithmetic (i.e. "addition" and "multiplication") as well as the laws, which they obey (e.g. both associative, both commutative laws, but only one distributive law) as relevant to all classes of all classes of any finite well-orderings since all natural numbers can be exhaustive defined in virtue of Peano's axioms.

In other words, the usual or implicit definition of natural numbers before Peano by means of their relevant algebra is rigorously deduced only from a few fundamental properties or relations featuring all natural numbers, furthermore restricted to be only *finite* by virtue of the axiom of induction, which are expressly listed in Peano's axioms as necessary and sufficient for arithmetic to be thoroughly inferable.

That is: the specific algebra of all natural numbers serving before Peano for arithmetic to be deduced is underlain by just several properties or relations shared by all finite well-orderings: this is the exact meaning or real sense of Peano's idea relying on Dedekind's one, which in turn indicates that the origin of well-orderings, whether finite (which alone are studied by arithmetic) or infinite, is Cantor's set theory: Dedekind's insight was reliably grounded about two decades later by means of a special new axiom by Zermelo, the axiom of choice, namely in order to justify the property of any set meant by set theory to be well-orderable. Then, Whitehead and Russell in *Principia mathematica* showed in an extremely simple way that the well-ordering "theorem", inferred by Peano from the newly introduced axiom of choice, in turn implies the latter therefore being equivalent to it.

Meaning that, the troubles about the foundations of mathematics in the first decades of the 20th century can be concisely depicted as follows. Any first-order logic (as almost all or all mathematics can be represented) is equivalent to (finite) arithmetic models. Arithmetic can be deduced from set theory as a theory relevant to all finite sets and their corresponding finite ordinals. However: (1) set theory admits true paradoxes starting from the famous one suggested by Russell in a letter to Zermelo (1902); the Gödel incompleteness statement (granted to be theorems: both "Satz VI" and "Satz X", 1931) states that arithmetic either does not imply in turn set theory, or does contradict it.

The real way out from those troubles about the foundations of mathematics, furthermore being obviously inconsistent as a whole, and advocated during the present and other papers consists in making clear that the Gödel incompleteness statement is a metamathematical axiom about the relation of mathematics to the world also representable properly mathematically as the relation of arithmetic and set theory if both are first-order logics on the basis of classical propositional logic.

However, the entire organization of cognition in Modernity (or using Michel Foucault's concept, "episteme") predetermines mathematics only to build models of reality, but being inherently and fundamentally different from "reality by itself" and thus within mathematics itself, as the Gödel incompleteness statement can be also realized. It is granted to be a "theorem" rather in virtue of philosophical considerations, at that, hidden and implicit, therefore preventing any investigation of its real status: to be an axiom, consequently assuming eventually its negation at least in the framework of mathematics.

The epistemological situation due to the confusion of mathematical and philosophical arguments[30] seems to be quite analogical to that in the beginning of the 19th century for the discovery of non-Euclidean geometry by the simple substitution of the Fifth postulate by its negation. Though properly mathematically researched, Euclidean geometry is formally equal to non-Euclidean geometry, the tradition, particularly philosophical, established the absolute privilege and thus supremacy of Euclidean geometry: even as a kind of self-censure not allowing for a so great mathematician such as Carl Friedrich Gauss to publish his investigation on non-Euclidean geometry.

A century later one might expect that any similar self-censure should weaken because of the advance of humankind to freedom. However, just on the contrary, it has strengthen even to such a degree the statement analogical to the Fifth postulate in its own formal and mathematical status to be "successfully proved" as a "theorem" uncritically accepted by almost all mathematicians to be a real theorem (while the Fifth postulate, maybe for Euclid himself called it a "postulate", gave an exceptionally powerful impetus for the development of mathematics, physics and science, after accepting its negation to be equally admissible).

Alas, an extremely disconcerting conclusion as to philosophy of science is plausible: human progress (at least nowadays) does not decrease dogmatism and conservatism in science, the blind belief in scientific authorities (such as Aristotle was in the Middle Ages), even increases them to the degree for any criticism to them to be excluded (and perhaps, this is an effect from the general organization of society not allowing for any really radical dissent just by means of the ostensibly and only seeming permissibility of any opinions therefore mutually neutralizing each other). One

[30] For example, following Reinhardt (1986).

might utilize the metaphor of "black hole": the volume of knowledge has exceeded a "critical threshold", after which the available knowledge is so much that it capsules into itself therefore not allowing for any fundamentally new cognition to appear.

About (3): arithmetic is not more than a doctrine about the structure shared by all well-orderings assignable to all sets to which the axiom of choice is applicable. The observation meant in (2) about the origin of Peano's axioms postulating for arithmetic to be a theory of all classes of all classes of all finite well-orderings (respectively, of all finite sets being well-orderable also without the assistance of the axiom of choice) can be generalized just by involving the axiom of choice, after which Peano arithmetic would be extendable to set theory and even equivalent to it (therefore eventually sharing the negation of the Gödel incompleteness statement) as long as there did not exist the contradiction of the axiom of induction, implying for all well-orderings in the scope of arithmetic to be finite, and the axiom of infinity.

In other words, and speaking loosely, set theory may be complemented by an "infinite arithmetic" to which the axiom of induction is suspended in order to be avoided the contradiction to the axiom of infinity so that both arithmetics together (i.e. the one being "finite", and the other one being "infinite") to be really equivalent to set theory therefore overcoming the Gödel incompleteness statement or sharing its negation. That suspension of the axiom of induction (though being always in the framework of its negation, i.e. the axiom of infinity) admits different interpretations, for example, involving a second induction relevant only to all infinite well-orderings, usually identified with "transfinite induction". Once "transfinite induction" has been introduced, one can question how it can be distinguished from the former me, the usual finite induction of Peano arithmetic.

The answer would be amazing: in nothing besides it is *necessarily* a *second* induction being only dual (or "complementary" in the sense of quantum mechanics) to the former, usual induction of Peano arithmetic so that it is consistent with the axiom of infinity as the negation of the axiom of induction. Using "complementarity" borrowed from quantum mechanics, one can say that induction and transfinite induction are (necessarily and sufficiently) only to exclude each other therefore being mutually (or idempotently) "complementary", thus each of them being consistent correspondingly with either the axiom of induction or its negation, what the axiom of infinity is.

The above observation that finite induction and transfinite induction do not differ from each other otherwise than being idempotently dual or complementary can be complemented by the statement that just the axiom of induction is the only obstacle for the unification of arithmetic and set theory (particularly implying either the Gödel incompleteness statement or its negation) since all the rest of Peano axioms are consistent with both arithmetic and set theory being equally relevant to any well-orderings whether finite or infinite.

Then, one can create a Peano arithmetic model (at all not the only possible one, but the only one if it is to be consistent with the qubit Hilbert space of quantum information) of both finite and transfinite inductions, thus a Peano arithmetic model of set theory, just by postulating for Peano arithmetic to be doubled by a dual (or complementary) counterpart, i.e. an "identical twin" so that what relates to the former does not relates to the latter and vice versa; that is: the two Peano arithmetics are to be really and idempotently dual or complementary. The so described idea can be embodied rigorously by the following construction: their duality or complementarity to be embedded in their mutual anti-isometry, by the by, an approach also borrowed from the complex Hilbert space of quantum mechanics. Then, their duality can be restricted to the following axioms:

(A) The least natural number of Peano arithmetic is a unit usually notated as “1”.

(A’) The least natural number of dual Peano arithmetic is the least countable ordinal usually notated as “ω” (necessarily existent in set theory in virtue of the axiom of choice, but now only postulated).

(B) The function successor of Peano arithmetic defined for any natural number “n” is “n+1” so that the sequence of natural numbers is: 1, 2, 3, …, n, … .

(B’) The function successor of dual Peano arithmetic defined for any dual natural number (notates also to be “n”) is its dual, anti-isometric counterpart, “$n - 1$” so that the sequence of natural numbers is: ω, $\omega - 1$, $\omega - 2$, $\omega - 3$, …, $\omega - n$,, … .

One can easily check that the dual counterpart satisfies all Peano axioms as well though containing different notations in order to be distinguishable (according to its interpretation) from Peano arithmetic (now interpreted to be the one “identical twin”) just as two human beings who are identical twins are always *called differently* to be distinguishable from each other.

That construction doubling Peano arithmetic is titled “Hilbert arithmetic in a narrow sense” since it can be also deduced from “Hilbert arithmetic in a wide sense” in turn identical with the qubit Hilbert space originating from the separable complex Hilbert space of quantum mechanics after any natural number of Hilbert arithmetic in a narrow sense is the class of equivalence of all values of the corresponding qubit of the qubit Hilbert space, i.e. that of Hilbert space in a wide sense, or in other words, an “empty” qubit in which can be “recorded” any value.

Thus, Hilbert arithmetic in a narrow sense can be a consistent complete basis of all mathematics, and the qubit Hilbert space can be the complete consistent basis of quantum mechanics and consequently, of all physics: Hilbert space in a wide sense is able to represent the inherent and fundamental unification of physics and mathematics in their shared foundations (just as quantum neo-Pythagoreanism needs).

The construction of Hilbert arithmetic in a narrow sense elucidates that Peano arithmetic and set theory can be related analogically to the pair of Euclidean and non-Euclidean geometry differing from each other only in the Fifth postulate sharing correspondingly either itself or its negation, on the one hand, being inconsistent, but on the hand, being unifiable by Riemann’s idea of space curvature (generalizable also to be a tensor, but this not essential in the present context). Then, set theory can be not less consistently discussed as a “non-Peano arithmetic” identical with Peano arithmetic in all other besides the axiom of induction also equivalent to the statement that “all natural numbers are finite”, respectively “all ordinals are finite”, and in the final analysis, that all sets are finite.

One can trace back that the axiom of induction can be equivalently interpreted as the axiom that “All sets are finite” accordingly only substituted by its negation in set theory, namely that “All sets are not finite” in turn equivalent to the axiom of infinity stating that there exists infinite sets (i.e. along with all finite sets meant by arithmetic by the concept of "natural numbers", all of which are postulated to be finite in virtue of the axiom of induction being equivalent to the negation of the axiom of infinity).

Even more, the axiom of infinity suggests a *constructive* procedure being isomorphic (respectively homomorphic from a proper algebraic viewpoint) to that meant by the axiom of induction, but on the contrary postulating that it defines the existence of an infinite set, or in other words, a statement equivalent to the literal and direct, formal and logical negation of the axiom of induction.

The difference between both relations of alternative axiomatics, namely Euclidean and non-Euclidean geometry, on the one hand, and arithmetic and set theory, on the other hand consists in the quite different historical pathways for each of both to appear rather than in the formal structure of those relation proved to be identical above. Indeed, non-Euclidean geometry originated directly from Euclidean geometry by investigations on the independence of the Fifth postulate (respectively, on the admissibility to be substituted by its negation in an alternative, but not less consistent axiomatics).

On the contrary, arithmetic and set theory appeared as two ostensibly absolute different mathematical theories thus referring to seemingly quite mismatching subjects, correspondingly natural numbers and sets. In addition, Gödel contributed to mystifying their relation by his famous paper (1931). The fact that his approach blurs rather than elucidates their real relation of a pair of axiomatics idempotent, dual, and complementary to each other differing in a single axiom just as Euclidean and non-Euclidean geometry can be made obvious if one projects it back, on the debate on non-Euclidean geometry during the 19th century, though counterfactually:

Its imaginary counterpart would be a (really inexistent) paper inferring the Fifth postulate from all the rest axioms of Euclidean geometry only in order to demonstrate for the dichotomy of the relation of Euclidean geometry to non-Euclidean geometry to be "either incompleteness or inconsistency" being wrongly interpreted to be two absolutely different theory with two, quite mismatching subjects.

Indeed, the last statement can be proved, but it is trivial to the pair of Euclidean geometry and and its non-Euclidean counterpart after Lobachevsky's works able to show their mutual inconsistency and Riemann's famous dissertation, which can be particularly interpreted to exhibit the incompleteness of Euclidean geometry to non-Euclidean geometry in virtue of the parameter (or tensor) of space curvature introduced by him: indeed, Euclidean geometry is the special case of zero space curvature therefore making possible Einstein's general relativity interpreting the *physical* theory of gravitation as the *mathematical* pseudo-Riemannian geometry.

However, the philosophical and ideological (in a wide sense) direction of that counterfactual counterpart of Gödel's paper (1931) in Riemann's age would be opposite: it would state that the unification of Euclidean and non-Euclidean geometries (as by means of Riemann's space curvature) is fundamentally impossible due to the "proved" relation of Euclidean geometry to be "either inconsistent or contradictory" to non-Euclidean geometry therefore preventing any scientific development able to reach Einstein’s general relativity (identifying geometry and physics, and thus, mathematics and physics in the final analysis).

Parallelly, but also counterfactually, one can imagine a counterpart of Riemann’s dissertation as to the relation of arithmetic and set theory therefore sharing a philosophical and ideological (in a wide sense) direction opposite to that of Gödel’s real paper (1931), furthermore able to give impetus to an analogue of Einstein’s general relativity in the foundations of mathematics relevant to both arithmetic and set theory as long as Gödel’s real paper (1931) did not exist (i.e. only in that counterfactual reality at issue, but not in ours).

It would consider arithmetic to be “flat” or only “local” in relation to set theory being the general case corresponding to any space curvature “globally” rather than only the particular case of arithmetic, corresponding to the zero space-curvature of Euclidean geometry. Then, the parameter of space curvature itself would be to be interpreted in terms of those counterfactually and Riemann-likely unifiable arithmetic and set theory as a newly-introduced “distance between finiteness and infinity” able to be equally well a negative quantity (corresponding to “dialectic mathematics” above) just as a positive quantity (corresponding to intuitionistic mathematics). The especially important special case of zero distance between finiteness and infinity would be occupied by “Gödel mathematics”, only to which the Gödel incompleteness statement would be valid but only because of being postulated.[31]

On the contrary, Gödel's real paper (1931) prevents any debate on non-Gödelian mathematics wrongly, but rather implicitly, stating that Gödel incompleteness statement is a theorem. The reasons are rather philosophical and ideological (in a wide sense), relying in the final analysis in the modern episteme itself predetermined for mathematics to create only models of reality[32] itself thus necessarily mismatching in definition and excluding any option of the coincidence of model and reality to any degree.

A frequent, but loose interpretation of space-time as pseudo-Riemannian space quests after the “real curvature of our world” establishable by physical experiments in exactly determined points of space-time. That interpretation is necessarily loose in the usual organization of cognition in Modernity gapped physics from mathematics (including geometry), however admitting geometry to be ambiguously considered both as an experimental physical science and as a deductive and axiomatic branch of mathematics after Euclid therefore ostensibly hiding the abyss of modern episteme within it.

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[31] Anyway one can consider propositions corresponding to the Gödel incompleteness statement in both dialectic and intuitionistic mathematics by mediation of its Gödel number: those possessing simultaneously two Gödel numbers in the former or no one as to the latter. The discussion above about the Gödel number of the Gödel incompleteness statement itself goes out of the framework of Gödel mathematics, in fact: in other words, the validity of the Gödel incompleteness statement is the equivalent to the postulate that its Gödel number is unambiguous, single and finite; then set theory accepts its Gödel number assigned by arithmetic to be in default or definitively.

[32] One can question whether the concept of model or, respectively, model theory in mathematics allow at all for completeness: a problem discussed by Read (1997) in the contexts of the works of Frege and Gödel.

Anyway, general relativity just as quantum mechanics already fundamentally refuting the premises of classical cognition and in a way opposite to Einstein's opinion admits to be interpreted also non-classically: by means of the identification of model and reality borrowed from quantum mechanics and fundamentally contradicting the attitude of classical science. Then, Einstein's theory of gravity offers the real geometry of reality where geometry is an only formal and mathematical discipline and it cannot but coincide with reality by itself just as quantum mechanics does in virtue of the theorems of the absence of hidden variables (Neumann 1932; Kochen, Specker 1967) and contradicting the classical scientific principles (in detail, in *Penchev 2021 June 8*).

The special property of general relativity to be interpretable both classically and after quantum neo-Pythagoreanism is due to its smoothness (or meaning mathematically a smooth manifold, to which it refers) implying the indistinguishability of the corresponding two options already contradicting each other in quantum mechanics because of the Planck constant. Model and reality in general reality are able both to coincide just as it is necessary in quantum mechanics and to mismatch as any classical physical theory needs but by infinitesimally small quantities therefore identifying experimentally those two alternatives.

So, one can speak of the *geometrical curvature of reality* in general relativity in both literal and metaphorical way since they are indistinguishable from each other in any empirical way thus generating inherent ambiguity, The same ambiguity is already divisible into two discernible parts if the pair of arithmetic and set theory is meant: what is relevantly discussable is the parameter of the distance between finiteness and infinity (interpretable also as the degree in which mathematical model and physical reality coincide) and granted to be zero without any debate or experiments in Gödel mathematics nowadays adopted to be the standard mathematics in default in Modernity.

The relevant physical experiments analogical to those following general relativity to establish the geometric curvature of reality, but now meaning the distance of finiteness and infinity (or that of model and reality) fall in the area of quantum mechanics and information and more especially, after investigating the phenomena of entanglement. That distance can be defined loosely as the degree of entanglement in a vacuum, which can be zero only in particular.

The pair of Euclidean and non-Euclidean geometry differable from each other in the Fifth postulate alone and initially seeming to be rather only an introductory metaphor or analogy to the pair of arithmetic and set theory relevant to the Gödel incompleteness statement turns out now to be linked to the latter pair of axiomatics in essence. Indeed, one can trace back and forward a formal and logical bridge able to connect the former and the latter pairs into a single syllogism, the stages of which are:

(1) The transition from Euclidean geometry to a variable manifold of non-Euclidean geometries following a smooth function of the Riemann (tensor) space curvature and called Rieman geometry (often distinguished from "Riemannian geometry") therefore being inherently representable in the generalized four-dimensional Euclidean space as a corresponding also four-dimensional "body" or manifold.

(2) The transition from Euclidean space to the imaginary domain of Minkowski space utilized by Einstein's theory of special relativity, furthermore interpreted by means of Poincaré's

conjecture proved by G. Perelman for the former to be topologically equivalent to the latter (i.e. topologically complementing the imaginary domain at issue to the 3-sphere meant by Poincaré's conjecture).

(3) Then, one can constitute the pseudo-Riemannian space of general relativity (at that, able to be topologically equivalent to Euclidean space by Poincare's conjecture) by synthesizing the previous two stages (1) and (2). Then, it can be complemented by its dual counterpart starting from the real domain of Minkowski space and following the same procedure as that from the corresponding imaginary domain.

(4) The axiom of choice is necessary to be involved in order to be mapped homeomorphically Minkowski space into the qubit Hilbert space originating from the separable complex Hilbert space of quantum mechanics. Then, the dual Hilbert space would correspond unambiguously to the real domain of Minkowski space being idempotent to the imaginary one.

(5) One can build an isomorphism of all possible world lines belonging to pseudo-Riemannian space and all wave functions represented by the qubit Hilbert space following the same unambiguous rule: any deformation of Minkowski space, which is relevant to a certain moment and equivalent to the state of pseudo-Riemannian space is mapped bijectively by a certain value of the corresponding qubit (for example, as a frequency reciprocal to the certain moment at issue, to which the deformation of Minkowski space refers).

(6) The qubit Hilbert space is identical to Hilbert arithmetic in a wide sense in turn corresponding to Hilbert arithmetic in a narrow sense as classes of equivalences of all values of each qubit of both dual spaces. Then, one can notice that the parameter of the "distance between finiteness and infinity" translated in the language of Hilbert arithmetic in a wide sense means either shared qubits (in the case of "dialectic mathematics') or missing qubits (in the case of "intuitionistic mathematics"), which also can be interpreted as shared ones, but eventually distinguishable from those in the former case by a conventionally addable "sign" (for example, "plus" in the former case, but "minus" in the later case) just as one certain space curvature of opposite signs means just one hyperbolic non-Euclidean geometry and just one spherical non-Euclidean geometry.

(7) The general case of pseudo-Riemannian space featured by the tensor quantity of variable space curvature assignable to each point of the manifold (along with the substitution of Euclidean space with Minkowski space as relevant to any infinitesimal and thus flat neighborhood of any point) accordingly translated in the language of Hilbert space in a wide sense would mean states of variable entanglement and then, i.e. in a narrow sense, states of variable distance between finiteness and infinity (i.e, a variable number of shared or missing units between the two dual arithmetic therefore a variable degree of the coincidence of mathematical model and reality in general).

Thus, one can trace back and forward, step by step, by means of successive transformations that the solution invented by Riemann in relation to a certain dual pair of axiomatics can be relevantly applied to another pair of those at the cost of destroying the prejudice that the Gödel incompleteness statement is a theorem (rather than an axiom analogical to the Fifth postulate) in turn relying on the much more fundamental organization of cognition in Modernity, establishing for mathematics to build only models of reality, inherently different from it by itself.

A few notices referring to (4): auxiliary algebraic considerations about the operations definable in arithmetic versus those as to set theory. The following observation can be the starting point: the definitions of the reverse arithmetic operations, namely "subtraction" and "division" need self-referential statements to the straight operations meant by Peano arithmetic, namely "addition" and "multiplication". Indeed:

"$a - b \Leftrightarrow (def)\ \exists x{:}\ x + b\ = a$" or "$a/b \Leftrightarrow (def)\ \exists x{:}\ x/b\ = a$"

Those definition are self-referential in relation to "x" in the following exact meaning. They can be considered to the pair of propositions: (1) "There exists 'x'"; (2) "'x' possesses a certain property in virtue of the definitions of "addition" or "multiplication" in Peano arithmetic". So, the latter proposition refers to the former sharing the same variable 'x'", but both are granted to constitute a single compose proposition, after which it is necessarily self-referential.

Then, one can further admit that the corresponding converse statement, namely: "Any (or at least, some) self-referential statement refers implicitly or explicitly out of Peano arithmetic due to its algebraic incompleteness (in comparison with the field of all rational numbers)", is to be investigated as an idea in relation to the Gödel incompleteness statement (being analogically self-referential) or to its Gödel number as well.

One may try to demonstrate that the Gödel incompleteness statement (just because of its self-referentiality) defines an object out of Peano arithmetic (similarly to the way of how "subtraction" or "division" are able to determine unambiguously objects out of Peano arithmetics being certain rational numbers in general). Indeed, Peano arithmetic is incomplete to all rational numbers, on the one hand, but all natural numbers are able to enumerate all rational numbers since both sets are countable, and thus all natural numbers are to be finite just as all natural numbers from the viewpoint of arithmetic (rather than that of set theory, for they both are infinite sets within it).

Then, the sense of the Gödel proof (now meaning the special ability of self-referential statements, to which the Gödel incompleteness statement belongs, to bring out of the class within which they have been initially defined) consists in the utilization of the algebraic incompleteness of Peano arithmetic to the the operation of "subtraction" or "division" (i.e. to field of all rational numbers) together with the enumerability of all rational numbers by all natural numbers both being finite in virtue of the axiom of induction in Peano arithmetic.

As a result, the Gödel incompleteness statement demonstrates the availability of at least an object out of Peano arithmetic, but definable by the class of statements belonging to Peano arithmetic: however, involving self-referentiality. Following the analogy of the definition of negative or fractal rational numbers by means of self-referentiality and which are out of all natural numbers, the incompleteness of Peano arithmetic to set theory in Gödel's meaning seems to be reducible to the algebraic incompleteness of Peano arithmetic to the field of rational numbers (among which the set of all natural numbers is a true subset).

Indeed, the fact that the Gödel incompleteness statement is able to bring out of Peano arithmetic just due to its self-referentiality can be easily shown considering all true and all false statements to Peano arithmetic. Obviously, the Gödel incompleteness statement being a true paradox (formally represented) belongs either to both (thus being contradictory by virtue of the law of noncontradiction and implying the contradiction of arithmetic and set theory) or to neither of both (thus demonstrating by itself the incompleteness of those two groups of statements to Peano arithmetic by virtue of the law of the excluded middle and further implying the incompleteness at issue as to set theory).

Then, the ability of the Gödel incompleteness statement (being self-referential) to define implicitly a mathematical entity out of Peano arithmetic may be not worse interpreted in order to determine explicitly and constructively that mathematical entity following an algebraic consideration or the pattern for the way out of Peano arithmetic to all rational numbers by the definition of “subtraction” or “division” and the condition for them to be always accomplishable.

That algebraic approach to the triple of arithmetic, set theory and propositional logic meant as to the Gödel incompleteness paper (1931) will be further developed (in the next *Part II* of the paper) in the framework of Hilbert arithmetic in a narrow sense also on the basis of Boolean algebra shareable by both propositional logic and set theory, but furthermore by the pair of two dual and anti-isometric Peano arithmetics (involved by Hilbert arithmetic in a narrow sense). In particular, the Gödel completeness paper (1930) is to be algebraically reinterpreted and seen to infer an almost obvious statement as its main result linking tautologically two different examples of the structure of Boolean algebra.

VII. AN INTERIM CONCLUSION ABOUT THE HORIZON OF “HILBERT MATHEMATICS” OPPOSED TO “GÖDEL MATHEMATICS”

Philosophy or rather common sense’s prejudice has predestined for mathematics to build models of reality while reality by itself is studied by quite different sciences such as physics, chemistry, biology, etc. only utilizing eventually models elaborated by mathematics, but first of all, that is physics, and chemistry sometimes: all other sciences rarely and accidentally use mathematical models to describe their subjects.

In fact, mathematics resolves mainly its own problems, without any relation to their applicability in other sciences for researching “reality by itself”. One might say that mathematics investigates a corresponding mathematical reality or a corresponding area of reality just as biology, chemistry, etc., being furthermore very similar to that studied by physics so that the gap between it and mathematics does not exist at all, but a smooth transition links them inherently continuously though the aforementioned prejudice absolutely separates them as belonging to two thoroughly different domains: correspondingly “mind” versus “body” in the Cartesian tradition of Western philosophy or according to the officially proclaimed organization of cognition in Modernity.

However, quantum mechanics predestined to research physical reality “by itself” was forced to rebel in the first decades of the 20th century if it wished to remain an objective science. Its revolution consisted in establishing that “mathematical model” and “reality by itself” are the same in its framework (in the sense of the theorems of the absence of hidden variables in it). However, its riot was not supported or shared by any other science even by mathematics thoroughly satisfied by the modern episteme. It remains an exotic and even inexplicable exception among all sciences continuing to obey reality by itself and the organization of cognition originating from the dictate of reality.

Though quantum mechanics cannot but state that mathematics and physics are the same within it, no other science tended to share its insurgent slogan. Science and philosophy got gradually used to the dissenting opinion of quantum mechanics, being unique even in the framework of physics and admissible as the extraordinary and shocking worldview of an insignificant minority: at that, being rejected by many physicists even so great as Einstein.

The present paper, on the contrary, not only shares the ridiculous or "shameful" philosophical idea of quantum mechanics, but demonstrates that it can be exceptionally fruitful also to the foundations of mathematics for elucidating the true relation of arithmetic and set theory as first-order logics to propositional logic: a problematic very investigated in the beginning of the 20th century (and chronologically parallel to the establishment of quantum mechanics needing its rebellious slogans) culminated in Russell and Whitehead's logicism[33], Hilbert's arithmetical finitism[34], Hilbert's program[35], intuitionism, constructivism, Gödel's papers, Gentzen's papers, etc.

However, that problematic has not been connected with the revolution of quantum mechanics regardless of being relative as stating the completeness of quantum mechanics and erasing the gap between physics and mathematics as a necessary condition for the completeness at issue. The correspondence of Hilbert arithmetic in a wide sense and in a narrow sense (two dual tools elaborated in other papers, cited above expressly, just for the transfer of the ideas about the foundations of quantum mechanics to the foundations of mathematics) can assist for reinterpreting Gödel's papers (1930; 1931) furthermore extracting lessons from the historical advance of mathematics and its philosophy.

Gödel's "incompleteness paper" is properly what is discussed from the new viewpoint borrowed from quantum mechanics in the present *Part I* of the paper conceptualizing two alternative meta-mathematical (and thus, derivatively philosophical) worldviews about the relation of mathematics and the world accordingly called "Gödel mathematics" versus "Hilbert mathematics". The former means the usual or "classical" foundations of mathematics, namely: arithmetic, set theory, and propositional logic ordered and subordinated by both meta-mathematical and mathematical results of Gödel's papers (1930; 1931) and naturally shares the aforementioned prejudice of common sense, but not less originating from philosophy and the general organization of modern cognition, that mathematics is inherently incomplete to the world

[33] One and maybe most frequent objection to logicism is that it cannot justify actual infinity inherent for set theory (Landini 2011; 1998). The philosophical context of that problem is discussed by Gandon (2012; 2008) or Clark (1993).

[34] Also relatable to Husserl's philosophy of mathematics (e.g. Majer 1997) or to his concepts of completeness (Silva 2000) as well as in relation to that of Hilbert (Silva 2016), furthermore inherently linkable to Husserl's phenomenology (e.g.: Tieszen 2005; Mahnke 1977) also in the context of Gödel's theorems (Niebergall, Schirn 2002).

[35] There exist doubts whether the Gödel incompleteness results refute Hilbert's program (e.g. Detlefsen 1990; 2014).

and thus able to build only models of reality in turn reflecting on the relation of arithmetic and set theory ostensibly proved by Gödel to obey the dichotomy “either incompleteness or contradiction”. One can realize that the relation of mathematics and the world obeys the same dichotomy, in fact borrowed from that prejudice of common sense. Meaning that, the rather philosophical or “ideological” sense of Gödel’s incompleteness paper tends to support and prevent the existent order and hierarchy against any riot to it.

Anyway if one dare rebel (analogically to Lobachevski’s innovation to Euclidean geometry), many arguments in favor of the thesis that the Gödel incompleteness statement (”Satz VI”) is an axiom rather than a true theorem deductible from those of (Peano) arithmetic, (ZFC) set theory and (*Principia mathematica*) propositional logic can be justified and supported. A series of them are enumerated in detail in the course of the paper.

Then granting that the Gödel incompleteness statement is really an axiom, one can assume (also following the historical precedent established by Lobachevski) an alternative mathematics, called Hilbert mathematics, in the framework of which it is substituted with its negation (just as Lobachevski did in relation to the Fifth postulate). The main feature of Hilbert mathematics consists in the statement opposite to the Gödel incompleteness statement, namely any proposition in it is soluble: either true or false.

One can continue the analogue to the pair of Euclidean and non-Euclidean geometry supplied by Riemann with the special parameter “space curvature” able to distinguish uniquely any non-Euclidean geometry furthermore including Euclidean geometry as a particular privileged case of zero curvature. One is to introduce the quantity of the “distance between finiteness and infinity” (also interpretable thoroughly philosophically as a corresponding distance between model and reality), after which Gödel mathematics can be considered as the particular of Hilbert mathematics feature by the parameter of *zero distance between finiteness and infinity*.

However, if that idea has been granted, a problem seems to appear: how one is to interpret statements to which Gödel numbers belonging to the interval of any nonzero distance between finitess and infinity, whether conventionally positive after the “convex” (called “dialectic”) mathematics or conventionally negative after the “concave” (called “intuitionistic”) mathematics, have been unambiguously assigned. Would not those statements be again insoluble therefore necessarily restoring Gödel mathematics as universal once that definitive contradiction to Hilbert mathematics has been ostensibly discovered? Quite not, and here is why:

The two idempotent anti-isometric Peano arithmetics, to which the parameter of the distance of infinity from finiteness is to be related are just *dual*, i.e. *complementary* to each other: so one is forced to choose definitively either the one or the other, but never both simultaneously, which is sufficient for the contradiction to be avoided (by the by, analogically to the measurement of two conjugate quantities in quantum mechanics).

Even more, that newly introduced “distance” is not able to influence on the nonstandard bijection even where the “length” of the two dual Peano arithmetics is different as the distance at issue can be also interpreted. Indeed, those are two countable sets implying a bijection between them in the Dedekind definition of “infinite set” as to the viewpoint of set theory. As to that of arithmetic, both mean all natural numbers being necessarily *equally finite* due to the *same* axiom of induction valid in both.

Consequently, the nonstandard bijection is again bijection even the "length" of the two dual Peano arithmetics is different because of the nonzero distance between them as that between finiteness and infinity should be seen, furthermore and therefore being able to support the illusion that Peano arithmetic is a single one (rather than two ones dual to each other), which Gödel mathematics needs for its seeming universality at first glance.

However, the second part will reinterpret Gödel's papers (1930; 1931) in a positive way, i.e. conversely to the criticism of the present first part, which might provoke some confusion. It can be prevented by elucidating the different viewpoint in either: in fact directed to their "synthesis" in the third part if one dare speak in a Hegelian manner, after which the first part would be a "thesis" opposed the "antithesis" advocated in the next second part intending to see the completeness of logic to set theory versus the corresponding incompleteness[36] of arithmetic as a quite relevant apology of logicism[37] against the literal finitism of arithmetic unable to be included in the foundations of mathematics consistently[38].

In other words, the extensional viewpoint featuring both arithmetic and set theory will be replaced by the intensional one inherent for propositional logic and even for logic at all: at that, admitting philosophical ("ontological") generalizations well known still yet since Hellada and restored by Hegel after Kant's revolution of transcendentalism. In a sense, Gödel advocating logicism can be reinterpreted to assist the implicit establishment of Hilbert mathematics, but in a traditional way originating from the inherent intensionality of logic[39] and then reinterpreted and generalized "ontologically" thoroughly and literally following the Western philosophical tradition.

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[36] A series of papers (e.g., Dean 2020; Schlöder, Koepke 2012; Mccarty 2008; Schlechta 1996; McAloon 1978) discusses completeness and incompleteness in a shared context, in fact suggested by Gödel himself (1932) or reflected by Plato (2020), McCarthy (2016), etc., but the ontological harmony shared by both him and logicism is usually neglected.

[37] The apology of logicism can be realized also as a "logic of logicism" suggested by Bird (1997), Feferman (1999), or MacFarlane (2002) but not worse, as "plural logicism" (Boccuni 2013). The papers about logicism are really a huge number: a few of them (Tennant 2014; Grattan-Guinness 2009; 1984; Hanson 1990; Hellman 1981) are more relevant to the context of the present paper.

[38] Being questioned by Magidor (2007; 2012).

[39] The inherent and fundamental intentionality of logic makes relative many forms of logicism, e.g. that of *Principia mathematica*, "Russell's early logicism" (Kraal 2014) or "Russell's reasons for logicism" (Proops 2006; Radner 1975), Frege's (e.g. Cook 2016; Reck 2013; Tabata (2000); Milne 1989), Dedekind's (e.g. Klev 2017; Reck 2013), "neo-logicism" (MacBride 2003; Zalta 2000), the "revisited logicism" of Musgrave (1977) or its "vindication" (Roeper 2016), etc. as well as opposing it (e,g. Doherty 2017; Ferreirós 2009) in a sense to Hilbert's ideas about the foundations of mathematics being extensional in the final analysis. The paper of Laundry (2001) relates intensionality to structuralism as well as extensionality to objectivity in the context of logicism. The relation of logicism and arithmetization is investigated by Gandon (2008), Kolman (2015), or Rumfitt (2003); the corresponding viewpoints of Peano and Russell by Hochberg (1956), their rather philosophical reflections, by Hodes (1984), Irvine (1989), Jeffrey (2002), or Korhonen (2013), Puryear (2014). or Kornai (2003) relating finitism to physics pioneered an implicit pathway to physical logicism, and thus though indirectly, to "strict finitism" (Wright 1982) in mathematics.

On the contrary and just for which, the predetermining first part is a criticism of Gödel based on a contemporary reading of Pythagoreanism after quantum mechanics and called "quantum neo-Pythagoreanism" utilizing the tool of Hilbert arithmetic. Its essence, meaning the converse apologizing design of the next second part, consists in highlighting the *extensional* viewpoint of arithmetic inherent also for Pythagoreanism as able to be universalized not only "onto-*logically*", but in a newly introduced "onto-arithmetic" way inherent for Hilbert mathematics properly and explicitly discussed in detail in the third part.

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