The relationship of arithmetic as two twin Peano arithmetic(s) and set theory: A new glance from the theory of information

Vasil Penchev, vasildinev@gmail.com
Bulgarian Academy of Sciences: Institute of Philosophy and Sociology:
Dept. of Logical Systems and Models

Abstract. The paper introduces and utilizes a few new concepts: “nonstandard Peano arithmetic”, “complementary Peano arithmetic”, “Hilbert arithmetic”. They identify the foundations of both mathematics and physics demonstrating the equivalence of the newly introduced Hilbert arithmetic and the separable complex Hilbert space of quantum mechanics in turn underlying physics and all the world. That new both mathematical and physical ground can be recognized as information complemented and generalized by quantum information. A few fundamental mathematical problems of the present such as Fermat’s last theorem, four-color theorem as well as its new-formulated generalization as “four-letter theorem”, Poincaré’s conjecture, “P vs NP” are considered over again, from and within the new-founding conceptual reference frame of information, as illustrations. Simple or crucially simplifying solutions and proofs are demonstrated. The link between the consistent completeness of the system mathematics-physics on the ground of information and all the great mathematical problems of the present (rather than the enumerated ones) is suggested.

Key words: Peano arithmetic, nonstandard interpretation of Peano arithmetic, two complimentary standard interpretations of Peano arithmetic, Hilbert arithmetic, consistent completeness of mathematics and physics, the unification of mathematics and physics, information, quantum information

I INTRODUCTION

Prehistory and background:

Cantor’s set theory involving the concept of actual infinity as a complete whole seemed to be a base of mathematics. However, a series of paradoxes were revealed in it, e.g. Russell’s (1902). One of the main directions for overcoming the crisis was set theory to be reduced to a model in arithmetic.

However, arithmetic is obviously finite. Utilizing the commonly accepted Peano axioms[1], one can prove that all natural numbers are finite:

1 is finite. Adding 1 to any natural number, which is finite, a finite natural number is obtained, again. Consequently, all natural numbers are finite according to the axiom of induction.

Arithmetic being thus finite seems not to have any access to actual infinity underlying set theory. Indeed, Gödel’s work (1931) demonstrates that any mathematical theory containing arithmetic, including the arithmetic itself as that theory, is necessary either incomplete or contradictory. If one admits to identify any mathematical theory with set theory as its base, complemented by additional axioms to the mathematical theory at issue, for philosophical consideration, the result of Gödel seems to be even obvious in the present context:

[1] In fact, it is offered by R. Dedekind (1888) as Peano himself pointed out expressly in his work (1889: 5).

Arithmetic is finite, and set theory needs the concept of actual infinity. Then, either arithmetic should be somehow complemented to be compensated missing actual infinity, or it is contradictory immediately adding actual infinite, just being finite as above.

In fact, the concept of actual infinity, e.g. meaning the commonly accepted ZFC axioms of set theory for certainty, is added by a consideration arithmetical in essence by the axiom of infinity. No contradiction arises in set theory, for the axiom of induction, necessary to be proved the finiteness of arithmetic as above, is not included in ZFC.

Arithmetic is anyway added to set theory, but "at last", by means of the axiom of choice. Indeed, that axiom is equivalent to the well-ordering principle, according to which any set is well-orderable and thus representable by some initial segment of natural numbers. If the axiom of induction holds, any set is representable by some finite set for all natural numbers are finite in that case.

If the axiom of induction does not hold (or particularly, it is replaced by the axiom of transfinite induction), any set is anyway countable in Cantor's sense.

Skolem (1922) was who noted that the axiom of choice implies the "relativity of 'set'" meaning those representations of any set either as finite or at least as countable.

Thus, arithmetic and set theory distinguish correspondingly the finite mathematics absolutely representable by arithmetical models from the mathematics of infinity needing actual infinity and irrepresentable by them both completely and consistently. The relation between the two kinds of mathematics might be regulated by both axioms of (transfinite) induction and choice. Any utilization of mathematical models in human activity, as far as being finite, thus needs arithmetizability.

In another branch of human knowledge, Shannon (1948) suggested the quantity of information to describe encoding, processing and transmitting of data. It can be featured as the measure of ordering or assigning a value (e.g. and usually a number) to a variable. The unit of that assignation is a bit definable as assigning either "0" or "1" to a variable. The quantity of information in any data can be considered also as the minimal length of that algorithm able to construct them (Kolmogorov 1965, 1968). Furthermore, its mathematical model is an immediate and natural generalization of the statistic and thermodynamic quantity of entropy expressing the "degree of chaos" of an ensemble of atoms, molecules, etc. Then, information admits still one interpretation as the measure of simile of two probability distributions.

Shannon's information means the record or reading of a number represented in some numerical system, e.g. binary or decimal. A natural generalization might be that to an infinite numerical system:

Quantum information, introduced by quantum mechanics to interpret itself in terms of the theory of information, is equivalent to that "infinite information". Thus, any physical process or phenomenon, being always quantum in its base, can be interpreted informationally. Quantum information might be interpreted as the classical one both as (1) the minimal length (eventually transfinite) of that algorithm able to construct the quantum data in question and as (2) the measure of simile of two probability distributions (represented by their characteristic functions as an operator in the complex Hilbert space).

One can summarize in a few words so:

Arithmetic being finite is insufficient to ground all mathematics, but sufficient to represent the human utilization of mathematical models. Set theory and arithmetic seem to be sufficient for that foundation, but the consistency of each of them is unprovable and even doubtful. Then, they together might found mathematics consistently if their relationships are relevantly formalized for both to coexist without reducing to the other.

Problem:

Can the theory of information represent exhaustedly the relationships of arithmetic and set theory in a non-reductionist way, and thus mathematics to be founded on the theory of information?

Thesis:

The concept of information is enough to represent both (1) the generation of arithmetic from set theory by the axiom of choice, and (2) the generation of set theory from arithmetic by the axiom of transfinite induction (equivalent to standard, but complementary induction): (1) involves the interpretation of information as a string length of an assigned value, encoding, or algorithm, and (2): as a relation of two probability distributions.

A few comments of the thesis:

1 The equivalence of those two interpretations of information mentioned in the thesis is not necessary, but consistent to the thesis.

2 The generation of arithmetic from the set theory by the axiom of choice seems to be both rather obvious and historically justified. The axiom of choice is equivalent to the well-ordering principle ("theorem"). The natural numbers are definable as the classes of equivalence of well-orderings as R. Dedekind did (1888), from which Peano took his axioms as he himself referred. Furthermore, the unit of information is interpretable as both an elementary choice between two equally probable alternatives and a binary digit[2] of any well-ordering represented by binary digits.

3 The generation of set theory from arithmetic by the axiom of induction is nontrivial:

3.1 It needs preliminarily the choice of some finite set as a nonstandard interpretation of any given infinite set in the sense of Skolem's "relativity" to be founded as necessarily random. Indeed, the axiom of choice guarantees a choice of an element of any set even if any constructive way to be chosen that element does not exist (in principle). One needs to postulate that non-constructiveness of the choice as its randomness: an element chosen in a nonconstructive way is chosen randomly. Further, no finite set can be mapped one-to-one to any infinite set. However, that finite set should exist "purely", i.e. non-constructively according to the Skolem argument about the "relativity of 'set'". This means that finite set exists, but no one can know which it is: It should be different after each one given choice, or in other words, it be randomly chosen each time.

Consequently, the Skolem nonstandard finite equivalent to any infinite set is necessarily random. This implies that any infinite set can be unambiguously represented by some probability distribution interpretable as the statistics of experiments for choosing a finite equivalent set for that infinite set.

3.2 One has to associate probability distributions to some sets of natural numbers by means of the axiom of induction unambiguously. If one utilizes the axiom of induction to one and the same property many times, the corresponding set is always finite, but the number of its elements might be as constant as variable as both cases are consistent to the Peano axioms. A probability

[2] The etymology of "bit" originates just from "binary digit".

distribution as the statistics of the number of the elements of that set after all experiments of induction in the latter case can be unambiguously associated with this set. One can say that the number of its elements is undetermined or uncertain and define as "infinite" in the sense of Peano arithmetic, where all natural numbers are finite. On the contrary, if the case is the former, i.e. the number of the elements of the set is constant after an arbitrarily series of experiments for induction, it is defined as "properly finite".

4. Quantum mechanics in terms of quantum information would be another interpretation of both arithmetic and set theory as above. The complex Hilbert space is a basic model as for quantum mechanics as for that pair. The theorems about the absence of hidden variables in quantum mechanics (Neumann 1932; Kochen and Specker 1968) would mean an internal proof of completeness and consistency as to the joint system of arithmetic and set theory.

The paper is organized as follows:

The new concepts are defined in Section II. Those are: "complementary Peano arithmetic"; "nonstandard interpretation of Peano arithmetic"; "generalized Peano arithmetic"; "Hilbert arithmetic"; "quantum neo-Pythagoreanism"; "physical and mathematical transcendentalism". The statement about the consistent completeness of Hilbert arithmetic is argued. It is identified, furthermore, as the separable complex Hilbert space of quantum mechanics. That identity is the ground of a few related fundamentally new conceptions: the unity of the physical and mathematical world; the absence of any boundary, and smooth transition between them; the option of the identity of mathematical model and physical reality; the quantum resurrection of Pythagoreanism as the quantum neo-Pythagoreanism; the arithmetical reduction of physical quantities, entities, and laws. The unity of the consistent completeness of both mathematics (by means of its foundations) and quantum mechanics is demonstrated. Links to paradoxes of the foundations of mathematics are elucidated and thus, ways for resolving them. The Gödel completeness (1930) and incompleteness (1931) papers are reinterpreted. The equivalence of the set the Gödel irresolvable statement and all statements satisfying Yablo's paradox is inferred. The idea of "physical and mathematical transcendentalism" on the ground of the philosophical "totality". Ways of inferring the concept of information from the totality are shown. "Quantum information" (as it is deduced in quantum mechanics) as a generalization or specification of "information" as to infinite series and sets is deduced. That generalized information underlies the unity of physics and mathematics (made visible by the new concepts of "Hilbert arithmetic" and "Hilbert mathematics' as the latter is opposed to "Gödel mathematics").

Section III exemplifies and verifies the new concepts in the case of Fermat's last theorem and its proofs. It is very suitable because of the following fact: being an arithmetical statement properly, it is simultaneously a Gödel irresolvable statement in the framework of both Peano arithmetic and (ZFC) set theory (i.e. the standard foundations of mathematics in the present). This may be demonstrated concisely by means of the interpretation of Fermat's last theorem in terms of Yablo's scheme and thus, of his paradox. Consequently, the proof of Fermat's last theorem as a corollary from the modularity theorem (Taniyama – Shimura – Weil conjecture proved by Weil) involves necessarily "inaccessible cardinals or ordinals", to which the Gödel number of any Gödel irresolvable statement belongs. However, one can admit the existence of a certain, direct arithmetical proof (thus, not involving any part of set theory whether explicitly or implicitly) being eventually elementary enough in order to have been accessible to Fermat himself, as his “lost proof”. That elementary, directly arithmetical proof of Fermat’s last theorem is demonstrated. It modifies Fermat’s “infinite descent” to link it to the Peano axiom of induction by means of *modus tollens*. Thus, an infinite stair, both “to and from infinity”, but in the rigorous frame of Peano arithmetic, is involved to be proved Fermat’s last theorem by induction (once the case for n = 3 has been proved).

The justification of that, purely arithmetical proof needs the wider framework of Hilbert arithmetic. One can show within it that the “infinite stairs” of *modus tollens* is invariant to which of both complementary standard Peano arithmetic is meant, and thus, it is valid to its nonstandard interpretation. Then, the “purely arithmetical proof” of Fermat’s last theorem can be understood as an arithmetical and logical proof, in fact, therefore involving by the pair of propositional logic and a single standard Peano arithmetic, furthermore, the complementary counterpart of the latter, implicitly. This means that the explicit reference to (ZFC) set theory and thus, to the Gödel irresolvability of Fermat’s last theorem (in other words, to “inaccessible cardinals or ordinals”) can be omitted or avoided. *Modus tollens* being invariant to both complementary counterparts of the standard Peano arithmetic, and furthermore, being a tautology of propositional logic is a sufficient tool for implementing that idea of arithmetical proof, however being at the same time verifiable in the framework of Hilbert arithmetic.

Summarizing metaphorically, Fermat’s original proof is possible for the innocent or naïve, unintentional and unconscious bypass of all “traps” of Gödel’s irresolvability. On the contrary, Wiles’s proof is impossible to bypass them for the actual cognition (“eating the apple”) of (ZFC) set theory and meaning it by the Taniyama – Shimura – Wail conjecture. Once this has been done, the only option is the way out of set theory (and thus, of the pair “Peano arithmetic & set theory”) whether explicitly or implicitly to “Hilbert arithmetic” as it is sketched here involving “inaccessible” cardinals or ordinals.

Section IV intends to extend the approach in Section III to other great mathematical puzzles of the present, therefore creating the ground of the conjecture that the consistent completeness of mathematics is the “problem of many (even all) problems” in the contemporary mathematics. The following three theorems are considered: the four-color theorem; Poincaré’s conjecture (proved by G. Perelman); the “P vs NP” problem (still one of the seven CMI millennium problems).

The idea of a “human proof” of the four-color theorem is suggested: any defect after coloring will reflect onto at least one of two orthogonal axes of the plane. Not to have any defect in any of both axes, two colors are enough for each of them, or four colors totally (2 colors x 2 axes). Further, the two orthogonal axes can be interpreted as the two complementary Peano arithmetics (both well-ordered), each of which needs two “digits”, or totally, four digits would be sufficient as to the alphabet for any mathematical entity to be recorded unambiguously. As far as both mathematical and physical world coincide in the totality (the thesis of the quantum neo-Pythagoreanism), four letters would be enough any entity whether mathematical or physical to be notated. The alphabet of natural language consists of four letters: thus, the four-color theorem can be generalized to the four-letter theorem ontologically, however, provable rigorously and mathematically by Hilbert arithmetic rather than only a way for the four-color theorem to be proved “humanly”.

Next, a physical interpretation of Poincaré’s conjecture demonstrates how it can be proved only arithmetically in the final analysis (therefore, revealing a direct topological meaning of both axiom of choice and principle of well-ordering). The unfolding of the 3-sphere is topologically equivalent to Minkowski space as follows:

Its imaginary domain (corresponding to the subluminal and physically observable area of special relativity) can be enumerated by one of the two complementary Peano arithmetics. Then, its real domain (corresponding to the superluminal and physically unobservable area of special relativity) can be enumerated by the Peano arithmetic being the complementary counterpart. If one removes the luminal barrier between them, they would mix therefore cancelling any well-ordering (for the two well-orderings of the two domains are inconsistent to each other) and resulting into the unorderable cyclic structure corresponding to the nonstandard interpretation of Peano arithmetic, on the one hand, and to the 3-sphere before unfolding (after unfolding it is represent topologically by the Minkowski space itself), one the other hand.

The coherent state of all 3-spheres is topologically Euclidean space, and thus Poincaré's conjecture can be proved only arithmetically in the above sense. This is due to the direct topological meaning of the axiom of choice: it is able to equate a cyclic topological state (such as the 3-sphere) and a "coherent" topological state (such as Euclidean space) by the mediation of the well-ordering "theorem" (resulting in both domains of Minkowski space).

There is still one and independent way to be expressed the above consideration. There exists an elementary and obvious homeomorphism of Euclidean space to each of both hemispheres of the sphere. So, one needs still one, but "nonstandard" homeomorphism being "two-to-one" (and thus as if contradictory as far as it seems obviously to be discrete). However, Hilbert arithmetic supplies that tool by the two complementary Peano arithmetics mergeable into the single one[3] of the nonstandard interpretation of Peano arithmetic, furthermore purely arithmetically.

At last, a class of examples demonstrating that "$P \neq NP$" in th "P vs NP" problem can be easily constructed by means of the Kochen and Specker theorem in quantum mechanics. Its elements can be visualizes by "Schrödinger's cat". Let the problem be: "Either alive or dead is the cat?" The problem is irresolvable by any Turing machine for any finite time (and particularly, it is not any P time being finite). However, it is NP for a Turing machine is able to resolve it if the door of the box is open checking up the state of the cat very fast. Thus, any problem analogical to "Schrödinger's cat" therefore involving Turing machine in a quantum superposition of any finite number of possible states would be "non-P, but NP".

That class of examples can be represented by Hilbert arithmetic. Of course, any Turing machine identifiable within a single standard Peano arithmetic cannot resolve a problem referring to Hilbert arithmetic (and thus, to both complementary Peano arithmetics) in general. Thus it not a P problem. However, it can examine the solution in a polynomial time if it known somehow because the check-up will be accomplished in the nonstandard interpretation of Peano arithmetic (or in other words, only logically). Consequently, it is a NP problem simultaneously being a non-P problem.

The final *Section V* formulates the conclusion and directions for future work:

[3] Quantum mechanics, chronologically first, has utilized the same kind of continuous (and even smooth) bijection "2:1" (being due to the identification, discussed above, of Hilbert arithmetic and the separable complex Hilbert space of quantum mechanics) because of the necessity to describe uniformly the quantum entity (commensurable with the Planck constant) and the microscopic apparatus obeying the smooth equations of classical physics.

Information by its generalization as quantum information is able to represent both completely and consistently the interrelations of arithmetic and set theory therefore suggesting a reliable ground of all mathematics. This implies further the unification of mathematics and quantum mechanics (and thus, all physics) as well as a form of neo-Pythagoreanism. However, the research here is philosophical rather than mathematical and do not offer complete and rigorous proofs, but only generalized consideration necessary to be "seen the forest for the trees".

Thus, the descriptions in each separate tree (anyway discernible in the present paper as well) is the generalizing direction for future work.

II BOTH STANDARD AND NONSTANDARD INTERPRETATIONS OF PEANO ARITHMETIC

There exists a curious "vicious circle", in which Peano defined the natural numbers by the five famous axioms called by his name. He mentioned expressively in his paper (Peano 1889: 5) that he had utilized the concept of set and well-ordering in the work of Dedekind (1888) when the paradoxes in Cantor's set theory starting e.g. from that of Russell (1902) had not been known yet.

In fact, Peano followed the then "fashion" for all mathematical theories to be underlain by set theory. Hilbert's formalism "however" granted Peano axioms for arithmetic as a finite reliable ground for set theory attacked by many paradoxes. And Gödel (1931) demonstrated that Peano arithmetic is inconsistent to (ZFC) set theory.

One can question, thus, retrospectively and rethorically how the "”Peano axioms" had been extracted or justified by set theory if they are inconsistent to it.

Meanwhile, Zermelo (1904; 1908) formulated a special axiom, the axiom of choice to infer the well-ordering theorem, and Whitehead and Russel[4] (i.e. very soon) demonstrated that the axiom of choice (more exactly, its equivalent meant by them) can be deduced from the well-ordering "theorem". Therefore, both are equivalent to each other.

Skolem (1922) reflecting on the theorem proved by him a few years ago, and known now as the "Löwenheim - Skolem theorem", showed that once the axiom of choice was involved, all infinite cardinal numbers can be considered as equivalent to each other. Even more, they are equivalent to Dedekind finite numbers (being finite as no true subset of them has the same cardinal number, a property shared by all infinite sets). This consideration was named Skolem's paradox after then. In fact, it is an argument as if contradicting common sense rather than a real paradox in a proper logical sense.

Summarizing all enumerated results, the Peano natural numbers are equally powerful to infinite sets (though in the meaning of Dedekind) by the meditation of the axiom of choice, on the one hand. However simultaneously, they are inherently inconsistent to (ZFC) set theory and thus, to concept of infinity as Gödel's incompleteness theorems made obvious, on the other hand.

One can localize the problem eventually in the relation of two concepts: "infinity" and "well-ordering". Their relation is managed in (ZFC) set theory by the axiom of choice.

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[4] The very beginning of third volume of *Principia mathematica,* even before the first enumerated statement in the volume. The proof is only a few strokes.

One can offer a very simple way of philosophical interpretation of Gödel's "incompleteness paper" (1931):

The axiom of induction implies for all natural numbers to be finite as it was demonstrated already ("1" is finite; if "n" is finite, "n+1" is finite, too; all natural numbers are finite according to the axiom of induction). On the other hand, the axiom of infinity in (ZFC) set theory implies for the set of all natural numbers to be infinite. Indeed, the one-to-one mapping of the construction in the axiom of infinity and the set of all natural numbers is obvious. After the axiom is postulated for that construction to represent an infinite set, and it is mapped by some bijection into the set of all natural numbers, the latter is infinite as well.

Thus, any mathematical structure containing substructure isomorphic correspondingly to Peano theory and ZFC (set theory) would be either incomplete or inconsistent for an obvious reason. Any infinite set cannot be represented only arithmetical since any natural number is finite. Furthermore, if the Peano axioms be complemented by any axiom corresponding to the axiom of infinity, it would contradict to the axiom of induction. Thus, the modified Peano arithmetic by an equivalent of the axiom of infinity would be inconsistent.

One can localize precisely where Peano arithmetic and (ZFC) set theory contradict to each other: the axiom of induction and the axiom of infinity. Both share the same structure. Indeed, the construction of infinity by the unary operation "{ ... }": {set}, {{set}}, {{{set}}}, etc., is isomorphic to the unary operation "successor" utilized in the axiom of induction. Thus, they can be considered as two versions of the same axiom, however stating disjunctively a proposition and its negation e.g. as the fifth postulate of Euclid and its counterpart in Lobachevski's geometry.

Indeed, the axiom of induction implies immediately that all natural numbers are finite, and the axiom of infinity states that a structure isomorphic to the set of all natural numbers is infinite. Thus, if one, as e.g. the pioneer Gödel, reveal a way for the same mathematical entity to be considered as both inductive scheme inherently referring to an unlimited series of natural numbers and infinite set simultaneously, the contradiction of the axioms would imply either incompleteness or inconsistency of Peano arithmetic and (ZFC) set theory.

One can visualize the scheme of how those axioms contradict to each other by the idea of Gödel's incompleteness as it is elucidated by himself (1931) by means of the "Liar paradox". The latter utilize the negation (for the "Liar") to Liar's statement referring to himself or herself. Gödel's innovation consists only in representing that statement simultaneously in two "reference frames": (ZFC) set theory and Peano arithmetic. The crucial link is the axiom of infinity in the former and its negative counterpart in the latter, namely, the axiom of induction:

Indeed, the negation of a statement referring to a finite set of natural numbers can be related to the complement of the same set to the set of all natural numbers. Thus, that complement is an infinite set. However, all natural numbers belonging to the complement are finite according to the axiom of induction in Peano arithmetic. Thus, the problem whether the complement contains finite or infinite numbers id irresolvable if both Peano arithmetic and (ZFC) set theory are granted. The some complement contains finite numbers according to the axiom of induction and Peano arithmetic, but infinite members according to the axiom of infinity in (ZFC) set theory. In other words, Gödel's innovation consists only in making explicit the implicit contradiction of the two axioms. He utilized self-reference as in the Liar paradox. However, it is only a sufficient, but not necessary condition:

Yablo's paradox demonstrates it convincingly. It does not involves self-reference (at least explicitly), but nonetheless generates a certain contradiction. It refers to the necessary condition for any paradox having granting both Peano arithmetic and (ZFC) set theory differently:

It addresses an infinite set of natural numbers as the next ones after a recursive proposition having a finite number in the recursive scheme. Indeed, the recursive scheme should contain a finite number of members in Peano arithmetic as far as the natural number "k" of any recursive member of that series is finite, but the set of the members is infinite in (ZFC) set theory.

One can build easily an isomorphism between all statements satisfying Yablo's paradox and all Gödel's irresolvable statements on the shared common essence of the necessary condition for the same class (in both cases of propositions) to be considered simultaneously as a class of natural numbers and as an infinite set of the same natural numbers.

The bijection of the two classes can be demonstrated furthermore constructively. One Yablo recursive scheme referring to all next members of a certain given one is sufficient. That certain given proposition considered in a self-reference proposition implies just one Gödel irresolvable statement.

One can suggest the idea of "semi-completeness" or "semi-consistency" utilizing the main idea of Gödel to prove incompleteness/ inconsistency of arithmetic & set theory by the "Liar" self-reference.

In fact, the Liar paradox appears never in the real use of language for the statement "I lie" is never used to itself, but to anything different form the statement itself. The self-referential use is artificial, intentional and only *ad hoc* in order to be created the paradox. So, many logicians starting from Russell suggested for any self-reference to be forbidden because of the implicit vicious circle it implies.

However, Yablo's paradox demonstrates that the self-reference is only an option for the same proposition to be considered simultaneously in two contexts: Peano arithmetic and (ZFC) set theory. However, the paradox is implied not by self-reference, but by unifying both contexts, which one may accomplish otherwise than by self-reference, e.g. by Yablo's scheme.

Further, one can demonstrate that the artificial technics, by which the "Liar" is constructed, can be applied in turn to the "Satz VI" incompleteness:

"Satz VI" satisfies all conditions in it just as the "Liar" does. Furthermore, on can isolate "Satz VI" into a theory consisting of a single theorem just as the "Liar" is isolated and therefore forced to self-reference. So, one obtains a kind of paradox, which might be called the "Gödel paradox" isomorphic to the "Liar", or accordingly, the "Gödel - Liar paradox".

Two series of recursive schemes appear; the first one:

(1) I lie …

(2) I lie that "I lie in the previous statement" …

(3) I lie that "I lie in the previous statement that "I lie in the previous statement" …

…

…

…

The second one:

(1) Arithmetic is either incomplete or inconsistent to set theory.

(2) If the above statement is true, it is either incomplete or inconsistent.

(3) If the above statement is true, it is either incomplete, or inconsistent.

…

…

…

The Liar paradox is avoided formally in both schemes since no statement refers to itself but to the closest previous one.

So, the Liar lies, considered in any odd level, but the Liar does not lie, considered in any even level. The paradox seems to be resolved if one distinguish disjunctively odd from even levels.

The construction of the "levels of lies" can be transferred to the second recursive scheme because of the noticed isomorphism.

Then, the second series can be transformed as follows:

(1) Arithmetic is either incomplete or inconsistent to set theory.

(2) Arithmetic is both complete and consistent to set theory.

(3) Arithmetic is either incomplete or inconsistent to set theory.

(4) Arithmetic is both complete and consistent to set theory.

…

…

Then, one can introduce the term "Gödel mathematics" to all odd levels, on which the Gödel incompleteness theorems are valid, and correspondingly the term "Hilbert mathematics" as to all even levels. Furthermore, arithmetic is semi-complete and semi-consistent to set theory, and thus to all mathematics if the above disjunctive distinction of "Gödel mathematics" and "Hilbert mathematics" is valid. In fact, it is semi-valid, too, and in the same rigorous meaning, but this is sufficient for the intention of the paper:

Indeed, both above schemes satisfies the conditions of Yablo's paradox. Thus, each of them corresponds to exactly one Gödel irresolvable statement. One can grant that the corresponding Gödel irresolvable statement is the "Satz VI" itself as to the second series, and consequently the scheme is confirmed again and independently: the concept of semi-completeness and semi-consistency is semi-complete and semi-consistent in turn.

What follows is a relevant generalization of Peano arithmetic able to embody the idea it to be the semi-complete and semi-consistent base of mathematics.

The concept "nonstandard interpretation" means usually an enumerable model in virtue of the Löwenheim – Skolem theorem in any of both directions: this means in the framework of set theory. It will be utilized now to Peano arithmetic to generate a model of it within set theory, namely the countable set of all natural numbers. The main problem for the new interpretation continues to be the contradiction between the axiom of induction and the axiom of infinity as it is sketched above.

One can observe that the well-ordering of all natural numbers is a sufficient, but not necessary condition of each of both axioms to be postulated. Furthermore, neither it, nor the rest four axiom implies a well-ordering. Actually, all the five are consistent to the well-ordering of all natural numbers, but an interpretation not requiring for all the natural numbers to be well-ordered is admissible because Peano axioms do no imply that well-ordering. It will be demonstrated explicitly and then, one can show that it satisfies the condition to be a nonstandard interpretation of Peano arithmetic in the above rigorous meaning.

The only difference from the standard interpretation consist in the "function successor":

One substitutes the series $"1, 1+1, 1+1+1, \dots., (n)x(1), (n+1)x(1), \dots"$ by:

$$1, 1=1, 1=1=1, \dots, (n)[1=1], (n+1)[1=1], \dots$$

Obviously, it satisfies the Peano axioms, including the axiom of induction. One is to notice that the definitive properties of the relation of equivalency together with the axiom of induction imply that interpretation as well. However, the reverse pathway is meant here: the properties of the relation of equivalence to be inferred from Peano arithmetic for it is considered as a base of mathematics.

Though both interpretations satisfy the Peano axioms, the former visualizes the well-ordering "theorem" (or "principle"), and thus, the latter needs the axiom of choice to be equated if one can prove that it is equivalent to the set of all natural number as it is intended.

The difference between the two interpretations consists in the openness of the former unlike the cyclic closeness of the latter, which is in virtue of the axiom of induction: $\forall n, 1 = n$; this means that all natural numbers are equal to a unit. The absence of well-ordering allows they to be considered as an infinite cycle, which is impossible as to the standard interpretation. Thus, the statement that all natural numbers are finite is not inferable for neither "1", nor any "n" is necessarily finite. Thus, the axiom of induction and the axiom of infinity does not contradict as to that, called "nonstandard interpretation of Peano arithmetic". Consequently, it is equivalent to the set of all natural members.

One can obtain the standard interpretation from the nonstandard one by means of the axiom of choice once the set of all natural numbers is granted (as to the nonstandard interpretation). The axiom of choice "cuts" the infinite cycle into the usual well-ordering of the natural numbers. Thus, the axiom of choice implies further the option of "cutting" for any cyclic structure, properly a topological ability.

The triple of axioms, INDUCTION, INFINITY, and CHOICE, turns out to be linked to each other by means of the standard and nonstandard interpretations of Peano arithmetic as it is described above.

The standard and nonstandard interpretations (and consequently the triple of axioms) imply still one interpretation, which can be called "complementary standard interpretation" necessary for reconciling the well-orderings and openness of the standard one with both absence of well-ordering and cyclic closeness of the nonstandard one.

The complementary standard interpretation can be considered as a well-ordering in the reverse direction once the set of all natural numbers is granted:

Its first element is greater than any element of the standard interpretation. Then, that first element exists necessarily being definable by the ordinal of the set of all natural numbers, usually notated as ω (the set of all natural numbers is due to the nonstandard interpretation of Peano axioms as it is shown above). The function successor is defined as $f(n) = n - 1$. By definition, both standard Peano arithmetics are complementary to each other in the following sense:

If one means a certain "finite n" in any of them, it is uncertainly big (roughly speaking, "infinite") in the other one. Particularly, this imply that any well-ordering of the set of all natural numbers can possess a certain finite bijective image just in one of them (roughly speaking, all infinite, more exactly transfinite natural numbers in the one are finite in the other, and vice versa).

If one consider both complementary standard arithmetics simultaneously, though each of them is well-ordered, they together result into a single one, which cannot be well-ordered for the two well-orderings contradict to each other.

One may visualize this by the relation of ordering " $\leq$ " and " $\geq$ " correspondingly in each of the two complementary directions. Both " $\leq$ " and " $\geq$ " imply the nonstandard interpretation.

One can explain the Gödel incompleteness/ inconsistency of Peano arithmetic (i.e. its standard interpretation) to set theory (i.e. the nonstandard interpretation of Peano arithmetic) by the complement of the former to the latter, namely the complementary Peano arithmetic.

Thus, neither the Gödel incompleteness/ inconsistency nor Yablo's paradox would be possible in the so generalized Peano arithmetic, consisting of two complementary standard interpretations and a nonstandard one. The conjecture that it is both complete and consistent to set theory, and therefore, a reliable ground of mathematics seems to be almost obvious.

Indeed, the Gödel proposition states that its number is the Gödel number of a false statement. However, that number cannot be finite and thus, a natural number (as far as all natural numbers are finite according to the Peano axiom of induction). It belongs to the set of all natural numbers (satisfying the alternative ZFC axiom of infinity). The self-reference of the "Liar" statement in the precise proof of Gödel is mediated by its unambiguous Gödel number, which cannot be a natural number, but belongs to the set of all natural numbers anyway and somehow with no contradiction.

Consequently, it is a finite natural number in the complementary Peano arithmetic, and thus resoluble in relation to it. Of course, there are statements irresoluble to it as well, but they are resoluble to the former Peano arithmetic. So, any Gödel statement is resoluble in the two complimentary arithmetics, which are identical to each other furthermore.

One may demonstrate more precisely as well that the self-reference of the "Liar" statement is impossible because it needs the mediation of a non-finite Gödel number, which turns out to be finite in one of both complementary and identical Peano arithmetics.

The two complimentary Peano arithmetics prevent Yablo's paradox by the same mechanism:

The statement in Yablo's scheme, which is necessarily both true and false, cannot be a natural number as far as all natural numbers are finite just as in the case of Gödel' "Liar" statement. As an isomorphic counterpart, it is a natural number in the complementary Peano arithmetic.

So, the mechanism for the paradox to be prevented in both cases is isomorphic, i.e. mathematically identical. It relays on securing the "dangerous" complement of the natural numbers to the set of all natural numbers correspondingly obeying two axioms contradicting to each other. The two complementary Peano arithmetics are that securing therefore excluding that complement, in fact.

The separable complex Hilbert space utilized by quantum mechanics suggests a model of that generalized Peano arithmetic including both the nonstandard and two complementary standard interpretations after the following considerations:

1. The separable complex Hilbert space is represented as a series of qubits (defined as usual: the normed superposition of two subspaces of it, thus orthogonal to each other), after one has meant that any two (successive) "axes" (of its) are two subspaces as well.

2 Any unit is the class of equivalence of the corresponding qubit (in other words, the class of equivalence of all "values" of it).

3. One means that the separable complex Hilbert equates its two interpretations; both well-ordered "infinitely dimensional" complex vectors and squarely integrable functions decomposable into a series of elementary functions unambiguously mapped into the components of the vectors. Thus, the latter is not orderable because of the commutativity of the elementary functions into the sum of any certain squarely integrable function.

Those two interpretations correspond accordingly to Heisenberg's matrix mechanics and Schrödinger wave mechanics, the unification of which results in the contemporary quantum mechanics and its mathematical formalism of the separable complex Hilbert space.

One can observe as above, that it exemplifies the generalized Peano arithmetic, or in other words, the latter underlies the former. That unification of the well-ordered version and non-ordered "coherent" version is conserved absolutely in the generalized Peano arithmetic and will be utilized further to elucidate the philosophical completeness of the totality and the mathematical completeness of the generalized Peano arithmetic together and in parallel. One can notice that the unification meant above is equivalent to the "theorem" (or "principle") of well-ordering, and thus, to the axiom of choice in set theory. The concept of "external reference frame" will be inferred, and a new generalized invariance of internal reference frames as well.

Once the concept of "Hilbert arithmetic' as the generalized Peano arithmetic has been introduced as above, one can question about its mapping, eventually bijection, onto the "usual" separable complex Hilbert space of quantum mechanics.

One should emphasize that Hilbert arithmetic is a "set-theory arithmetic" building a model of Peano arithmetic in the set-theory mathematics, properly by a special vector space. Thus, the axiom of infinity (ZFC) or its equivalent is granted. Thus, the complement of all natural numbers (finite, according to the axiom of induction) and the set of all natural numbers (infinite, according to the axiom of infinity) should be interpreted in a relevant way for the contradiction to be resolved consistently:

Once the axiom of choice for ZFC is granted, all "wave functions" (or "points" of the separable complex Hilbert space) can be enumerated necessarily "transfinitely", i.e. "after" all natural numbers of Peano arithmetic, therefore suggesting a relevant consistent interpretation of that problematic complement above.

The one-to-one transformation between the two complimentary Peano arithmetic can be represented by the complex bijection consisting of the real part bijection of all natural numbers of Peano arithmetic into all wave functions and the reverse bijection as the imaginary part. For the idempotency, the reverse one-to-one transformation of the two complimentary Peano arithmetics only will exchange the real and imaginary part of that complex bijection.

Here is still one and equivalent representation of the same mapping:

One can order-well the elementary functions (i.e. the "axes" of the complex separable Hilbert space) of any squarely integrable function of that Hilbert space into a vector of the same. That well-ordering is a bijection furthermore for the axiom of choice. Or vice versa: the isomorphism of Heisenberg's matrix mechanics (i.e. the vector interpretation) and

Schrödinger's wave ("undulatory") mechanics (i.e. the function interpretation) implies the axiom of choice implicitly (explicitly, the well-ordering "theorem")

The same equivalence implies the unambiguous transformation of Hilbert arithmetic into that Hilbert space (and thus, vice versa): both are isomorphic to each other. If one defines formally "external state" as the corresponding wave function according to its assigned natural number as "internal state", the same isomorphism implies a generalized invariance to the exchange of external and internal states:

The choice of the terms "external and internal states" is forced by a relevant teleological intention to be generalized the concept of reference frame (e.g. in special and general relativity) to "external reference frame", and a corresponding generalized invariance of external and internal reference frames to be investigated. A well-ordered set as an internal trajectory is equated to an external and even unorderable (coherent) set obtained by the axiom of choice. The equivalence of the well-ordering "theorem" and the axiom of choice underlies that invariance to external and internal frames. Indeed, the axiom of choice acts on any set "outside" generating a well-ordering inside it. Thus, it is able to transform a continuous or even smooth trajectory inside a physical system in a discrete, 'quantum" leap outside it. The "viewpoint" to the system from any *external* reference frame is discrete necessarily. Accordingly, the "viewpoint" to the system from any internal reference frame is continuous (smooth) necessarily.

The fundamental postulate of quantum mechanics (formulated by Niels Bohr), according to which it studies the system of continuous macroscopic apparatus described by the smooth differential laws of classical mechanics and quantum microscopic entities by the readings of the former, implies that invariance at least as to quantum mechanics as well as its mathematical formalism (the separable complex Hilbert space) to describe it relevantly. It can be called "quantum invariance".

However, the conception of generalized reference frames and corresponding invariance extends it to general relativity as a fundamental and set-theory-arithmetical, mathematical approach to the problem of "quantum gravity". Quantum invariance is isomorphic to the invariance to external and internal reference frames implying for the gravity of general relativity to be quantum gravity simultaneously in virtue of the same invariance called whether "quantum invariance" or "quantum gravity invariance".

That invariance is due to the fundamental mathematical axioms or statements and representable well in terms of Hilbert arithmetic as above rather than to any physical laws confirmable experimentally.

Meaning "Dedekind finiteness" (i.e. any set that cannot be mapped by any bijection into its true subset is "Dedekind finite") as a set-theory finiteness, and thus, being relevant to the approach of Hilbert arithmetic, the wave function as generalized "natural numbers" can be interpreted as follows:

The bijective mappings of the set of all natural numbers (granting ZFC) into the class of all natural numbers (granting the Peano axioms) are meant. Thus, the former is actually infinite, and the latter is uncertainly finite, or an equivalent of the set of all finite subsets of the former. Both sets are countable and thus bijections between them exist. Now, one considers the set of all those bijections grouped into classes of equivalence according to the same element of the latter set, which is equivalent, in fact, to a natural number. The same natural number (thus finite) is considered in the complementary Peano arithmetic where it corresponds to just one certain wave function after all of them have been ordered well in virtue of the axiom of choice. As to the former, initial Peano arithmetic, the pair of a finite natural number in the complementary Peano arithmetic and a certain wave function corresponds unambiguously to just one transfinite "natural number" being out of Peano arithmetic consisting only of finite natural numbers. One can identify the elements of the triple, namely a finite natural number in the complementary Peano arithmetic, a transfinite "natural number" out of the initial Peano arithmetic, and a certain wave function.

The same triple corresponds unambiguously to the triple where the third element is replaced by the corresponding element of the nonstandard Peano arithmetic. The equating of the two triples in virtue of the identifying of two of their elements implies the interpretations of the elements of the set of all nonstandard Peano natural numbers as wave functions and the identification of classical information (measured in bits) and quantum information (measured in qubits).

One considers the mappings of the set of all natural numbers into the class of all natural numbers as above, again. They will be identified as all wave functions ordered well as all in-between transfinite "natural numbers" between the ordinal of the set of all natural numbers and all natural numbers themselves. Any wave function of them (thus, any corresponding transfinite "natural number") corresponds to just one normal distribution[5] of natural numbers in virtue of the central limit theorem as far as its conditions are satisfied by the meant mapping.

Explicitly, the construction for a "one-to-one" mapping of the infinite set of all natural numbers into the "finite" class of all natural numbers (according to Peano arithmetic) by mediation of the Dedekind set-theory finiteness is as follows:

The Dedekind finiteness of the class of all natural numbers is conserved in virtue of the absence of any mapping of the set of all natural numbers into the former class. The mapping is fundamentally random. The infinite set is "mapped one-to-one" into a certain finite set, but the latter is different after each case (in general) being fundamentally random.

One can prove that the set of all possible cases of those mappings can be distributed into disjunctive subsets, each of which is determined unambiguously by the two parameters of a certain normal distribution, thus in turn corresponding to a certain wave function as its characteristic function. And vice versa: any wave function corresponds to just one of those disjunctive subsets by the mediation of a certain normal distribution of all natural numbers.

From a mathematical viewpoint (rather than a philosophical one as here), that proof would be the central result:

There exists a bijection of all mappings of the set of all natural numbers into the class of all natural numbers (as all possible normal probability distributions after the fundamentally random choices of the elements of the set of all natural numbers in virtue of the axiom of choice) and all elements of the separable complex Hilbert space (all wave functions).

The proof is not too difficult, but bulky technically. From a philosophical viewpoint, only one point in it is worth to notice:

The bijection, i.e. the availability of both forward and inverse function needs necessarily the two complementary Peano arithmetics. For example, if the one function is proved in the one Peano arithmetic, the other function cannot be proved in it, but only in the complementary

[5] The wave function is the characteristic function of the corresponding probability distribution.

Peano arithmetic (and vice versa). Thus, any of the two arithmetics obeys Gödel's incompleteness, but two ones together, as a whole, does not: the generalized Peano arithmetic (Hilbert arithmetic) is both complete and consistent.

The introduction of Hilbert arithmetic involves furthermore the conception of "bosons", "fermions" and "classical particles" even arithmetically. Indeed, any natural number in both complementary standard interpretations satisfies the definitive condition of "fermion": only two ones (being due to the well-ordering of both arithmetics) can share the same wave function (corresponding to the same state in quantum mechanics). Thus, the Fermi-Dirac statistics is valid for any ensemble of them. Further, any natural number in the non-standard interpretation satisfies the definitive condition of "boson": any number of particles can share the same wave function (being due to the necessary absence of well-ordering and corresponding of the same state of bosons in quantum mechanics). Thus, the Bose – Einstein statistics is valid for any ensemble of them. The elements of the two transfinite complements of each of the two complementary Peano arithmetics satisfy the definite condition of "classical particle": the unit (as the Planck constant to classical physics) is infinitely small "zero" to any element of them. The, any their ensemble satisfies the Maxwell – Boltzmann statistics.

The observation that a physical quantity such as "spin" origins directly and only from arithmetical and set-theoretical considerations referring to the foundations of mathematics admits the generalization that the same property is valid to other quantum quantities, even to classes of them, e.g. all quantum quantities of mechanical motion and thus, those of thermodynamics, etc.

From a philosophical viewpoint, the following conjecture seems to be admissible:

A reduction of quantum mechanics (and even physics) to mathematics is possible (implying a new kind of reductionism). Many physical properties and laws might originate from mathematics directly. A quantum form of Pythagoreanism, resurrecting the ancient one, makes sense to be discussed. The further consideration of that hypothesis will be postponed for a future paper.

Hilbert arithmetic as it is defined rigorously above allows for both classical and quantum information to be defined uniformly. Thus, the relation between them is articulated and becomes obvious:

Classical information is defined in Peano arithmetic, and quantum information is defined in Hilbert arithmetic, which generalizes the former.

For example, classical information is measured in units of "bits", and quantum information is measured in units of qubits. A bit is defined as one elementary choice between two equally probable alternatives. Thus, any choice in the framework of Peano arithmetic would correspond to a certain value of classical information measured in bits.

A qubit is defined as a choice between all elements of an actually infinite set, which is equivalent to its standard definition as the normed superposition of (orthogonal) subspaces of the separable complex Hilbert space, Thus, any choice in the framework of (ZFC) set theory would correspond to a certain value of quantum information measured in qubits, i.e. to a certain wave function.

As for as the concept of Hilbert arithmetic is created to regulate (and particularly, to make consistent to each other) Peano arithmetic and (ZFC) set theory in a single mathematical structure, it relates the two kinds of information: classical and quantum.

Particularly, Hilbert arithmetic subordinates them cyclically as follows:

A "global" structure of classical information implemented in the complementary Peano arithmetics, in which any enumerated bit (particularly, a cell in a Turing machine) is the choice between the natural numbers of the same name (number) in both complementary arithmetics. The local structure of any of those bit is the complementary pair of the qubits in the two dual and complementary separable complex Hilbert spaces.

The cyclicality consists in the fact that the global structure of two complementary Peano arithmetics represents a single pair of two complementary qubits. In other words, the global structure as if it is contained in each given exemplification of local structure (within a qubit), but differently represented in each of them as a value of its.

For example, one can interpreted the Schrödinger equation, fundamental for quantum mechanics, as relating the global structure of Hilbert arithmetic (the left part of the equation) and the local structure of it (the right part) in a certain way valid for our universe. Thus, all possible universes can be described by the class of all possible ways for the global and local structure to be equated to each other (or roughly speaking, as the class of all possible "Schrödinger equations").

The operation "addition" and "multiplication" are defined standardly in both Peano arithmetics. However, this is due to the well-ordering, which is not available in the nonstandard interpretation. So, the following question arises: how should those operations be defined if the fundamental operation of successor is determined as "= successor"?

One can show, that they are equivalent correspondingly to the Boolean operations "disjunction" and "conjunction", and the unary operation "negation" can be defined as well. The negation of any nonstandard Peano "number" is its unambiguous "mirror" counterpart at the "other end". More precisely, that "mirror' counterpart can be defined by the same natural number in the complementary standard Peano arithmetic as the counterpart of the last one in the nonstandard Peano arithmetic consisting of the elements of both complementary standard Peano arithmetics.

A seeming contradiction between Hilbert arithmetic and intuitionist arithmetic appears:

The "excluded middle" is provable (if one defines "negation' as here) in Hilbert arithmetic including to actual infinity. Furthermore, it is complete and consistent to set theory. On the contrary, intuitionist arithmetic postulates that the "excluded third" is not valid as to infinity (interpretable in (ZFC) set theory as "actual infinity") being complete and consistent to set theory as the former.

The contradiction is seeming only since "infinity" defined in an intuitionist way admits to be identified in both arithmetic and set theory. What is the complementary "finite" Peano arithmetic as a sub-arithmetic in Hilbert arithmetic corresponds to the implicit area of the middle between the two forms of infinity admissible in intuitionist mathematics: arithmetical and set theoretical. That implicit domain of the "infinite middle" is explicit in Hilbert arithmetic as the complementary Peano arithmetic.

So, the algebraic structure of the nonstandard Peano arithmetic is isomorphic to Boolean algebra and thus can be interpreted as propositional logic, which in turn is considered as the "zero-order logic". Thus the standard Peano arithmetic as a first-order logic (because the propositional zero-order logic is identified here as the nonstandard interpretation of Peano arithmetic) implies both complementary counterpart of it and Hilbert arithmetic, further. That relation of Peano arithmetic and Hilbert arithmetic can be rather useful practically as it will be illustrated a little bellow by means of a thoroughly arithmetical proof of Fermat's last theorem.

The relation of both Gödel papers about logical completeness (1930) versus arithmetical incompleteness (1931) to (ZFC) set theory can be explained easily in the framework of Hilbert arithmetic. As to the arithmetical inconsistency of eventual completeness to set theory, its validity is restricted in general (only to the half levels of a hierarchy, e.g. odd ones), and particularly, invalid to Hilbert arithmetic (referring to the even levels in the same scheme).

One can demonstrate very easily that the generalized Peano arithmetic (also being called Hilbert arithmetic here) is equivalent to the separable complex Hilbert space of quantum mechanics as follows:

If one considers both transfinite complements of both complementary standard Peano arithmeticc as separately as unified as a single nonstandard Peano arithmetic, they constitute as Hilbert arithmetic, too, complementary in turn to the initial Hilbert arithmetic generated it, as a separable complex Hilbert space identical to that utilized of quantum mechanics. The same observation implies that *Hilbert arithmetic is isomorphic to the separable complex Hilbert space*, or as a metaphorical philosophical conclusion:

Hilbert arithmetic is the real arithmetic of nature: particularly, all physical entities, processes or quantities can be understood and reduced to arithmetical if the natural arithmetic is generalized as Hilbert one. A new form of both scientific and philosophical, furthermore, very heuristic reductionism appears: the reductionism form quantum mechanics (and thus physics thoroughly) to arithmetic.

Its fruitfulness can be exemplified by the following consideration. The separable complex Hilbert space (or at least, its "bosonic subspace") turns out to be equivalent to the nonstandard interpretation of Peano arithmetic and thus, to propositional logic. As the latter is both complete and consistent to set theory and therefore, to all mathematics in the sense of Gödel's "completeness paper" (1930), the separable complex Hilbert space is both complete and consistent in the same sense. As a direct corollary, one can deduce the completeness of quantum mechanics (Kochen – Specker's theorem) as equivalent to the completeness of logic to set theory (Gödel 1930) after utilizing Hilbert arithmetic as equivalent to the separable complex Hilbert space.

Hilbert arithmetic as well as the concept of information (by which mathematics and physics can be merged) can be inferred philosophically from the totality, a unique entity possessing an unusual definitive property, "to be all", and thus, to contain its "externality" "within" itself.

A course of thought, invented by Kant and applied by him to philosophy, can be utilized analogically to the foundations of both mathematics and physics where they as merge into each other as merge into philosophy. This is Kant's transcendentalism, which can specified particularly as "physical and mathematical transcendentalism". In fact, it was reduced to an elementary scheme, similar to a logical one involving the "contradiction" as a fundamental base, however forbidden by classical logic by the restriction known both as the "law of contradiction" and as the "law of non-contradiction". This is Hegel's famous idea of "dialectic logic", a philosophical logic applicable to the philosophical "totality" (unlike classical logic). The impossibility to be utilized consistently both classical logic and "dialectic logic" caused furthermore the rejection of the philosophical "totality" as implying inconsistency to science and empirical experience.

It resulted in the "schism" of philosophy and science, not less fundamental that one between religion and science. As the latter is due to the inconsistency of the fundamental religious concept of "God" to science as the former is due to the analogical inconsistency of the fundamental philosophical concept of the "totality' to science (visible by the direct contradiction of classical logic and "dialectic logic").

Anyway, the "totality" will be used here in relation to science and its physical and mathematical foundations rather than to philosophy, and therefore that the initial scheme of "dialectic logic" can be modified into the concept of information, thus, made secure and consistent to classical logic.

One can notice that Hegel's "triad" (thesis – antithesis – synthesis) is isomorphic to the concept of an elementary choice (i.e. to a bit of information) between two equally probable alternatives, *however "upside down"* (i.e. in the opposite direction). Indeed, the concept of information as well as the underlying one of choice (such as that in the axiom of *choice*) are usual scientific ideas consistent to classical logic.

So, one need interpret those "opposite directions" as to the totality in both cases to elucidate why the one is secure, but the other, opposite one generates contradictions at least to classical logic:

The concepts of choice and information *remain within the totality*[6], but the idea of 'dialectic logic" (being directed oppositely) *goes out of the totality* (into its 'antithesis') therefore generating a contradiction to the definition of the totality[7]. Two strategies to the totality (for it to be consistent) appear then:

(1) The definitive property of the totality to be postulate therefore prohibiting any way out from it.

(2) Physical and mathematical transcendentalism to deduce a relevant kind of transcendental invariance, according to which the totality (by itself or by its definition) generates an invariance of the domain within it ant that out of it. Then, the totality would be any entity, to which that "transcendental invariance" takes place.

The former is able to find certain ways to be avoided the paradoxes in the foundations of mathematics. Its obvious disadvantage is that it is *ad hoc* and accordingly, it cannot be proved in turn or justified otherwise than "empirically".

Physical and mathematical transcendental invariance can be visualized loosely and more freely by the CPT invariance in quantum mechanics if the charge "C" is interpreted as referring (or "name") to the one of the two domains: either "within the totality"; or "out of the totality. Furthermore, the space "P" is interpreted correspondingly as the space within it or alternatively, the space out of it, and the time "T", as either the normal, "forward" course of time or the

[6] The usual "*ad hoc*" approach for set theory to be consistent demonstrates the same idea however postulated rather than proved as here on the base of formal transcendentalism Any set is necessary to be an implicit subset of some other set: thus, any set is necessary in the framework of the set of all sets, which is postulated to be that set, to which the class of all sets which do not belong to themselves does not make sense and there is not any relation between the former and the latter. The same property only postulated in set theory ad hoc to be avoided the irresolvable contradiction is well-founded in the formal definition of the totality after transcendentalism,

[7] This is visible formally e.g. in Russell's paradox, where the 'totality' is exemplified by the "set of all sets". Then, its "antithesis" of the "set of all sets which belong to themselves", namely the "set of sets which do not belong to themselves" is what generates the irresolvable contradiction of the paradox.

opposite, “backward” direction of it. Then, the CPT invariance itself can be interpreted as a physical transcendental invariance: if one changes simultaneously the name of the domain, “C”, the space of the domain, “P”, and the direction of time, “T”, to the opposite ones, noting will change, in fact.

One can demonstrates that Hilbert arithmetic possesses the same kind of invariance, which can be called initially “mathematical transcendental invariance” (in order to be able to be proved its identity with the “physical” one into an intended “physical and mathematical” one):

Indeed, if one changes the standard Peano arithmetic into its complementary counterpart, the space of the finite domain of natural numbers into that of the “infinite” (or “transfinite”) one, and the direction of the function successor from “$+1$” to “-1”, nothing will change, analogically. The nonstandard interpretation of Peano arithmetic represent both complementary Peano arithmetics therefore cancelling necessarily the well-ordering of each of them and identifying both with the usual standard, or “naïve” Peano arithmetic under the additional condition of whether any relevant well-ordering is available or not.

So, three particular symmetries or invariances, analogical and even isomorphic to those “C”, “P”, and “T” in the physical case above can be observed again. Even more, the equivalence of Hilbert arithmetic to the separable complex Hilbert space (also explicated above) implies that the three symmetries or invariances are the same rather than only similar or isomorphic. Thus, the two separate kinds, accordingly “physical transcendental invariance” and mathematical transcendental invariance” can be unified rigorously into “physical and mathematical transcendental invariance”, or merely “transcendental invariance”, because mathematical, physical and philosophical transcendental invariance are unified into it rather than only physical and mathematical ones. In other words, as physical transcendental invariance as mathematical transcendental invariance as well as philosophical transcendental invariance are only different interpretations (or exemplifications) of that most fundamental transcendental invariance. And vice versa, the existence of that most fundamental transcendental invariance implies the unification of the three ones.

Particularly, the paradoxes in the foundations of mathematics can be explained by partial symmetries and invariances, which are not valid in general and thus they are able to produce contradictions. The postulated prohibition of any way out of the totality, as in (1), excludes the antinomies practically, however without any proof that no antinomies are possible in principle. On the contrary, transcendental invariance as in (2) guarantees the consistent completeness of mathematics as necessary.

III. AN ILLUSTRATION BY FERMAT’S LAST THEOREM AND ITS PROOFS

Fermat’s last theorem and its proof supply a wonderful visualization about the relations and accessibility of the actual infinity of set theory from the inductive finiteness of arithmetic.

Fermat’s last theorem is proved by Andrew Wiles (1995) as a corollary from the modularity theorem (known as the “Tanyiama – Shimura – Weil conjecture” before that) and therefore involving at least set theory (or even a relevant generalization of it). Some authors admit that inaccessible cardinals are useful implicitly in the proof of the modularity theorem. Other authors refuse this.

If one does not impose the condition for the inaccessible cardinals to be uncountable, therefore admitting enumerable inaccessible cardinals commensurable with the cardinal of the set of all natural numbers, their necessity availability in any proof of Fermat's last theorem, which involves furthermore (at least) set theory as that of Andrew Wiles can be demonstrated rather simply:

If one proves that Fermat's last theorem is a Gödel irresolvable statement, this would be equivalent to the statement in the previous paragraph.

Indeed, let Wiles's proof be granted as correct and thus, Fermat's last theorem proved as a corollary from the modularity theorem. As far as the modularity theorem (unlike Fermat's last theorem) needs set theory to be formulated as well as the Peano arithmetic itself, the Gödel irresolvable statements make sense to that proof.

Further, one can prove that Fermat's last theorem is a Gödel irresolvable statement by the mediation of Yablo's paradox demonstrating that any proof of Fermat's last theorem, which has involved set theory as necessary for the proof implies for it to be an exemplification of Yablo's scheme and thus, a Gödel irresolvable statement, i.e. unprovable in the framework of (ZFC) set theory & (Peano) arithmetic.

The same observation would not reject the following two options: (1) Fermat's last theorem is provable in (ZFC) set theory & Hilbert arithmetic (as Hilbert arithmetic is a crucial generalization of Peano arithmetic) (2) Fermat's last theorem is provable in Hilbert arithmetic, which, however, is equivalent to Peano arithmetic as a first order logic on the following reason:

The nonstandard interpretation of Peano arithmetic can be identified as propositional logic for their isomorphism. Then, the usual standard interpretation of Peano arithmetic & propositional logic as the nonstandard interpretation of Peano arithmetic are equivalent to Peano arithmetic as a first-order logic, on the one hand, and to the complementary standard interpretation of Peano arithmetic, and thus to Hlibert arithmetic (as the triple of the two complimentary Peano arithmetic and the nonstandard interpretation of Peano arithmetic), on the other hand.

As to Wiles's proof, it should generalize (ZFC) set theory or (Peano) arithmetic. The investigation of what exactly is generalized to be reached a correct proof of the modularity theorem (as far as it in turn implies Fermat's last theorem) is far out of the framework and intention of the present work. Its purpose will be restricted to the explicit demonstration of its proof in Hilbert arithmetic.

Statement: Fermat's last theorem is a Gödel irresolvable statement (under conditions, under which the later makes sense).

Fermat's last theorem is proved as a corollary from the modularity theorem (Wiles 1995). Thus, (a part of) set theory is necessary for the proof (besides Peano arithmetic for the Fermat theorem itself), and thus, the statement (A) that "Fermat's last theorem is a Gödel irresolvable statement" make sense (it is either false or true).

One can prove that "A" is true as follows. One notates by "$FLT(> n)$" the statement that Fermat's last theorem is true for any (each) natural number greater than "n", and "$YP(n)$" means "$FLT(> n)$". Then, "$YP(n)$" is a true statement "$\forall n > 1$" (and "n" is a natural number) in virtue of Wiles's proof of Fermat's last theorem.

Furthermore, the series by "n" of "$YP(n), \forall n > 1$" constitutes the scheme of Yablo's paradox, and thus, implies it: Wiles's proof of Fermat's last theorem implies Yablo's paradox, and the latter in turn implies for Fermat's last theorem to be a Gödel irresolvable statement as it has been demonstrated above.

Statement is proved.

A corollary from the Statement: if Wiles's proof is correct, it generalizes necessarily Peano arithmetic or (ZFC) set theory (in order to be able to avoid Yablo's paradox).

There is another way out for Yablo's paradox to be sidestepped as to Fermat's last theorem: involving set theory as a necessary condition of the proof to be prevented, for example, by an only arithmetical proof of Fermat's last theorem (Penchev 2020a).

The cited only arithmetical proof will be reinterpreted here in terms of Hilbert arithmetic. Hilbert arithmetic generalizes Peano arithmetic, and furthermore, it is consistent to set theory, however, without involving set theory in Hilbert arithmetic to be necessary.

That reinterpretation need explain only the meaning and sense of the proof in terms of the *complimentary standard Peano arithmetic* after the only arithmetical proof of Fermat's last theorem has utilized already the Peano arithmetic as a first-order logic. That sense consists in the explanation of how the complimentary standard Peano arithmetic (and thus, Hilbert arithmetic by means of it) is able to avoid Yablo's paradox for Fermat's last theorem (i.e. how the latter not to be a Gödel irresolvable statement):

One can reinterpret the "semi-validity" of Gödel's incompleteness (as it has been introduced and explained above) in terms of Hilbert arithmetic. The two complementary standard Peano arithmetics can be considered both as identical to each other (in virtue of "transcendental invariance" as above) as complimentary. Gödel's incompleteness is valid only in the latter case, and thus, only in the half of cases: just "semi-validity". If one uses the "CPT" metaphor or visualization of that transcendental invariance, the latter case of Gödel's incompleteness is featured by the partial, only "CP" invariance, but the "T" component is not available: the direction of the function successor is not changed to the opposite one, "-1", but remains the same "$+1$", therefore constituting or transforming the complementary standard Peano arithmetic into the transfinite complementation to the first, "forward" standard Peano arithmetic. The Gödel numbers of all Gödel irresolvable statements belong to it, and thus, they can exist only under the condition of that "partial invariance".

Next, if one suggests a proof of Fermat's last theorem which would be invariant to the change of the direction of the function successor (i.e. between "$+1$" and "-1"), it would be valid as after the "naïve" identification of both complimentary Peano arithmetics to the usual standard interpretation of Peano arithmetic as in the case of Gödel's incompleteness meaning a transfinite forward continuation of the "naïve" Peano arithmetic.

In other words, that proof can be continued forward in the domain of transfinite induction in virtue of its validity in the domain of the "finite" induction meant by the axiom of induction, or in virtue of the invariance of the proof to the change of the direction of the function successor.

Therefore, the utilized "modified Fermat descent" for the only arithmetical (or "naïve") proof of Fermat's last theorem satisfies that condition to be invariant to the change of the direction of the function successor. This is due to involving *modus tollens* being equivalently valid in both opposite directions of the function successor.

Thus, the "naïve", only arithmetical proof of Fermat's last theorem implies the proof in Hilbert arithmetic, and further, the option of Wiles's proof to be valid after a necessary relevant generalization of the pair of Peano arithmetic and (ZFC) set theory (once the pair is necessary for the modularity theorem to be formulated).

If one considers Peano arithmetic as a first-order logic from the viewpoint of Hilbert arithmetic, the former is incomplete “to itself” for it implies Hilbert arithmetic and thus, the complimentary counterpart. One may say that Peano arithmetic as a first order logic is not incomplete to set theory, but to itself. Hilbert arithmetic, therefore, is that generalization of Peano arithmetic which is not incomplete to itself. Furthermore, Hilbert arithmetic possesses the extraordinary property Hilbert arithmetic as a first-order logic and Hilbert arithmetic “by itself” to be isomorphic to each other because Hilbert arithmetic contains a structure, namely the nonstandard interpretation of Peano arithmetic, which is isomorphic to propositional logic for both share the structure of Boolean algebra as isomorphic to each other.

Furthermore, Hilbert arithmetic is not only complete, but also consistent to set theory. This is due to the fact that Gödel’s either incompleteness or inconsistency of Peano arithmetic to set theory is suspended as to Hilbert arithmetic. Thus, its completeness does not implies inconsistency for the validity of Gödel’s incompleteness (and thus, inconsistency) is restricted to its semi-validity to Peano arithmetic (and thus, invalidity to Hilbert arithmetic).

Particularly, Peano arithmetic as a first order logic from the viewpoint of Hilbert arithmetic is only incomplete, but not inconsistent to set theory once it has been completed. The only arithmetical proof of Fermat’s last theorem is able to demonstrate that, in virtue of which it is valid in both Peano arithmetic and Hilbert arithmetic, and even the former validity implies the latter validity for the utilization of the (modified) Fermat descent being symmetrical to both directions of function successor by means of involving *modus tollens*.

However, Hilbert arithmetic by means of the nonstandard interpretation Peano arithmetic even implies set theory rather than only being consistent to it. Thus, “transcendental doubling” or transcendental invariance, and thus the concept of information itself are implied by Hilbert arithmetic: it is the relevant way for Hilbert’s program, i.e. the total and complete arithmetization of mathematics, however after one has managed to interpret Peano arithmetic as equivalent to Hilbert arithmetic in virtue of the nonstandard interpretation of the former. The crucial philosophical obstacle for Hilbert’s program is the necessity to be radicalized to a form of neo-Pythagoreanism, by which the total world is to be considered as mathematical, particularly implying the philosophical concept of totality and transcendentalism relevant to it.

The identification of Hilbert arithmetic and the separable complex Hilbert space (as above) implies the following:

Quantum information and classical information are complimentary to each other (for example, as the naïve ideas of finiteness and infinity are). Nonetheless, their change in time (respectively, their corresponding quantities of time derivatives) can be equated to each other. This due to the identity of well-ordering or to that of the function successor even where it is defined oppositely. That identity underlies the Schrödinger equation: its essence is to equate classical information (in its “left side”) and quantum information (in its “right side”).

From the equivalent viewpoint of the foundations of mathematics, the same essence is the equivalence of well-ordering (i.e. the principle or “theorem” of well-ordering applied to a continuous or smooth trajectory *within* a mechanical system, respectively for the classical description of the macroscopic “apparatus” in quantum mechanics) and the “free will” or fundamentally random choice (i.e. the axiom of choice applied to the discrete *out of* a mechanical system, respectively for the quantum description of the microscopic entity of quantum mechanics).

The same equivalence or equating (being informational in essence and thus referring to the foundations of mathematics by the foundations of physics) is represented in general relativity as a certain curving therefore equating two different (in general) distances: the straight distance without gravity (corresponding to the mathematical "choice" between *discrete* alternatives) and the curved distance with gravity (corresponding to the mathematical 'well-ordering" of a *continuous* or smooth trajectory).

In other words, general relativity describes the same as quantum mechanics only in a different, but equivalent way. The reason not to be seen that physical equivalence or mathematical isomorphism is the initial "blinding" of our age: the fundamental difference between "model" and "reality" (particularly, between mathematical "model" and physical "reality") therefore excluding their unifying informational "reference frame" within which, properly and particularly, the unification of general relativity and quantum information can be noticed and described. It implies a few much more radical and fundamental unifications such as that of mathematics and physics or that of model and reality at all.

IV STILL A FEW ILLUSTRATIONS BY POSSIBLE PROOFS OF IMPORTANT THEOREMS

The thesis, which is intended to be illustrated, is the following:

The consistent completeness of the foundations of mathematics (and thus, that of mathematics at all, furthermore unified with quantum mechanics, and by with, with physics at all) is the core of many great puzzles and problems in the contemporary mathematics. Its resolution allows for them to be resolved also.

Besides Fermat's last theorem and the only arithmetical proof of, ones of the great enigmas of mathematics in Modernity, discussed already in the previous section, still a few ones will be unraveled in a link to "Hilbert arithmetic" and that consistent completeness implied.

Those are: four color theorem proved "humanly" (i.e. rather than an enumerating a huge number of cases, and thus, accomplishable actually only by computers: the humans can check only their software programs to be correct); a generalization of the four-color theorem based on its "human" proof, namely, the "four letter theorem"; Poincare's conjecture (proved by G. Perelman), its physical interpretation and purely arithmetical proof; a class of examples demonstrating that "$P \neq NP$" in the "P vs NP" problem.

The beginning is the four-color theorem: it states that any two-dimensional map consisting of many areas can be colored by means of only four different colors so that no neighboring (i.e. sharing a common boundary consisting of more than one point) have the same color.

Let one admit that there exists a color defect in the map so that some two neighboring areas are colored equally (i.e. by the same color). That defect implies that the one one-dimensional projection or the other one of the two-dimensional map contains a one-dimensional defect (possibly both as well).

That one-dimensional defect is defined analogically: two neighboring one-dimensional areas (i.e. sharing only a single point as their boundary) are colored equally. *Modus tollens* implies that if no one-dimensional defect exists in any of both projections, no two-dimensional defect exists in the map. If no defect exists in any pair of two one-dimensional projections, no defects exist in the map at all. The obvious sufficient condition for no one-dimensional defect in any projection are two colors for it.

Two different projections of the same direction (e.g. either horizontally or vertically) cannot share any point. So two colors are sufficient (and necessary) for no one-dimensional defect in each of them. Two different projections of different directions (i.e. both horizontally and vertically) share necessarily a common point because any vertical projection crosses any horizontal projection in just one point. Though, all horizontal projections, on the one hand, and all vertical projections, on the other hand, need only two colors sufficient to be colored without any defect, both groups of projections need totally four colors for no-defect coloring because projections belonging to the two different group can share a common point (for any pair consisting of a vertical and a horizontal projection).

In other words, all vertical projections need two colors, but all horizontal projections need two different colors (to be avoided any defect in the case of crossing), thus, four colors are sufficient totally. Then, no two-dimensional defect can exist in the map if no one-dimensional project exists in any projection both vertical and horizontal. Four colors are sufficient for the latter, and thus, for the former.

The four-color theorem is proved "humanly".

Two notices to the proof are necessary furthermore:

1. The term "projection" used above might be misleading. It does not mean that all the map to be projected on a single straight line. It means that all two-dimensional map is decomposed into two groups (or by two alternative ways) of one dimensional "sub-maps" parallel to each other and thus not sharing any common point. The one group is defined to consist of all the "vertical projections", and the other one, of all the "horizontal projections'. The term "projection" means an element of those two groups (classes).

2. One has to pay a special attention to the case of no two-dimensional defect though two areas sharing just one single point are colored equally. Then, the absence of two-dimensional defect would imply the availability of one-dimensional defect for any projections defined as follows:

Two neighboring one-dimensional areas of that kind of (”defective”) projections share that one single point at issue. The one neighboring one-dimensional area belongs to the one of the two-dimensional areas at issue, and the other one-dimensional one, accordingly, to the other two-dimensional one.

That consideration is to be interpreted as follows. The exhibited proof refers to a stronger case, which includes the usual weaker formulation of the four-color theorem admitting the case where two neighboring areas are colored equally if they share only a single point as their boundary. So, the proof is valid for the weaker case.

In fact, that weaker case makes sense only under a topological interpretation as what the four-color theorem is considered usually. However, the theorem will be generalized to the "four-letter theorem" therefore not being interpreted topologically, and that weaker case does not make sense after the new generalized interpretation (being properly in the framework of the foundation of mathematics: arithmetic & set theory):

How many letters does the "alphabet of nature" need?

The "alphabet of nature" is meant as a finite tuple of symbols allowing for any existing or non-existing entity to be notated unambiguously by their combination. If those entities are the areas of a map, the four-letter theorem is the solution after reducing. Thus, the four-letter theorem refers to the corresponding conjecture as a generalization, stating that four letters (as four colors for a map) are sufficient for that alphabet.

The proof of the generalization is based on the exhibited "human" proof of the four-color theorem after corresponding reinterpretation (first of all, the reinterpretation in terms of Hilbert arithmetic of those two groups of either horizontal or vertical projections):

They are interpreted correspondingly as the two complimentary Peano arithmetics as follows:

Each of them needs the tuple (alphabet) of at least two symbols (as the notation system is positional or not) for any natural to be notated unambiguously. The two alphabets are necessarily different as the well-orderings of the two arithmetics are different (properly, directed oppositely). If one consider the same element as belonging to each of them, it has to be notated differently in each of them for their well-orderings are inconsistent to each other (e.g. "$(A < B)\&(C < D)$", where "B" & "C" (accordingly, "A" & "D") are the different notations in the two arithmetics. If the notations were not different, a contradiction would appear.

So, four letters are sufficient for both alphabets totally, or in other words, for the total alphabet. Further, the two complimentary arithmetics are interpreted equivalently as subarithmetics of Hilbert arithmetic representing it exhaustively. Since Hilbert arithmetic is equivalent to the separable complex Hilbert space (as this is demonstrated above), the tuple of four symbols is sufficient to designate unambiguously any wave function (i.e. any element of the separable complex Hilbert space), and further, any entity for a wave function corresponds to any entity: the four-letter theorem is proved.

Poincaré's conjecture can suggest another visualization and exemplification of the idea that the consistent completeness of mathematics reflects on the deepest and most difficult problems of the contemporary mathematics: indeed, it is the only one from the CMI Millennium Prize problems, the solution of which is recognized officially. Gregory Perelman's proof will not be used here. It corresponds the present one by the utilization of the quantity of information. However, the links are too hidden and sophisticated mathematically so that their revelation remains far out of the scope and objectivity of the present paper.

The proof here will be based on Hilbert arithmetic, however its identification as the separable complex Hilbert space will be omitted intentionally. Instead of that, Poincare's conjecture will be proved by two independent ways (properly, granted as independent here): (1) by Hilbert arithmetic; and (2) by the separable complex Hilbert space (interpreted as equivalent to Minkowski space, and thus by it, to our usual three-dimensional Euclidean space of empirical experience).

Consequently the identity of Poincaré's conjecture proved by those two ways utilized as independent of each other will supply still one, i.e. independent proof of identity of Hilbert arithmetic and the separable complex Hilbert space of quantum mechanics:

The proof in the way (1) of Hilbert arithmetic is the following:

Poincaré's conjecture states the homeomorphic transformation of an infinite three-dimensional vector structure, namely Euclidean space, and a finite four-dimensional vector structure such as a unit 3-sphere. So, the problem can be generalized philosophically and mathematically that "infinity" is equivalent to a new dimension, at least topologically. If one proves that last statement, the proof of Poincaré's conjecture would be a direct corollary from it as to the particular case at issue.

That generalization can be rearticulated as follows. There exists a homeomorphism able to transform a discrete topology corresponding to a new dimension (the forth one in the case of a unit 3-sphere) into a continuous "infinite" topology corresponding to the absence of any new dimension (i.e. the conservation of the three "infinite" dimensions of Euclidean space in the case in question). At first glance, that homeomorphism seems to be absurd, a "mistake in definition" for the direct contradiction of the continuous and the discrete, in virtue of which any homeomorphism between them is ostensibly impossible.

In fact, the discrete and continuous ate not inconsistent as contradictory to each other in topology: the discrete can be consider as a particular case of the continuous after adding a closed topology to an open one therefore transforming it into "clopen" and thus, into a discrete one. In other words, the extraordinary consistency of continuity and discreteness in topology is due to the not less extraordinary, but well-known consistency of open and closed topology, even available simultaneously in discrete topology. Thus, a homeomorphism of the open continuous topology into both open and closed discrete topology turns out to be admissibly possible under a certain condition: infinity, which is topologically "open" to be substituted by finiteness, which is topologically "closed" by means of a new additional dimension, therefore by implicit reference to discrete topology (implied by the discontinuity of the new dimension).

Even more, the topological consideration as to that homeomorphism can be avoided at all by its definition only in set theory & arithmetic by the mediation of Hilbert arithmetic (fundamentally involved to make consistent the actual infinity of set theory and the finiteness of arithmetic, to each other).

Indeed, Poincaré's conjecture defined properly topologically can be generalized in terms of set theory, if the homeomorphism be substituted by a corresponding identity (i.e. an identical transformation) as far as any identity is a homeomorphism (the reverse statement is false obviously). So, one intends to reveal the deep origin of Poincaré's conjecture as set-theoretical rather than only topological. It exemplifies and illustrates that essence of a special kind of identity:

That extraordinary identity will be built by means of the identity of the two complimentary standard interpretations of Peano arithmetic, both embedded in Hilbert arithmetic.

Indeed, any two or more interpretations are identical to each other by definition for sharing a certain underlying mathematical structure. As to Hilbert arithmetic, that identity of standard and nonstandard interpretation equates a single element of the nonstandard interpretation to two complementary elements of the standard interpretation, a property inherited by Hilbert space, particularly, by the separable complex Hilbert space of quantum mechanics.

Thus, that identity of the standard and nonstandard interpretation defines and can be considered in relation to the concept and quantity of information, and even, to its unit: a bit of information.

Further, that set-theoretical identity implies Poincaré's conjecture in virtue of the identity of the identity itself. However, this identity cannot bed defined only in the framework of set theory for it needs Peano arithmetic (at least implicitly) to be able to be distinguished the standard and nonstandard interpretation from each other. That identity can be defined not worse as the identity of the same element as being enumerated (therefore involving Peano arithmetic) and as not being enumerated (therefore no needing Peano arithmetic). In other words, that identity states that the counting of entities is only "external" or accidental to the essence of each of them. That property is postulated as underlying Hilbert arithmetic and seems to be implied or even maybe equivalent to the equivalence of the axiom of choice and the well-ordering "theorem": both choice and counting do not change the essence and identity of what has been chosen or enumerated.

In fact, the above continuous logical pathway from Poincaré's conjecture to the identity of both enumerated and non-enumerated, on the one hand, and both chosen and unchosen, on the other hand, is the searched only arithmetical and set-theoretical proof of it.

Nonetheless, the "only arithmetical proof" can be visualized once again directly and independently in properly topological terms:

There exists an obvious homeomorphism of Euclidean space to any open topological subspace homeomorphic to the space of the "unfolding" of the unit 3-sphere as well as another, second and absolutely independent homeomorphism of the same Euclidean space to the complement of the "unfolding" to the same unit 3-sphere. So, if there exists a homeomorphism able to unify those two homeomorphisms (discrete to each other), it would suggests s proof of Poincaré's conjecture: well, the identity of the nonstandard interpretation of Peano arithmetic into the nonstandard one(s) supplies the necessary homeomorphism, and thus the proof (called here "only arithmetical" as restricted rigorously to the framework of Hilbert arithmetic).

The only arithmetical proof is referred immediately to "mathematical and physical transcendentalism" as follows. One can defined "transcendental identity" as a property definitive to the totality in both philosophical and mathematical meaning:

The totality (by itself) generates an equivalent doubling of both "externality' and "internality" of it within it. Hilbert arithmetic (as well as quantum mechanics by means of the separable complex Hilbert space equivalent to the former) embodies that definitive property of the totality as to mathematics and "mathematical transcendentalism" (thus, to physics and "physical transcendentalism' as well). Then, Poincaré's conjecture is still one immediate corollary form that "transcendental identity" by the mediation of Hilbert arithmetic.

And now, the proof in the way (2) by the separable complex Hilbert space (interpreted as equivalent to Minkowski space, and thus by it, to our usual three-dimensional space of empirical experience):

The separable complex Hilbert space is homeomorphic to Minkowski space under the condition that the choice meant in the axiom of choice is a homeomorphism.

Indeed, each of the two dual Hilbert spaces is homeomorphic either to the real domain or to the imaginary domain of Minkowski space. Each "qubit ball" of Hilbert space is homeomorphic to a ball in Minkowski space at a certain time.

However, the balls of Hilbert space are discrete to each other, unlike those of Minkowski space, which are continuous to each other. Furthermore, the two dual Hilbert spaces are also discrete to each other again unlike the two domains of Minkowski space smoothly (and thus, continuously) touching each other and therefore sharing the light cone.

That double "however" can be overcome just under the single condition for choice to be a homeomorphism. The condition will be proved bellow without referring to the eventual homeomorphism of the separable complex Hilbert space of quantum mechanics and Minkowski space (of special relativity). Anyway that homeomorphism will be granted here only for the simplification of the exhibition.

In turn, Minkowski space is homeomorphic to a unit 3-sphere for any three-dimensional (i.e. usual) ball belonging to the former is homeomorphic to any three-dimensional ball belonging to the latter. The (four-dimensional) 3-sphere consists of two equal "glued unfoldings", the one of which corresponds to the one domain of Minkowski space, and the other one, to the other Minkowski one (whether the imaginary or the real one).

Each of the two Minkowski domains is well-ordered by the pseudo-axis interpreted as the quantity of time in special relativity; however, the two corresponding well-orderings are opposite (and thus inconsistent) to each other. That inconsistency does not reflect to Minkowski space for the two domains are disjunctive to each other sharing the light cone as identical in both.

If one "mixes" the two domains (therefore removing the two opposite well-orderings inconsistent to each other and thus, the fourth pseudo-axis of time) the result will be a usual, three-dimensional Euclidean space. The result "after mixing" is expected as far as Minkowski space should be an equivalent model of Euclidean space as to special relativity.

However, one has to prove that the mixing is a homeomorphism in the rigorous mathematical meaning. Furthermore, it is equivalent to the proof that the choice meant in the axiom of choice is a homeomorphism (already granted in the conjecture that the separable complex Hilbert space is topologically equivalent to Minkowski space).

Indeed, the two domains of Minkowski space are well-ordered as to its homeomorphism to the unit 3-sphere, but they are "mixed coherently" in Euclidean space. The well-ordering "theorem" is applied in the former case, and the axiom of choice as to the latter case. As far as the well-ordering "theorem" and the axiom of choice are equivalent to each other, and both act onto the same Minkowski space, the two results, correspondingly the unit 3-sphere and Euclidean sphere are identical to each other, and thus, homeomorphic.

Poincaré's conjecture is proved.

The homeomorphism of the separable complex Hilbert space and Minkowski space is proved independently. Being due to both, the identity of Hilbert arithmetic and the separable complex Hilbert space is proved once again and independently.

A comment to the utilization of "coherent mixing" is possible. The equivalence of the well-ordered state to the coherent state, which is implied by the equivalence of the well-ordering "theorem" and the axiom of choice, is well-known after the problem of the epistemological relevance of the fundamentally random quantum measurement. Indeed, the quantum measurement equates epistemologically a fundamentally unorderable coherent state to a well ordered series of single quantum measurements: if our quantum cognition is relevant, they must be identical.

The publication (Penchev 2020) demonstrates a wide class of examples that "$P \neq NP$" in *"P vs NP" problem* (one of the seven CMI millennium prize problems). That class can be represented by the well-known "cat of Schrödinger": no Turing machine might guess for any finite time (and thus, for any polynomial time) whether the cat would be alive or dead once the door of the box would be been open. However, once the door is open, a Turing machine can resolve the problem whether the cat is alive or dead for a polynomial time. Any generalization of the quantum superposition of any natural number of quantum states belongs to the class of the “non-P, but NP” problems.

The proof involves the Kochen – Specker (1968) theorem about the absence of hidden variables in quantum mechanics: if a Turing machine might guess whether the cat is alive or dead in advance, those hidden variables would exist. At the same time, if a result is given (i.e. the door of the box is open), it can be checked for a polynomial time (and this is elementary to be shown after the rigorous formulation of the CMI problem).

Still a few conjectures are suggested as verifiable, but without rigorous proofs in the same publication:

1. Any “non-P, but NP” problem belongs to that class of examples.
2. A “P” algorithm of Turing machine exists necessarily for any problem resolvable for any finite time by a Turing machine
3. The proof of the above statement “2” is non-constructive necessarily.

The probable verifiability of those conjectures involves the concept of set-theoretical (i.e. non-arithmetical), or “Dedekind” finiteness consistent to the axiom of infinity rather than to the axiom of induction in Peano arithmetic. The same concept in the present publication is utilized in the first proof of the identity of Hilbert arithmetic and the separable complex Hilbert space. That identity will be used now to be reinterpreted the class of the “non-P, but NP” problems as well as for suggesting of a loose proof (or additional reasonable arguments) of the three statements based on Hilbert arithmetic in the more intuitive, philosophical and mathematical intention of the present publication:

The interpretation of the “P vs NP” problem in Hilbert arithmetic is the following:

Any natural number in the standard Peano arithmetic (being finite necessarily) can be reached in a certain polynomial time by a certain Turing machine. However, no transfinite number can be reached by the same Turing machine in any finite time (and thus, in any polynomial time). Any certain transfinite number in a standard Peano arithmetic is a natural number in the complementary standard Peano arithmetic as it is proved above. Thus, a “complementary” Turing machine (which means that it processes in the complementary standard Peano arithmetic) will reach that natural number (corresponding to a certain transfinite number in the initial standard Peano arithmetic and thus, to the former Turing machine needing infinite time for processing) for a certain polynomial time.

One need demonstrate that that the pair of a certain transfinite number in the initial standard Peano arithmetic and a certain natural number in its complementary counterpart is equivalent to a “non-P, but NP” example: it is a “non-P” example as to any Turing machine associated to the former Peano arithmetic, and a “NP” example as to any Turing machine associated to the latter Peano arithmetic. The properties of Hilbert arithmetic imply that the mapping of the set of all “non-P, but NP” examples and the set of all pairs defined above is one-to-one necessarily. Thus, *Statement 1* is proved.

The metaphor of “Schrödinger’s cat” about how one should interpret the difference between the “P”/ “non-P” viewpoint, on the one hand, and the “NP”/“non-NP” viewpoint, on the other hand is rather instructive:

The transfinite “non-P” state and its finite counterpart, “NP” state correspond to “opening the door of the box”, in which Schrödinger’s cat is situated, and thus, to a certain quantum measurement, therefore being fundamentally random:

After any measurement ("opening the door"), a different natural number as a "NP" state would be observed implying a generalizing probability density distribution (and a certain wave function as its characteristic function) linked one-to-one to a certain transfinite number. Thus, the set of all transfinite numbers can be mapped unambiguously into the wave functions of the separable complex Hilbert space.

Consequently, the "P vs NP" problem is able to visualize the duality of mathematical/ physical (meant philosophically in the "quantum neo-Pythagoreanism") once again.

As to the second conjecture above, any finite time for a certain problem to be resolved by any Turing machine implies either (1) a certain maximal number, "b", to be reached in the processing of the problem (for any step in the processing needs a certain nonzero time) or (2) at least a step in the processing to be accomplished for zero time (loosely speaking, by a quantum leap).

The alternative (1) implies a polynomial time, $t = f(a^m)$, for a relevant algorithm such that for the latter to exist the exponent "m": $\forall b, \exists m, a: b < a^m$ where a, b, m are natural numbers, but the third conjecture states for the algorithm at issue only to exist and thus, it implies the absence of any common constructive method for the algorithm to be written explicitly.

The alternative (2) implies for the processing machine not to be a Turing machine.

The third conjecture is a corollary from Gödel's incompleteness of Peano arithmetic to set theory. The meant general constructive method for the algorithm to be written explicitly is equivalent to an arithmetical proof of the completeness of Peano arithmetic, and thus, that general method is necessary not to exist.

Indeed, the proof of those second and third statements seems to be almost obvious in Hilbert arithmetic. There exists a relevant polynomial time for any natural number, but the proof of the general constructive method to be described explicitly the relevant algorithm has to be greater than any natural number, and thus, a transfinite number, which is equivalent for the corresponding proof to be nonconstructive.

III CONCLUSIONS & FUTURE WORK

The concept of information as the quantity of elementary choices is able to reconcile arithmetic and set theory in the foundations of mathematics. The consistent completeness of mathematics turns out to be possible and provable within the framework of mathematics.

A nonstandard interpretation of Peano arithmetic is involved: the function of successor is interpreted as "equal to the next" rather than "$+1$" as in the standard interpretation. The term of "nonstandard interpretation" refers usually to countable or finite models of arbitrary mathematical structures. The direction of its use here is opposite: it means actually infinite (i.e. set-theoretical) models of Peano arithmetic. Furthermore, it implies a complementary counterpart of the standard Peano arithmetic (for example, the function successor in the latter can be interpreted as "$-\ 1$" starting of the least ordinal of actual infinity).

The nonstandard and both complementary standard interpretations of Peano arithmetics can be generalized as Hilbert arithmetic, furthermore, being consistent to set theory (unlike the standard Peano arithmetic after Gödel's incompleteness, 1931). The complementary standard Peano arithmetic is equivalent to the arithmetic of transfinite numbers being well-ordered reversely. A certain wave function (i.e. an element of the separable complex Hilbert space of quantum mechanics) is assigned unambiguously to any transfinite number by the mediation of the Dedekind (i.e. set-theoretically rather than arithmetically defined) finiteness.

The isomorphism of Hilbert arithmetic and the separable complex Hilbert space can be proved. It implies the complementarity of physics (underlain by the separable complex Hilbert space of quantum mechanics) and mathematics (underlain by Hilbert arithmetic), and thus philosophically, a form of Pythagoreanism called "quantum neo-Pythagoreanism".

The consistent completeness of mathematics is considered as a powerful tool for resolving a few of the most difficult contemporary mathematical problems: an only arithmetical proof of Fermat's last theorem; a "human" proof of the four-color theorem, and its generalization as the four-letter theorem on the same base of the "human" proof; an only arithmetical (in the framework of Hilbert arithmetic) of Poincaré's conjecture and another proof of it, physically interpretable; the "P vs NP" problem.

The following, rather philosophical conjecture is formulated by means of those examples of very difficult mathematical problems:

The consistent completeness of mathematics is the problem of all problems in the contemporary mathematics: thus, its solution by Hilbert arithmetic can serve as a "key" for many unresolved mathematical problems.

The hypothesis of "physical and mathematical transcendentalism" as a generalization and rigorously formulated counterpart of Kant's philosophical transcendentalism and Hegel's dialectic doctrine is suggested. It can be postulated as a definitive property of the totality (philosophically) or that of the universe (physically), or that of the consistent completeness (mathematically).

The paper is philosophical rather than narrowly mathematical or physical: it is directed to "see the forest for the trees": many proofs are outlined, but their details are omitted as unessential to the general idea of the "forest".

On the contrary, the future work suggests to be described the "trees" in detail, namely complete rigorous mathematical proof for each fundamental problem considered in the general philosophical context in the present paper.

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