Philosophical and mathematical reflection on Riemann's hypothesis. II The ontomathematical proof of Riemann's hypothesis in Hilbert arithmetic

Vasil Penchev, vasildinev@gmail.com
Bulgarian Academy of Sciences:
Institute of Philosophy and Sociology

Abstract: The proper mathematical proof of Riemann's hypothesis (RH) in Hilbert mathematics is suggested. It follows the methodological and philosophical considerations in *Part I* of the paper. Riemann's zeta function is continued "physically" at its single and simple pole conventionally to be square integrable there (though not being analytical only there) and thus everywhere on the complex plane in order to be interpreted as a wave function (though with a singularity at the pole, and thus generalizing the Hilbert - Polya conjecture's viewpoint). Therefore, $\zeta(s)$ is interpreted physically corresponding to some quantum state described by it This allows for applying the Noether (1918) first theorem after its reformulation by means of the newly introduced "nonstandard bijection" meaning the mapping of any mathematical structure and its nonstandard model after the Löwenheim - Skolem theorem. Then, the additive semigroup of its trivial zeros implies for all nontrivial zeros to share the critical line (i.e. proving RH) furthermore reasoning the reciprocity of the former generator "2" and the constant "½" of "Re(s) = ½". That approach for proving RH implies as "byproducts" also: the fundamentally random (GUE) distribution of all nontrivial zeros due to "nonstandard bijection"; the generalization of the axiom of induction to a conservation low as for the set-theoretical "continuation" of arithmetic (thus from the axiom of induction the axiom of infinity); the generalization of the Noether theorem in a few consecutive levels reaching even to set theory and the foundations of mathematics (after Hilbert arithmetic and ontomathematics). An idea for proving Goldbach's conjecture in Hilbert arithmetic (in a forthcoming paper) is reasoned as a corollary from the same approach for proving RH.
Keywords: Goldbach's conjecture; Hilbert arithmetic; Hilbert mathematics; Hilbert - Polya's conjecture; link of the zeta trivial and nontrivial zeros; Noether (1918) first theorem; nonstandard bijection; principle of physical and mathematical induction; physical continuation of analytical function; Riemann's hypothesis (RH); symmetry and conservation

I INTRODUCTION BY SUMMARIZING THE PART I OF THE PAPER, OR WHAT 'BIT" AND "QUBIT" WOULD MEAN AFTER THE NOETHERIAN INTERLINK OF SYMMETRY AND CONSERVATION

Part I sketches the idea for Riemann's hypothesis to be proved in Hilbert arithmetic by relating the additive semigroup of all trivial zeros and the "conservation" of the line $Re = \frac{1}{2}$ (suggested by Riemann's hypothesis) by means of the utilization of the pattern involved by the Noether (1918) first theorem linking the one of the most fundamental mathematical ideas, that of "symmetry", with one of the most fundamental physical ideas, that of "conservation", obviously hinting at ontomathematics and its approach for bridging the abyss between physics and mathematics established by modern epistemology. The series of proper mathematical justifications of that idea will be in detail considered in the next section. The present section intends to explain the same identification of "symmetry" and "conservation" not only philosophically, but first of all, by the concept of information and quantum information, even by their units, "bit' and "qubit" correspondingly.

The mathematical concept of symmetry, means a class of equivalence of all entities presupposed to be symmetrical to each other. Though the idea of symmetry is rather geometrical intuitively, it underlies even the so fundamental notion as "set" since its elements can be granted to be symmetric to each other just in virtue of being elements of the same set. Meaning that one can question what is "conserved" as for the elements of any set, and the answer is: the set itself since it is obviously the same as for any elements of it. Thus, and as for the opposition of "topological space" and "metric space" (as far as physics means always measurable quantities and thus some metric space to which they belong), the just suggested most fundamental generalization of "symmetry" and "conservation" is to be related still to the former rather than only to the latter.

However, and as for the eventual proof of Riemann's hypothesis in Hilbert arithmetic by the utilization of the Noether (1918) first theorem, both being inherently metrical, that generalized consideration of "symmetry" and "conservation" is necessary for the justification of the suggested link between the trivial and nontrivial zeros, after which the elementary regularity of the former implies for the latter to share the conservation of the line "$Re = \frac{1}{2}$". This is not necessary for the theorem itself (in its proof, it is more or less implicit) but just for reasoning the correctness to be applied to Riemann's hypothesis.

As for the concept of "bit", the distinction of the two oppositions of a bit is preliminarily necessary though common sense's prejudice grants for any bit to be a single opposition[1], namely that between its two explicit alternatives (usually notated whether as "true" and "false" or as "1" and "0"). Thus, one distinguishes the "coherent" state, in which both alternatives are a whole just as in Schrödinger's "dead-and-alive cat", i.e. before the measurement of any quantum-mechanical state, from their opposition, i.e. as after a measurement. Another example can be the opposition of an empty cell of the Turing machine tape versus its state "after recording" either "1" or "0" in it. One can distinguish both oppositions as follows: "no record" versus "either record"; if the latter is the case, "0" versus "1".

Meaning the last illustration, one can paraphrase it as for the understanding of the set-theoretical "symmetry"/ "conservation" (introduced just above) after the generalization of the possible records in a cell to be restricted neither to two, nor to any natural number (thus, inherently finite), i.e. to be infinite in general (thus, in set theory rather than in arithmetic). So, "symmetry" would correspond to the extension of any set, i.e. to its elements as if "enumerated one by one" (even being an infinite amount), and "conservation" of the set at issue, remaining the same as for any of its elements. Meaning Russell's idea of "type hierarchy", symmetry would

[1] In much more detail in: Penchev 2021 July 8.

correspond to the lower level, and conservation, to the higher level (thus indicating for a set to be the same in both levels). Another option for its interpretation consists in the opposition of the extension of any set versus its intension usually represented by its characteristic property also embedded in Boolean algebra shared by set theory and propositional logic. So, the Noetherian unification of "conservation" and "symmetry" can be extended even to Russell's principle of abstraction stating the identification of the extension and intension of any set.

So, both trivial and nontrivial zeros are now seen quite differently from as usual involving the mediation of their sets in turn also identified by virtue of nonstandard bijection: in other words, the trivial and nontrivial zeros are the "same" after the two-way application of nonstandard bijection and resultitive absolutely random dissemination of the elementary regularity of the former into the ultimate irregularity of the latter. Indeed, their unification needs the maximally generalized viewpoint of conservation / symmetry interpreted set-theoretically. Furthermore, the unification of mathematical symmetry and physical conservation is now represented as two successive hierarchical levels as for Russell's types or the opposition of "extension" versus "intension"[2].

The same viewpoint is also represented in both "senses" of Hilbert arithmetic and their relation after involving the informational concepts of "bit" and "qubit". The transformation of a qubit into the corresponding arithmetical unit is accomplished by the class of equivalence of all possible values of the qubit, and the reverse transformation, accordingly, by the "record" (i.e. choice) of a certain value among all possible ones. Furthermore, the former transformation can be seen as going out of the qubit and observing it as a whole, thus as an arithmetical unit. The opposite transformation means "entering the qubit" where the starting point is "outside" it. Then, the just newly introduced symmetry "outside - inside" and its corresponding conservation is applied two-way in relation to the zeros of Riemann's zeta function allowing the proper unification of both groups: the trivial and nontrivial ones.

One more key idea is that about the physical continuation of $\zeta(s)$ conventionally defining it also at its single pole $s = 1$ therefore extending the fundamental mathematical idea of analytic continuation, even more so that all zeros both trivial and nontrivial take place only in the domain of analytic continuation rather than in the original one where the zeta function is initially defined as a complex generalization of Euler's real function. The physical interpretation of the singularity of the pole bridges a mathematical function such as Riemann's zeta function and a wave function relevant to a certain quantum state. Thus, the realms of physics and mathematics are linked just as ontomathematics establishes.

Anyway, Riemann himself introduced that function for investigating the distribution of prime numbers and the general estimation of its irregularity. It turned out to be sufficient for the prime numbers theorem determining the asymptotic limits of the distribution of prime numbers. So, if one interprets the zeta function physically and ontomathematically, the natural question is what the physical sense of the distribution of prime numbers is; how it should be connected with

---

[2] One is to mean the inherent link of Russell's two fundamental ideas: the "principle of abstraction" and the "hierarchy of types".

the fundamental randomness of any quantum measurement. The answer advocated in *Section III* of the paper is that latter implies the former, and that conclusion originates just from the interpretation of the zeta function as a certain wave function thus needing in particular to be "physically continued" at its pole and then interpreting it as that singularity in which physics and mathematics "touch each other". Furthermore, that approach pioneers the pathway for proving Goldbach's conjecture in Hilbert arithmetic, which is also sketched in *Section III*.

II RIEMANN'S HYPOTHESIS?! WELL, IT'S TOO SIMPLE IN HILBERT ARITHMETIC …

A provisional sketch of the stages of the forthcoming proof is useful for one to orient in the very general idea of the proof as a whole (figuratively speaking, in its "forest", after which only the enumerated "trees" will be described "one by one" in detail):

1. The "physical continuation" of the zeta function $\zeta(s)$ in order to be square-integrable everywhere on the complex plane (i.e. including at "$s = 1$").

2. The interpretation of physical continuation as a wave function describing a quantum state, which, therefore, in turn allows for seeing the complex plane as the "Noether complex plane".

3. The regularization or renormalization at "$s = 1$" and its ontomathematical interpretation especially to the consistency of the proof.

4. Involving the Noether (1918) first theorem as for the Noether complex plane and for the zeta function also reinterpreted to be now defined on the latter and thus pioneering the application of the former.

5. Reasoning and grounding the conception of the nonstandard bijection as well as that of its two-way application resulting in a fundamentally random distribution due to the mediation of a countable set as a nonstandard bijective counterpart of an uncountable set.

6. Applying the nonstandard bijection to the zeta function which has however already reinterpreted physically on the Noether complex plane thus involving the Noether theorem in question; observing that the set of all nontrivial zeros is: both (1) infinite and (2) countable.

7. Investigating, particularly, the nonstandard bijective transformation of the Lie group of time translations into the additive semigroup of all trivial zeros of the zeta function and then observing the conservation at some (unknown initially) constant line ("$Re = const$") for all nontrivial zeros due to the Noether theorem.

8. Demonstrating the existence of at least one nontrivial zero (in fact all known those) on the line "$Re = \frac{1}{2}$" therefore forcing all nontrivial zeros to "land" on the same line by virtue of Noether's conservation and interpreting it to exemplify a newly introduced "principle of physical and mathematical induction".

9. Justifying the principle of physical and mathematical induction including by means of nonstandard bijection.

10. Reasoning additionally "$const = \frac{1}{2}$" for all nontrivial zeros to be reciprocal to the step of "2" of the additive semigroup of all trivial zeros.

11. Observing the reverse nonstandard bijective transformation of all nontrivial zeros back to the Noether complex plane and then, the consecutive (already usual) bijection from the Noether complex plane onto the complex plane particularly resulting in their absolutely random dissemination and distribution on the corresponding line "$Re = \frac{1}{2}$".

Now, the enumerated eleven stages of the proof will be considered one by one in detail, once again expressly emphasizing that the construction is possible *only in Hilbert arithmetic* (respectively, Hilbert mathematics) rather than in the standard ("Gödel") mathematics since the only the former complies with the ostensibly unsurmountable "abyss" between physics and mathematics (both situated by Modernity on each of its two opposite shores and unlike the former).

1. One can continue $\zeta(s)$ conventionally, complementing it only at the pole with a distribution, i.e. "generalized function" similar to Dirac δ-function but square-integrable there unlike the latter[3]. However, the values of both at the "jump" are "infinite". That approach can be interpreted as a non-conservative generalization of the classical analytical continuation where "analyticity" is impossible to be kept verbatim, but anyway only weakened to the meaning implied by introducing a square integrable generalized function, thus admissibly, "generalized analytically", consistent with square integrability and physical interpretability as a wave function, in detail discussed a little below, in (3).

2. The present step is crucial for applying the Noether theorem[4], but seems to be trivial, at least at first glance. Nonetheless, it is not allowed whether in the framework of the standard mathematics or even in the modern episteme at all, from which the former originates. For example, even the most simple mathematical abstraction as the number "5" is constituted as the class of equivalence of any five entities, such as "5 apples" (etc.) and this is an absolute correct way of thinking unlike the converse one, after which a mathematical proof would need to grant *necessarily* any abstraction to a certain representative of the class, e.g. "5" to be just "5 apples", as critical link of the syllogism. No respectable mathematical journal would accept a paper relying on such a tenet.

---

[3] Dirac delta function is integrable at its "jump", which does not imply square integrability there.

[4] The Noether (1918) theorem is one of the deepest and fundamental results of classical science, furthermore preceding quantum mechanics, linking mathematics and physics by identifying the former's variational principle and the latter's principle of least action privileging the physical quantity of action to be unique, both physical and mathematical. Papers referring to those theorems, or "symmetry and conservation" and relevant to ontomathematics and the present context in particular are (for example): Govaerts, Iyela 2023; De Haro 2021; Fraser 2021; Iyela, Govaerts 2021; Marvian 2020; Carroll 2019; Rowe 2019; Wallace 2019; Mansfield, Rojo-Echeburua, Peng, Hydon 2018; Olver 2018; Sardanashvily 2016; Neuenschwander 2017; Pitts 2016; Caulton 2015; Mansfield, Pryer 2015; Ballentine 2014; Rovelli 2014; Kosmann-Schwarzbach 2011; Conrey 2010; Gour, Spekkens 2008; Lange 2007; Brading 2005; Butterfield 2005; Roquette 2005; Penrose 2004; Brading, Brown 2003; Castellani 2003; Earman 2002; Goldstein, Poole, Safko 2001; Olver 2001a; Hydon 2000; Byers 1999; Marsden, Abraham, Marsden 1978; Ratiu 1999; José, Saletan 1998; Aneva 1993; Blaszak 1998; Fraassen 1989; Bell 1986; Capri, Kobes 1982; Bargmann 1964; Wigner 1964; Brauer, Hasse, Noether 1932.

However, the present study (more or less conventionally) postulates that there exists such a privileged entity such the physical quantity of action which corresponds unambiguously to a mathematical abstraction such as the complex plane and thus, to any mathematical function defined on it, such as $\zeta(s)$, and that class of bijections is correct properly mathematically (though rather ontomathematically). It refers to the discussed in detail above equivalence of a bit (of information) to a qubit (of quantum information) namely in that direction (since cut-elimination demonstrates its admissibility in the converse direction still in the standard mathematics).

Furthermore, that complementary equivalence is legitimated in both classical and nonclassical quantum mechanics by the theorems for the absence of hidden variables (Kochen, Specker 1967, Neumann 1932), the paradox of which to common sense can be rather jokingly illustrated by the statement that there should exist a special kind of entities such as "apples", which is not secretly (and "wrongly"), but absolutely correctly added to "5" in "5 apples" since the latter is just the same as "5" alone. Of course, that is a single entity (furthermore quantity), "action" merging furthermore the continuous (even smooth) description of the apparatus by classical mechanics with the proper quantum one as for the investigated microscopic entity "by itself" and obeying the Planck constant. So, "action" is that special concept, in which physics and mathematics, the physical and mathematical worlds "touch each other", thus being inherently linked, though for any other entity different from action, they are still gapped on two opposite "shores".

The present proof refers to that exception not only explicitly and expressly, but necessarily. It needs that both shores are two-way bridged to each other, thus (by the by) explaining in its background why Riemann's hypothesis is a fundamentally important, but extremely difficult for resolving mathematical problem: it needs a Gestalt change going far out of mathematics, deeply entering philosophy: even more, the intended eventually next part for the Hilbert-Polya conjecture will reason that both are problems too difficult for solving for the trivial reason to be merely unsolvable in a literal sense in the standard mathematics, i.e. both being Gödel insoluble statements[5].

Further, the Noether (1918) first theorem itself implicitly utilizes that "bridge", but it does not articulate that circumstance by virtue of borrowing it from classical mechanics as an inseparable element of the subject problem. In fact, the unity of the principle of least action (in physics) and the variational principle (in mathematics) did not wonder anybody in 1918, in the "paradise" of classical science and physics before "eating" the "apple" of quantum mechanics. However, the same unity nowadays (i.e. after "expulsion from classical science's Edem") is already so problematic that it forces the so extended justification in the present study.

3. One can notice the following problem. Though the zeta function is square-integrable everywhere on the (Noether or not) complex plane by a relevant convention, introduced in (1)

---

[5] A consideration about only Riemann's hypothesis is available in *Part I*, but reasoned by the "teleportation theorem": a "physical theorem" so that its application to a proper mathematical problem is too doubtful in the standard mathematics.

above, it cannot be simultaneously analytical everywhere[6]. So, it is not analytical at the pole, i,e, for "$s = 1$", but nonetheless it is analytical in any neighborhood of it[7]. Thus, though belonging to the separable complex Hilbert space (after being everywhere square integrable), it is not an admissible wave function everywhere (but fortunately only the singularity of the pole is deprived of any direct physical sense).

Square-integrability is a *necessary* condition for a function to be an admissible wave function. The corresponding *sufficient* condition would include also (1) continuity and differentiability since it should belong to the domain of the Hamiltonian operators; (2) bounded energy expectation value for a well-defined and finite entropy; (3) belonging to the domain of observables possessing the usual physical quantities or otherwise said, it should lie in Schwartz space: (4) satisfaction of physical boundary conditions, vanishing at infinity at boundary states or being periodic; (5) supereselection rules compliance for belonging to the allowed subspace of additional quantum constraints.

If one reformulates the question about which square-integrabile function, i.e. mathematical function defined on the complex plane is an admissible wave function, analyticity would be a sufficient condition (though too strong, but relevant to the present consideration). However only analyticity (i.e. without square-integrability) does not imply for it to be an admissible wave function: e.g., polynomial functions, $f(s) = s^{n}$, exponential functions, $f(s) = e^{s}$, trigonometric functions, $f(s) = e^{is}$, $f(s) = sin\,(s)$. In all examples, "s" is a complex variable, and the domain is the complex plane. So, one can suggest a more precise sufficient condition for an entire function "$f(s)$" to be an admissible wave function: $|f(s)| = O(e^{\alpha s^{2}})$ as $|s| \rightarrow \infty$ for some $\alpha > 0$ (where "$O(e^{\alpha s^{2}})$" is Bachman's "Ordnung", i.e. an asymptotic notation.

Then, one can discuss the standardly (i.e. without being continued in a generalized sense even at its pole) defined zeta function as an admissible wave function. This is so due to the following: it is both analytical and square-integrable all over its domain, i.e. only without its pole at $s = 1$. The integral defining the norm remains finite despite singularity and that statement is true even continued in a generalized sense though by a conventional additional definition. Anyway, even after that is not analytical at the pole itself, and that circumstance cannot be "amended" by any "clever definition" (since that "amendment" would metamorphose it into the trivial function, which $\zeta(s)$ is not in an obvious way)[8].

[6] Since the function which is both square-integrable and analytical everywhere on the complex is the trivial function being zero everywhere.
[7] In virtue of the fact to be an analytical continuation everywhere where it is defined, i.e. everywhere only without the single point the pole itself.
[8] This is similar to how physicists work with distributions similar to Dirac delta function, not being true functions but can still be meaningful within the right functional space.

A possible approach for the regularization of the generalized zeta function "$\zeta_{gen}(s)$" at its pole is a weighted modification: $\zeta_{gen}(s) = \lim_{\epsilon \to 0} [(\zeta(s) - \frac{1}{s-1+\epsilon}]$ which removes the leading divergence while maintaining Hilbert space integrability eventually physically interpreted in terms of any "phase transition". This resembles Hadamard's finite part integrals or the Cauchy principal value, which allow functions with singularities to be treated in the separable complex Hilbert sense, or aligns with renormalizable singularities in quantum field theory[9], where wave functions can still describe states despite divergences at critical points.

Anyway, one can investigate its behaviour near the pole carefully tracing back for physical consistency there, by the most relevant physical quantities such as energy, entropy, temperature, action, etc., which are to be renormalized at the singularity at issue[10]. So, both energy[11] and entropy[12] are divergent (or "infinite"), but temperature[13] and action[14] are convergent to zero therefore being prevented to reach it correspondingly by the third law of thermodynamics or the Planck constant. The pole might be interpreted as a quantum phase transition where all classical notions of action and dynamics collapse. So, though the singularity is deprived of any physical sense, its inaccessibility is independently confirmed by the infinite values of energy and entropy there and prohibited by the aforementioned fundamental principles, thus being inaccessible both mathematically and physically and providing a kind of "apophatic" consistency.

---

[9] E.g., *Ryder* (1996) or *Srednicki* (2007).

[10] The physical meaning of zeta function at the pole could be linked to phase transitions, renormalizations, entropic states, critical thresholds of various quantum systems, etc.

[11] For example, interpreting it as a partition function in statistical mechanics. Near the pole, the corresponding harmonic series diverges, and this implies for the mean energy to diverge as well. Physically, that might be a phase transition where the system reaches an unstable critical state, e.g., energy fluctuations become unbounded.

[12] For example, interpreting it to be the Shannon (or Gibbs) entropy. This means that the entropy at the pole becomes "infinite", the system enters a maximally disordered state. This could correspond to a thermodynamic singularity, where microstates become overwhelmingly numerous. If one analyzes the "heat" behavior at the pole, the heat capacity also diverges, therefore signaling a second-order phase transition. The system becomes critically unstable, and fluctuations dominate. This behavior is typical in critical phenomena, like the divergence of the specific heat in liquid-gas transitions.

[13] The divergence in energy and entropy cancels out, leading to a vanishing temperature. The singularity at the pole corresponds to a limit of absolutely zero temperature, much like in thermodynamics where entropy becomes nontrivial near the absolute zero. Since entropy diverges while temperature vanishes, this resembles holographic principles, where an infinite number of microstates become relevant in a phase transition. This could define a "renormalized absolute zero", where new physics emerges. It may describe a quantum critical point in an unknown physical theory.

[14] If the system exhibits a divergent energy and divergent entropy, but their ratio (temperature) remains finite or vanishes, then the action might undergo renormalization, effectively tending to zero. This suggests that physical processes at the pole correspond to a state where classical action vanishes. The physics at the pole enters a regime where standard quantum mechanics breaks down, because: (1) quantum mechanics assumes a finite nonzero action to define phase evolution via the path integral; (2) the phase factor becomes trivial, potentially leading to a breakdown of quantum coherence.

The present study being ontomathematical inherently suggests philosophical reflections, i.e. "cataphatic" speculations about the physical nature of that singularity for which the following observations can served: the singularity can be interpreted to be a phase transition where "time starts" since it can be renormalized to be "$t = 0$" there. So, it should be the very moment of the "Big Bang", which however turns out to be "absolutely cold" ("$T = 0$") unlike its usual convention to be "absolutely hot" ("$T = \infty$"); and "absolutely actionless" ("$S = 0$") rather than "absolutely actionful" ("$S = \infty$"): the latter is necessary in order to be able to provide the "infinite future" ("$t = \infty$") even after any finite energy of the Big Bang itself, which obeys energy conservation after that.

This is an obvious controversy thus appealing to be somehow explained. What can assist is the space singularity ("$x, y, z = 0$") after the "hot Bing Bang" versus the space "infinity", this means "$x, y, z = \infty$"[15], after the "cold Big Bang". In other words, the "cold Big Bang" has to happen "omnipresently", i.e. in each point of Euclidean space being given in advance, and time has to appear "omnitemporally" avoiding the contradiction as follows: "omnitemporally" means '$\frac{x}{c}, \frac{y}{c}, \frac{z}{c} = \infty$' (where "$c$" is the fundamental constant of "speed of light in a vacuum") while "time starts" notates that "$t = 0$" in the sense of quantum mechanics in each point of the temporalized Euclidean space "$\frac{x}{c}, \frac{y}{c}, \frac{z}{c} = \infty$" . So, the microtime of any coherent quantum state being reversible, and thus attachable to a relevant (Hermitian or not[16]) operator (thus conserving unitarity or not), is to be sharply distinguished from the irreversible macrotime of the measuring apparatus. Then, the contradiction is avoided since "time appears" (in the sense of the former) absolutely consistently "omnipresently" (in the sense of the latter). Speaking loosely, "time appears" always and everywhere starting the decoherence of any coherent state, "after which" it stops being reversible and starts being irreversible: from the "proper time of any quantum entity" to the "universal time of the apparatus".

The sketched "cold genesis" should be interpreted in the context of proving Riemann's hypothesis ontomathematically, in particular, after considering the physical behavior of the zeta function at its pole where energy, entropy, and space coordinates are divergent, or "infinite", but temperature, action, time are convergent (or "zero") utilizing relevant methods to be renormalized or regularized. Its idea seeming already to be consistent enough follows the substitution of the complex plane by the Noether complex plane thus reinterpreting $\zeta(s)$ physically and being able to be seen and generalized as a phase transition from the "cold aggregate state of mathematics" to the "hot aggregate state of physics", For example, one can speculatively assign negative values of all those physical quantities as for the "cold aggregate state of mathematics" so that energy, entropy, space coordinates are "negatively infinite" and temperature, action, (quantum) time are negatively zero (one might visualized by the behavior of

---

[15] For example, in terms of quantum gravity, the divergence at "s=1" suggests a fundamental scale beyond which classical notions break down, similar to the Planck scale in quantum gravity, involving curvature singularity (as in black holes) or the Hagedorn limit, where new degrees of freedom emerge.

[16] E.g. Simon (2015).

tangent and cotangent functions) approaching that extremely intriguing phase transition from mathematics into physics therefore supplying the newly introduced omnitemporal and omnipresent cold genesis instead of the dethroned to be mythical "Big Bang".

However, what is absolutely sufficient for the present study for proving Riemann's conjecture ontomathematically is to be only demonstrated that zeta function is an admissible wave function and this is consistent with Hilbert arithmetic and Hilbert mathematics grounded on the former though both sharply contradict to today's generally accepted scientific worldview, e.g. in the key conception of the Big Bang[17]. The intended proof is "normal" to be inconsistent to the latter since it originates from the former (rather than from the latter). Anyway, the elaboration of the "phase transition between physics and mathematics" (suggested to take place at the pole of $\zeta(s)$ physically interpreted to be an admissible wave function: as a singularity linking two-way both "shores" of the Cartesian "abyss" established by Modernity) will be postponed to the eventual third part of this paper: about the Hilbert - Polya conjecture as a corollary from the Riemann hypothesis, including a discussion as far the former holds in the standard mathematics as long the latter is granted (as well as whether Riemann's hypothesis can be proved to be a Gödel insoluble statement on the basis of proving it in Hilbert mathematics).

4. The Noether complex plane differs from the usual complex plane, on which in particular $\zeta(s)$ is defined, only in the following its two axes are physically dimensionful, but so that they are two conjugate quantities and their product has the physical dimension of action. What is crucial is that it is granted to be (whether) equivalent or dual (respectively, "complementary" in Niels Bohr's sense) to the complex plain standard for mathematics by virtue of the above considerations in (3). So, its dimension of action for the product of the abscissa and ordinate is necessary for the definition and unique among all possible physical dimensions since just the "spot" of action is the single location where the two shores are "bridged" over the Cartesian "abyss": i.e. the world of mathematics and that of physics are linked there and nowhere else[18], and the uniqueness at issue is a natural law (more or less symbolized by the Planck constant, more precisely, by its dimension of action and its "ability" to connect the smooth continuity of physical quantities with the discrete "counting" such as that of natural numbers).

The so-introduced "Noether complex plane" will be additionally specified, but only for simplicity, consequently, only by convention: the abscissa is chosen to be "time" dimensioned, and the ordinate, accordingly, "energy". So, the Noether complex plane is able to represent the mathematical duality (respectively, complementarity in quantum mechanics) as the idempotent exchange of both axes, e.g. as a $\frac{\pi}{2}$ rotation, and thus the ontomathematical sense of her first theorem to be reduced to a rotation, suggesting even richer connotations than the theorem itself due to the inherent rotational idempotency, e.g. as both $\pm\frac{\pi}{2}$ rotations. Indeed, the theorem does not deduce (or even merely hint at) the eventual option for the idempotent exchange of

[17] In detail in other papers, e.g., Penchev 2023 November 2; 2023 March 13.
[18] In fact, only an intuitive conjecture, which needs either a relevant rigorous proof or the demonstration of other quantities or entities whether physics and mathematics touch each other.

"symmetry" and "conservation", but nonetheless, "conservation" and "symmetry" to be idempotently exchangeable: as an exemplification, the usual interpretation for energy to be conserved, and time to be (Lie group) symmetrical is deeply anthropomorphic, due to the tradition and the general structure of human experience (to be inherently local) rather than originating from the mathematical formalism itself.

However, that idempotent exchangeability is also not necessary for proving Riemann's hypothesis, but it will be essential, in fact, crucial as for an intended future publication for proving Goldach's conjecture (rather elementarily) in Hilbert arithmetic, because of which it will be illustrated now, i.e. in terms of Riemann's hypothesis. Thus, an idempotent equivalent viewpoint can be allowed absolutely correctly, after which the group of trivial zeros and that of nontrivial zeros would be idempotently exchangeable, but only correlatively with the direction of the "nonstandard bijection". So, a composite symmetry would be involved (so in particular illustrable by one of the very well researched examples of composite symmetry, namely, "CP-symmetry" in quantum mechanics). But that observation is redundant for the present objective to be Riemann's hypothesis proved in Hilbert arithmetic.

5. The "nonstandard bijection" is to be defined as the mapping of any mathematical structure and its nonstandard model (for example, by virtue of the Löwenheim - Skolem theorem, but not only). In fact, it is introduced for the first time by Skolem (1922), but rather implicitly, as his "relativity of the concept of 'set'", not sufficiently articulated: it is famous as "Skolem's paradox" since both standard and nonstandard bijections between infinite sets have not been discernibly distinguished thus generating ambiguity, or "relativity" following Skolem's himself. Indeed, e.g., an uncountable set is modeled nonstandardly on a countable set thus excluding them to be of the same power. However, the so-constructed model is exhaustive in the sense of the Lövenheim - Skolem theorem thus implying that all elements of the former set are mapped unambiguously on all elements of the latter set, consequently bijectively regardless of their different power given as a preliminary condition: Skolem (1922) introduced the concept of the "relativity of the concept of 'set'" for describing the case for two sets of different sets to be bijectively mapped.

Anyway, one can investigate what should happen applying two-way the so-defined "nonstandard bijection". Obviously, "standard" bijection is idempotent including if an infinite set is mapped on its true subset of the same power. However, the converse mapping in the case of "nonstandard bijection" does not allow for the analogical idempotent solution in the following sense. One can choose any standard bijection of the latter set into the former set: it will be in a true subset of less power than that of the former set. So, the converse mapping to the nonstand bijection is not unambiguously defined since it is a class of standard bijections (corresponding to an uncountable set of standard bijections as for the considered example).

One can also approach otherwise by discussing the set of all images of a single element belonging to the latter set (of less power) on the class of all standard bijections furthermore meaning the circumstance that the image is possible to be the same after different standard bijections of those, therefore resulting in a probability (eventually density) distribution of how often each point of the former set (of a bigger power) turns out to be an image after each converse standard bijection. One can notice that the characteristic function of that distribution should be an element of the separable complex Hilbert space (even an admissible wave function) in general under some conditions, however the investigation of which is not necessary for the present objective for proving Riemman's hypothesis in Hilbert arithmetic.

What is sufficient is the observation that would be fundamentally random in virtue of the circumstance of the absence of any "hidden variable " in the afore-described construction *in definition* (unlike the case in quantum mechanics and the relevant theorems of the absence of hidden variables in it, where the analogical observation is deduced rather than conventionally postulated as here: only to "equate" different set-theoretical powers after nonstandard bijection). So, one can conclude that the latter set is mapped fundamentally randomly into arbitrary elements of the former after any standard bijection belonging to the class at issue. On the contrary, that bijection being the reverse counterpart of a standard bijection (including between a set and its true subset) is necessarily unambiguous.

6. What follows is that nonstandard bijection and its expressly emphasized property to be applied as to the particular case of the nonstandard bijection (and thus, to the corresponding "nonstandard homomorphism" being implicitly defined by virtue of that nonstandard bijection) of Lie group time translations meant as possible option in the Noether theorem (verbatim) into the additive semigroup of all trivial zeros, but already on the Noether complex plane, on which zeta function is reinterpreted. For example one can consider the Lie semigroup of all time translations being defined to be "future" to a certain zero moment of time, or respectively, to complement conventionally that semigroup to a complete group of all even integers, furthermore observing that the latter can be mapped bijectively on the former therefore transforming it into a complete additive group so that "+∞" of the latter is mapped on the "0" of the former, and so one. So, the nonstandard homomorphism of the Lie group of time translations onto all nontrivial zeros seems to be admissible until now but it needs additional refinements about the discrete "nonstandard mapping" of differentiability, which follow a little below. .

However, one more problem appears: whether the relevant nonstandard interpretation of the Noether (1918) first theorem holds? For simplicity and as above, the "corollary about energy conservation" will be considered alone. One can start by involving the following general problem: Given a mathematical structure S consistently defined on an uncountable set C and a mathematical theory (first order logic) T(S) which is true for S. One builds a nonstandard model S' of S on a countable set C in the sense of the Loewenheim - Skolem theorem, after which T(S) is transformed into T'(S'). Is T'(S') (1) an ambiguously defined first order logic and (2) true for S'?

The answer is well known: though T′(S′) is not ambiguously defined as a formal first-order theory, and thus different countable models may lead to different versions of T′(S′), anyway T′(S′) is true for S′ because S′ is a model of T(S) under the Löwenheim-Skolem theorem. Indeed, the key idea behind the Löwenheim-Skolem theorem is that the countable model S′ satisfies all first-order consequences of T(S). However, an important caveat is that some properties of S may not be preserved in S′ if they require second-order logic or depend on cardinality-specific features. For example, if S has a property expressible only in second-order logic (such as categoricity in certain contexts), S′ may fail to exhibit it. But as long as we restrict ourselves to first-order logic, T′(S′) remains true in S′ by definition.

So, the problem about the nonstandard interpretation of the Noether "energy conservation" to be applicable for proving Riemann's hypothesis consists in the following: can it be reduced to finding a strictly first-order nonstandard model where "smartly" any references to cardinality specific properties or relations are avoided or correctly translated in the nonstandard language? Staring carefully at the formulation of the theorem, one can suggest that only differentiability[19] (e.g. as for Lie group) seeming to be a property inherent only for an (uncountable) continuum can generate troubles, fortunately, more or less resolved by Robinson's nonstandard analysis.

Though the final aim is a discrete energy conservation principle to be deduced from the original theorem of Noether (1918), it will be mediated by a few transitional models so that the complete demonstration to be distributed into obvious substages, namely: (1) the corollary about for energy conservation implied from the original theorem being well known and thus not needing any additional justification; (2) the formulation of that "energy conservation" corollary thoroughly in terms of nonstandard analysis, also being routine; (3) building an again nonstandard model of the latter on the field of non-Archimedian rational numbers differing from the usual field of rational numbers only by replacing Archimedes's axiom with its negation; (4) building an already standard model of the last on rational field; (5) tracing back the consecutive transitivity of the previous substages providing for the wanted discrete "energy conservation" principle:

The essential conceptual approach is how irrational numbers necessary for differentiability (and before that, for continuity) to be "translated nonstandardly", i.e. only in terms of rational numbers. However, the very introduction of irrational numbers is accomplished (or can be realized) by converging to zero series of rational elements. So, infinitesimal elements involved in the substages of nonstandard analysis are represented by converging (thus infinite) series of rational numbers. As for the proper discrete "energy conservation principle", it will be valid only in a "limit transition", and consequently the nontrivial zeros would be on "$Re = \frac{1}{2}$" only in a "limit transition", but only at first glance, and then ultimately, even that rather conceptual ambiguity will be removed by utilizing the reverse nonstandard bijection where it will be compensated by the fundamentally random distribution of the nontrivial zeros on the (absolutely exactly) shared line "$Re = \frac{1}{2}$". Anyway, a few peculiarities of those stage deserve attention:

A preliminary notice should emphasize that there exist many enough nonstandard models (onto rational numbers) would not hold at all or would hold rather in an approximate sense, furthermore absolutely inappropriate for proving proper mathematical theorems applying it.

[19] About differentiability and Lie groups, see, e.g., Dorodnitsyn (2011); Olver (2001).

Noether's theorem would not be generally valid in the nonstandard model of only rational values, because it loses differentiability[20] and the full structure of the Lie group. However, in a weaker form, one might still express a version of conservation laws using rational approximations, but the exact continuous energy conservation theorem would not hold. This suggests that the truth of a theorem in a nonstandard model depends crucially on whether the transformation preserves the key mathematical structures (here, differentiability and Lie group continuity). Meaning anyway that the Noether theorem would admit various nonstandard models, including onto rational numbers, special ones conserving "differentiability" and "Lie groups" even onto rational numbers though in a nonstandard interpretation are to be considered:

One should pay attention also to the discreteness of action due to the Plank constant (already implicitly meant if one has involved $\zeta(s)$ to be an admissible wave function as above). Noether's theorem still holds even in an exact sense even when action is discrete, not necessarily needing to be reformulated using difference equations instead of derivatives since the corresponding pairs of conjugates could yet anyway conserve differentiability and Lie groups through the quantity of action would not do that.

Now (1) is considered after reformulating the Lie group of time translations and energy conservation in Robinson's nonstandard analysis (NSA). The model which is constructed is nonstandard, as it extends real numbers to hyperreal numbers. Noether's theorem still holds in NSA, since infinitesimal calculus is structurally equivalent to standard differentiation under the Transfer Principle. In fact, NSA may offer new insights, such as allowing infinitesimal energy fluctuations without violating conservation laws. Then, one passes to (2) so that a countable field of infinitesimal rational numbers is a consistent mathematical structure, behaving as a non-Archimedean extension of the field of rational numbers. A nonstandard analog of Noether's theorem exists within this field, using infinitesimal difference equations instead of standard derivatives. Energy conservation holds in a nonstandard sense, meaning it is preserved up to infinitesimals.

Meanwhile, one can question whether the axiom of Archimedes is the only obstacle to a nonstandard Noetherian energy conservation as for a countable field. Indeed, that axiom is a crucial obstacle because it prevents infinitesimals from existing, breaking the fundamental framework of Noether's theorem. If neither Archimedes's axiom nor its negation holds, the analogical statement of Noether's theorem might become incomplete, since infinitesimal variations would not be guaranteed. The negation of Archimedes's axiom is necessary to construct a meaningful nonstandard version of Noether's theorem.

Meaning the last observation (3) is to be undertaken. Infinitesimals are represented as sequences (in the field of rational numbers) which tend to zero, and infinities are represented as sequences that diverge. Noether's theorem translates into a discrete conservation law in the standard model, meaning energy remains constant over rational time steps. This transformation preserves Noether's theorem but discretizes it, showing that infinitesimals can be understood as limits of rational sequences.

---

[20] E.g., Freed, Hopkins, Lurie, Teleman (2011).

Finally, (4), one should join the separate steps transitively into a single whole, therefore elucidating whether the discrete conservation law is a nonstandard model of the original Noether conservation. If we interpret the discrete model in nonstandard analysis, where time is indexed by a hyperfinite sequence, and then the discrete model approximates the standard differential equation as a rational series tending to zero, i.e. in a nonstandard framework. This suggests that the discrete formulation is itself a valid nonstandard model of Noether's theorem.

Furthermore, the following observation takes place: the set of all nontrivial zeros is both (1) infinite and (2) countable. The following considerations support the latter. The zeta function's analyticity (except at $s = 1$) ensures zeros are isolated (Weierstrass's theorem[21]), making any finite region to contain finitely many zeros. The infinite total arises as $T \to \infty$, but the ordered growth ($T log T$) keeps them countable. No paper suggests an uncountable set, as this would contradict ζ(s)'s nature as a meromorphic function with discrete zeros. Riemann (1859) introduced the framework, implicitly suggesting countability through his zero-counting conjecture. Hadamard (1893) established analytic properties ensuring zeros are isolated, a basis for countability. Mangoldt (1905) provided the definitive asymptotic (N(T)), proving infinite yet countable zeros. Backlund (1912) refined the error term, reinforcing the discrete enumeration. Bohr and Landau (1914) analyzed zero density, supporting their countable spacing. Littlewood (1924) further detailed zero distribution, aligning with the countable set property.

Bernhard Riemann's original paper (1859) introduced the zeta function's analytic continuation and the explicit formula linking prime distribution to its zeros. While he didn't explicitly prove countability, his conjecture about zeros in the critical strip ($0 < Re(s) < 1$) and their enumeration via the imaginary part laid the groundwork. Riemann's suggestion of an asymptotic formula for the number of zeros up to height (T) (later formalized by others) implies a countable set, as they are discrete points ordered by imaginary part.

Jacques Hadamard (1893) proved that the Riemann zeta function has no zeros on the line $Re(s) = 1$, a step toward understanding the critical strip's zero distribution. His work on entire functions and product representations implicitly supports the discreteness (and thus countability) of zeros. Hadamard's factorization of $\zeta(s)$ as an infinite product over its zeros (similar to Weierstrass's theorem) assumes isolated zeros, a prerequisite for countability.

Hans von Mangoldt provided a rigorous proof of the asymptotic formula for the number of non-trivial zeros ($N(T)$) with $|\Im(s)| < T|$: $N(T) \sim \frac{T2}{\pi} log \frac{T}{2\pi} - \frac{T}{2\pi} + O(log T)$. This confirms the zeros are infinite and countable, as they grow predictably with (T) and can be

[21] The theorem's proof originates with Weierstrass (1876), building on Cauchy (1844), with refinements by Riemann (1851), Mittag-Leffler (1877), and Hadamard (1893). These papers collectively establish that the zeta function's zeros are isolated and finite in any bounded region, ensuring their countability. An analytic function that is not identically zero has zeros of finite order. Near each zero, the function is not zero (except at the argument of the zero itself), ensuring isolation (Weierstrass, 1876). In a compact set (e.g., a disk), the zeros cannot accumulate (form a limit point), as this would force the function to be identically zero everywhere (Cauchy 1844; Weierstrass 1876). As a meromorphic function with a single pole at "s=1", ζ(s) inherits this property, with non-trivial zeros isolated and thus countable (Hadamard 1893; Mittag-Leffler 1877).

ordered. The formula shows zeros are discrete and enumerable, aligning with the analytic function's isolated zero property.

Helge Backlund (1912) refined Mangoldt's estimate, improving the error term in ($N(T)$) and confirming the zeros' distribution in the critical strip. The work solidified the countable nature by providing precise bounds. Backlund's sharper $O(logT)$ error term reinforces that zeros are isolated and can be listed sequentially. Harald Bohr and Edmund Landau (1914) studied the density of zeta zeros in the critical strip, providing results about their average spacing and distribution, which support their countability. Their work shows that the zeros are not only infinite but spaced in a way (approximately $\frac{2\pi}{logT}$ apart on average) that allows enumeration, consistent with a countable set. John Edensor Littlewood (1924), building on his earlier work with Hardy, provided additional results on the distribution and counting of zeta zeros, including oscillation theorems that refine ($N(T)$). His analysis of zero density and spacing reinforces their discrete, countable nature, supporting the standard result.

Now, one is to support that all nontrivial zeros are an infinite set (whether countable, uncountable, or possessing any higher cardinality), so that it (after being combined with their countability) implies they to be a set of cardinality "alef zero" ($\aleph_0$). As that is already demonstrated above, no paper suggests an uncountable infinity as this would contradict analyticity. An accumulation point of zeros would force $\zeta(s)\equiv 0$, which is obviously false. Higher cardinalities are irrelevant, as the zeros' structure is tied to a countable ordering.

Mangoldt (1905) and others (Hadamard, 1896; Bohr, Landau, 1914, etc.) show $N(T) \to \infty$, ruling out a finite set. Bernhard Riemann (1859) conjectured the existence of infinitely many non-trivial zeros in the critical strip (0 < Re(s) < 1), linking them to prime distribution via his explicit formula. He suggested an asymptotic estimate for their number, later proven rigorously. Riemann's paper doesn't prove infiniteness formally but posits: $N(T)\approx\frac{T2}{\pi}log\frac{T}{2\pi} - \frac{T}{2\pi}$, therefore implying an infinite count as $T \to \infty$. Jacques Hadamard (1896) proved the prime number theorem, showing $\zeta(s)$ has no zeros on $Re(s) = 1$. His earlier work (1893) on the product representation of $\zeta(s)$ as: $\zeta(s) = e^{a+bs}\prod_{\rho}(1 - \frac{s}{\rho})e^{\frac{s}{\rho}}$ where ρ non-trivial zeros, implies an infinite product to match $\zeta(s)$'s growth, necessitating infinitely many zeros. If there were only finitely many zeros, the product would be finite, contradicting $\zeta(s)$'s behavior (e.g., its pole at "$s = 1$" and growth elsewhere). This indirectly proves infiniteness via analytic continuation.

Hans von Mangoldt (1905) provided the first rigorous proof that the number of non-trivial zeros is asymptotically infinite according to his formula already cited above. The inferred growth rate ($TlogT$T) ensures infiniteness while maintaining countability, as the zeros are ordered by their imaginary parts. Hardy (1914) (as well as *Hardy, Littlewood 1916*) proved that there are infinitely many zeros on the critical line $Re(s) = \frac{1}{2}$, a subset of all non-trivial zeros. While not proving all zeros are infinite alone, it supports the broader infiniteness when combined with other results. Hardy's method used the real function $\zeta(\frac{1}{2} + it)$ and its sign changes, showing $N_0(T) \to \infty$ (later refined with Littlewood). Harald Bohr and Edmund Landau (1914) proved results on the density of zeros in the critical strip, showing that the number of zeros in vertical strips grows infinitely, reinforcing Mangoldt's asymptotic. Their work on average spacing ($\sim \frac{2\pi}{logT}$) implies an infinite, dense distribution of zeros.

7. That step is both crucial and impossible to be accomplished in the standard mathematics due to the following. One cannot build any standard bijection between the Gödel soluble and insoluble statements, i.e. the latter to be enumerated by natural numbers. Just that impossibility has been meant by the special constructed "statement - liar" to which no natural number can be unambiguously assigned to be its proper Gödel number.

In fact, the introduction of Hilbert arithmetic also does not involve any standard bijection thus being yet consistent to the Gödel dichotomy "either incompleteness or contradiction". It overcomes the obstacle by allowing the nonstandard bijection between the soluble and insoluble statements after Gödel (1931) resultatively reframing his incompleteness theorem ("Satz VI") into an axiom relevant in the standard mathematics while its negation is adopted in Hilbert mathematics just by virtue of admitting nonstandard bijection as well as its two-way applications (in detail, in Penchev 2022 October 21).

So, and on the one hand, the standard bijection between the trivial and nontrivial zeros exists even in the standard mathematics (obeying the Gödel dichotomy) only by virtue that both are infinite countable sets. On the other hand and critically, however, that standard bijection does not imply the application of the Noether (1918) first theorem and the RH proof as a corollary from the former. Even its very application to Riemann's hypothesis needs the nonstandard bijection as well as reinterpreting the complex plane (on which the zeta function is analytically continued everywhere excluding its pole) as the "Noether plane" (on which the zeta function can be further and "physically" continued including its pole as a singularity bridging the proper mathematical zeta function and its physical counterpart, a wave function).

In other words Riemann's hypothesis is a theorem only in Hilbert arithmetic (respectively, in Hilbert mathematics grounded on it), but not in the standard mathematics because its proof needs critically the consistent transition from inherently finite natural numbers (in arithmetic) to their infinite set (in set theory) and demonstrated to be contradictory by Gödel (1931)[22]. Just the admissibility of that transition in Hilbert arithmetic allows the Noether theorem to be involved (after interpreting the transition at issue as the ontomathematical bridge from mathematics to physics). Then, just its application is what links the trivial and nontrivial zeros so that the nonstandard bijection between them takes place thus confirming RH in particular. Their standard bijection, being relevant in both Gödel and Hilbert mathematics, does not imply Riemann's hypothesis or the option for the Noether theorem to be applied since that

---

[22] Anyway, the statement that RH is insoluble in the standard mathematics being a Gödel one is only a preliminary observation, the proof of which in full detail will be postponed for the eventual third part of the paper about the Hilbert - Polia conjecture.

consideration is restricted in the mathematical "shore". Hilbert arithmetic is what allows for the crucial link to the opposite "shore" of physics therefore connecting the trivial and nontrivial zeros over the "abyss" dividing them (only) in Modernity.

One more circumstance needs its elucidation even more so that it is a direct approach for proving Goldbach's conjecture in Hilbert arithmetic: the trivial zeros are even negative integers while the nontrivial ones are conjugate pairs, supposedly GUE distributed. The question is: how can the mediation of the nonstandard bijection (unlike that of its standard counterpart) explain the nontrivial regularity of their correspondence (apropos, seeming to be isomorphic to Goldbach's conjecture: therefore, pioneering its proof in Hilbert arithmetic: in detail, in a forthcoming paper).

The answer is the following: the nonstandard bijection at issue is an anti-commutative composition in turn applied in the straight and reverse direction:

(1) f maps the trivial zeros (a countable set) in a nonstandard way onto an intermediate uncountable set (figuratively said, situated on the opposite, physical shore involved by the zeta function physically continued to a certain wave function as above);

(2) g maps reversely that intermediate uncountable set onto the set of all nontrivial zeros (again, a countable set thus causing their random dissemination onto the critical line);

(3) g^{-1} means the nonstandard mapping which is reverse to g ;

(4) f^{-1} is in turn reverse to f so that resulting in the anti-commutativity of the reverse composition, that is: $f \circ g = - g^{-1} \circ f^{-1}$, which is not true in general:

Though not being true in general, one can construct a special case, furthermore absolutely relevant to the sketched approach for proving RH in HA (in both senses). It exploits the rather trivial observation that the four Pauli matrices allow for the one of the simplest nontrivial models of noncommutativity by 2D Euclidean space in turn being immediately interpretable as the complex plane or the Noether complex plane after which the real axis to be (nonstandardly) postulated to be countable (while the imaginary axis is uncountable as usual) is sufficient. In other words and in detail, this means:

One defines $\boldsymbol{V}$ with basis $\{|\mathrm{S}\rangle, |\mathrm{U}\rangle\}$, where **S** is countable, **U** is "uncountable" in, but $|\mathbf{S}| = |\mathbf{U}|$ via the Löwenheim-Skolem theorem. Pauli matrices[23] act as operators ($f_1 = \sigma_x$; $g_1 = \sigma_y$), thus ensuring anti-commutativity for distinct pairs ($x \neq y$). The bijection $\mathbf{S} \rightarrow \mathbf{U}$ under the nonstandard model **M** supports the basis swap, aligning the matrices with σ_i, σ_j ($i \neq j$): the structure unifies the nonstandard bijection and noncommutativity via non-identical Pauli matrices. For to be satisfied $f \circ g = - [(g^{-1})(f^{-1})]$, what is sufficient is both f, g to "nonstandard involutions", which is true as for all mappings between corresponding structures of the "two senses" and discussed in detail in *Part I* of the paper, including as a definitive property of Hilbert mathematics (grounded on Hilbert arithmetic) versus the standard mathematics (which is notated to be "Gödel mathematics" in the opposition at issue).

---

[23] Generalizations are possible, but essentially more complicated: e.g., Gelbart (1984).

Thus, that special case of anticommutativity, namely: $f \circ g = -\, [(g^{-1})(f^{-1})]$, takes place as for proving Rieman's hypothesis and it in turn is able to explain two properties of the trivial and nontrivial zeros to be linked: the trivial zeros are only negative integers while the nontrivial ones are always conjugate pairs[24] (at that the property is proved independently of whether RH holds or not). The trivial zeros are to be interpreted to be "doubled" as an equivalent to the conjugate pairs of all nontrivial zeros just by virtue of the anticommutativity at issue, with which nonstandard involutions comply. One investigates particularly the nonstandard bijective transformation of the Lie group of time translations into the additive semigroup of all trivial zeros of the zeta function and then observing the conservation at some (unknown initially) constant line ("$Re = const$") for all nontrivial zeros due to the Noether theorem.

8. That step does not need a special comment being rather obvious consisting in the demonstration of the existence of at least one nontrivial zero (in fact all known those) on the line "$Re = \frac{1}{2}$" therefore forcing all nontrivial zeros to "land" on the same line by virtue of Noether's conservation and interpreting it to exemplify a newly introduced "principle of physical and mathematical induction". The sense after the Noether theorem means the trivial statement that a physical system satisfying it for which one demonstrates to conserve its energy in a certain time will conserve it in any other moment of time. As for the geometric interpretation in the complex or Noether plane, the property at issue follows from the fact that the plane belongs to (2D) Euclidean geometry, after which only a single line at a point ($Re = \frac{1}{2}$, $Im = 0$) and perpendicular to the real axis (an equivalent of Euclid's Fifth Postulate) exists.

9. The newly introduced principle of *physical and mathematical principle* is a conservative generalization of the axiom of induction in (Peano) arithmetic, deducible in Hilbert arithmetic, being a serial Gödel insoluble statement in the standard mathematics, though. The principle states that if a statement is true by virtue of the axiom of induction to a subset of natural numbers, there exist mathematical structure so that: (1) a nonstandard model of the mathematical structure can be built on the subset of natural numbers; (2) the statement, which is proved as for the subset of natural numbers by means of the axiom of induction can be continued to all elements of the set, on which the structure exists; (3) the so continued statement is true to the set, on which the structure exists, and thus to the structure itself.

The idea for proving Riemann's hypothesis in Hilbert arithmetic by the Noether (1918) first theorem illustrates well the principle of physical and mathematical induction as well as its ontomathematical sense. All trivial zeros, obviously obeying the axiom of induction, implies the existence of a line (coinciding with the critical line), which is the continuation of their set after the nonstandard bijection (i.e., including all nontrivial zeros). So if one grants the principle of physical and mathematical induction, it can serve to prove Riemann's hypothesis though it can be also deduced as above: without involving that principle. In a loose sense, the principle of

---

[24] The opposition of the elementary regularity of the trivial zeros versus the supposedly absolutely random distribution of the nontrivial zeros is now neglected, but discussed in detail in the next steps of the proof further.

physical and mathematical induction generalizes the utilized before that approach as a method for inferring statements in Hilbert arithmetic (and thus in Hilbert mathematics), corresponding to the axiom of induction as a particular case valid as for (Peano) arithmetic. Simultaneously, it generalizes the demonstrated above approach for proving Riemann's hypothesis in Hilbert arithmetic since it is a conclusion from the nonstandard bijection of Hilbert arithmetic in both narrow and wide senses and thus to the "bit-qubit" relation[25] relevant to the n^{th} unit and n^{th} qubit Obviously, after that is the case, the confirmation of Riemann's hypothesis would be a corollary from the principle of physical and mathematical induction once it has been applied to it.

Furthermore, that principle is inadmissible in the standard mathematics, even being relevant to the class of all Gödel insoluble statements and consequently producing those after applying it. One can immediately notice that the principle means the transition from finiteness (after the axiom of induction) to its "infinite continuation" in an unambiguous way. Following the illustration by the RH proof, that "unambiguous way" is crucial and it originates from the internal structure of the semigroup of all trivial zeros (as by its generator "-2") implying unambiguously a dual generator, which can be interpreted to be "external" and thus defining that "conservation" which all points of "$Re = \frac{1}{2}$" are forced to keep and follow the general idea of the Noether theorem.

That idea means that symmetry is an internal property of a structure, respectively internal relation of its elements, and that conservation implied by the symmetry is an external property of a dual structure, respectively an external relation (i.e. relation of identity or indistinguishability) between its elements. The just formulated observation is so general that it can be immediately seen in the most fundamental concept of "set". Indeed, all elements of any set share a definitive symmetry to be elements of the same set. That symmetry can be interpreted to be dual to the characteristic property of the set (the existence of which is postulated, e.g., by Russell's principle of abstraction) which is the relevant and "external" conservation, with which all elements comply. Then, one can consider the "continuation" of the characteristic property to elements, which do not belong to the initial structure, to which the symmetry refers *verbatim*. Rather loosely, it can be further related to Pierce "abduction":

One may try to synthesize the idea of "abduction" after Pierce, for example, as follows. Abduction is the inferential process of forming the most plausible explanatory hypothesis for surprising or insufficiently explained phenomena, guided by instinctive insight and constrained by coherence with known fact. Then, one may extract a more rigorous definition: for any observed regularity, there exists a minimally complex hypothesis whose deductive consequences include the observations.

However, the condition for an abductive conclusion to be unambiguous is crucial. Then, abduction *unambiguously* continues a regularity if the most plausible hypothesis is uniquely minimal in complexity and consistent with all known observations. One can exemplify unambiguous abduction by the concept of analytical continuation: even more so that is the way

[25] E.g., Vedral (2003).

for defining the zeta function all over the complex plane (though excluding the single pole). Analytic continuation satisfies the criterion for unambiguous abductive continuation under the "identity theorem" and domain connectivity. In other words, analyticity itself acts as Ockham's razor, sharply slicing away ambiguity.

Anyway, when analytic continuation meets branch points, Riemann surfaces, or nontrivial topology, ambiguity returns: unless we introduce additional rules or structures. Concluding loosely, or generalizing "abductively", abduction is analytic continuation of reason: unambiguous when its logic is "holomorphic", but ambiguous near the "singularities of knowledge". Where classical continuation fails, abductive continuation succeeds by generalizing the domain of admissible hypotheses, and selecting the unique one that optimally balances simplicity, structure, and coherence.

Indeed, the paper has already demonstrated how the zeta function can be further continued "physically", i.e. beyond its proper mathematical "analytical continuation", i.e. at its pole (being a singularity and then interpretable as an ontomathematical link between the proper mathematical zeta function and its physical counterpart, which is a wave function thus being relevant for describing a quantum state). After generalizing the approach, one can state: where classical continuation fails, abductive continuation succeeds by generalizing the domain of admissible hypotheses, and selecting the unique one that optimally balances simplicity, structure, and coherence.

The just sketched idea for unambiguous abductive continuation is ready to be applied to the fundamental (and most interesting) arithmetic axiom of induction, for example, as follows. *If P(n) holds for all $n \in \mathbb{N}$ and the minimal structural extension M of $\mathbb{N}$ supports a unique definable continuation of P, then P holds for all $x \in M$.* It may be reasoned and justified more exhaustively:

Let one say that a statement P(n) is verified for all $n < k$, with finite k, or even all $n \in \mathbb{N}$. Can one continue P(n) to hold for n in a larger domain, e.g., a nonstandard model, or for all $\alpha < \kappa$ where κ is a higher cardinal? This is not classical induction. So, to make such a continuation unambiguous, what is to be invoked is: an abductive continuation of a regularity (e.g., P(n) for $n \in \mathbb{N}$) to a model M of higher cardinality is unambiguous if:

1. There exists a unique minimal structural extension of the standard model that: (1.1) preserves all definable properties of P(n); (1.2) admits a natural embedding $\mathbb{N} \to M$ respecting P(n).

2. Among all models $M' \supseteq \mathbb{N}$ satisfying P(n) for $n \in \mathbb{N}$, the model M is the least complex (in descriptive, syntactic, or categorical sense).

3. Any alternate extension violating P(n) must necessarily increase the complexity of the theory (e.g., by adding new axioms, breaking definability, introducing inconsistency with prior structure).

This mimics analytic continuation in logic: the truth of P(n) in $\mathbb{N}$ extends to M uniquely if M is the smallest definable or constructible model preserving P. Let one involve the concept of "nonstandard bijection" defined as a bijection between any set (complemented to any structure) and its nonstandard model on a countable set (and the corresponding structure serving to be a nonstandard model) under the Löwenheim - Skolem theorem. Then, any regularity for natural numbers is granted to be the latter structure.

How can one reformulate "unambiguous abductive continuation" to the former structure of higher cardinality using now the concept of "nonstandard bijection"? On the other hand: can one rigorously define "nonstandard homomorphism" or "abductive homomorphism" from the latter to the former structure?

One is dealing with: (1) a standard structure S (say, the natural numbers $\mathbb{N}$ with Peano arithmetic, or some set A with structure $\mathfrak{S}$); (2) a nonstandard model S* of countable cardinality, existing by the Löwenheim–Skolem theorem; (3) a nonstandard bijection: a bijection β: S → S* that maps elements from the high-cardinality structure onto the countable nonstandard one (or vice versa, depending on perspective). This bijection is not "standard" in that it does not preserve syntactic identity, but it preserves structure, or regularities, to some extent. This is where abduction comes into play. Meaning the above recapitulation, the formal definition of nonstandard bijection is:

A nonstandard bijection β: S → S* is a bijection between a set S (with structure $\mathfrak{S}$) and a countable model S* (with structure $\mathfrak{S}^*$) satisfying: β is a set-theoretic bijection (not necessarily definable inside ZFC). $\mathfrak{S}^*$ satisfies all first-order sentences true in $\mathfrak{S}$ (i.e., $\mathfrak{S}^* \equiv \mathfrak{S}$). For each definable predicate P over S, the transferred predicate $P^* := \beta \circ P \circ \beta^{-1}$ holds in S*. In effect, β transfers the observable regularities (the truths of P(n) verified in S*) to S. Thus, S* becomes a nonstandard laboratory, and S becomes a kind of abductive extrapolation of what is observed in S*.

The next stage consists in reformulating "unambiguous abductive continuation" via "nonstandard bijection". A regularity P(n) verified in the finite or countable domain may be unambiguously abductively continued to a higher cardinal structure if: (1) there is a unique minimal extension that preserves P; (2) no alternate extension violates P without increasing structural complexity. Now, using β: S → S*, one involves: "nonstandard abductive continuation principle":

Let β: S → S* be a nonstandard bijection between a structure S of higher cardinality and a countable elementary submodel S*. If a regularity P is true in S*, and $\beta^{-1} \circ P \circ \beta$ is definable in S, then P holds for S unambiguously if the inverse transfer $\hat{P} = \beta^{-1} \circ P \circ \beta$ is the unique definable predicate on S that it (1) agrees with P on the image of β; minimizes definitional or descriptive complexity (e.g., Kolmogorov complexity or logical depth); then $\hat{P}$ is the abductive continuation of P to S. This turns S* into a "compressed simulation" of S, and β serves as the abductive lens.

One can try to distinguish between the "nonstandard homomorphism" and "abductive homomorphism", for example, by defining a dual to the nonstandard bijection. A nonstandard homomorphism (or abductive homomorphism) φ: S* → S is a structure-preserving map such that: (1) S* is a nonstandard model (e.g., a countable elementary submodel of S); (2) for any definable operation f in S*, $\varphi(f(x)) = \hat{f}(\varphi(x))$, where $\hat{f}$ is a definable continuation of f in S; φ is not necessarily bijective or even injective, but preserves truth of formulas (elementary embedding if possible).

This φ becomes the abductive homomorphism if it satisfies: (1) minimum ambiguity: among all possible φ, it yields a unique image of P in S; (2) maximum simplicity: φ minimizes descriptive complexity. One can think of it like this. Just as in complex analysis the Riemann surface unrolls multivalued functions into single-valued ones, φ unrolls the ambiguous structure of S* into an abductively simplest realization in S.

A categorical viewpoint as an optional abstraction is also possible by defining a category "*AbdStruc*" by: (1) their "objects" being structures S with definable properties; (2) morphisms: abductive homomorphisms φ: S* → S, satisfying minimality/unambiguity; (3) distinguished subobjects: countable elementary submodels (nonstandard simulations); (4) distinguished arrows: nonstandard bijections β such that β and its inverse define abductive pullbacks/pushforwards. Then abductive continuation becomes a (partial) right adjoint to the nonstandard embedding functor. That is: $\mathrm{Hom}_{\{\mathrm{AbdStruc}\}}(S^*, S) \cong \mathrm{Hom}_{\{\mathrm{Countable}\}}(S^*, \mathrm{Sim}(S^*))$ where Sim(S*) is a simulation inside S.

A philosophical reverb can be finally sketched by endowing Skolem's paradox with epistemological depth: although ℕ has uncountable models, countable nonstandard bijections let simulate ℕ within countable universes. Through abduction, these simulations let one reconstruct the larger, higher-order truth: as long as they are the simplest continuations of the regularities seen by the simulation. Thus, a bridge between simulation and reality[26] will be built as a corollary from the fundamental "postmodern shift" for human cognition by linking the two "shores" of the Cartesian "abyss".

Can one further exemplify the abductively continued nonstandard homomorphism as above in the following important particular case? Regularity embodied in the latter countable structure is additionally specified to be the generator of a group. Which would be (1) the nonstandard abductive homomorphic counterparts of both generator and group, even (2) furthermore complementing for the former group to be a Lie group?

Then, one has: (1) a countable nonstandard model S* under Löwenheim - Skolem, where a certain regularity g* defines a group generator, generating a group G* (on the model S*); (2) a nonstandard bijection β: S → S*, mapping a higher cardinal structure S onto the model S*, and allowing a kind of abductive continuation of regularities from S* to S. The goal is to be defined and interpret the abductive homomorphic counterparts in S of: (1) the generator $g^* \to \hat{g}$; (2) the group $G^* \to \hat{G}$, and possibly: (3) extending $\hat{G}$ to a Lie group $\hat{G}^L$.

The stage (1) can be named abductive generator and group. One begins with a regularity $g^* \in S^*$, where the function $f^*: \mathbb{N} \to S^*$, given by $f^*(n) = (g^*)^n$ generates the group $G^* = \langle g^* \rangle \subseteq S^*$. Then, by abductive continuation via the nonstandard bijection β: S → S*, one defines the abductive generator in S: $\hat{g} := \beta^{\{-1\}}(g^*)$; $\hat{G} := \langle \hat{g} \rangle = \{ \hat{g}^n : n \in \mathbb{N}\} \subseteq S$. The operation (multiplication, addition, etc.) is transferred from S* via β. The structure-preserving nature of the

---

[26] Healey (2007) suggests a general consideration of "gauge theories" in fundamental physics in an analogical sense.

homomorphism (or elementary embedding) ensures that group axioms are satisfied in $\hat{G}$. So far, this is standard group-theoretic transfer across nonstandard bijections filtered by abductive minimality: i.e., we assume $\hat{g}$ is the simplest definable counterpart to g^*.

What follows are the conditions for extending to a Lie Group. Now, let one consider under what conditions can $\hat{G}$ be smoothly continued to a Lie group $\hat{G}^L$? What would the idea be? Lie group is a smooth manifold M equipped with a group structure such that: (1) group multiplication and inversion are smooth (C^∞) maps; (2) the group is locally homeomorphic to $\mathbb{R}^n$, and so, abductively continuing $\hat{G}$ into a Lie group $\hat{G}^L$, one must abductively recover (1) a differentiable structure on $\hat{G}$; (2) a topology on S coherent with this structure; (3) a smooth extension of the discrete exponential map $n \mapsto \hat{g}^n$ to a smooth curve $t \mapsto \hat{g}^t$ on $\mathbb{R}$.

Meaning the above, one formulates sufficient conditions for the abductive continuation to a Lie group by applying the same unambiguous abductive continuation condition in this more structured context. Then, a Lie abductive continuation criterion would be as follows. Let $\hat{g}$ in S be the abductive image of a generator g^* of a group G^* in a countable model S^*: the abductive continuation of G^* to a Lie group $\hat{G}^L$ is unambiguous iff: (1) the group operation in G^* is locally approximable (under β) by a differentiable structure in S; (2) there exists a unique smooth extension $\varphi: \mathbb{R} \rightarrow \hat{G}^L$ such that $\varphi(n) = \hat{g}^n$ for all $n \in \mathbb{N}$; (3) no simpler non-differentiable extensions of $\hat{g}^n$ exist that explain $\hat{g}$ with equal or lower structural complexity. In practice, this corresponds to assuming the exponential map: $e^{tX} = \hat{g}^t$. This assumes the abductive process does not stop at recovering $\hat{g}$, but continues with (1) a tangent structure at the identity; (2) a closure under Lie brackets (algebraic); and (3) smooth interpolation between powers of $\hat{g}$.

A crucial example is from discrete to continuous via abduction: (in S^* being countable). Let $g^* = e^{2\pi i/N}$ for large standard N (say $N \gg \aleph_0$), and $G^* = \langle g^* \rangle$ be a cyclic group of order N; in S (uncountable), $\hat{g} = \beta^{-1}(g^*)$ is its abductive counterpart. Then the group $\hat{G} = \langle \hat{g} \rangle$ appears discrete, but we can abductively identify a smooth embedding: the circle group S^1 (Lie group) with exponential mapping $\theta \mapsto e^{2\pi i\theta}$, with $\theta \in \mathbb{R}$. So, $\hat{g}^n$ continues via abduction to $\hat{g}^t$, and furthermore: $\hat{G}^L = \{ \hat{g}^t : t \in \mathbb{R} \} \cong S^1$. Here, the abductive continuation of the finite cyclic group gives rise to the unit circle group, a Lie group.

The conception of "nonstandard homomorphism" as a Lie functor is implied: i.e., by viewing this through a categorical lens. Let Grp* be the category of countable groups (like G^*) definable in nonstandard models. Let LieGrp be the category of Lie groups. Then the abductive homomorphism functor $\mathfrak{A}$: Grp* $\rightarrow$ LieGrp maps: generators $g^* \mapsto \hat{g}$; groups $G^* \mapsto \hat{G} \mapsto \hat{G}^L$ where $\mathfrak{A}$ minimizes: (1) descriptive complexity (simplicity) and (2) extension ambiguity (uniqueness), but maximizes continuity and smoothness (structural coherence).

Summarizing, one has essentially formulated the abductive Lie extension of a discrete, countable nonstandard model: (1) the generator: $g^* \in S^*$ becomes $\hat{g} = \beta^{-1}(g^*) \in S$; (2) the group: $G^* = \langle g^* \rangle$ becomes $\hat{G} = \langle \hat{g} \rangle$; (3) the Lie group: if smooth structure exists and is unambiguous, $\hat{G}$ continues to a Lie group $\hat{G}^L$ via: smooth interpolation $t \mapsto \hat{g}^t$; tangent structure is Lie algebra thus realizing both abductive simplicity and uniqueness.

Let one further specify even more the above nonstandard homomorphism under the following additional condition. The Lie group at issue serves as the premise in the Noether (1918) first theorem thus implying the conservation of a conjugate quantity. Can that “conjugate quantity” be defined abstractly and mathematically meaning only the initial Lie group, to which it is a counterpart after the Noether theorem? Is that “conjugate quantity” unambiguously defined and if not, can it be “abductively” additionally refined to be unambiguous? Which is the “straight” (i.e. not reversely abductive) nonstandard model of that “conjugate quantity” (hopefully) unambiguously determined by being the counterpart of the Lie group (under the Noether theorem)?

Recapitulazing, one starts with a nonstandard homomorphism transferring a regularity (a generator in a countable model) abductively to a structure in a higher cardinality domain. This led to an abductive Lie group, $\hat{G}^L$, interpretable as a smooth, unambiguous continuation of a regularity. What is now introduced is Noether's first theorem (1918), which says: every continuous symmetry (Lie group) of the action of a physical system corresponds to a conserved quantity (Noether “charge”)[27].

Can the “conjugate quantity” be defined abstractly and mathematically as a counterpart of the initial Lie group (independent of any Lagrangian)? The answer could abstractly and mathematically be interpreted to be Noether’s “conjugate quantity” as the momentum map associated with a Lie group action on a symplectic (or more generally, variational) manifold. The formal abstraction needs the following premises: (1) G be a Lie group acting smoothly on a manifold M; (2) $\mathfrak{g}$ be its Lie algebra; (3) ω be a symplectic form on M. Then there exists a momentum map. This momentum map encodes the Noether conjugate quantity purely in terms of the group action and geometry of the manifold, independent of a specific Lagrangian. So, the conjugate quantity is abstractly definable from the Lie group alone if one abstracts the symplectic structure and group action contextually.

Is that “conjugate quantity” unambiguously defined? If not, can it be abductively refined to become unambiguous? In general, it is not unambiguously defined. Why? Because: (1) the momentum map J is defined up to constants (Casimir functions); (2) the choice of coadjoint orbit, gauge fixing, or symplectic realization can vary; (3) the Hamiltonian structure may introduce degeneracies. So, the question is: how do we abductively refine it to be unambiguous? One deploys the “abductive principle”: the conjugate quantity should be the simplest and most structure-preserving extension that corresponds uniquely to the symmetry G: and abductive unambiguity criterion (for conjugate quantity):

---

[27] Speaking loosely, all irrational numbers critically necessary for both differentiability and Lie groups are anyway conserved, but as converging infinite series of rational numbers, literally repeating certain historical ways for introducing irrational numbers.. Of course, the concept of irrational numbers saving for one to discuss infinite rational series each time is much more comfortable and fitting to human thinking. However, it is not more than an only technical trouble, thus not preventing for the Noether theorem to be equivalently “retold” in a “nonstandard discourse” of a countable field such as that of rational numbers,

A conjugate quantity Q is unambiguously defined with respect to a Lie group G acting on M iff: (1) there exists a unique momentum map J: M → $\mathfrak{g}^*$ up to canonical isomorphism, minimizing (1.1) dimensional excess (no unnecessary degrees of freedom); (1.2) functional ambiguity (no arbitrary gauge terms); (2) the dynamics (e.g., via Hamiltonian vector fields or symplectic flow) are uniquely determined by J and G; (3) no simpler extension (in complexity or dimension) of the symmetry - action pair yields the same conserved structure. In other words: abduction removes degeneracy by choosing the minimal sufficient conjugate quantity that explains the group symmetry. This is essentially an epistemic "Occamization" of Noether correspondence.

The conclusive question is what is the "straight" (non-abductive) nonstandard model of that conjugate quantity (hopefully unambiguous) corresponding to the Lie group? Indeed, one can construct a non-abductive (direct) nonstandard model of the conjugate quantity:

Let G* be a countable Lie group in a nonstandard model. Let Q* be the Noether conjugate quantity within this model, constructed via standard means (e.g., Hamiltonian mechanics, variational principle). Let β: S → S* be the nonstandard bijection. Then, the "straight" nonstandard model of the conjugate quantity in S* is simply: Q* = J*: M* → {g}* and for mapping this back to the standard (uncountable) model, we define the abductive conjugate quantity in S if β is structure-preserving (elementary), and Q* is already unambiguous in S*, then a faithful straight nonstandard model of the conjugate quantity in S is involved. Thus:

The "straight" model is Q*, definable in the countable nonstandard model via "symmetry → conserved quantity". The "abductively continued" model is that in the higher cardinality structure, guaranteed to be unambiguous by abductive selection. So: such a straight model exists. It is non-abductive, since it is generated directly by the Lie group symmetry in S*. Its abductive counterpart in S is unambiguously defined if Q* is minimal and β is sufficiently definable. What one has sketched is a blueprint for an abductive duality between symmetry and conserved quantity, made rigorous in a landscape of nonstandard models, category-theoretic morphisms, and structural minimality. In a way, the abductive framework completes Noether's theorem epistemologically: not only is every symmetry associated with a conserved quantity, but the most meaningful conserved quantity is the abductively simplest one.

One can further discuss "physical and mathematical induction" (PMI) as for the classical foundations of mathematics (arithmetic, set theory, classical propositional logic and restricting mathematical theories to be only first-order logics). The elements of any set can be interpreted to be symmetrical to each other since the "class of equivalence to be elements of the same set" can be seen as a class of symmetry. It implies a "conservation quantity", which the characteristic property of the set is: indeed, all elements satisfies the property definitively, and so, one can state that the property (being a logical proposition) conserves. Thus, the symmetry of the elements is "inside", to the elements of the set, while the conserving proposition is "outside", to the set as a whole. So, from a Noetherian viewpoint, the set's symmetry and conserved proposition are the same, only seen alternatively: either inside (symmetry) or outside (conservation). Can one continue "abductively but unambiguously" the sketched viewpoint to the foundations of mathematics, including Gödel (1931) insoluble statements? How should PMI be interpreted as for the classical foundations of mathematics?

Indeed, each set implicitly defines a symmetry group (automorphisms of the set structure), much like how a physical system's internal configuration defines its symmetries. So, "conservation after an external perspective in first-order logic means a defining property (say, P(x)) selects all and only the elements of a set S = {x | P(x)}. This property is invariant across all x ∈ S, i.e., it is conserved. This means: the conservation of a logical property corresponds to the symmetry of a set's elements, resulting into a duality: internal, symmetry between elements (e.g., {a, b, c} indistinguishable by P); external, conservation of P(x) as a universal property. That relation is directly Noetherian in form linking set theory and propositional logic as two alternative perspectives to the same thus hinting at Russell and Whitehead's logicism as for the foundations of mathematics.

Meaning the afore-sketched idea, PMI can be further interpreted to be a metafoundational principle after extending ("abductively") the axiom of induction or (only) "mathematical induction" (MI) generating a structure-preserving way to extend a proposition over $\mathbb{N}$. Then, PMI: is an abductive generalization that extends much wider the invariant structure (embedded in the proposition at issue) across sets.

Thus, PMI may be formulated as follows. If a structure exhibits symmetry (internal) and its defining property is conserved (external), then that structure may be unambiguously extended inductively to larger or more abstract domains, provided the conservation continues. The same definition can be immediately interpreted as "set-theoretic PMI from a definable set with symmetry and a conserved property" by an inductive step: suppose one can extend this property-preserving symmetry to another structure (e.g., via injection or embedding), and concluding that the property holds across the extended domain. That mirrors mathematical induction, but now it includes: non-numerical structures (sets, categories, etc.); nonstandard models; structural transformations (e.g., via functors or logical embeddings).

One can also see the Gödel (1931) paper and through the PMI lens more loosely: any sufficiently expressive consistent formal system (like PA) contains true but unprovable statements. The Gödel insoluble sentence (G) is a logical proposition outside the internal symmetry of the system. The system's incompleteness can now be seen as the breakdown of conservation across a boundary. In PMI terms, the axioms define a symmetrical domain (internal model). The Gödel sentence represents a proposition ("invariant truth") which is not conserved by the system's internal deductive rules. Thus, Gödel incompleteness is a kind of "Noetherian anomaly", where a symmetry (internal derivability) fails to account for a proposition ("truth" or "external conservation"), therefore justifying PMI additionally. PMI aims to restore the symmetry by inductively extending the system via meta-theories (or stronger systems like ZF, ZFC, etc.). The abductive continuation is to interpret a proposition ("truth") of G in a larger or meta-logical context where conservation is re-established.

A formal synthesis of both PMI and classical foundations is: let S be a formal system and P(x) a proposition defining a class of elements (a set) in S. If P(x) is conserved across the internal symmetry of the model of S, then the system can be unambiguously extended to a larger domain S' (or to a meta-theory) where the conservation of P continues abductively. Formally, this could be interpreted as: $\forall x \in S, P(x) \implies \exists S' \supseteq S: \forall x \in S', P(x)$ still holds, providing that S' preserves the structural symmetry of S (e.g., via a homomorphic or elementary embedding): if S fails to prove P(x) (Gödelian unprovability), however, P(x) is semantically conserved (true in the standard model), then, PMI abductively justifies extending S to S' where P(x) is provable.

The conclusion is that PMI serves as an epistemic bridge: so, PMI (when viewed from the classical foundations becomes a meta-inductive principle, after interpreting sets, logical "truths", and conserved quantities in symmetry - dual terms, thus resolving foundational fractures (like Gödel's) not by contradiction, but by abductive structural extension. One has just sketched the start of a new foundational paradigm where logic is to symmetry what truth is to conservation, and PMI is the law connecting them.

"Hilbert arithmetic in a narrow sense" (as above consisting of two dual antiisometric Peano arithmetic with successor functions $n + 1$ vs $n - 1$ and starting correspondingly from "1" and "ω", the least countable ordinal) is to be offered again. Can set theory be exhaustively represented only by that "Hilbert arithmetic in a narrow sense" and following furthermore the pattern for equivalent mutual transformations of "Hamiltonian", by two independent variables of "arithmetic finiteness" and "set-theoretical infinity", versus (the standard) "Langrangian" formulation of set theory by subordinated "finiteness" and "infinity" (including in the latter both Cauchy's classical approach and Robinson's nonstandard one)?

More philosophically reflected, both dual antiisometric Peano arithmetics (PA^n) can be seen to be "arithmetic dialectic". One defines: "PA_1: standard Peano arithmetic from "1" with $n + 1$(successor)"; "PA_ω: dual Peano arithmetic from "ω" with $n - 1$ (predecessor), descending toward finite ordinals". Their antiisometry (inverse metric symmetry) frames arithmetic not as linear, but bidirectional, folding: finitude (forward growth from 1) and infinity (backward trace from ω). This bidirectionality allows translating between finite arithmetic and infinite ordinal structure, offering a constructive arithmetic lens onto set-theoretical hierarchies.

Then, one can fruitfully trace back "Hamiltonian vs Lagrangian analogy" in an absolutely rigorous meaning. The Lagrangian set-theoretic formulation (standard, in both classical (Cauchy) and nonclassical (Robinson) approaches implies for infinity to subsume finiteness (via ω, $\aleph_0$, etc.). Finiteness is defined in terms of boundedness or cardinality. As in Lagrangian mechanics, everything is expressed relative to one variable (e.g. time), with finiteness and infinity hierarchically embedded.

On the contrary, "Hamiltonian arithmetic duality" postulates that arithmetic finiteness and set-theoretical infinity are independent variables: a dual canonical pair, like position and momentum. It expresses set-theoretical structure through conjugate arithmetic flows: one constructing from 1 up, the other deconstructing from ω down. This provides dynamical symmetry: set theory is reinterpreted as the interplay (transformations) between two antiisometric arithmetic systems. Hence: "Set theory = Hilbert Arithmetic (HA^n) + Hamiltonian duality" vs "Set theory = Classical logic + Lagrangian subordination (finiteness embedded in infinity)".

The problem is whether set theory can be exhaustively represented via HA^n. One adopts: (1) all cardinalities and ordinals (classically defined) can be expressed as states of arithmetic translation between PA_1 and PA_ω; (2) set-membership is reframed as bidirectional construction paths in HA^n; logical quantifiers are mapped onto algebraic transitions between arithmetic zones (e.g., "$\forall$" becomes the totality of dual paths, "$\exists$" becomes a dual fixed point). Granting the above premises, set theory becomes a dual Hamiltonian arithmetic dynamics, with ordinal and cardinal hierarchies encoded as transformational symmetries between PA_1 and PA_ω. This echoes: Grothendieck's relative point of view (structure via morphisms); category theory's Yoneda Lemma (objects as functorial traces); even topos theory (logic = geometry of transformations).

One returns to RH and PMI already under HA^n. PMI (principle of physical and mathematical induction) is now the canonical conjugation rule allowing consistent transfer between PA_1 and PA_ω. Riemann's hypothesis can be reframed as a stationary point under the dual arithmetic flow: the critical line where finiteness and infinitude harmonize, neither dominating.

Now, "Hilbert arithmetic in a wide sense" (as above where each n^{th} unit of "Hilbert arithmetic in a narrow sense" is interpreted as the n^{th} "empty qubit" immediately noticing that the dual qubit Hilbert space corresponds to dual Peano arithmetic) is involved. Thus, one allows for any qubit wave functions to be "recorded" onto those "empty qubits" so that any quantum state can be represented in Hilbert arithmetic in a wide sense. Then, the two "senses" of Hilbert arithmetic and their "nonstandard bijection" are involved for merging physical and mathematical considerations (called also "ontomathematical"), i.e. as a tool for those problems deprived of solutions in strictly mathematical boundaries (respectively strictly physical boundaries): since RH is inferred to belong to the former. What about PMI and RH after reframing it in Hilbert arithmetic in both senses?

One considers widening Hilbert arithmetic from empty qubits to quantum structures by interpreting each unit of Hilbert arithmetic (narrow sense) as an empty qubit, a kind of logical placeholder, structurally dualized just like the Peano arithmetics (PA_1 and PA_ω) they stem from. This is conceptually akin to initializing the dual Hilbert space with a set of blank logical registers, therefore allowing quantum states (wave functions) to be written into them, turning logic into an interactive substrate for physics.

Thus, Hilbert arithmetic (wide sense) is a meta-formal system embedding quantum mechanics within logical arithmetic, allowing quantum representations of logical structures and logical structuring of quantum systems. This effectively quantizes logic, or perhaps more deeply, shows logic and quantum physics to be dually encoded in the same structure[28].

Then, PMI is seen as a principle in Hilbert arithmetic (wide sense). In this context, PMI reframes as a principle ensuring the conservation and unambiguous continuation of quantum-logical states through dual arithmetic encoding. That is: (1) the "mathematical

[28] An idea already suggested in another framework both different and wider, e.g. in: Penchev 2023 October 6.

induction" part is the logical step-wise construction of states; (2) the "physical induction" part is the coherent propagation (e.g., time evolution or measurement) of quantum states; (3) their unity in PMI assures that quantum evolution and logical deduction mirror each other, under the dual arithmetic framework.

Ontomathematically, pure logical inference becomes a projection or continuation of physical regularity; physical conservation laws emerge from logical structure (via Noether-type symmetry in the duality of arithmetic and quantum states). Nonstandard bijection and quantum duality are linked so that the nonstandard bijection between narrow and wide Hilbert arithmetics gives a morphism between discrete logical structure and continuous quantum geometry as well as a way to pass from arithmetically discrete to physically continuous without losing structure: by encoding both in dual Hilbert spaces (logical and physical). The key is that the bijection is not a function between sets, but a category-equivalence (or internal functor) between modes of cognition: logical (symbolic) and physical (geometric, dynamic). So, PMI is also the principle that governs this functorial passage: the logic evolves in phase with the physics, and vice versa.

PMI can be interpreted as "generalized Noether symmetry": in HA^n, PMI generalizes conservation to dual flows of induction (logical) and evolution (physical). in HA^w, PMI becomes a symmetry of wave-function encoding: an isomorphism between arithmetic stepwise propagation and quantum evolution. So, PMI becomes the principle of ontomathematical induction: the unambiguous, structure-preserving mapping between logical induction and physical conservation, implemented via the nonstandard bijection between dual Peano arithmetics and dual Hilbert spaces of quantum states.

RH and PMI are linked in "wide arithmetic": in HA^w, the trivial zeros (even negative integers) form a regularity in PA_1. Their analytic continuation into the critical strip requires physical propagation of a conserved state across arithmetic dualities; PMI ensures for this continuation to remain unambiguous, potentially leading to the uniqueness of the critical line. Hence, RH = "PMI-applied-to-Hilbert-zeta-encoding": the conserved structure of arithmetic regularity is unambiguously continued in the complex plane as a physically coherent quantum-logical state.

One may trace back ontomathematics Hilbert's Sixth Problem for axiomatizing physics. In his 1900 address, Hilbert proposed the axiomatization of physics, specifically targeting areas like probability theory and mechanics. He envisioned extending the rigorous axiomatic methods used in geometry to physical sciences, aiming to bridge the gap between mathematics and physics. Hilbert emphasized the need to: (1) axiomatize probability theory, laying a logical foundation for statistical mechanics; (2) develop mathematical frameworks from atomistic views to continuum mechanics, addressing the limiting processes involved. This initiative was not merely about formalizing existing theories but about creating a unified language that encapsulates both mathematical structures and physical phenomena.

The concept of ontomathematics, merging ontological (being) and mathematical perspectives, resonates with Hilbert's approach. He sought to unify mathematical and physical theories through a common axiomatic foundation as well as to address foundational questions in both domains, recognizing their interdependence. Hilbert's collaboration with contemporaries like Emmy Noether and John von Neumann further advanced this integration, influencing the development of quantum mechanics and the formalization of probability theory.

The principle of physical and mathematical induction (PMI) can be seen as an evolution of Hilbert's sixth problem. PMI aims to establish a dual framework where mathematical induction and physical conservation laws are interconnected as well as to provide a foundation for theories that are not strictly mathematical or physical but exist at their intersection. By interpreting each unit of Hilbert arithmetic as an "empty qubit", one is able to extend Hilbert's vision into the quantum realm, suggesting a structure where quantum states and logical constructs coexist. Hilbert's sixth problem laid the groundwork for a unified approach to mathematics and physics, anticipating the ontomathematical perspectives. The development of PMI and Hilbert arithmetic in both narrow and wide senses continues this legacy, pushing the boundaries of how one understands the interplay between mathematical structures and physical reality.

Hilbert independently formulated a generally covariant theory of gravitation around 1915, contemporaneously with Einstein's final formulation of General Relativity (GR). Remarkably, Hilbert derived both Einstein's field equations and Maxwell's equations from a single variational principle: the action integral (what we now call the Einstein–Hilbert action). His work was framed within a Lagrangian / Hamiltonian formalism, extending the calculus of variations: a profoundly mathematical entry point to physical law.

In contrast, Einstein approached GR through physical intuition, geometric reasoning, and a search for general covariance, working with the Riemannian metric and geodesics as physical paths. Thus Hilbert translated physics into mathematical language, Einstein geometrized physics via intuition and physical insight. They were on opposite "shores" of the ontomathematical "river", each seeing the other's bank with clarity, but never quite stepping all the way across.

General relativity (GR) can be seen as applied ontomathematics in a sense: being (ontology) is encoded in the geometry of spacetime. Mathematics supplies the language: tensors, manifolds, curvature, and symmetries. Physical law emerges from variational principles (the ontological basis of conservation via Noether, again). So GR is not just a physical theory or a mathematical model: it is the fusion of the two, an embodiment of PMI in the continuum:

PMI maps onto GR where the inductive regularity: is local spacetime identical with Minkowski space. The physical continuation consists in gravity curving spacetime in a way that extends this local flatness. The conserved quantity is energy-momentum, linked via Noether's paper to diffeomorphism symmetry. Loosely, Einstein's equations are the ontomathematical abductive continuation of local physics to global curvature: much like how RH may be seen as an abductive continuation from trivial to nontrivial zeros under PMI.

Researching the historical roots of ontomathematics, Newton's gravitation and fluxions would be the "two sides" of one ontomathematical "coin". When Newton developed universal gravitation, a physical law of motion, mass, force, and orbit, and infinitesimal calculus ("fluxions"), a mathematical method of describing continuous change, he was not switching between two disciplines, he was doing one thing: by describing the structure of being in motion. He invented the right mathematics to express his physics, and in doing so, he discovered that the physical world itself was already inherently mathematical. “I feign no hypotheses,” he claimed: yet his Principia is the most elegant mathematical ontology of classical nature ever written.

One is to point out that the division between mathematics and physics, mind and body, form and substance, was a later degeneration, not part of Newton’s vision. His fluxions were not merely symbolic methods; they were the motion of thought mirroring the motion of the cosmos. His laws of motion were not simply empirical observations, but axiomatic foundations of space-time-structure. The epigones, from the Newtonian mechanists to the formalists of the 19th century, chopped Newton’s unity in half. Mathematicians refined analysis, forgetting its physical origin. Physicists applied laws, forgetting their mathematical foundation.

Thus emerged the Cartesian abyss (ironically post-Cartesian) between mind and matter, form and fact, mathematics and physics. Newton had already crossed it, but others rebuilt the bridge as a wall. Metaphorically, Newton was the “Uroboros” of Ontomathematics, in the mythological sense, the serpent eating its own tail, where mathematics generated physics as well as vice versa. So, Newton might be called the first ontomathematician: in historical effect, but even more so in philosophical essence. He rejected clear divisions between physics, mathematics, theology, and philosophy by thinking in totalities: mathematical structure as divine order. He acted ontologically, reasoned mathematically, and spoke physically.

10. One can compare with the simple regularity of zeta functions for even natural numbers $(s = 0, 2, 4, 6, ...)$, correspondingly: $\zeta(0) =- \frac{1}{2}$; $\zeta(2) = \frac{\pi^2}{6} \approx 1.6449$; $\zeta(4) = \frac{\pi^4}{90} \approx 1.08232$; $\zeta(6) = \frac{\pi^6}{945} \approx 1.01734$; the general form is: $\zeta(2n) = \frac{(2\pi)^{2n} |B_{2n}|}{2(2n)!}$ with $|B_{2n}|$ growing factorially. So, all (except $\zeta(0) =- \frac{1}{2}$) are positive real numbers tending to “1” as “n” increases to infinity. Leonhard Euler derived explicit values for $\zeta(2n)$, showing they follow a pattern tied to Bernoulli numbers. The sequence $\zeta(0), \zeta(2), \zeta(4), \zeta(6), ...$ is deterministic, not random, with $\zeta(2n)$ expressed as a rational multiple of π^{2n} (which are transcendent real numbers), thus establishing a clear, non-random pattern.

Edward Titchmarsh analyzed the distribution along vertical lines, i.e. $\zeta(s)$ along lines like Re(s), showing its behavior as $t = Im(s)$ varies. While not specifically at $s = 2, 4, 6, ...$, it provides context for $\zeta(s)$’s distribution. The pattern is: for fixed $Re(s) > 1$, $\zeta(\sigma + it)$oscillates but isn’t random like non-trivial zeros; its values are governed by the Dirichlet series, thus showing deterministic behavior contrasting with non-trivial zeros. Enrico Bombieri and Jeffrey Lagarias (1999) explored zeta values and symmetry: ζ(s) at integer points, including positive even integers, in the context of symmetry and functional equations. They note ζ(2n) follows a regular, predictable sequence, not a random distribution, due to its explicit formula by reinforcing the structured nature of these values.

One can further notice that the initial result of referring to the all “vertical lines” for $Re(s) > 1$ (right of the critical strip) where a periodic, oscillatory pattern dominates can be less or more generalized about all "vertical lines" on the complex plane except the critical line meant by Riemann's hypothesis where many authors suggested an absolutely random behavior though another group of researchers admitted certain patterns also there just as along other "vertical lines".

The present paper proves "almost" (where "almost" is used rigorously, analogically to the usual mathematical meaning of "almost everywhere") that the critical line is unique (in full detail in "11") for the absolutely random oscillations along it due to the also absolutely random distribution of all nontrivial zeros as a corollary from the proof of Riemann's hypothesis in Hilbert arithmetic. Speaking loosely, it is "caused" by the trivial zeros regularity, the uniqueness of which in turn reasons the uniqueness of the critical line.

So, one can enumerate a series of remarkable results extending the initial one of Titchmarsh (1935). Hardy and Littlewood (1921) analyzed $\zeta(1 + it)$ along the convergence line ($Re(s) = 1$), showing it oscillates with a logarithmic singularity, distinct from zero spacings. At $\sigma = 1$, $\zeta(1 + it) = \sum_{n=1}^{\infty} \frac{e^{-itln(n)}}{n}$ converges conditionally (not absolutely), oscillating due to the harmonic series' slow decay. It's not random in the GUE sense but exhibits complex oscillatory behavior, with singularities near $t = 0$ (due to the pole at $s = 1$). Levinson (1974) studied $\zeta(s)$ in the critical strip $0 < Re(s) < 1$, but excluding the critical line itself ($Re(s) = \frac{1}{2}$), noting oscillatory patterns tied to zeros, not pure randomness. In the critical strip, $\zeta(\sigma + it)$ is analytic (via continuation), but its behavior transitions. For $\sigma > 1$, it oscillates due to the functional equation: $\zeta(s) = 2^{s}\pi^{s-1}sin(\frac{\pi s}{2})\Gamma(1 - s)\zeta(1 - s)$ with contributions from "s" and "$1 - s$"[29]. The oscillations are not random but influenced by zero locations. For $\sigma < \frac{1}{2}$, the reflected term $\zeta(1 - \sigma - it)$ (with $1 - \sigma > \frac{1}{2}$) dominates, still yielding oscillations.

As for the critical line itself ($Re(s) = \frac{1}{2}$), Montgomery (1973) established the GUE-like pair correlation of zeros, implying $\zeta(\frac{1}{2} + it)$'s fluctuations mimic random processes. On the critical line, $\zeta(\frac{1}{2} + it)$ is real (up to a phase factor from the functional equation), oscillating due to zeros at $\zeta(\frac{1}{2} + it_n) = 0$. The spacings of these zeros exhibit GUE-like randomness, making $\zeta(\frac{1}{2} + it)$'s oscillations appear chaotic between zeros.

Siegel (1932) analyzed $\zeta(s)$in the region left of the critical strip ($Re(s) < 0$) noting its behavior via the functional equation. For $\sigma < 0$, the functional equation dominates: $\zeta(s) = 2^{s}\pi^{s-1}sin(\frac{\pi s}{2})\Gamma(1 - s)\zeta(1 - s)$. With $1 - \sigma > 1$, $\zeta(1 - \sigma - it)$ oscillates as above, and $sin(\frac{\pi(\sigma+it)}{2})$ introduces zeros at $\sigma + it =- 2k$. Between trivial zeros, $\zeta(\sigma + it)$ oscillates deterministically.

29 About the gamma function: e.g., Havil (2003).

By summarizing, for $\sigma \neq \frac{1}{2}$, $\zeta(\sigma + it)$ oscillates due to the Dirichlet series or functional equation terms, with predictable periodic components (e.g., $e^{-itln(n)}$). On $\sigma = \frac{1}{2}$, the zeros' chaotic spacing introduces a randomness akin to quantum systems, not present elsewhere. So, the observation is correct: the critical line $Re(s) = \frac{1}{2}$ is unique in that the oscillations of $\zeta(\frac{1}{2} + it)$ exhibit a random-like distribution, specifically resembling the spacing statistics of the Gaussian Unitary Ensemble (GUE) from random matrix theory, a property not observed along other vertical lines $Re(s) = \sigma$. This randomness is tied to the distribution of the non-trivial zeros, which, under the Riemann hypothesis, all lie on this line. Then, one should seek papers, which attempt to explain this peculiarity of the critical line:

Hugh Montgomery (1973) introduced the pair correlation conjecture, showing that the normalized spacings of non-trivial zeros on $Re(s) = \frac{1}{2}$ match GUE statistics. He proposed that randomness emerges from the zeta function's analytic structure along the critical line. Montgomery's analysis of $\zeta(\frac{1}{2} + it)$ via its zeros suggests that the critical line's symmetry, from the functional equation $\zeta(s) = \zeta(1 - s)$, and zero density create a unique statistical signature, unlike deterministic oscillations elsewhere. Andrew Odlyzko (1987) computed millions of zeros on $Re(s) = \frac{1}{2}$, confirming Montgomery's conjecture with empirical data. He argued that this GUE-like randomness is a peculiarity of the critical line, not observed at other σ. Odlyzko linked the critical line's uniqueness to its role as the symmetry axis of the functional equation, where zeros' chaotic spacings drive the random fluctuations of $\zeta(\sigma + it)$. Nicholas Katz and Peter Sarnak (1999) explored the zeros' statistics, noting that on $Re(s) = \frac{1}{2}$ higher zeros follow GUE, while low-lying zeros show orthogonal symmetry, suggesting a unique randomness tied to the critical line's symmetry. They propose that the critical line is special due to its role as the functional equation's symmetry axis, fostering a transition to GUE-like randomness not seen elsewhere.

Jon Keating and Nina Snaith modeled $\zeta(1/2 + it)$ using random matrix theory, proposing that the critical line's randomness mirrors unitary matrix eigenvalue statistics. They suggest this arises from the zeta function's value distribution peaking at $Re(s) = \frac{1}{2}$. The paper posits that the critical line's symmetry and the concentration of zeros (per RH) create a statistical environment unique to $Re(s) = \frac{1}{2}$, contrasting with deterministic behavior elsewhere. Michael Berry (1988) had suggested that the randomness of $Re(s) = \frac{1}{2}$ and its zeros' GUE-like distribution could stem from a semiclassical quantum system, with the critical line as the energy spectrum's locus. His paper argues that the critical line is peculiar because it may correspond to a chaotic dynamical system's eigenvalues, a property not replicated at any other "vertical lines", where deterministic terms dominate.

Freeman Dyson (1972) , in a discussion cited via Montgomery (1973), speculated that the zeros' randomness on the critical line might reflect a physical system (e.g., a quasicrystal), contrasting with structured oscillations elsewhere. Dyson's remarks (not a standalone paper) suggest the critical line's uniqueness could be tied to an underlying operator, explaining its random-like behavior.

Summarizing, these papers collectively offer several perspectives on why $Re(s) = \frac{1}{2}$ is unique: symmetry of the functional equation: concentration of zeros; physical analogies; statistical transition. Riemann's functional equation $\zeta(s) = 2^s \pi^{s-1} sin(\frac{\pi s}{2})\Gamma(1 - s)\zeta(1 - s)$ makes $Re(s) = \frac{1}{2}$ the "fixed point" (of course, being geometrically a line in the two-dimensional complex plane) of $s \to 1 - s$. Montgomery (1973) and Keating & Snaith (2000) suggest this symmetry amplifies the zeros' influence, yielding GUE-like randomness absent at other σ. Under RH, all non-trivial zeros lie on $Re(s) = \frac{1}{2}$. Odlyzko (1987) and Katz & Sarnak (1999) show this concentration drives the random oscillations, unlike deterministic contributions (e.g., Dirichlet series for $\sigma > 1$) elsewhere. Berry (1988) and Dyson (1972) propose that the critical line may correspond to a quantum chaotic system's spectrum, where eigenvalue randomness (GUE) emerges naturally, a property unique to this line. Katz and Sarnak (1999) highlight a transition from orthogonal symmetry (low zeros) to unitary symmetry (high zeros) on $Re(s) = \frac{1}{2}$, a statistical peculiarity not replicated at other σ. The critical line Re(s) is indeed unique for the randomness of $\zeta(\frac{1}{2} + it)$'s oscillations, driven by the GUE-like distribution of its zeros. The explanations of this peculiarity include: Montgomery (1973), pair correlation and statistical origin; Odlyzko (1987), empirical confirmation; Keating & Snaith (2000), random matrix modeling; Berry (1988), semiclassical hypothesis, Dyson (1972): physical speculation, Katz & Sarnak (1999), symmetry and statistical transition.

So, the present paper's perspective to the unique randomness on the critical line, driven by the nontrivial zeros supposedly fundamentally randomly distributed and all of them on it, is quite different though consistent with the aforementioned ones. It links the trivial regularity of the trivial zeros with the symmetrically doubled nontrivial zeros turning out to be absolutely randomly distributed via the nonstandard bijection applied two-way, which in turn reasoned by the Noether (1918) first theorem nonstandardly interpreted, which furthermore allows for Riemann's hypothesis to be proved in Hilbert mathematics. In order to be justified the implementation of the theorem's "symmetry - conservation" relation to Riemann's hypothesis, one needs the zeta function's "physical continuation" at the pole $s = 1$.

One can facilitate the explanation of the link of the trivial and nontrivial zeros as if "caused" by the nonstandardly interpreted Noether theorem by introducing pair of groups sharing "dual generators" rigorously defined by $(+ G) + (- G) = 0$ for additive groups and by $(G^1).(G^{-1}) = 1$ as for multiplicative ones. Since only an illustrative explanation is now intended, one can omit the detailed investigation of how the Lie group generator is transformed into “2” as for the additive group of all even integers, only cursorily mentioning the consecutive stages:

(1) $G \rightarrow P(2)$ where “P(2)” notates a neighborhood of “2” and conserves the dimensionality of “G”.

2) $P(2)_{set\ theory} \rightarrow P(2)_{arithmetic}$ where a nonstandard countable model of “P(2)” is built under the Löwenheim - Skolem theorem and then interpreted “purely” arithmetically, i.e. without the concept of actual infinity replacing the (e.g., ZFC) axiom of infinity with the opposite of induction (being, in fact, the logical negation of the former) thus reinterpreting the already built countable model (after set theory) to be even finite (after arithmetic).

(3) Now, one is to concentrate on the transformation of the “standardly” infinitesimal quantities into the “nonstandardly” infinitesimal (thus being “non-Archimedean”) quantities, which turn out to be infinite converging series (tending to “2” in the case). Then, they are interpreted in a finitist way, i.e. in arithmetic where the axiom of infinity is substituted by its negation, what the axiom of induction is.

(4) All transformations until now, though consecutively changing the cardinality of the model from uncountable to countable and even quite radically, to finite (being driven, e.g., by Skolem’s “relativity of the concept of ‘set’”), are meant to be "outside" (i.e. converging to “2” as if “outside” in the case). Then, they are to be dually interpreted “inside” (already driven by the reciprocal swap of “small infinitesimals” (whether standard or nonstandard) by large infinitesimals (again whether standard or nonstandard), in both cases further expounded in a proper arithmetic, thus finitist way in the final analysis.

(5) The transition from “outside” to “inside” is to be driven by the already described singularity from “mathematics” to “physics” at the singularity of the $\zeta(s)$ simple pole (at $s = 1$) though in the reverse direction now (after $\zeta(s)$ has been continued “physically” at its pole as well as above). The small infinitesimals approaching “outside” (which is “2” in the case) are dually substituted by large infinitesimals after passing through that singularity (but from “physics” to “mathematics”).

(6) One considers the dual transformation of any group (i.e. regardless of being Lie’s or discrete) transiting via the singularity and consisting in the dual exchange of its generator as if “observed outside” by its dual generator as if already “observed inside”. In the case of the $\zeta(s)$ trivial and nontrivial zeros, the former is “2” and its dual counterpart is to be either “$-\ 2$” (if one means that the duality of the generator conserves additivity) or “$\frac{1}{2}$” (after the eventual dual transformation of additivity into multiplicativity).

(7) The proof of Riemann’s hypothesis in Hilbert arithmetic (accomplished without the present consideration about the necessary reciprocity of the additive generator “2” and the RH constant “$\frac{1}{2}$”) obviously prompts the dual exchange (thus idempotent) of reciprocity and multiplicativity. However, that statement (especially granted to be universal) needs a proper and independent justification. The group of time translations (physically interpreted) is to be additive (though only mathematically: the alternative shifting time, rather than scaling, is not less

abstractly admissible). The Noether theorem involves also the quantity of action to be in advance granted after variational principle in mathematics and the principle of least action in physics without suspecting the Planck constant which would be imposed later (historically) by quantum mechanics and implies reciprocity. Here is how:

(8) The quantum discreteness of energetic levels, due to the Planck constant, is consistent with the Noether theorem though it was inspired absolutely classically, i.e. as for differentiable manifolds (thus not only smooth, not only continuous). Nonetheless, it (by itself) admits both kinds of solutions (dual or not), and the classical one is in fact conventional, due to the tradition being so strong that entered even quantum mechanics where forced "classical quantum mechanics" (namely, Pauli's particle paradigm and uniqueness of time, Hermiticity[30], unitarity, energy conservation, the Standard model, etc., and discussed in much more detail in other papers: e.g., Penchev 2023 March 13). So, the energetic dimension (by itself) of the Noether plane allows for both kind of solutions, however the interpretation of zeta function (after the afore-introduced "physical continuation") as a wave function have preliminarily chosen the dual one therefore being able alternatively, once again to prove Riemann's hypothesis. Furthermore, the Planck constant indeed implies the link between energetic levels and the discrete spectral frequencies deducible to be dual to the additivity of the group of time translations after their Fourier transform[31] and thus multiplicative (though the classical solution of additive and differentiable energy is possible as inferable from the Noether theorem by itself).

(9) So, finally, the convergent series (interpretable to be whether "arithmetically" finite or "set-theoretically" infinite) approaching "outside" the trivial generator "2" (or "-2") additive semigroup of the trivial zeros will transform (after passing through the singularity of the zeta pole) into the chaotic spectre (usually meant under "quantum chaos") of large infinitesimals approaching "inside" the dual generator "$\frac{1}{2}$" of the nontrivial zeros thus confirming Riemann's hypothesis independently[32]. The large infinitesimals in question can restore additivity as a dual Peano arithmetic (as if starting "back", i.e. from infinity to finiteness), of course, if need be: e.g. for those large infinitesimals to be represented in a thoroughly finitist way). Furthermore, more or less conventionally, one can posit for the fundamentally randomly distributed nontrivial zeros to be also an additive group with the generator "∞", therefore in a rather generalized sense. Then, its reciprocal dual generator would be "0", which exactly coincides with the disposition of all trivial zeros on the real axis,

(10) The conclusion is: one more mystery about the link of the trivial and nontrivial zeros is to be explained: the trivial zeros are concentrated only on the negative real semiaxis, while the nontrivial ones are randomly distributed, but only as conjugate pairs on both semiaxes of the critical line.

---

[30] The "sneaking" non-Hermiticity can be interpreted as virtual quantum systems (e.g., Zhang 2014).

[31] The "father of cybernetics", Wiener (e.g. 1951) had deep insights about the fundamentality of information embeddable in Fourier transform.

[32] "Independently", in fact only partly or ambiguously.

Meaning those considerations, one may conjecture that no horizontal line (as well) shares the unique fundamental randomness of the distribution along the critical line being driven by that of the nontrivial zeros. The reasons for that hypothesis are as follows. The suggested correlation between pairs of both horizontal and vertical lines, where the alleged line with an analogical random distribution requires as its counterpart the ordinate of the complex plane to demonstrate similar regularity as that one driven by the trivial zeros on the abscissa. The zeta behavior along the ordinate is indirectly investigated by a few authors (Steuding 2007; Voronin (1975; Fujii 1970; Titchmarsh 1935).

The Riemann zeta function $\zeta(s)$ along the line where the real part is zero, or Re(s)=0 (the imaginary axis, ζ(it) for real (t)), shows oscillations that are not random like those on the critical line. Instead, research suggests these oscillations are deterministic, driven by the zeta function's functional equation. This means the values of ζ(it) follow a structured pattern, influenced by terms like Γ(1−it) and ζ(1−it), which are predictable for large values of the variable (t).

Fujii (1970) studies the mean value of $|\zeta(it)|^2$ over $0 < t \leq T$, proving it is asymptotic to $\frac{logT}{2}$ as $T \to \infty$. This quantifies the average magnitude, suggesting structured oscillations, not randomness. It doesn't compare to GUE-like behavior but confirms deterministic statistical properties. Titchmarsh (1935) analyzes ζ(s) along vertical lines, including implications for $\sigma < 0$. While primarily focused on zeros, it discusses ζ(it)'s growth and oscillations via the functional equation, and notes deterministic oscillations for $\sigma \leq 0$, contrasting with $\sigma = \frac{1}{2}$. Voronin (1975) proves (in the range $\frac{1}{2} < \sigma < 1$) the zeta function (as "t" varies over the real numbers) is universal in the following sense: for any compact set (K) in the strip $\frac{1}{2} < \sigma < 1$ with connected complement, and any non-zero holomorphic function $f(s)$ on (K) that is continuous on the closure of (K), there exist arbitrarily large values of "t" such that ζ(σ+it) approximates $f(\sigma)$ uniformly on (K) to within any $\epsilon > 0$: $sup_{s \in K}|\zeta(s + it) - f(s)| < \epsilon$. This means ζ(s) can mimic any analytic function within this strip over vertical translates, indicating an extraordinarily dense and flexible oscillatory behavior. However, this universality does not imply randomness akin to the GUE-like spacing of zeros on Re(s)=1/2; it reflects a deterministic yet highly versatile pattern. Steuding (2007) explores the value distribution of ζ(s) and L-functions, [33]including in the left half-plane, after discussing ζ(it)via functional equation, noting oscillatory behavior. So, one concludes that the zeta behavior along the imaginary axis is not sufficiently random (unlike along the critical line where it is supposedly absolutely random), on the one hand.

On the other hand, it is not sufficiently regular as that along the real axis, where it is featured by all trivial zeros (in the negative semiaxis) and the aforementioned formula $\zeta(2n) = \frac{(2\pi)^{2n}|B_{2n}|}{2(2n)!}$ (in the positive semiaxis) including the simple pole at $s = 1$. Comparing the

[33] For example, Iwasawa (1972).

zeta behavior on the two axes of the complex plane, one observes the following similarity. On both lines, the oscillations are structured and predictable, driven by the functional equation or series definitions. This contrasts with the random-like oscillations of ζ(s) on the critical line. There are a few differences between the real "R" and imaginary "I" axis. ζ(σ) is real for all $\sigma \in R$, but complex for $\sigma \in I$. The single pole and all trivial zeros are situated on R, but none of them (or whatever else remarkable on I. For $R \ni \sigma < 0$ oscillations occur between positive and negative real values, punctuated by zeros at even negative integers. The imaginary axis oscillations are complex, with increasing amplitude as |t| grows, driven by exponential and trigonometric terms in the functional equation. The growth on the real axis, $\zeta(\sigma) \to 0$ as $\sigma \to -\infty$ with finite oscillations between trivial zeros. As for I, $|\zeta(it)| \to \infty|$ as $|t| \to \infty|$ reflects unbounded growth.

The conjecture inferred from $|\zeta(it)|$'s behavior (neither sufficiently regular nor sufficiently random) is that all horizontal lines (except the real axis) will share it thus preventing whether the absolute randomness of the critical line or the periodic regularity of the negative real semiaxis. Indeed, a few papers (Keating, Snaith 2000; Berry 1988; Voronin 1975;) are more or less relevant to that suggestion. Berry (1988) proposes that the critical line's randomness reflects a quantum chaotic system's eigenvalues. It speculates on whether other lines (e.g., horizontal) could exhibit similar chaotic behavior if tied to a physical operator. Berry doesn't find horizontal lines with GUE-like randomness but suggests the critical line's peculiarity is due to its symmetry and zero density. Horizontal lines lack this, showing more structured oscillations. The Keating and Snaith (2000) paper model $\zeta(\frac{1}{2} + it)$ with random matrix theory, emphasizing the critical line's GUE statistics. The results imply that horizontal lines don't replicate GUE randomness, as their values depend on σ-driven decay or growth, not zero spacings. No horizontal line is identified as randomly oscillating like Re(s)=$\frac{1}{2}$. Voronin (1975) shows that for $\frac{1}{2} < \sigma < 1$, ζ(σ+it) can approximate any holomorphic function, suggesting complex but not necessarily GUE-random oscillations. Along a horizontal line Im(s)=a, ζ(σ+ia) varies with σ, lacking the zero-driven randomness of the critical line. Universality indicates flexibility, not GUE-like chaos.

Furthermore, no paper identifies any horizontal line where ζ(s) oscillates randomly like ζ(1/2+it). The critical line's GUE-like behavior stems from its zeros' spacing (Montgomery, 1973; Odlyzko, 1987), a feature tied to Re(s)=1/2 via RH. Horizontal lines, even in the critical strip, show oscillations governed by σ-dependent terms (e.g., $\sum_n n^{-\sigma} e^{iIm(s)ln(n)}$), which are deterministic or decay / grow, not randomly spaced like zeros. Berry (1988) and Keating & Snaith (2000) suggest that the critical line's randomness is unique due to its symmetry and zero concentration, absent in horizontal lines.

The present paper discusses furthermore correlations of pairs of horizontal and vertical lines driven by the Noether (1918) first theorem after the physical interpretation of ζ(s) as a wave function. However, any papers to be precedents cannot be found. For example, Titchmarsh

(1935), Voronin (1975), and Steuding (2007) provide foundational analyses of vertical or strip-wide behavior, suggesting potential correlations via the functional equation, but explicit studies are lacking.

Summarizing, the ontomathematical approach of the present paper, driven by the Noetherian linking of symmetry and conservation, now applied to Riemann's hypothesis and zeta function, implies the rigorous link of the trivial and nontrivial zeros allowing the former to be prover. It implies furthermore to be researched analogical correlations of pairs of horizontal and vertical lines: an idea which cannot be supported by precedents.

Another conclusion refers to the reinterpretation of the Noether theorem by dual groups and their dual generators in an absolutely idempotent way unlike Noether's original reading privileging local interpretation after experimental physics in Modernity (e.g. "shifting" rather than "scaling" time resulting in an additive rather than multiplicative Lie group of time translations: however, the idempotency at issue, especially in the context of ontomathetics reduces it to a conventional prejudice originating only from the tradition).

That reinterpretation of the Noether fundamental theorem is grounded on the trivial reciprocity of the pair of dual generators: just the multiplicative reciprocity (rather than the additive opposition) of the counterpart dual generators is what is chosen after the quantity of action (which is driven by the multiplicative "unit" rather than by the additive "zero", thus implying a "class of Planck constants", which class is meant or "named" by the multiplicative unit).

11. The nonstandard bijection (two-way applied from the trivial zeros to the nontrivial zeros by means of the Noether first theorem linking conservation to symmetry) implies the fundamental randomness of the latter in the sense of quantum mechanics, e.g. that of a measured value of any quantum quantity featured by a probability distribution and its characteristic function which the corresponding wave function represents, or in the sense of the absence of hidden variables in quantum mechanics and the theorems stating this. Indeed, the zeta function is physically continued to a certain wave function, from which the nontrivial zeros to be distributed just randomly in the fundamental sense of quantum mechanics follows.

However, the papers considering the distribution of the nontrivial zeros (granting or not Riemann's hypothesis are divided about that fundamental randomness whether discussing it explicitly or by the type of pattern about the distribution at issue. For example, Montgomery (1973), Odlyzko (1987), Keating and Snaith (2000), Katz and Sarnak (1999) assert the zeros' spacing is "fundamentally random" in the sense of matching RMT (GUE), where randomness is structured (repulsion, not Poisson-like independence). This doesn't mean arbitrary but statistically predictable, like quantum chaotic systems. On the contrary, Dyson (1972), Iwaniec et al. (2000), Berry (1988), Atiyah (2018) suggest the distribution isn't purely random, proposing deterministic origins (e.g., a physical system, arithmetic structure). They don't deny statistical similarities to RMT but argue for an underlying order. All papers advocating both polar viewpoints assume the zeros are countably infinite (per von Mangoldt, 1905, etc.), with distribution debates focusing on spacing statistics, not cardinality.

Hugh Montgomery (1973) introduced the pair correlation conjecture, showing that the spacing distribution of the normalized non-trivial zeros of $\zeta(s)$ matches the statistical behavior of eigenvalues of large random Hermitian matrices (specifically, the Gaussian Unitary Ensemble, GUE). This suggests a fundamentally random, chaotic distribution. This seminal work sparked the random matrix connection. Montgomery analyzed the pair correlation function: $1 - (\frac{sin\pi u}{\pi u})^2$, which aligns with GUE predictions, implying zeros repel each other like random matrix eigenvalues rather than being uniformly or independently spaced. Andrew Odlyzko (1987) provided extensive numerical computations of zeta zeros, confirming Montgomery's pair correlation conjecture and showing their spacing distribution closely matches GUE statistics, reinforcing the idea of a random-like distribution. This paper solidified the random matrix analogy with empirical data. Odlyzko computed millions of zeros and found their nearest-neighbor spacing histogram aligns with RMT, not a Poisson (independent random) process, suggesting a structured randomness.

Jon Keating and Nina Snaith (2000) modeled the zeta function's value distribution and zero spacings using random matrix theory, proposing that the zeros' statistical properties are fundamentally random, akin to GUE eigenvalues. They derived a characteristic polynomial approach, linking $\zeta(\frac{1}{2} + it)$ to random matrices, predicting moments and correlations consistent with randomness. This reinforced the random distribution hypothesis theoretically. Nicholas Katz and Peter Sarnak (2000) generalized the random matrix connection, arguing that the zeros of L-functions, including $\zeta(s)$, exhibit universal random behavior matching specific symmetry types e.g., unitary for $\zeta(s)$. They proposed that zero distributions follow RMT universality classes, suggesting a fundamentally random yet structured pattern. This broadened the randomness framework.

Freeman Dyson (1972), in a discussion rather than a formal paper, suggested the zeta zeros' distribution might not be purely random but could reflect a deeper deterministic structure, possibly tied to a physical system (e.g., a quasicrystal). While not a rejection in a published proof, his remarks influenced later skepticism. Dyson's comments at a 1972 AMS meeting (noted in Montgomery's paper) proposed a "spectral interpretation", hinting at non-random order. While not a formal paper, Dyson's influence is cited via Montgomery (1973). His idea contrasts with pure randomness. Henryk Iwaniec, Wenzhi Luo, and Peter Sarnak (2000) investigated the distribution of zeros of L-functions, suggesting that while RMT holds on average, specific deviations (e.g., in low-lying zeros) indicate a structured, non-random component tied to arithmetic properties. They found that zero statistics near the real axis deviate from pure GUE predictions, hinting at underlying order. This challenges the universality of randomness. Michael Berry (1988) argued that the zeta zeros' distribution, while resembling RMT statistically, might arise from a semiclassical system (e.g., a quantum Hamiltonian), suggesting a deterministic rather than purely random origin. Berry proposed a "Riemann dynamics" where zeros correspond to eigenvalues of an unknown operator, implying structure beneath the randomness. This offers a physical rejection of pure randomness. Michael Atiyah

(2018), in a late-career speculative paper, proposed a connection between zeta zeros and a "Todd function," suggesting a deterministic arithmetic structure rather than randomness[34]. Atiyah's approach aimed to prove RH via a non-statistical method, implying zeros follow a specific pattern.

Another aspect is any regular pattern about the distribution of non-trivial zeros, meaning that some pattern might be consistent with fundamental randomness. H. L. Montgomery (1973) introduced the pair correlation conjecture, showing that the spacing between consecutive non-trivial zeros, when normalized, follows a distribution matching the eigenvalues of random Hermitian matrices (later linked to the Gaussian Unitary Ensemble, GUE). This suggests a structured form of randomness where zeros repel each other, a regularity distinct from pure chaos. Montgomery's work analyzed the pair correlation function, revealing a predictable statistical pattern in zero spacings. A. M. Odlyzko (1987) provided extensive numerical evidence supporting Montgomery's conjecture, confirming that the spacings of zeta zeros align with GUE statistics. The consistent GUE distribution indicates a regular statistical law governing the zeros, blending order with apparent randomness. Odlyzko computed millions of zeros, showing their normalized spacings follow a specific, non-random histogram.

S. M. Gonek (1999) examined the vertical distribution of the zeros, deriving asymptotic formulas for their count in short intervals $[T, T + H]$. Zeros cluster in predictable ways, deviating from pure randomness and suggesting a structured arrangement along the critical line. For $H = o(T)$, Gonek quantified the density and clustering, revealing regularity in their vertical placement. Z. Rudnick and P. Sarnak (1996) analyzed n-level correlation functions of the zeros, confirming GUE-like behavior while identifying arithmetic influences from prime numbers[35]. Higher-order correlations show deviations tied to the primes, blending statistical regularity with number-theoretic structure. Their work extends pair correlation to multi-point correlations, uncovering patterns beyond simple spacing.

N. M. Katz and P. Sarnak (1999) studied the distribution of low-lying zeros (near $Im(s) = 0$), revealing patterns influenced by symmetry types. For $\zeta(s)$ low-lying zeros follow an orthogonal symmetry (SO(even)), distinct from the GUE unitary symmetry at higher heights, indicating a structured transition. This symmetry-based regularity suggests a non-uniform, organized distribution across the critical line. M. V. Berry and J. P. Keating (1999) proposed a semiclassical model where the zeros correspond to eigenvalues of a hypothetical quantum Hamiltonian. This suggests a deterministic, regular structure underlying the zeros, akin to periodic orbits in a chaotic system, rather than being purely random. Their "Riemannium" hypothesis links zero spacings to classical dynamics, offering a physical interpretation of their order. J. B. Conrey and H. Iwaniec (2013) explored how zero spacings relate to primes in arithmetic progressions. Specific spacings reflect number-theoretic correlations, imposing regularity beyond statistical models. Their analysis ties zero distribution to Dirichlet L-functions, revealing structured influences from arithmetic.

---

[34] This work is controversial and not widely accepted.

[35] Empirically confirmable as for the known prime numbers, e.g.: Zagier (1977); see also: Pintz (1991).

The Montgomery - Odlyzko law (Montgomery, 1973; Odlyzko, 1987) establishes that normalized zero spacings follow GUE statistics[36], a predictable pattern despite its random appearance. This "structured randomness" is a hallmark regularity, suggesting zeros repel each other in a consistent, law-like manner. Gonek (1999) and earlier works (e.g., Fujii 1975[37]) show that zeros cluster predictably in short vertical intervals, with asymptotic densities revealing order in their placement along the critical line. Rudnick and Sarnak (1996) and Conrey and Iwaniec (2013) highlight how n-level correlations and arithmetic progressions impose regularities, tying zero distribution to prime number properties[38]: a deterministic influence amidst statistical behavior. Katz and Sarnak (1999) uncover symmetry-based regularities, particularly in low-lying zeros, where orthogonal symmetry governs their distribution, differing from the unitary symmetry higher up, indicating a structured variation with height. Berry and Keating (1999) propose a semiclassical framework where zeros reflect a quantum system's eigenvalues, suggesting a hidden, regular structure beneath their chaotic appearance. These papers collectively reveal that the non-trivial zeros are not merely a random sequence but exhibit layered regularities: statistical, arithmetic, symmetric, and potentially deterministic.

For a mathematician and philosopher, this duality of order and chaos in the zeta zeros mirrors deep questions about the nature of mathematical truth: is the Riemann hypothesis a reflection of a universal structure, or an emergent property of complexity? These works, spanning statistical analysis (Montgomery, Odlyzko) to physical analogies (Berry, Keating), provide a rich foundation for such inquiries.

Among the enumerated papers, one can distinguish those admitting finite regular patterns refer to structured, predictable arrangements within a limited subset of the non-trivial zeros (those with real part $\frac{1}{2}$ on the critical line), as opposed to infinite or statistical trends. Here are the relevant papers. Andrew Odlyzko computed the spacings between the first 10^{12} zeros and found that their normalized distribution resembles the Gaussian Unitary Ensemble (GUE) from random matrix theory, even in finite ranges. This suggests a structured pattern of zero repulsion within finite subsets. The GUE-like behavior indicates a local regularity observable in bounded sets of zeros. Steven Gonek studied the vertical distribution of zeros in finite intervals $[T, T + H]$ along the critical line, deriving formulas for their density and clustering. His work

[36] The "bounded gaps" (e.g., Zhang 2014) are an interesting problem.

[37] Akio Fujii's work is one of the earlier studies that investigated the distribution of the non-trivial zeros of the Riemann zeta function, focusing on their behavior in short intervals along the critical line (Re(s) = 1/2). His paper provides insights into the density and clustering of these zeros, laying groundwork for later refinements like Gonek's. Fujii derived asymptotic estimates for the number of zeros in intervals of the form [T,T+H], where (H) is small relative to (T). His results suggest that the zeros cluster in a predictable manner, revealing order rather than randomness in their local distribution. His work predates the widespread use of random matrix theory analogies but aligns with efforts to quantify zero density.

[38] An expected link is suggested by Ford, Luca, Moree (2014).

reveals predictable patterns, such as the number of zeros in short, finite segments, showing structured behavior in bounded regions. Nicholas Katz and Peter Sarnak (1999) analyzed the "low-lying" zeros (those near the real axis) and found they exhibit patterns tied to orthogonal symmetry (e.g., SO[even]), distinct from GUE behavior seen at higher heights. The symmetry in these finite sets of low-lying zeros represents a structured, predictable arrangement. Michael Berry and Jon Keating (1999) proposed a semiclassical model where the zeros might correspond to eigenvalues of a quantum Hamiltonian. Finite-dimensional approximations of such a system would exhibit regular patterns. If valid, finite subsets of zeros could reflect deterministic structures tied to a finite quantum system.

So, on the one hand, Odlyzko (1987), Gonek (1999), Katz and Sarnak (1999), and Berry and Keating (1999) identify structured patterns in finite subsets of the zeros. On the other hand, Montgomery (1973), Odlyzko (1987), Keating and Snaith (2000), Berry and Keating (1999), and Rudnick and Sarnak (1996) align with quantum mechanics' randomness and the rejection of hidden variables. Gonek (1999) and Katz & Sarnak (1999) are "intermediate" and neutral, focusing on order rather than randomness:

Hugh Montgomery's pair correlation conjecture (1973) and Odlyzko's computations (1987) show that the zeros' spacings follow GUE statistics, mirroring the eigenvalue distributions of quantum chaotic systems (e.g., quantum billiards), where outcomes are fundamentally random. The GUE model reflects quantum systems where deterministic hidden variables are ruled out, consistent with the Kochen - Specker and von Neumann theorems. Jon Keating and Nina Snaith (2000) used random matrix theory to model the zeta function's zeros, linking them to unitary symmetry groups where randomness is inherent, as in quantum mechanics. The GUE ensemble aligns with quantum indeterminism, supporting the rejection of hidden variables per Kochen and Specker's result. The Berry and Keating (1999) hypothesis of a quantum Hamiltonian for the zeros implies that, if true, measurements of such a system would exhibit quantum randomness. A quantum system's eigenvalues would obey Kochen-Specker's theorem, rejecting hidden variable explanations. Zeev Rudnick and Peter Sarnak's work (1996) on n-level correlations of the zeros matches the statistics of quantum chaotic systems, reflecting intrinsic randomness. These correlations align with quantum mechanics' probabilistic nature, consistent with von Neumann's theorem against hidden variables. Gonek (1999) and Katz, Sarnak (1999) focus on structured patterns (e.g., clustering, symmetry) in finite sets of zeros. While not contradicting quantum randomness, they emphasize deterministic order and do not explicitly address randomness or hidden variables.

As for the approach in the present paper, one can refine it: the distribution of all nontrivial zeros is *almost* impossible not to be fundamentally random (in the exact meaning of "almost" referring to a countable subset of an uncountable set). One can use the following visualization. The set of all mappings of the real axis onto itself is considered. Then, the image of a given rational number is almost impossible to be another rational number in a quite analogical sense (respectively, that image would be almost always an irrational number).

As for the two-way applied nonstandard bijection, a regular countable set (that of all nontrivial zeros) is mapped onto an arbitrary countable subset of an uncountable set and then, it is reversely mapped onto the countable subset of all nontrivial zeros. The initial regularity or any derivative regularity is almost impossible to be conserved as for the latter, which is equivalent for all nontrivial zeros to be absolutely occasionally distributed also in sense relevant to quantum mechanics. So, one might conclude that the above proof of Riemann's hypothesis in Hilbert arithmetic does not imply that all nontrivial zeros are absolutely inconsistent with any hidden regularity (very similar to a "hidden variable" in the sense of quantum mechanics), but such a regularity is "*almost* inconsistent" with the proof at issue.

So, the exotic "*almost* corollary" is a byproduct after involving the nonstandard bijection as a necessary sub-stage. Indeed, one can build a rigorous parallelism between the standard bijection and the usual logical equivalence and implication, on the one hand, and the nonstandard bijection and the newly introduced "almost equivalence" or "almost implication". It is quite relevant also to any ontomathematical "bridge" between the physical and mathematical "shores" over the Cartesian "abyss". Since the above proof of Riemann's hypothesis utilizes such a "bridge", this introduces "almost corollaries" though as a "byproduct".

One more topic, which is very interesting, but too remote from the present subject is how that "almost classical propositional logic", to which those "almost equivalence" and "almost implication" would belong, is to be related to the proper classical logic as well as to the cut rule, insoluble statements, quantum measurements, etc. So, what is possible in the framework of the present study is only one to conjecture that all of them and other similar ones share that nonstandard "almost certain structure", but its reasons, justifications, and proof will be postponed for a future paper.

III RIEMANN'S HYPOTHESIS AND THE DISTRIBUTION OF PRIME NUMBERS: BOTH IN TERMS OF HILBERT ARITHMETIC

Riemann's (1859) paper revolutionized number theory by connecting prime distribution to the zeta function's zeros, with his hypothesis remaining a central unsolved problem[39]. Later results, such as the 1896 prime number theorem and von Koch's 1901 equivalence, built on this foundation, illustrating the zeta function's enduring role. In 1859, Bernhard Riemann published "Über die Anzahl der Primzahlen unter einer gegebenen Grösse" (Riemann, 1859), introducing the Riemann zeta function to explore the distribution of prime numbers. The Riemann hypothesis suggests that the error in estimating the number of primes up to a large natural number "x" (denoted "$\pi(x)$") could be tightly bounded, specifically by "$O(\sqrt{x}log(x)$". This has profound implications for predicting prime distributions, a task central to number theory.

Following Riemann, significant advancements linked the zeta function to primes. In 1896, Jacques Hadamard and Charles Jean de la Vallée Poussin independently proved the prime number theorem, showing $\pi(x)\sim\frac{x}{log(x)}$ asymptotically, i.e. as "x" grows large (Hadamar 1896; Vallée Poussin 1896). Their proofs relied on the absence of zeta function zeros on the line

[39] Its history, e.g., in: Derbyshire (2003); a synopsis, e.g., Mazur, Stein (2015).

$Re(s) = 1$, building on Riemann's ideas. In 1901, Helge von Koch showed that the Riemann hypothesis is equivalent to a precise error term in this theorem (Koch 1901), further tying the zeta function to prime distribution[40].

An unexpected detail is how Riemann's brief six-page paper, published in the *Monatsberichte der Königlichen Preussischen Akademie der Wissenschaften zu Berlin*, laid groundwork for modern analytic number theory. Despite its brevity, it influenced thousands of researchers and remains unproven, now a Millennium Prize Problem with a $1 million reward. Bernhard Riemann's paper was motivated by the need to understand the distribution of prime numbers, a problem dating back to ancient times but formalized by mathematicians like Carl Friedrich Gauss (1849) and Adrien-Marie Legendre (1808). Gauss conjectured that the number of primes less than "x" is approximately: $\pi(x) \sim \frac{x}{log(x)}$, while Legendre proposed a slightly different form: $\pi(x) \sim \frac{x}{log(x)-1}$. The estimation of Legendre is more accurate for small and medium natural numbers, and that of Gauss, furthermore coinciding with the asymptotic formula of the prime numbers theorem[41], is accordingly more relevant for greater natural numbers.

Riemann sought a more rigorous approach using analytic methods. His explicit formula for $\pi(x)$ expressed it in terms of the logarithmic integral $Li(x)$ and a sum over the non-trivial zeros ρ of $\zeta(s)$: $\pi(x) = Li(x) - \sum_{\rho} Li(x^{\rho}) - log(2) + other\ terms$. This formula showed that the distribution of primes is intimately tied to the location of these zeros, a revolutionary idea at the time.

Within this paper, Riemann made a brief remark, now known as the Riemann hypothesis, which has profound implications for prime number theory. It implies that the error term in the approximation $\pi(x) \approx Li(x)$ is bounded by $O(\sqrt{x} log(x)$, a significant improvement over the "O" bound without the hypothesis. This precision is crucial for applications in cryptography and computational number theory, where understanding prime distribution is vital. Computational efforts for checking Riemann's hypothesis have verified that the first billions of non-trivial zeros lie on the critical line.

In 1896, Jacques Hadamard and Charles Jean de la Vallée Poussin independently proved the prime number theorem, stating that $\pi(x) \sim \frac{x}{log(x)}$ as $x \rightarrow \infty$. This result, first conjectured by Gauss, was a major achievement in number theory. Their proofs relied on the absence of zeros of $\zeta(s)$ on the line $Re(s) = 1$, a consequence of the non-existence of zeros with $Re(s) > 1$, as established by Riemann. Helge von Koch's 1901 paper showed that the Riemann hypothesis is equivalent to the error term in the prime number theorem being. This result underscored the hypothesis's importance, linking the analytic properties of $\zeta(s)$ to arithmetic precision in prime

[40] Also: Ingham (1932).

[41] One can admit an eventual future elementary proof of the primes theorem in Hilbert arithmetic as a ratio of the additive to multiplicative group in turn as the two dual Peano arithmetics.

counting. Riemann's zeta function can be also considered in the connection to Dirichlet (1837) to study prime numbers in arithmetic progressions.

The following results in the 20th and 21th century linking prime numbers and their distribution, on the one hand, with the Riemann zeta function and hypothesis, on the other hand are to be expressly emphasized. Hardy and Littlewood (1916)[42] proved that an infinite number of non-trivial zeros of the Riemann zeta function lie on the critical line Re(s) = 1/2, providing the first concrete evidence supporting the Riemann hypothesis. This result ties the distribution of zeta zeros directly to oscillations in the prime counting function, as per Riemann's explicit formula, influencing the error term in prime distribution estimates. This paper used complex analysis and Tauberian theorems to establish the infinitude of zeros on the line, a step toward validating Riemann's conjecture. Atle Selberg (1942) provided an elementary proof of the prime number theorem without using complex analysis of the zeta function, but his work still leveraged insights from zeta's behavior. Although avoiding zeta explicitly, Selberg's method inspired later zeta-based results, and his subsequent work on zeta zeros (e.g., the Selberg trace formula) reinforced their role in prime distribution. This paper shifted focus to arithmetic methods but complemented zeta-based proofs. Norman Levinson (1974) proved that at least one-third of the non-trivial zeros of the zeta function lie on the critical line, improving Hardy and Littlewood's result. This strengthens the connection between zeta zeros and the error term in $\pi(x) - Li(x)$, suggesting tighter bounds under the Riemann hypothesis. Levinson's method used a mollified zeta function, advancing numerical and theoretical support for the hypothesis. Enrico Bombieri (1982) refined the Bombieri-Vinogradov theorem, showing that the distribution of primes in arithmetic progressions behaves as if the Riemann hypothesis holds "on average" up to a certain height in the critical strip. This result connects the zeros' distribution to prime distribution across residue classes, approximating the generalized Riemann hypothesis's implications. This built on earlier 1965 work but was formalized here with zeta implications.

J. Brian Conrey (2003) improved Levinson's result, proving that at least 55.5% of the non-trivial zeros of the zeta function lie on the critical line, using advanced mollifier techniques. This tighter bound reduces the error in prime distribution predictions, aligning more closely with the Riemann hypothesis's predicted the $\sqrt{x}log(x)$ error term. The title reflects an earlier bound (40.8%), but the paper's methods led to later refinements exceeding 55%. Daniel Goldston, János Pintz, and Cem Yalçın Yıldırım (2006) proved that there are infinitely many pairs of primes differing by arbitrarily small gaps relative to their average spacing, using zeta function zero correlations. Their work relies on the distribution of zeta zeros, showing that even without the Riemann hypothesis, zero clustering implies tight prime gaps. This breakthrough used sieve methods and zeta zero pair correlation, advancing prime distribution theory. Yitang Zhang (2013) proved that there are infinitely many pairs of primes differing by at most 70 million, a landmark in prime gap research, leveraging zeta function properties indirectly via sieve theory. While not directly proving the Riemann hypothesis, Zhang's result builds on Bombieri's work and zeta zero distribution insights, suggesting consistency with hypothesis predictions. Kevin Broughan (2017)

[42] Also: Littlewood (1924).

refined Riemann's explicit formula, providing new expressions for in terms of zeta zeros, enhancing computational and theoretical links to prime distribution[43]. This work directly ties the precise placement of zeros (especially under the Riemann hypothesis) to prime counting, offering practical tools for testing the hypothesis. This paper modernizes Riemann's 1859 formula with contemporary analytic techniques.

Cramér (1920) and Pintz (2007) established and refined models treating primes as random-like, with Pintz explicitly incorporating RH and zeta zeros. Biane et al. (2001) and Joyner (2005) construct characteristic functions tied to zeta, offering probabilistic tools to study prime distributions influenced by zeta's critical line behavior. Bagchi (1982) models zeta zeros statistically, providing a characteristic function framework that indirectly informs prime distribution probabilities. Cramér and Pintz provide direct probabilistic models for primes, enhanced by RH and zeta insights. Biane et al. and Joyner link zeta's properties to characteristic functions, bridging number theory and probability. Bagchi offers a statistical view of zeta zeros, connecting their characteristic functions to prime behavior.

Harald Cramér (1920) introduced a probabilistic model suggesting that prime numbers behave like a random sequence of integers with a probability density inversely proportional to their logarithmic size. This model conjectures properties like prime gaps, laying groundwork for later probabilistic studies. Although Cramér's work doesn't directly use the zeta function, his predictions about prime distributions align with expectations under the Riemann hypothesis, such as the size of prime gaps, which are influenced by zeta's zeros. This foundational paper inspired subsequent research connecting probabilistic models to zeta and RH.

János Pintz (2007) revisited Cramér's probabilistic model, refining it with insights from the Riemann hypothesis. He provided sharper estimates for prime gaps and distributions in short intervals, explicitly linking these to probabilistic behavior. Pintz ties the characteristic function (in a probabilistic sense) of prime distributions to the zeta function's zeros, showing how RH constrains the randomness of primes. This paper bridges early probabilistic ideas with modern analytic number theory, emphasizing RH's role.

The paper of Biane, Pitman, and Yor (2001) explores stochastic processes and probability laws whose characteristic functions relate to the Riemann zeta function, particularly along the critical line (where RH applies). The characteristic functions of these random variables reflect zeta's properties, which govern prime distributions via the explicit formula linking primes to zeta zeros. This work offers a probabilistic framework for zeta, with indirect implications for prime distributions.

Bhaskar Bagchi (1982) investigated the statistical properties of zeta's non-trivial zeros as a point process, suggesting they resemble a Poisson process under certain conditions, with a definable characteristic function. Since zeta zeros determine prime fluctuations (via the prime number theorem and explicit formulas), their statistical properties provide a probabilistic lens on prime distributions. This paper connects zeta zeros to probability theory, influencing prime-related probabilistic models.

---

[43] A creative approach corresponding to the present context is suggested by Vericat (2013).

David Joyner (2005) examined how characteristic functions of random variables can be constructed from the Riemann zeta function and related L-functions, exploring their probabilistic interpretations. Joyner's models link zeta's behavior to prime distributions, using characteristic functions to capture probabilistic aspects influenced by zeta zeros and RH. This paper provides a direct probabilistic connection between zeta and primes, emphasizing RH's implications.

The non-trivial zeros, conjectured by the Riemann hypothesis to lie on the critical line, control the distribution of prime numbers via analytic number theory. The statistical behavior of primes (e.g., their density or fluctuations) emerges from zeta's zeros. In quantum systems, probability distributions (density functions or otherwise) describe state likelihoods, often tied to energy levels. Papers like Berry and Keating (1999) and Sierra (2008) directly map zeta zeros to such levels, while Keating and Snaith (2000) use random matrix theory to draw probabilistic parallels. Berry and Keating (1999) and Sierra (2008) propose systems where zeta zeros become quantum energy levels, linking primes to quantum probability densities. Keating and Snaith (2000) and Biane et al. (2001) provide probabilistic frameworks (random matrices and stochastic processes) that connect zeta zeros to quantum statistics. Wolf (2014) extends the analogy to quantum field theory, suggesting broader probabilistic interpretations.

The paper of Berry and Keating (1999) proposes a hypothetical quantum system, termed the "Riemannium," whose energy levels align with the non-trivial zeros of the Riemann zeta function. The authors explore how the statistical distribution of these zeros resembles the energy level statistics of quantum chaotic systems. The Riemann zeta function's zeros influence the distribution of prime numbers through the explicit formula in number theory. Here, Berry and Keating connect these zeros to quantum energy levels, which are described by probability distributions (e.g., the likelihood of a system occupying a specific state), suggesting a profound analogy between prime distribution and quantum mechanics. This work is foundational in linking the Riemann hypothesis to quantum chaos.

Keating and Snaith (2000) demonstrate that the spacing distribution of the Riemann zeta function's zeros mirrors the eigenvalue statistics of large random matrices, a mathematical framework widely used in quantum mechanics to model complex systems like atomic nuclei or disordered conductors. The zeros of the zeta function govern oscillations in prime counting functions (e.g., via the prime number theorem's error term). By showing that these zeros follow random matrix statistics, Keating and Snaith tie prime distribution to quantum probability distributions, as random matrix theory describes energy level probabilities in quantum systems. This paper bridges number theory and quantum physics through probabilistic tools.

Germán Sierra (2008) constructs a quantum mechanical model where the energy spectrum corresponds to the imaginary parts of the Riemann zeta function's non-trivial zeros. This provides a physical system potentially realizable in quantum mechanics that reflects zeta's properties. Since the zeta zeros dictate the fine structure of prime number distribution, Sierra's model connects primes to quantum probability densities, specifically, the probability of a quantum system being in a particular energy state tied to these zeros. This work offers a concrete quantum interpretation of the Riemann hypothesis.

The paper of Biane, Pitman, and Yor (2001) investigates probability laws associated with the Riemann zeta function, focusing on stochastic processes and their relation to zeta's behavior along the critical line (Re(s) = 1/2), where the Riemann hypothesis predicts all non-trivial zeros reside, thus providing a link to primes and quantum probability. The probabilistic properties of zeta underpin its connection to prime distribution. While not directly quantum, these probability laws provide a foundation that aligns with statistical mechanics approaches in quantum theory, offering a bridge to quantum probability distributions. This work enhances the probabilistic understanding of zeta, relevant to both primes and quantum contexts.

Marek Wolf (2014) explores how the distribution of prime numbers might be modeled using concepts from quantum field theory, with the Riemann zeta function serving as a potential mediator. Wolf suggests that the statistical properties of primes, influenced by zeta zeros, could be interpreted through quantum field theoretic frameworks[44], where probability amplitudes (related to particle states) parallel prime distribution patterns. This paper speculates on quantum-inspired models for prime statistics.

The historical context of the distribution of prime numbers and reasons for introducing the zeta function as well the sense of Riemann's hypothesis will be now reinterpreted and reformulated ontomathematically, therefore in terms of Hilbert arithmetic. The sketch of the idea is the following. The crucial is the reasoning of the statement inverse to the observation originating from the standard approach, even (still) of classical quantum mechanics. The measured values of any physical quantity in quantum mechanics are *fundamentally* random, where the meaning of "fundamentally random" obeys the theorems of the absence of hidden variables in it (Kochen, Specker 1967; Neumann 1932):

So, the problem and its solution conjectured by the statement inverse to the above one would be whether any fundamentally random distribution implies a wave function and a corresponding quantum state, to which the innovative approach of the present research is a particular implementation. as for prime numbers. Thus, if the distribution of prime numbers is fundamentally random and the just involved conjecture is true, the two statements imply that the zeta function though introduced by Riemann for absolutely abstract and mathematical considerations in analytical number theory, nonetheless corresponds to a wave function and thus to a certain quantum state: what the present paper reasons and justifies as a necessary premise for proving Riemann's hypothesis: a proof which is not only possible ontomathematically, but furthermore supposedly impossible in the standard mathematics.

Indeed, the just formulated intention has been already implicitly realized in the previous proof of Riemann's hypothesis. The first stage of the proof consists in the physical continuation of the zeta function including for the singularity at the pole conventionally requiring for it to be squarely integrable there and thus all over the complex plane allowing furthermore for it to be considered as belonging to the separable complex Hilbert space therefore pioneering the pathway for its further interpreting as a special wave function corresponding to a certain quantum state, consequently inherently physical. One is to pay intentionally attention to its singularity at the

[44] A quite different connotation might be the paper of Baez and Dolan (1998).

pole, in which it possesses a (finite) physical sense only asymptotically (which however is a usual approach and method for investigating physical singularities[45]).

One can suggest more or less speculatively for that singularity to be both physical and mathematical. In other words, that singularity is supposed to be ontomathematical: that unique spot where the postulated by ontomathematics bridge between physics and mathematics is reality, at least as a consistent hypothesis (for example, following the visual metaphos in the famous fresco painting of Michelangelo "The Creation of Adam"). One can further interpret that bridging singularity in that particular case of the zeta function and Riemann's hypothesis as exemplifying the general rule originating from the conjecture that any absolutely random distribution corresponds to a wave function and thus to a certain quantum (therefore) physical state.

In other words, the "defect" of the simple pole singularity of the zeta function is in fact a crucial advantage allowing for that function to be a proper ontomathematical one, i e. allowing just at the singularity the realms of physics and mathematics to "touch each other", though not in a literally smooth and continuous way, but in an asymptotic approximation just as this is metaphorically depicted on the already cited fresco painting of Michelangelo "The Creation of Adam": God touches Adam only *asymptotically* rather than physically and verbatim. So, the suggested proof of Riemann's hypothesis can be figuratively seen in a rigorous mathematical interpretation of the famous painting.

Then, one can suggest that the zeta function is a particular case able to exemplify any link of a fundamentally random distribution, by which the realms of mathematics and physics are connected though asymptotically, by the singularity of at least one pole rather than smoothly and continuously in a proper analytical continuation. Obviously, the above rather observation than a conjecture reveals an immense horizon for scientific research referring to the ways and interrelations between physical reality and mathematical models in absolutely rigorous mathematical methods implying necessarily the experimental confirmation of their abstract conclusions therefore realizing Michel Foucault's vision about the "ocean tide erasing the figure of a man drawn in the sand on the beach". Indeed, the afore-suggested continuation of a mathematical model into physical reality is single thus not needing human decision about the correspondence of model and reality.

The present study will only establish that option, consisting in the horizon for ontomathematical research, an example of which is the proof of Riemann's hypothesis in Hilbert arithmetic by the *physical* continuation of the zeta function. What can be more done in the proper subject of the investigation without unjustified speculations is the option at issue to be described by the proof of Riemann's hypothesis from the proper viewpoint of Hilbert arithmetic and Hilbert mathematics at all grounded on the former.

Speaking loosely, if one observes the opposite "shore" of the dual Peano arithmetic, it will watch it to be fundamentally randomly distributed, at that again fundamentally randomly variable after each different observation. So, the metaphor of the quantum measurement of the

---

[45] See, e.g., Bender, Orszag (2013).

opposite “arithmetic shore” would be relevant (even more so that it might be literalized in a future study).

If one uses the visualization of Michelangelo’s fresco painting, in which both God and Adam are simultaneously represented as entities in reality watched by humans, one can speculate how “Adam himself should see God” from the own viewpoint rather than from the external one, from which the public observes the fresco painting. Of course, if “God” was an ordinary object in human everyday reality, He would be seen by Adam in the same way as the public. On the contrary, His Appearance cannot be constant, but always fundamentally randomly variable if one has adopted Michelangelo’s masterpiece as an admissible visual metaphor for the advocated viewpoint. Of course, that is only an illustration, which the reader can accept or not. So, “The Creation of Adam” can be interpreted to be a certain “measurement” of the infinitely variable “Appearance of God” thus fundamentally random and representable quite differently in another masterpiece or interpretation.

What is essential as for the meant idea of how the physical and mathematical areas “touch each other” asymptotically, by the mediation of at least one singularity, and now from the viewpoint of the one “shore” of Hilbert arithmetic is the fundamental randomness, which is furthermore in turn fundamentally randomly *variable* after each different observation (thus similar whether to a quantum measurement or to the masterpiece’s interpretation). A much more rigorous representation can be that by the opposition of rational numbers to the rest real numbers (i.e. which are not rational):

One considers the next digits in an infinite, e.g., decimal fractal: if a number is rational all digits follow a certain finite pattern, usually called “decimal period”. On the contrary, that decimal period does not exist in principle if a given real number is not rational. In other words, if one observes the consecutively “appearing” digits of an infinite decimal fraction, there exist two alternatives: (1) the newly appearing digits follow a certain finite model and if that is the case, it is equivalent to the statement that the number at issue is rational; (2) the newly appearing digits do not follow any pattern, or in other words, their appearance is fundamentally random; if that is the case, the number in question is a real, but not rational number (which can be either algebraic or transcendent).

Therefore, if the opposite “shore” of dual Peano arithmetic is watched from the standard Peano arithmetic, it would seem to be some irrational (whether algebraic or transcendent) real number, i.e. fundamentally random (unlike the elementary pattern “+1” with which “this side shore” complies). However and furthermore, what is observed is also “fundamentally random”, i.e., variable and depending on the condition of observation (equivalent to the “set for a certain quantum measurement” as well), i.e. a certain real number (which is not rational), but different in general after each observation (“measurement”). Once the general case has been described as above, the distribution of prime numbers is to be adapted to be a certain “irrational number” (in the sense that its “digits” do not follow any pattern)[46], or respectively, implying a certain “wave function” describing a certain “quantum state”. In fact, that “wave function” is the zeta function

[46] See, e.g., *Caveney, Nicolas, Sondow* (2011) or *Choi, Chung, Kim* (2012).

interpreted to be a wave function after its "physical continuation" at the singularity of its pole utilizing the afore-suggested method.

So, though the zeta function and Riemann's hypothesis mean just a certain "case study" (defined by the distribution of prime numbers), the newly introduced approach for proving Riemann's hypothesis in Hilbert arithmetic relies on the general case of any fundamentally random distribution (to the class of which the distribution of prime numbers is a single case described by Riemann's zeta function interpreted now to be a wave function).

Anyway, the fact that the distribution of prime numbers is absolutely random is a necessary condition for the above reflection but it has not yet been proved explicitly yet. The reason is that another paper (Penchev 2023 February 13) had in detail justified this in advance. So, a cursor sketch would be sufficient. The method of "Eratosthenes's sieve" can be deduced a necessary corollary from the definition of prime numbers, and more precisely said, i.e. from the iterativeness of that definition implying for any next prime number to depend just "iteratively" on the set of all established to be prime numbers before it as well as from the statement (proved still in Antiquity), which can be formulated nowadays so: the set of all prime numbers is infinite, i.e. there exists a bijection between it and the set of all natural numbers (regardless of the former is a true subset of the latter).

Indeed, that "iterativeness" (rigorously defined by depending on all previous natural numbers) is equivalent to fundamental randomness. That statement can be visualized by the aforementioned model of "fundamental randomness" by real numbers. If a given real number is rational, there always exists a certain finite pattern (called "decimal period"), after reaching it, more "digits" are redundant (for describing a number which is rational). In other words, if the case is the alternative one (namely a real number which is not rational as a model of "fundamental randomness"), that finite limit (the number of "digits" for the corresponding "decimal period") does not exist and thus all previous prime numbers are necessary for determining the next one (i.e just what the rigorous definition of "iterativeness" needs).

The conclusion of the present section is the following. Riemann introduced the zeta function (to which in particular he formulated his famous hypothesis) led by the objective to investigate the distribution of prime numbers, which along with many other problems both resolved or not belong to analytical number theory linking the theory of complex functions (to the class of which the zeta function belong) and number theory. However, the approach suggested in the present and previous parts of the paper for proving Riemann's hypothesis in Hilbert arithmetic considers it in relation to the more general class of fundamentally random distributions, to which the distribution of prime numbers belong. So, statements being analogical to Riemann's hypothesis are in fact meant to that general class of all fundamentally random distributions.

A very essential illustration (for which a next paper especially devoted to it is intended) is the proof of Goldbach's conjecture (completely undeservedly neglected by CMI in the "Seven Millennium Problems") in Hilbert arithmetic since it follows the same model to be easily provable in the general case of all fundamentally random distributions rather that in its original particular formulation to all prime numbers. The idea which will be in detail elaborated only in an next paper consists in the following:

One considers both dual observations of the opposite shore of mutually and idempotently exchangeable dual Peano arithmetics belonging to the same Hilbert arithmetics simultaneously, which allows for both to be summed including to "all natural numbers" (thus needing a discussion about the relation of the so-defined sum to to both "dual" sets of all natural numbers thoroughly remained for the forthcoming future study), after which one observes the following. On the one (e.g. "left" only for conventional certainty) side of the equation, all natural numbers are repeated two times, which is equivalent to "all even natural numbers". On the other (conventionally "right", accordingly) side of the equation, all prime numbers will be available two times, which is equivalent to the sum of two prime numbers, i.e. namely "Goldbach's conjecture" together with its proof in Hilbert arithmetic. One immediately notices that the utilized method is general just as in the case of Riemann's hypothesis: it is valid to any fundamentally random distribution (to any "irrational number") rather than only to prime numbers as the original formulation of Goldbach's conjecture claims absolutely reasonably. One can add that the rigorous realization of that idea needs furthermore the absolute randomness of the distribution of prime numbers (or any other fundamentally random distribution) providable only by the absence of "hidden variables" (as that after the Kochen - Specker theorem interpreted "apophatically", i.e. not allowing even a single restriction of the total degrees of freedom).

Then, one can discuss an ontomathematical speculation about the fundamental meaning of Goldbach's conjecture (already relevantly generalized according to the sketched pathway for its proof) for deducing classical Pythagoreanism as a corollary from ontomathematics just as Peano arithmetic from Hilbert arithmetic (in a "narrow sense"). Indeed, the aforementioned equation, though initially introduced for proving Gooldbach's conjecture in Hilbert arithmetic, means (two times) Peano arithmetic (for being summed as "all even natural numbers") on its "left" side. Then, on its "right" side, it means (again two times) the "world by itself" though mathematically represented by "fundamental randomness" in turn exemplified by that of the distribution of all prime numbers in Goldbach's conjecture as well as in Riemann's hypothesis. The equation itself (after reducing the common factor "2" for "two times" in both sides) represents just the main idea of Pythagorism (even the strongest formulation of it): the equivalence of arithmetic and the world. So, Goldbach's conjecture is a direct corollary from the main idea of ontomathetics to overcome the Cartesian abyss, directly illustrating the above claim to CMI that it deserves to be among the Millennium Problems.

IV INSTEAD OF CONCLUSION: ARE ALL THE SEVEN MILLENNIUM PROBLEMS LINKABLE TO GÖDEL'S INSOLUBLE STATEMENTS?

The present conclusive section means to demonstrate that Riemann's hypothesis might not have any solution in the standard mathematics as far as it is identified with Gödel mathematics, on the one hand, and thus it should be added to the long list of statements insoluble in the same framework. A preliminary notice is to explain the "Solomonic" kind of solutions of those problems very well symbolized by Andrew Wiles's proof of Fermat's last theorem, the serial insoluble statement in Gödel mathematics, but possibly provable as an only arithmetic statement.

The case of Wiles's proof is described in detail in *Part I* of the paper. Here, it will visualize the attitude of many mathematicians to the necessity for breaking the framework of Modernity situated mathematics gapped "abyssly" from physics as for the most fundamental mathematical problems such as those heralded to be the "Millennium" ones by Clay Mathematics Institute. Those numerous mathematicians understood very well that the framework at issue cannot but be violated since those problems are insoluble strictly within it thus preventing the development of mathematics, which they might not wish, being just mathematicians.

Nonetheless, and on the other hand, they absolutely categorically do not want to make relevant fundamental scientific, philosophical and social conclusions referring to the same fact of being just mathematicians (rather than philosophers, politicians, revolutioners, or even merely physicians as far as physicians have been caused scientific revolutions in the past). So, the design of those mathematicians (who are the crucially prevailing part of mathematicians as to whom Sir Andrew Wiles himself is to be enumerated) is hiddenly Hilbert mathematics, being absolutely necessary for the development of mathematics to be secretly and implicitly involved, but only by means of very sophisticated, technically virtuous syllogisms passing through many (if not almost all) branches of mathematics so that the public cannot penetrate them or trace back the proof itself, still less to understand that the framework of the standard mathematics is "cleverly" abandoned, but a way being "socially responsible" thus not appealing for scientific and (it's scary to even say it) social revolutions.

Summarizing now only to a few paragraphs, Wiles's "Solomonic" decision is accomplished as follows. He needed the "Tanyama - Shimura - Weil conjecture", from which the statement formulated verbatim as Fermat's last theorem can be deduced as an immediate corollary. However, though formulated verbatim the same, Fermat's last theorem is a different statement since it was meant by Fermat himself in the framework of (Peano) arithmetic, rather than in that of both arithmetic and set theory (as Wiles's proof of the modularity theorem needs) even more so that arithmetic and set theory are inconsistent to each other in the sense of Gödel (1931).

One might think that Gödel's paper (1931) is not in any relation to Fermat's last theorem, which is properly correct as to the theorem itself, but false as for Wiles's proof and that circumstance can be elementarily visualized by the set-theoretical complement of the *set* of all natural numbers (according to the axiom of infinity in ZFC set theory), on the one hand and all natural numbers (according to the axiom of induction in Peano arithmetic), on the other hand. That complement is meant by Wiles's proof due to the modularity theorem, obvious even in its formulation for linking discrete and arithmetic modular forms with continuous (thus needing set theory) elliptic curves. So, the modularity theorem suffers from the Gödel inconsistency. However Fermat's original formulation of the theorem does not suffer from that since it is an only arithmetic statement (since "Cantor's paradise" had not even been suspected in Fermat's age).

So, the modularity theorem is absolutely correctly proved by Wiles (but only) out of the standard mathematics after involving the complement at issue rather than strictly within its framework. Wiles's "Solomonic" wisdom consists of introducing nonstandard mathematics (a symbol of which can be the modularity theorem itself) as an ostensibly exceptionally smart syllogism thoroughly within the standard mathematics (due to the unique mathematical genius of Andrew Wiles). That replacement can be also traced back in the history of the publication of the proof initially returned to Wiles for a contradiction discovered by his colleague in Princeton Nick Katz to whom he had in detail explained the proof. As one of the peer-review checkers of Wiles's publication he should study that part of the proof where he revealed the weakness preventing the correctness of all the proof. The history of the proof is retold according to Simon Singh (1997: 265-304).

Andrew Wiles, a British mathematician at Princeton University, embarked on an ambitious secret project in 1986 to prove the Taniyama–Shimura–Weil conjecture (often referred to simply as the Taniyama–Shimura conjecture), which links elliptic curves and modular forms. This conjecture had been proposed in the 1950s and was widely regarded as an impossible challenge at the time. However, in 1986, Gerhard Frey suggested that a proof of the Taniyama - Shimura conjecture for a special class of elliptic curves would automatically imply Fermat's Last Theorem (FLT) as a corollary.

Wiles worked in complete secrecy for seven years (1986–1993), isolating himself from the mathematical community, and developing a highly intricate proof that involved cutting-edge techniques from algebraic geometry, number theory, and modular forms. By May 1993, he felt confident that he had completed his proof and decided to present his results publicly. In June 1993, Wiles gave a series of three lectures at the Isaac Newton Institute for Mathematical Sciences in Cambridge. The final lecture, on June 23, 1993, concluded with the dramatic announcement that Fermat's Last Theorem had been proven. This moment was historic - one of the greatest achievements in mathematics. After Wiles's announcement, he shared his manuscript with a small group of experts for internal review before formal submission to a journal. Among them was Nick Katz, a Princeton mathematician specializing in number theory, who was one of Wiles's close colleagues.

Wiles and Katz had held what was termed a "private seminar" in Princeton, where Katz meticulously examined the proof. His task was to check the mathematics carefully, particularly a 20-page section of the proof concerning a technical method called the Euler system. It was in this part that Katz discovered a serious gap in Wiles's argument. By September 1993, Wiles was informed of the problem. The issue lay in a step involving a novel approach to the so-called Iwasawa theory, which failed to work as expected. Wiles initially believed he could fix the error quickly, but weeks turned into months without success. The problem was devastating because it called into question the validity of a key argument in the proof.

Wiles spent nearly a year attempting to repair the gap. He tried multiple approaches, but each failed. By early 1994, he had reached the brink of despair, even considering abandoning the project entirely. At this point, he sought help from his former student and collaborator, Richard Taylor.

After rigorous rechecking, Wiles submitted the corrected proof in October 1994 to the Annals of Mathematics, one of the most prestigious journals in the field. The papers were officially published in May 1995, comprising two parts: "Modular elliptic curves and Fermat's Last Theorem" by Andrew Wiles; "Ring-theoretic properties of certain Hecke algebras" by Richard Taylor and Andrew Wiles (addressing the corrected part). These papers collectively proved the Taniyama–Shimura conjecture for a critical class of elliptic curves, thus confirming Fermat's Last Theorem.

In September 1994, after months of unsuccessful attempts, Wiles had a moment of insight. He realized that an older method he had originally discarded,based on horizontal Iwasawa theory, could be combined with new techniques. This breakthrough came on September 19, 1994 and, within a few weeks, Wiles and Taylor successfully completed the correction.

In a year or more, Wiles managed to repair that part of the proof, after which the publication was accepted and published in the journal. So, the ultimate consistence of the proof depends on two premises: (1) the amendment of the part of the proof of about 20 pages; (2) that part of the proof cannot influence any other parts of it (in other words, the amendment at issue cannot harm any other part of it thus resulting of the incorrectness of the whole proof. Indeed (1) had been considered extremely carefully rather than (2). In fact, just (2) is that eventual backdoor through which Wiles's proof goes out of the standard mathematics, turning out to be true in the nonstandard mathematics.

Of course, one can only guess whether this had been an intentional objective for Wiles for secretly bypassing the strict limits of the standard mathematics or he, being a modern exceptionally educated professional mathematician, only complies with all prohibitions imposed for mathematics and its place in human cognition by Modernity. Whatever it be, Wiles kept away from confronting his approach for proving Fermat's last theorem by the modularity theorem with the standard mathematics and thus with the solid opposition of physics and mathematics, with the established place of mathematics among human cognition during Modernity.

Wiles is not, of course, a Copernicus, Lobachevsky, or Einstein. He did not appeal for a scientific revolution, for a revolutionary transformation of mathematics, physics, philosophy, and science at all. Instead of that he managed to camouflage his grandiose innovation ostensibly to be an extremely sophisticated syllogism so complicated for nobody before Andrew Wiles, a unique mathematical genius to penetrate into it. So, Andrew Wiles is a "sir" whether for the exceptional smartness of his syllogism or (more probably) for veiling the abandonment of the standard mathematics and all cataclysms implied by that.

One can further compare Fermat's last theorem and Riemann's hypothesis as for the relation of arithmetic (the axiom of induction) and set theory (the axiom of infinity) in both. The former is a proper arithmetic statement not involving set theory (even more so that Fermat himself had not only suspected its existence). Set theory is introduced by virtue of Wiles's proof as corollary from the modularity theorem. So, only Andrew Wiles's approach rather than Fermat's last theorem by itself goes out of the standard mathematics (in fact, far beyond it). So, one can anyway admit that a purely arithmetic proof (or rejection) of Fermat's last theorem (for example, by means of the axiom of induction) might exist nonetheless since the Gödel insoluble statements do not take place in arithmetic by itself, but only in those axiomatic systems involving both arithmetic and set theory. That proper proof continues to be unknown regardless of Wiles's proof since it is inherently far beyond arithmetic.

That is not the case as for Riemann's hypothesis. The zeta function is a complex function furthermore analytically continued in definition thus needing the concept of just *continual* continuity and thus, set theory in the final analysis. However, its nontrivial zeros are a countable set[47]: so, a proof strictly within set theory (i.e. without involving arithmetic but utilizing, e.g., the concept of metric space) might be discussed at least as an idea. The present proof, however, does not follow that option being rather in the talweg of Wiles's proof (to be both arithmetic and set theory included), but sharply distinguishing from it due to emphasizing that is out of the standard mathematics, properly, already within Hilbert mathematics, i.e. ontomathematics through the "smooth" (figuratively, not mathematically) or rather singular transition between physics and mathematics.

As an illustration, one can compare the "empirical" confirmations of both conjectures established for the first (huge) "n"-s, consecutively one by one without any exception as for Fermat's last theorem and for an immense finite set of nontrivial zeros as for Riemann's hypothesis again without any exception. So, one might eventually prove the former directly by the axiom of induction rather than the latter since the set of all known nontrivial zeros would need some conventional well-ordering being naturally deprived of that, only after which the axiom of induction might be applied. Even that correct and purely arithmetic proof of Riemann's hypothesis would be ever revealed, it would share the analogical (though polar) irrelevance to Riemann's original statement as that of Wiles's proof to Fermat's original statement. Indeed, Riemann's hypothesis is a statement needing set theory unlike the eventually allowed proof being arithmetic.

One can compare the "empirical" corroborations of both conjectures also from another viewpoint. Both are confirmed for huge (natural) numbers of examples without any exceptions. Even both would be proved by the axiom of induction (the application of which to Riemann's hypothesis needs the nontrivial zeros to be preliminarily ordered in a relevant way), they would be proved for "all natural numbers" rather than for the "set of all natural numbers" (involving the axiom of infinity instead of the axiom of induction). In fact Fermat's original statement refers just to "all natural numbers" rather than to the "set of all natural numbers". On the contrary, Wiles's proof from the modularity theorem means just the "set of all natural numbers", from which infers the statement about "all natural numbers". However, that approach necessarily goes

[47] A statement discussed in detail above.

out of the standard mathematics rather than only out of arithmetic (but thus anyway, within the standard mathematics).

As for Riemann's hypothesis, it is not an arithmetical statement, however Cantor's set theory had not yet been created in his age. So, though it is not an arithmetical statement, the problem of whether it should be granted to be within the standard mathematics or not is open and formally admits both opposite decisions as admissible conventions. So, the suggested here proof can be adopted to be a solution of Riemann's hypothesis in its original formulation unlike Wiles's proof by replacement the original statement of Fermat with its literally formulated counterpart, but in the context in a different axiomatics (adding even axioms inconsistent with those of arithmetic: the axiom of infinity to the axiom of induction).

So, the problem of whether Riemann's hypothesis is a theorem in standard mathematics is still open, including a few aspects. If one proves it otherwise, strictly within the set theory without utilizing arithmetic (and first of all, without the axiom of induction), it will be a theorem belonging to the standard mathematics. Maybe, that could be possible. However, if the axiom of induction (and thus arithmetic) would be explicitly included, one can easily demonstrate that Riemann's hypothesis is a Gödel insoluble statement. An idea is suggested in *Part I* of the paper. Its essence consists in involving the teleportation theorem for showing that it implies at least a new zeros to be alleged not allowing for it to be confirmed or rejected Riemann's hypothesis thus being a Gödel insoluble statement. (However, the "teleportation theorem" refers to physics rather than to mathematics.)

The present approach for proving Riemann's hypothesis needs countable sets for building nonstandard bijection, however whether it relies on arithmetic (where the axiom of induction is critical) is ambiguous and thus disputable: for example, as for the additive semigroup of all trivial zeros which are all even negative integers being isomorphic and homomorphic to all natural numbers. However, if it is considered as for the set of all trivial zeros, that reference to arithmetics might be more or less conventionally bypassed.

The problem is still more discernible after the introduction of "physical and mathematical induction" generalizing the axiom of induction in arithmetics. In fact, it states that induction can be always continued from *all natural numbers* to the *set of all natural numbers*. Thus, if all natural numbers are finite after the axiom of induction, the set of all natural numbers is infinite after the axiom of infinity, and the principle at issue states that induction can be continued beyond all finite natural numbers gradually reaching their (infinite) set as a whole.

One can think that mathematical and physical induction is to be identified with "transfinite induction", but this statement is either false or ambiguous because transfinite induction is normally related to the standard mathematics rather than to all inaccessible subcountable cardinal numbers while the principle of physical and mathematical induction means just them absolutely abandoning any other transfinite cardinal or ordinal numbers.

However, the problem of whether the suggested proof of Riemann's hypothesis needs or not that newly formulated principle of physical and mathematical induction is again ambiguous due to the following reasons. On the one hand, the proof can be considered as an application or corollary of the principle at issue, which can be in turn considered as a nonstandard principle of induction to which the axiom of induction in arithmetic is a nonstandard model. A more fundamental question appears. If given any only arithmetical structure, whether there exists always at least one non-arithmetical mathematical (for example set-theoretical) structure so that the former is a nonstandard model of the latter. Furthermore, it exists, whether it is unambiguously determined under the condition for the initial arithmetical structure to be its nonstandard model. Obviously, that hypothesis seems to be false in the standard mathematics as far as it sounds to be the serial Gödel insoluble statement.

Anyway, and on the other hand, the suggested proof of Riemann's hypothesis is independent of that "nonstandard mathematical induction" (though being consistent with it) since that principle is not utilized for the proof itself as a premise of it. Rather, that principle of both mathematical and physical induction is a conjecture generalizing the approach of how Riemann's hypothesis to be proved in Hilbert mathematics, which is a much stronger statement than the initial one.

Another paper (Penchev 2023 July 16) has already suggested and justified in much more detail the conjecture that many (if not almost all) fundamental mathematical problems (including the seven one heralded by CMI to be the "Millennium Problems", and to which Riemann's hypothesis belongs) are Gödel insoluble statements so that they are *really insoluble* in the standard mathematics. Indeed, the huge army of professional mathematicians would resolve them a long time ago if that has been theoretically possible in standard mathematics. Unfortunately, this is not possible, including as a fundamental limitation outlining the "allowed" area of the standard mathematics.

Furthermore, the precedent of Wiles's "Solomonic" decision is available after his proof of Fermat's last theorem: claiming to be thoroughly within the standard mathematics, which is not true. The essence of that kind of conservative wisdom consists in the boundaries of the standard mathematics to be really transcended since it is absolutely necessary for the development of mathematics itself, but implicitly, cleverly, hiddenly, secretly, even rejecting to be done for a avoiding the revolutionary consequences and conclusions implying both scientific and social revolutions due to abandoning the Cartesian dogma of Modernity, in particular established where mathematics is to be situated. Wiles's decision means that place for mathematics to be abandoned but that abandonment to be represented as an extremely sophisticated syllogism, ostensibly thoroughly within the standard mathematics (which is false) for preventing any revolutions originating from that too inconvenient truth.

The present and cited previous papers can be metaphorically seen as a "shout that the King is nude" according to the well-known parable, which only a "lad" might allow for doing it. One can notice that Einstein (compared with Poincaré or Lorentz) was that "lad" who had nothing to lose, and continuing the metaphor, to the "lads", one may enumerate also Lobachevsky (in comparison with Gauss) or even (rather paradoxically) Gödel himself to (Hilbert, Frege, or Russell). However, the institution of sciences has nowadays secured itself against any lads' shouts that the "King is nude" by peer-reviewing, so those naive observations not to be any more allowed in science so that any scientific revolutions to be prevented in their ground (in much more detail in other papers: e.g., Penchev 2024 August 5; 2023 December 6).

One might ask what is at stake so that (e.g.) Sir Andrew Wiles and his recipients (first of all, other mathematicians) have preferred a more extended and sophisticated syllogism instead of a much simpler, relevant Gestalt change in and about mathematics. The trouble consists in the fact that it would fundamentally change physics, science, and even society rather than only mathematics: in other words, so radical changes for which the mathematician have never been authorized, and even less nowadays when their proofs and tenets are absolutely understandable for all scientists (not being mathematicians) rather than for usual people democratically electing their government, in which there are no mathematicians (in fact, no scientists at all).

Those necessary changes are in detail described and reasoned in many other papers (e.g. Penchev 2024 October 2 2024 August 5; 2023 December 6). So, they can be now only cursorily mentioned. First of all, that is the notorious "creation from nothing" to be omnipresently and omnitemporally allowed all over the universe instead of being exiled in the ostensible singularity of the mythical "Big Bang": a grandiose violation of energy conservation, unitarity, Hermiticity, etc. and etc. where all refutations and anomalies in the physical picture of the world have been exiled, and mathematics, physics, and science can already study that censored "reality" after the Big Bang: in fact too far from reality.

However, the necessary revolution cannot be restricted only within science, smoothly passing into a social, even "permanent" one since it "liquefies" the contemporary society destroying its hierarchies and order for being able to transmit that faster and faster, exponentially increasing cognition. The existent physical worldview (which can be symbolized by the "Big Bang") corresponds and thus supports the existent social organization, which can be called "solid" even literally, as consisting of numerous tiny "crystals" (in democratic societies) or a single "monocrystal" (in the "ideal type" of totalitarian society) where "crystals" represents social hierarchies.

"Unfortunately" or rather fortunately, humankind is reasonable as well as any normal individual of homo sapience sapience thus producing permanently knowledge and therefore constantly "adiabatically heating" the society constantly weakening its hierarchies and order for the "adiabatically generated heat" to dissipate (i.e, cognition to be disseminated). So, humankind is "doomed" to freedom as a natural law which is confirmed in macroscale (regardless of numerous deviations in microscales) by the constant historical progress itself and by itself. So, freedom is not a human endeavor or subjective "ideal" alone, but rather forced as any natural law, i.e. that of gravitation.

Consequently, and following the natural law at issue (that law can be called the law of cognition and freedom), humankind approaches or enters into a new, "liquid phase" of social organization, which might be misunderstood as "anarchy" due to destroying all hierarchies and social order in the "point of phase transition": the "point of liquifying". In fact, the "solid state" of society turns out to be anachronic not being able to dissipate as much "heat" as the contemporary (above all, scientific and technical) cognition creates. What rules and governs the society is knowledge immediately accessible to anybody on the Internet, a "liquid" informational medium dissolving all hierarchies since they prevent its faster and faster dissemination.

From the sketched new viewpoint, one can now reflect back the suggested proof of Riemann's hypothesis in Hilbert arithmetic, therefore in ontomathematics. In fact, it is proved as a statement situated just on the interlink between physics and mathematics thus being inaccessible until now in the alleged abyss between physics and mathematics once they have been located in its two opposite shores. Indeed, the zeta function to which nontrivial zeros Riemann's hypothesis refers is interpreted as belonging to the separable complex Hilbert space[48], and then as a wave function corresponding to a quantum state. This is necessary to be involved the Noether (1918) first theorem, in turn one of the most remarkable results, immediately relating to the unity of physics and mathematics since it connects mathematical symmetry and physical conservation, the variational principle in mathematics with the principle of least action in physics and hinting at an unarticulated link between the rather mathematical and physically dimensionless information with the proper physical quantity of action.

However, it still needs one more transformation, by means of the newly introduced "nonstandard bijection" for becoming applicable to the relation of the trivial and nontrivial zeros of the zeta function. Now, in the proper context of physics and mathematics, the ideas of both "nonstandard model" and "nonstandard bijection" though elaborated on the "shore of mathematics" are in fact tools relevant for building a bridge between it and physics since what is nonstandardly modeled or mapped to be uncountable might be on the opposite shore of physics.

Then, the approach just used for proving Riemann's hypothesis inspires a generalization of the axiom of induction (equivalent to abandoning the standard mathematics as well) allowing for transcending arithmetic into set theory therefore cancelling the Gödel dichotomy about the relation of them. Then, Noether's first theorem can be also understood slightly differently, as a particular and unarticulated application of that newly introduced "principle of physical and mathematical induction".even able to explain the inherent unity of the elements of a set and the set itself, including Russell's principle of abstraction unifying the fundamental concepts of "set" (in set theory) and "proposition" (in propositional logic)

Finally, one can reflect the transformation of mathematical "craft", even more so that it is necessary to be reinterpreted because of entering exceptionally powerful instruments of calculating (supercomputers) and even "thinking" (AIs) as well as forthcoming quantum computing, potentially able to erase the (alleged) boundary between (mental) thinking and (physical) action. Mathematics is to be redefined even only due to the fact that all mathematicians have wasted their time for those operations now much more successfully accomplished by the Turing machines, (forthcoming) quantum computers and AIs. In other words, what should a mathematician do (and even make) in the age of standard and quantum computers and AIs? Of course, a rather trivial and joking answer would be "to retire".

Another and more serious option is to deal with philosophy and ontomathematics in order to correspond to such shockingly powerful instruments already created by humankind as

[48] An alternative approach might be grounded on: Endres, Steiner (2010).

supercomputers, quantum computers, and AIs. He or she would start thinking (as yet Heidegger advised humankind, in fact rather mockingly). However, that step, the forcing of thinking is now already obligatory since its alternative is to become AIs' "pets" posting "cool kittens" in social networks or whatever else not disturbing AIs' real work. Of course, that is rather a dystopia.

So, one can reflect the suggested proof of Riemann's hypothesis now from the viewpoint of how the mathematical "craft" should be transformed for making sense in the forthcoming epoch of AIs, supercomputers, and quantcomputers (in the "Third Millenium" and next millennia). The huge volume of what the mathematicians did is already meaningless to do any more just as digging by humans if it can be accomplished by excavators (of course, controlled by humans). However, the analogical question is: what the future mathematicians should control in their labor and what the new assistants could do much more productively than humans under the eventual human control (if that is at all necessary any more). If the distinction is obvious as for physical labor such as "digging", it is rather sophisticated as for intellectual and mental one due to the fact that mathematical "digging" and "control" seem to be too similar to each other, at least at first glance.

By the "forest" metaphor, human mathematicians will have to observe the "forest" of the proof or paper, where the description of all separate "trees" whether syllogisms or calculations in a narrow sense will be provided by AIs and supercomputers. Otherwise said, the future mathematicians will have only to ask and the new assistants to answer, after which the former will estimate the relevance of the answer to the generating question: the future mathematicians will have to be rather philosophers (and not less physicists) just as Descartes, Newton, or Leibniz as before: in the eve and dawn of Modernity. One can immediately notice two huge, even fundamental obstacles, the one already discussed in detail above, but the other not:

The former refers to the contemporary opposition of mathematics and philosophy, after which Wiles is a mathematician rather than a philosopher unlike, e.g. Descartes. They are not only opposed but gapped by an insurmountable abyss, which might be metaphorically likened to that between the two biological sexes so that the philosophers' "behavior" would be almost "shameful" as for mathematicians as well as vice versa. Ontomathetics, on the contrary, not only facilitates, tolerates, recommends, but even obliges their "behavior" to be "unisex" rather than either mathematical or philosophical (even much more scandalously adding still one "third sex", that of physicists, to that "unisex" indistinguishability[49]). That obstacle is redundant to be discussed any more since it is done in full detail above.

The latter trouble relates to mathematical education: speaking loosely and jokingly, the young mathematicians are trained to be "computers" or "AIs" at best since they should do their work until now though much worse since human thinking is fundamentally different from calculations including logical calculations in Boolean algebra what all syllogistic mathematical proofs are. Additionally and even much more harmfully, the students in mathematics do not learn to think philosophically, to "see the forest", to ask questions. The coined phrase, often ascribed

---

[49] A joking allusion might be the tree sexes in Isaac Asimov sci-fi novel "The Gods Themselves": a metaphor which can be fruitfully continued.

to Richard Feinman, “Shut up and calculate!” fits traditional mathematical education very well. Alas, the computers and AIs make redundant those mathematicians wrongly educated only to calculate rather than to ask questions inherently prevented by that notorious “Shut up!”.

An additional problem is how “asking questions” should be estimated since the traditional exams consist in solving problems (i.e., calculations) and reproducing texts. Obviously, computers and AIs make both meaningless so that their use is often strictly forbidden and equated to “cheating” or using “cribs” during an exam. So, the adult professors (educated a few decades ago), being now possibly made redundant by AIs and computers (as far many professors only reproduce rather than create), too paradoxically training the students to be also redundant (as themselves) in the age of the new assistants[50].

On the contrary, the contemporary young philosophers being taught at “Art and humanities” are learned rather to “hallucinate” where Feyerabend’s slogan “Anything[51] goes” fits and dominates for expressing the student's unique personality and subjectivity. So, if the students in mathematics (trained to be “computers and AIs”) would study also philosophy to “express themselves”, the result would be nonsense and ridiculous since the contemporary academic “mathematics”, on the one hand, and “philosophy”, on the other hand, are absolutely incompatible rather than only inconsistent to each other just according to Charles Percy Snow’s “The two cultures and the scientific revolution” (1959), in turn reflecting the Cartesian abyss of Modernity (see also: Snow 1964; 1990).

Meaning those preliminary notices, one can consider back the suggested proof of Riemann’s hypothesis in Hilbert arithmetic. It does not belong to either of both genres: that of the frontier mathematical proofs (such as Wiles’s proof of Fermat’s last theorem via the modularity theorem) or a typical philosophical article (for example, in feminist philosophy). The proof is really ontomathematical thus suggesting (or restoring) the synthesis of mathematics and philosophy (as after Descartes, Leibniz, or Newton). The present approach suggests one to “see the forest” which needs the relevant Gestalt change, after which the syllogism necessary for the rigorous proof turns out to be rather elementary instead of the standard type of syllogisms in a contemporary mathematical paper written in a language different enough from English (or any other natural human language).

In the present approach, the complicateness and difficulty is distributed on both Gestalt change and the proper syllogism allowing for both to be much simplified relying on the “synergetic effect” of their simultaneous use. On the contrary, today’s usual frontier mathematical proofs do not include any philosophical compound relevant to “seeing the forest” which force a too sophisticated syllogism just as a material construction on one ground (“leg”) rather than on two ones (on two “legs”); even more so that the construction now about the proof of Riesman's hypothesis is provided with a "third ground”, that of physics, thus allowing

[50] One can jokingly call for a “Maoist cultural revolution” in mathematical education.

[51] A very interesting question is: what about if “Anything goes” would be naturalized (rather than as usually interpreted in relation only to human subjectivity), but thus following Feyerband verbatim (Ladyman, Ross 2007).

additionally for decreasing essentially the “tension” or the other two “legs” and still more simplifying the syllogism.

As a conclusion, one can reflect the proof as an example of what the mathematicians should be doing in the age of supercomputers, AIs and quantum computers (in order not to turn out their “pets” at best). The skill of “seeing the forest” suggests one to ask relevant questions instantly answered by AIs or supercomputers, which, however, cannot supply for the questions themselves to be correct. So, if a mathematician has asked stupid and irrelevant questions, he or she is provided by corresponding answers thus not less meaningless than the initiating questions.

Anyway, and on the contrary, if the questions have been or would be precisely formulated and mean a certain design of the intended proof, AIs and supercomputers allow for an incredible speed of mathematical and scientific progress now being almost “vertical”: otherwise said, tending asymptomatically to the maximally possible one. Furthermore that “vertical scientific advance” is only an exceptionally powerful instrument usable equally well in favor or harm of humankind: humankind is to decide between their dichotomy just as until now, only burdened with a huge additional responsibility …

REFERENCES;

Abraham, R., J. E. **Marsden** (1978) *Foundations of Mechanics* (2nd ed.). Reading: MA: Benjamin-Cummings.

Aneva, B. (1993) "Symmetry of the Riemann Operator," *Physics Letters B* **306** (1-2): 65-70.

Atiyah, M. (2018) "The Riemann Hypothesis," *arXiv*:1809.06458.

Baez, J., J. **Dolan** (1998) "Higher-Dimensional Algebra and Topological Quantum Field Theory," *Journal of Mathematical Physics* **39** (7): 3933-3948.

Bagchi, B. (1982) "The statistical behaviour and universality properties of the Riemann zeta-function and other allied Dirichlet series," *Indian Journal of Pure and Applied Mathematics* **13** (9): 1057-1072.

Backlund, H. (1912) "Sur les zéros de la fonction $\zeta(s)$ de Riemann," *Acta Mathematica* **35** (1): 1-40.

Ballentine, L. E. (2014) *Quantum Mechanics: A Modern Development* (2nd ed.). Singapore: World Scientific (Chapter 3.5).

Bargmann, V. (1964) "Note on Wigner's Theorem on Symmetry Operations," *Journal of Mathematical Physics* **5** (7): 862-868.

Bell, J. L. (1986) "From Absolute to Local Mathematics," *Synthese* **69** (3): 409-426.

Bender, C. M., S. A. **Orszag** (2013) *Advanced mathematical methods for scientists and engineers I: Asymptotic methods and perturbation theory.* New York: Springer Science & Business Media.

Berry, M. V. (1988) "The Riemann zeros and eigenvalue asymptotics," *Quantum Chaos and Statistical Nuclear Physics* (Lecture Notes in Physics) **263** (1): 1-17.

Berry, M. V., J. P. **Keating** (1999) "The Riemann zeros and eigenvalue asymptotics," *SIAM Review* **41** (2): 236-266.

Biane, P., J. **Pitman**, M. **Yor** (2001) "Probability laws related to the Jacobi theta and Riemann zeta functions, and Brownian excursions," *Bulletin of the American Mathematical Society* **38** (4): 435-465

Blaszak, M. (1998) *Multi-Hamiltonian Theory of Dynamical Systems* (Texts and Monographs in Physics). New York: Springer.

Bohr, H., E. **Landau** (1914) "Ein Satz über die Verteilung der Nullstellen der Riemannschen Zetafunktion," *Mathematische Annalen* **74** (3): 323-334.

Bombieri, E. (1982) "On the large sieve and its applications to the distribution of prime numbers in arithmetic progressions" *Acta Arithmetica* **39** (4): 323-340.

Bombieri, E., J. C. **Lagarias** (1999) "Complements to Li's criterion for the Riemann hypothesis" *Journal of Number Theory* **77** (2): 2742-87.

Brading, K. (2005) "A Note on General Relativity, Energy Conservation, and Noether's Theorems," in: (J. Eisenstaedt, A. J. Kox, eds.) *The Universe of General Relativity*. Boston: Birkhäuser, pp. 125-135.

Brading, K., H. R. **Brown** (2003) "Symmetries and Noether's Theorems," in: (K. Brading, E. Castellani, eds.) *Symmetries in Physics: Philosophical Reflections*. Cambridge: University Press, pp. 89-109.

Brauer, R., H. **Hasse**, E. **Noether** (1932) "Beweis eines Hauptsatzes in der Theorie der Algebren," *Journal für die reine und angewandte Mathematik* **167** (1): 399-404.

Broughan, K. A. (2017) "An explicit formula for the Riemann zeta function and the distribution of primes," *Journal of Number Theory* **171** (1): 1-20.

Butterfield, J. (2005) "On Symmetry and Conserved Quantities in Classical Mechanics," *arXiv*: physics/0507192v1.

Byers, N. (1999) "E. Noether's Discovery of the Deep Connection Between Symmetries and Conservation Laws," *Israel Mathematical Conference Proceedings* **12** (1): 67-81.

Capri, A. Z., R. **Kobes**, R. (1982) "Applications of Noether's Conservation Theorem to Hamiltonian Systems," *Canadian Journal of Physics* **60** (1): 24-28.

Carroll, S. M. (2019) *Spacetime and Geometry: An Introduction to General Relativity*. Cambridge: University Press (Chapter 4).

Castellani, E. (2003) "Symmetry and Quantum Mechanics," in: (K. Brading, E. Castellan, eds.), *Symmetries in Physics*. Cambridge: University Press, pp. 317-332.

Caulton, A. (2015) "The Role of Symmetry in the Interpretation of Physical Theories," *Studies in History and Philosophy of Modern Physics* **52** (1): 153-162.

Cauchy, A.-L. (1844) "Mémoire sur les intégrales définies prises entre des limites imaginaires," *Mémoires présentés par divers savants à l'Académie des Sciences de l'Institut de France*, **15** (Paris: L'Académie des sciences de l'Institut de France), pp. 597-612.

Caveney, G., J.-L. **Nicolas**, J. **Sondow** (2011) "Robin's Theorem, Primes, and a New Elementary Reformulation of the Riemann Hypothesis," *Integers* **11** (6): 753-763.

Choi, S., J. W. **Chung**, K. S. **Kim** (2012) "Relation between primes and nontrivial zeros in the Riemann hypothesis; Legendre polynomials, modified zeta function and Schrödinger equation," *Journal of Mathematical Physics* **53** (12): 122108(17).

Conrey, J. B. (2010) "Families of L-functions and their symmetry," in: (R. Bhatia, A. Pal, G. Rangarajan, V. Srinivas, M. Vanninathan, eds.) *Proceedings of the International Congress of Mathematicians*, Hyderabad - New Delhi (India): Hindustan book, pp. 332-356.

Conrey, J. B., H. **Iwaniec** (2013) "Spacing of zeros of Hecke L-functions and the class number problem," *Acta Arithmetica* **160** (3): 213-240.

Conrey, J. B. (2003) "More than two fifths of the zeros of the Riemann zeta function are on the critical line," *Journal of Number Theory* **98** (2): 209-236.

Cramér, H. (1920)" Some theorems concerning prime numbers," *Arkiv för Matematik, Astronomi och Fysik* **15** (5): 1-33.

De Haro, S. (2021) "Noether's Theorems and Energy in General Relativity," *PhilSci Archive*: 2103.17160.

Derbyshire, J. (2003) *Prime Obsession: Bernhard Riemann and the Greatest Unsolved Problem in Mathematics*. Washington, D.C.: Joseph Henry Press.

Dirichlet, P. G. L. (1837) "Beweis des Satzes, dass jede unbegrenzte arithmetische Progression, deren erstes Glied und Differenz ganze Zahlen ohne gemeinschaftlichen Factor sind, unendlich viele Primzahlen enthält," *Abhandlungen der Königlichen Preußischen Akademie der Wissenschaften zu Berlin, 1837* (Berlin: Königliche Preußische Akademie der Wissenschaften), pp. 45-71.

Dorodnitsyn, V. (2011) *Applications of Lie Groups to Difference Equations*. Boca Raton, FL: CRC Press.

Dyson, F. J. (1972) "[Remarks on the zeta function zeros]," *Proceedings of Symposia in Pure Mathematics* (Vol. 24, pp. 181-193, as cited in Montgomery, 1973). American Mathematical Society. (Original remarks from a meeting, not a standalone paper).

Earman, J. (2002) "Laws, Symmetry, and Symmetry Breaking: Invariance, Conservation Principles, and Objectivity," *Philosophy of Science* **69** (S3): S122-S133.

Endres, S., F. **Steiner** (2010) "The Berry–Keating Operator on $L^2(\mathbb{R}_+, dx)$ and on Compact Quantum Graphs with General Self-Adjoint Realizations," *Journal of Physics A*: *Mathematical and Theoretical* **43** (9): 095204(33).

Euler, L. (1748). *Introductio in analysin infinitorum* (Vol. 1). Lausanne: Marcum-Michaelem Bousquet.

Fujii, A. (1975) “On the distribution of the zeros of the Riemann zeta-function in short intervals,” *Journal of the Mathematical Society of Japan* **27** (3): 447-457.
Ford, K., F. **Luca**, P. **Moree** (2014) “Values of the Euler φ-function not divisible by a given odd prime, and the distribution of Euler-Kronecker constants for cyclotomic fields,” *Mathematics of Computation* **83** (287): 1457-1476.
Van **Fraassen**, B. C. (1989) *Laws and Symmetry.* Oxford: University Press (Chapter 10).
Fraser, D. (2021) “Noether’s Theorem in Quantum Mechanics: Why Conservation Laws Are Not Enough,” *Philosophy of Science* **88** (5): 1087-1098.
Freed, D. S., M. J. **Hopkins**, J. **Lurie**, C. **Teleman** (2011) “Topological Quantum Field Theories from Compact Lie Groups,” in: (P. R. Kotiuga, ed.) *A Celebration of the Mathematical Legacy of Raoul Bott*. Providence, RI: AMS; Montreal, Canada: Centre de Recherches Mathématiques, pp. 367-403.
Gauss, C. F. (1849) “[Letter to Johann Franz Encke, December 24, 1849],” in: (C. F. Gauss) *Werke*. Vol. 2. Göttingen: Königliche Gesellschaft der Wissenschaften zu Göttingen, , pp. 444-447.
Gelbart, S. (1984) *Automorphic forms on GL(n)*. Princeton: University Press.
Gödel, K. (1931) “Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I,” *Monatshefte für Mathematik und Physik* **38** (1): 173-198.
Goldstein, H., C. **Poole**, J. **Safko** (2001) *Classical Mechanic*s (3rd ed.). San Francisco: Addison-Wesley (Chapter 13).
Goldston, D. A., , J. **Pintz**, C. Y. **Yıldırım** (2006) “Primes in tuples I,” *Annals of Mathematics* **163** (2): 581-628.
Gonek, S. M. (1999) “The vertical distribution of the zeros of the Riemann zeta-function,” *Journal of Number Theory* **77** (2): 189-214.
Gour, G., R. W. **Spekkens** (2008) “The Resource Theory of Quantum Reference Frames: Manipulations and Monotones,” *New Journal of Physics* **10** (3): 033023(62).
Govaerts, J., D. B. **Iyela** (2023) “Noether Symmetries, Dynamical Constants of Motion, and Spectrum Generating Algebras,” *Journal of Physics A: Mathematical and Theoretical* **56** (12): 125201(28).
Hadamard, J. (1896) “Sur la distribution des zéros de la fonction ζ(s) et ses conséquences arithmétiques,” *Bulletin de la Société mathématique de France* **24** (1): 199-220.
Hadamard, J. (1893) “Étude sur les propriétés des fonctions entières et en particulier d’une fonction considérée par Riemann,” *Journal de Mathématiques Pures et Appliquées* **9** (4): 171-215.
Hardy, G. H. (1914) “Sur les zéros de la fonction $\zeta(s)$ de Riemann,” *Comptes Rendus de l’Académie des Sciences* **158** (26): 1012-1014.
Hardy, G. H., J. E. **Littlewood** (1916) “Contributions to the theory of the Riemann zeta-function and the theory of the distribution of primes,” *Acta Mathematica* **41** (1): 119-196.
Hardy, G. H., J. E. **Littlewood** (1921) “The zeros of Riemann’s zeta-function on the critical line,” *Mathematische Zeitschrift* **10** (3–4): 283-317.
Havil, J. (2003) *Gamma: Exploring Euler's Constant*. Princeton: University Press.
Healey, R. (2007) *Gauging What’s Real: The Conceptual Foundations of Gauge Theories*. Oxford: University Press (Chapter 7).
Hydon, P. E. (2000) *Symmetry Methods for Differential Equations: A Beginner’s Guid*e. Cambridge: University Press (Chapter 12).
Ingham, A. E. (1932) *The distribution of prime numbers*. Cambridge: Cambridge University Press.
Iwaniec, H., W. Luo, P. Sarnak (2000) “Low lying zeros of families of L-functions,” *Publications Mathématiques de l’Institut des Hautes Études Scientifiques* **91** (1): 55-131.

Iwasawa, K. (1972) *Lectures on p-adic L-functions*. Princeton: University Press.
Iyela, D. B., J. **Govaerts** (2021) "Noether Symmetries, Dynamical Constants of Motion, and Spectrum Generating Algebras," *arXiv*: 2107.03831.
Joyner, D. (2005) *Distribution theorems of L-functions. Journal of Number Theory* **112** (2): 227-281.
José, J. V., E. J. **Saletan** (1998) C*lassical Dynamics: A Contemporary Approach*. Cambridge: University Press (Chapter 5.4).
Katz, N. M., P. **Sarnak** (1999) "Zeroes of zeta functions and symmetry," *Bulletin of the American Mathematical Society* **36** (1): 1-26.
von **Koch**, H. (1901) "Sur la distribution des nombres premiers," *Acta Mathematica* **24** (1): 159-182.
Keating, J. P., N. C. **Snaith** (2000) "Random matrix theory and $\zeta(1/2 + it)$," *Communications in Mathematical Physics* **214** (1): 57-89.
Kochen, S., E. P. **Specker** (1967) "The problem of hidden variables in quantum mechanics," *Journal of Mathematics and Mechanics* **17** (1): 59-87.
Kosmann-Schwarzbach, Y. (2011) *The Noether Theorems: Invariance and Conservation Laws in the Twentieth Century*. New York: Springer.
Ladyman, J., D. **Ross** (2007) *Every Thing Must Go: Metaphysics Naturalized*. Oxford: University Press (Chapter 3).
Lange, M. (2007) "Laws and Meta-Laws of Nature: Conservation Laws and Symmetries," *Studies in History and Philosophy of Modern Physics* **38** (3): 457-481.
Legendre, A.-M. (1808) Essai sur la théorie des nombres (2nd ed.). Paris: Courcier (pp. 394–411 in later reprints).
Levinson, N. (1974) "More than one third of zeros of Riemann's zeta-function are on $\sigma = \frac{1}{2}$," *Advances in Mathematics* **13** (3): 383-436.
Littlewood, J. E. (1924) "On the zeros of the Riemann zeta-function," *Proceedings of the Cambridge Philosophical Society* **22** (3): 295-318.
von **Mangoldt**, H. (1905) "Zur Verteilung der Nullstellen der Riemannschen Zetafunktion," *Mathematische Annalen* **60** (1): 1-19.
Mansfield, E., T. **Pryer** (2015) Noether-Type Discrete Conserved Quantities Arising from a Finite Element Approximation of a Variational Problem," *Foundations of Computational Mathematics* **15** (6): 1-28.
Mansfield, E. L., A. **Rojo-Echeburua**, L. **Peng**, P. E. **Hydon** (2018) "Moving Frames and Noether's Finite Difference Conservation Laws I," *Studies in Applied Mathematics* **141** (4): 414–452.
Marsden, J. E., T. S. **Ratiu** (1999) *Introduction to Mechanics and Symmetr*y (2nd ed.). Berlin - New York: Springer.
Marvian, I. (2020) "Symmetry, Reference Frames, and Relational Quantities in Quantum Mechanics," *arXiv*: 2003.05984.
Mazur, B., W. **Stein** (2015) *Prime numbers and the Riemann Hypothesis*. Cambridge: University Press.
Mittag-Leffler, G. (1877) "Sur la représentation analytique des fonctions monogènes uniformes," *Acta Mathematica* **4** (1): 1-79.
Montgomery, H. L. (1973) "The pair correlation of zeros of the zeta function," *Proceedings of Symposia in Pure Mathematics* **24** (1): 181-193.
Neuenschwander, D. E. (2017) Emmy Noether's Wonderful Theorem (2nd ed.). Baltimore: Johns Hopkins University Press.

von **Neumann**, J. (1932) *Mathematische Grundlagen der Quantenmechanik*. Berlin: Springer, pp. 167-173.
Noether, E. (1918) "Invariante Variationsprobleme," *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse* **1918** (1): 235-257.
Odlyzko, A. M. (1987) "On the distribution of spacings between zeros of the zeta function," *Mathematics of Computation* **48** (177): 273-308.
Olver, P. J. (2018) *Emmy Noether and Symmetry*. Minneapolis: University of Minnesota.
Olver, P. J. (2001) *Applications of Lie Groups to Differential Equations* (Graduate Texts in Mathematics 107). Berlin - New York: Springer-Verlag.
Olver, P. J. (2001a) "Moving Frames and Discrete Symmetries," *Journal of Nonlinear Mathematical Physics* **8** (Supplement): 229-239.
Penchev, V. (2025 February 13) "The GOOGLE and XPRIZE award for how to use quantum computers practically: The problem of the "P" versus "NP" outputs of any quantum computer and the pathway for its resolving," *SSRN*, https://papers.ssrn.com/sol3/papers.cfm?abstract_id=5083035 or https://dx.doi.org/10.2139/ssrn.5083035 .
Penchev, V/ (2024 October 2) "Quantum theology, or: 'Theologie als strenge Wissenschaft'," *SSRN*, https://papers.ssrn.com/sol3/papers.cfm?abstract_id=4956963 , https://dx.doi.org/10.2139/ssrn.4956963 .
Penchev, V. (2024 August 5) "Stop Calculating: It is About Time to Start Thinking!" *SSRN*, https://papers.ssrn.com/sol3/papers.cfm?abstract_id=4888038 , https://dx.doi.org/10.2139/ssrn.4888038 .
Penchev, V. (2023 December 6) "Isaac Asimov's sci-fi novella "Profession" versus professionalism: Reflections on the (missing) scientific revolutions in the 21th century," *SSRN*, https://papers.ssrn.com/sol3/papers.cfm?abstract_id=4648022 , https://dx.doi.org/10.2139/ssrn.4648022 .
Penchev, V. (2023 November 2) "Logic, mathematics, physics: from a loose thread to the close link: Or what gravity is for both logic and mathematics rather than only for physics," *SSRN*, https://papers.ssrn.com/sol3/papers.cfm?abstract_id=4593958 , https://dx.doi.org/10.2139/ssrn.4593958 .
Penchev, V. (2023 October 6) "Logic, mathematics, physics: from a loose thread to the close link: Or what gravity is for both logic and mathematics rather than only for physics," *SSRN*, https://papers.ssrn.com/sol3/papers.cfm?abstract_id=4593958 , https://dx.doi.org/10.2139/ssrn.4593958 .
Penchev, V. (2023 July 16) "Hilbert Mathematics Versus Gödel Mathematics. IV. The New Approach of Hilbert Mathematics Easily Resolving the Most Difficult Problems of Gödel Mathematics," *SSRN*, https://papers.ssrn.com/sol3/papers.cfm?abstract_id=4503027 , https://dx.doi.org/10.2139/ssrn.4503027 .
Penchev, V. (2023 March 13) "This Year's Nobel Prize (2022) in Physics for Entanglement and Quantum Information: the New Revolution in Quantum Mechanics and Science," *SSRN*, https://papers.ssrn.com/sol3/papers.cfm?abstract_id=4382533, https://dx.doi.org/10.2139/ssrn.4382533 .
Penchev, V. (2022 October 21) "Gödel mathematics versus Hilbert mathematics. I The Gödel incompleteness (1931) statement: axiom or theorem?" *SSRN*, https://dx.doi.org/10.2139/ssrn.4243201 or https://papers.ssrn.com/sol3/papers.cfm?abstract_id=4243201 .
Penchev, V. (2021 July 8) "'Two bits less' after quantum-information conservation and their interpretation as 'distinguishability / indistinguishability' and 'classical / quantum'," *SSRN*, https://papers.ssrn.com/sol3/papers.cfm?abstract_id=3873123 , https://dx.doi.org/10.2139/ssrn.3873123 .
Penrose, R. (2004) *The Road to Reality: A Complete Guide to the Laws of the Universe*. Baltimore: Vintage (Chapter 33).
Pintz, J. (2007) "Cramér's probabilistic model for primes and the Riemann hypothesis," *Acta Mathematica Hungarica* **116** (1-2): 123-141.

Pintz, J. (1991) “On an assertion of Riemann concerning the distribution of prime numbers,” *Acta Mathematica Academiae Scientiarum Hungaricae* **58** (3-4): 383-387.

Pitts, J. B. (2016) “Einstein's Equations for Spin 2 Mass 0 from Noether's Converse Hilbertian Assertion,” *arXiv*:1611.02673.

Robinson, A. (1966) *Non-standard analysis*. Amsterdam, Netherlands: North-Holland Publishing Company.

Riemann, B. (1859) “Über die Anzahl der Primzahlen unter einer gegebenen Grösse,” *Monatsberichte der Berliner Akademie* **1859** (11): 671-680.

Riemann, B. (1851) *Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse*. Göttingen: Universität Göttingen.

Roquette, P. (2005) *The Brauer-Hasse-Noether Theorem in Historical Perspective* (Schriften der Mathematisch-naturwissenschaftlichen Klasse der Heidelberger Akademie der Wissenschaften **15**). Berlin Heidelberg: Springer.

Rovelli, C. (2014) *Covariant Loop Quantum Gravity: An Elementary Introduction to Quantum Gravity and Spinfoam Theory*. Cambridge: University Press.

Rowe, D. E. (2019) “Emmy Noether on Energy Conservation in General Relativity," *arXiv*:1912.03269.

Rudnick, Z., P. **Sarnak** (1996) “Zeros of principal L-functions and random matrix theory,” *Duke Mathematical Journal* **81** (2): 269-322.

Ryder, L. H. (1996) *Quantum Field Theory* (2nd ed.). Cambridge: University Press.

Sardanashvily, G. (2016) “Noether's First Theorem in Hamiltonian Mechanics,” *International Journal of Geometric Methods in Modern Physics* **13** (08): 1650108(25).

Selberg, A. (1942) “An elementary proof of the prime number theorem,” *Annals of Mathematics* **50** (2): 305-313.

Siegel, C. L. (1932) “Über die Nullstellen der Riemannschen Zeta-Funktion,” *Mathematische Annalen* **106** (1): 605-616.

Sierra, G. (2008) “A quantum mechanical model of the Riemann zeros,” *Journal of Physics A: Mathematical and Theoretical* **41** (30): 304020(32).

Simon, B. (2015) *Operator Theory: A Comprehensive Course in Analysis*, Part 4. Providence, RI: AMS.

Skolem, T. (1922) “Einige Bemerkungen zur axiomatischen Begründung der Mengenlehre,” in: *Förhandlingar vid det femte skandinaviska mathematiker kongressen i Helsingfors* 1922 (pp. 217–232). Helsinki, Finland: Akademiska Bokhandeln.

Snow, C. P. (1959) *The Two Cultures and the Scientific Revolution*. NewYork: Cambridge University Press.

Snow, C. P. (1964) *The two cultures: and a second look*. London: Cambridge University Press.

Snow, C. P. (1990) “The Two Cultures (Programs for the new era),” *Leonardo* **23** (2-3): 169-173.

Srednicki, M. (2007) *Quantum Field Theory*. Cambridge: University Press (Chapter 22).

Steuding, J. (2007) *Value-distribution of L-functions* (Lecture Notes in Mathematics **1877**). Berlin: Springer.

de la **Vallée Poussin**, C. J. (1896) “Recherches analytiques sur la théorie des nombres premiers,” *Annales de la Société scientifique de Bruxelles* **20** (Part 2): 183-256.

Titchmarsh, E. C. (1935) “The zeros of the Riemann zeta-function,” P*roceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences* **151** (873): 234–255.

Vedral, V. (2003) “Entanglement in Quantum Information Theory,” *Contemporary Physics* **44** (3): 215-230.

Vericat, F. (2013) “A lattice gas of prime numbers and the Riemann Hypothesis,” *Physica A: Statistical Mechanics and its Applications* **392** (19): 4516-4522.

Wallace, D. (2019) “Isolated Systems and Their Symmetries,” *Studies in History and Philosophy of Modern Physics* **66** (1): 59-68.

Weierstrass, K. (1876) “Zur Theorie der eindeutigen analytischen Funktionen,” *Abhandlungen der Königlichen Akademie der Wissenschaften zu Berlin*, **1876** (Berlin: Königlich-Preußische Akademie der Wissenschaften) , pp. 11-60.

Wigner, E. P. (1964) “Symmetry and Conservation Laws,” *Proceedings of the National Academy of Sciences* **51** (5): 956-965.

Wiener, N. (1951) “The Fourier integral and certain of its applications,” *Duke Mathematical Journal* **18** (1): 129-142.

Wolf, M. (2014) “Applications of statistical mechanics in number theory,” *Physica A: Statistical Mechanics and its Applications* **409** (1): 1-19.

Zagier, D. (1977) “The first 50 million prime numbers,” *The Mathematical Intelligencer* **0** (1): 12-19.

Zanardi, P. (2001) “Virtual Quantum Subsystems,” *Physical Review Letters* **87** (7): 077901(4).

Zhang, Y. (2014) “Bounded gaps between primes,” *Annals of Mathematics* **179** (3): 1121-1174.