Two strategies to infinity: completeness and incompleteness. The completeness of quantum mechanics

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Abstract. *Two strategies to infinity are equally relevant for it is as universal and thus complete as open and thus incomplete. Quantum mechanics is forced to introduce infinity implicitly by Hilbert space, on which is founded its formalism. One can demonstrate that essential properties of quantum information, entanglement, and quantum computer originate directly from infinity once it is involved in quantum mechanics. Thus, these phenomena can be elucidated as both complete and incomplete, after which choice is the border between them. A special kind of invariance to the axiom of choice shared by quantum mechanics is discussed to be involved that border between the completeness and incompleteness of infinity in a consistent way. The so-called paradox of Albert Einstein, Boris Podolsky, and Nathan Rosen is interpreted entirely in the same terms only of set theory. Quantum computer can demonstrate especially clearly the privilege of the internal position, or "observer", or "user" to infinity implied by Henkin's proposition as the only consistent ones as to infinity.*

Key words: *entanglement, EPR, Henkin's propostion, Löb's theorem, EPR, quantum information, quantum computer, qubit*

Both completeness and incompleteness are well distinguishable as to finiteness: Completeness supposes that any operations defined over any finite sets do not transcend them while incompleteness displays that they can do it sometimes. This legible boundary turns out to be unclear and even inconsistent jumping into infinity.

One may say that there are two strategies or "philosophies" after that leap has been just made and any orientation in the unknown infinity is necessary for the thought to survive: one should keep either to completeness or to incompleteness for the infinity seems both complete and incomplete being as universal as open. Henkin and Gödel tempt to be recognized correspondingly as the symbolic personages embodying those two opposite types.

The paper retraces the strategy of completeness originating from Henkin's proposition [1] via Löb's proof [2] to the alleged incompleteness of quantum mechanics [3], the "no hidden variables" theorem [12-13], entanglement and quantum information and to the endpoint of a quantum computer [4-5], which can solve any problem only if its user is situated inside it [6-7].

An introductory reformulating roof sketch in the most concise relevant language is necessary for the above tentative itinerary to be able to be seen as a progressing whole rather than a heterogenic and eclectic mix. The man and scientist Leon Henkin can be a reliable guide implicitly:

Many efforts address the constructiveness of the proof in the so-called Gödel first incompleteness theorem [8]. However it is not only redundant but putting obstacles for the above objectivity. One should use the axiom of choice instead of this and abstract from that the theorem is formulated about statements, which will be replaced by the arbitrary elements of any infinite countable set. Furthermore all four theorems [2], [8-10] about the so-called metamathematical fixed points induced by the Gödel incompleteness theorems will be considered jointly and in a generalized way.

The strategy is the following:

An arbitrary infinite countable set "A" and another set "B" so that their intersection is empty are given. One constitutes their union "$C=A\cup B$", which will be an infinite set whatever B is. Utilizing the axiom of choice, a one-to-one mapping "$f=A\leftrightarrow C$" exists. One designates the image of B into A through f by "B_f" so that "$B_f\subset A$". If the axiom of choice holds, there always an internal and equivalent image like B_f for any external set like B. Thus, if one accepts that "$B_f\equiv B$", whether an element "$b\in B$" belongs or not to A is an undecidable problem as $b_f\equiv b$.
However if the axiom of choice is not valid, one cannot guarantee that f exists and should display how a constructive analog of "f" can be built. If one shows how f to be constructed at least in one case, this will be a constructive proof of undecidablity as what Gödel's is.

Another option is to prove that no constructive analog exists for any f guaranteed by the axiom of choice. In a sense, this would be an analog of the theorems about the absence of hidden variables in quantum mechanics [12-13].

Coring the reformulation, the problem of (in)completeness can be generalized as a property of all infinite sets. An infinite set unlike any finite one can be both complete (universal) and incomplete (open) in a sense reminiscent to the "clopen" (both closed and open) sets in topology such as all discrete sets are. *There is a hidden, but intimately link between discreteness and infinity*. Quantum mechanics, forced to introduce quanta and thus discreteness, has therefore introduced infinity in an experimental and exact science such as physics. What is that bridge is what leads from Henkin's proposition to quantum computer, both being from the "internal side" of completeness.

1. Henkin's proposition

This is the "formula" defined in "a problem concerning provability":

"If Σ is any standard formal system adequate for recursive number theory, a formula (having a certain integer q as its Gödel number) can be constructed which expresses the proposition that the formula with Gödel number q is provable in Σ. Is this formula provable or independent in Σ?" [1]
It is commonly interpreted as the opposite to the Gödel proposition stating the proper unprovability under the same conditions.

The statements can be also reformulated in two ways, which are not self-referential:

–"Jeroslow's proposition": a formula stating the provability of another under the same conditions [9].

– "Rogers' proposition": a formula stating the unprovability of another under the same conditions [10].

The reformulation of all four cases in the present context is the following:

An element "a" of an infinite countable set "A" and an element "b" of another set "B" are given, and "b_f" designates the image of b after "$A\cup B$" is counted by "A" ["$f=(A\cup B)\leftrightarrow A$"]:

The reformulation of Henkin's case: "$(a \equiv b_f) \wedge (B \subset A)$"

The reformulation of Gödel's case: "$(a \equiv b_f) \wedge (B \cap A = \emptyset)$"

The reformulation of Jeroslow's case: "$(a \not\equiv b_f) \wedge (B \subset A)$"

The reformulation of Roger's case: "$(a \not\equiv b_f)\ (B \cap A = \emptyset)$"

The reformulation is more general for it abandons:

– The specification of the elements both of "A" and "B" as any propositions

– The specification of "f" as any constructive encoding of the elements of "A∪B" including the Gödel numbering.

However, it needs the axiom of choice to guarantee the existence of "f" in any case.

Consequently, the proof of the, reformulation of Henkin's case would be a proof of the ZFC consistency ("ZFC" means the set theory founded on the axioms of Zermelo, Fraenkel plus the axiom of choice). Indeed the strategy of ZCF involves as "sets" those entities ("classes"), which are true subsets or parts of another entity ("class"). Then one can say about any proposition in ZFC that it has an image as an element of some set definable in ZFC and thus the ZFC set theory represents its proper consistency just as Henkin's proposition states its provability.
Even more, any other axiomatics of set theory should be necessarily equivalent to it as Gödel's, Jeroslow's, and Rogers' propositions are undecidable and one can conjecture the same about their reformulations as above.

2. Martin Löb's theorem

Martin Löb proves that Henkin's proposition is provable [2]:

"If $\mathfrak{S}$ is any formula such that $\tilde{\mathfrak{B}}(\{\mathfrak{S}\}) \rightarrow \mathfrak{S}$ is a theorem, then $\mathfrak{S}$ is a theorem"

The corollary is: "The particular formula $\mathfrak{S}$ of Henkin's problem, which is the same as $\tilde{\mathfrak{B}}(\{\mathfrak{S}\})$, is a theorem."

One can reformulate in a trivial way his proof so that it holds in the reformulation of Henkin's case as above. Indeed almost all proof of Martin Löb concerns the Gödel encoding to be validated under the conditions of the theorem. If one accepts the above generalizing reformulation, its statement seems to be obvious:

"b_f implies b" (as "$b_f \equiv b$") implies "$b \in A$" (as "$b \in B \subset A$"). This is not true in the rest three cases (Gödel's, Jeroslow's, and Rogers' proposition).

The only positive result (about Henkin's propostion) as well as the rest three, which can be considered as negative, can be interpreted also thus: The only consistent position to infinity is the internal one. As set theory involves infinity, its axiomatics such as ZFC should guarantee just the internal position for itself to be consistent.

3. The Einstein – Podolsky – Rosen "paradox" in quantum mechanics [3]

However the set theory is not the single one, which involves infinity in science. The same does quantum mechanics utilizing the infinite-dimensional Hilbert space to describe both quantum leaps and continuous trajectories uniformly.

On the other hand, Einstein's theories of relativity includes the notion of an internal or external position by means of the concepts of an "observer" or a "reference frame" yet in relation to finite sets as what special and general relativity discuss in general.

So the collision of relativity and quantum mechanics was unavoidable and it was articulated in an especially clear and instructive way in [3]: Indeed, if one introduces an external observer to a quantum system, the incompleteness of quantum mechanics can be easily demonstrated. After reformulating above, this should be obvious as a quantum system implies infinity to be adequately described, and thus excludes an external position to it to be consistent and therefore complete. The close friendship between the Princeton refugees Einstein and Gödel [11] could address the fact that the Einstein, Podolsky and Rosen's argument is another interpretation of the famous Gödel incompleteness argument [8] in terms of quantum mechanics. This is obvious as soon as ones look at them by the above reformulation of Gödel's case as their common structure:

Indeed there is an initial quantum system Q, which is divided to two other systems P and S moving with some relative speed to each other in space-time. For Q, P, and S are quantum systems and they are represented by three infinite-dimensional Hilbert spaces, the EPR argument can be bared to the following set-theory core: There is an initial infinity Q, which is divided into two infinities P and S, each of which suggests an external viewpoint to the other. So each of the two pairs (P, Q) and (Q, P) models the "reformulation of Gödel's case" above and can serve to demonstrate the "incompleteness of quantum mechanics", which produces that description of reality. However the cause is the paradoxical properties of infinity rather than the description of quantum mechanics once forced to introduce infinity in itself.

4. The theorems about the absence of "hidden variables" in quantum mechanics

As Einstein, Podolsky, and Rosen could paraphrased the Gödel incompleteness argument in terms of quantum mechanics, as John von Neumann initially [12] and Simon Kochen and Ernst Specker afterwards [13] also managed to paraphrase the internal consistency of infinity corresponding to the reformulation of Henkin's proposition in the same terms. Neumann deduced it from the availability of non-commutative operators in the mathematical formalism of quantum mechanics founded on Hilbert space while Kochen and Specker did the same on the ground of wave-particle dualism even if the corresponding operators are commutative.

Furthermore the corollaries of these theorems can demonstrate the intimate and extraordinary link between quantum mechanics and the axiom of choice: one link inherited in fact yet from the infinity admitted in quantum mechanics:

According to them, any well-ordering in any quantum system is excluded before measurement being a coherent and inseparable whole. Nevertheless it represents a well-ordered statistical ensemble of results after measurement. The basic epistemological postulate requires the states before and after measurement to be equated and thus the well-ordering theorem equivalent to the axiom of choice to be utilized necessarily to equate an unorderable set (before measurement) with a well-ordered set (after measurement).

However the coherent state before measurement excludes the axiom of choice just as these theorems are valid. Thus the mathematical structure used by quantum mechanics should be invariant to the axiom of choice in a sense, and this is just so after one involves the so-called probabilistic interpretation of wave function [14-18]. Indeed Hilbert space together with its identical dual space can describe equally successfully a well-ordered infinitely-dimensional vector and the unorderable characteristic function of the probabilistic distribution of the values of its components.

Thus quantum mechanics interpreted as a theory about infinity can transfer rather instructive conclusions into set theory concerning a possible and undeveloped yet probabilistic interpretation of infinity and therefore, that of the set theory itself. All this allows the further discussed entanglement, quantum information, and quantum computer to be interpreted in terms of set theory and seen from the viewpoint of the generalization of Henkin's propositions.

5. Entanglement and quantum information

The EPR situation will be discussed in detail. The system "$P\cup S$" is mapped one-to-one on "P" or on "S" by "f" and to choose "P" for definiteness. Being quantum systems, they are represented by wave functions and thus by two corresponding Hilbert spaces "$H_{P\cup S}$" and "H_P". H_P can be or not a subspace of $H_{P\cup S}$. If it is not, P and S are entangled. Here, the only essential properties of Hilbert space are its infinite dimensionality and the so-called invariance to the axiom of choice. So entanglement can be well represented in a set-theory way as a special kind of relation between infinite sets within the frameworks of that invariance. Hereinafter "P", "S" and "$P\cup S$" can be thought as infinite sets only, and entanglement will be defined as that case where "P_f" is not a subset of "$P\cup S$", however "$E=P_f\cap P\cup S$" can be or not empty. The apparent contradiction will be elucidated a little below on the ground of that invariance.

The interpretation is the following: Entanglement of P and S means that any of them is sited in an intermediate position between the two "extremes" of the internal or external position to the other set. However that intermediate position would be impossible if the following dichotomy held: either the axiom of choice is valid or it is not. For it implies the dichotomy between the "extremes": the "internal extreme" where the axiom of choice holds vs. the "internal extreme" where it does not. Since Hilbert space is invariant to the axiom of choice, it implies the option of that intermediate position, in which the axiom of choice holds only in relation to the true subset of P and thus only a true subset of P can be mapped by f. In other words, entanglement represents the choice or the action of the axiom of choice as a process yet unaccomplished ultimately and thus incomplete.

To be involved that process of choice in a consistent way, the elements of the sets "P", "S" and "P∪S" being quantum systems are gifted by an internal structure of the components of a vector in Hilbert space, "$C_n e^{in\omega}$", where "C_n" is a complex number, which is the n^{th} component of the vector, and "$e^{in\omega}$" is the n^{th} unit vector of the basis.

That internal structure can be equivalently represented as a quantum bit (qubit) defined as "$\alpha|0\rangle + \beta|1\rangle$" where α and β are two complex numbers such that "$|\alpha|^2 + |\beta|^2 = 1$" and "$|0\rangle, |1\rangle$" are two mutually orthogonal subspaces of Hilbert space, e.g. two successive "axes" of Hilbert space such as "$e^{in\omega}$" and "$e^{i(n+1)\omega}$". A qubit is isomorphic to a unit 3D ball, in which two points (or vectors) are chosen under certain conditions: they should be on two orthogonal great circles of the ball so that their sum is a point of the surface of the ball.

It can be considered as a generalization of the notion of "bit", the unit of information, where the choice is between a continuum of alternatives rather than two ones as in a bit. That generalization allows the choice to be seen as a process as it is necessary above. Furthermore all physical processes, which are represented by certain transformations of wave functions according to quantum mechanics, can be thought as the computations on a quantum computer processing qubits rather than bits. Thus a bridge is built to connect physics with mathematics by means of two its branches: the one is applied (the theory of information) and the other is fundamental (set theory), therefore erasing those borders.

One can object that entanglement does not involve necessarily infinity for it can exist between finitely-dimensional subspaces of Hilbert space. However, the so called Banach-Tarski paradox [19] implies that any qubit (being isomorphic to a 3D sphere as above) is equivalent to two ones and thus to the entire Hilbert space under the condition of the axiom of choice. Consequently, once the elements of a set are gifted by the structure of qubits, the infinity is already involved, and entanglement is definable even if the set is finite.

6. Quantum computer

The quantum Turing machine [4] and quantum circuits [5] describes how a quantum computer can be defined. Those two ways are equivalent to each other [20]. One can utilize another model of a quantum Turing machine where all bits are replaced by qubits, and a quantum Turing tape can content infinitely many qubits so that it is equivalent to a wave function. The operations, which can be accomplished on a single qubit, are the same: "Write!"; "Read!"; "Next!", and "Stop!" That model is quite enough for it includes the essential features as to the present context: the infinity and "invariance to the axiom of choice".

One can object that the computation of a quantum computer is reversible unlike a Turing machine or unlike a quantum Turing machine if it is defined in thus. However the computation within each separate qubit is reversible and equivalent to the computation of the entire quantum Turing tape containing infinitely many qubits if or to the extent in which the axiom of choice holds.

Consequently a quantum computer is reversible to or from a boundary defined by that the axiom of choice is valid to or from it.

What is the criterion for an infinite computation to have a certain result? That is the following: if the limit of all successive partial computations exists. If the axiom of choice is valid as to all those members, which is equivalent to an internal position to infinity as above, that series can be always reordered in two (increasing and decreasing) monotonic ways so that it converges necessarily though to two different limits in general, which are complementary to each other: Indeed if one chooses to reorder in an increasing way, this excludes it to be simultaneously reordered in a decreasing one, and vice versa.

The internal position to infinity as to a quantum computer means the following: Any potential "user" should be sited inside the quantum computer before the beginning of the computation so that user's observation on the quantum computer is included in advance in the result which will be obtained ultimately. Under that condition, the quantum computer can resolve any problem, and it cannot "hang up" ever. As any quantum computation is equivalent to a wave function of the universe in general and thus to one state of it, mankind observes constantly this property of a quantum computer: Indeed the universe, to which mankind is sited always inside, never "hangs up" and any physical process finishes in a finite time. A quantum computer to an internal observer knows its state and resolves the "halting problem" as its computation ends always with a result. The same David Albert [6-7] proved in another way coining the term "quantum automaton" for a quantum computer.

What about a quantum computer locked in a "Chinese room" therefore forcing an external position to it? First of all a quantum "Chinese room" should admit an arbitrary degree of entanglement between it inside and the interviewers outside, a kind of "quantum telepathy". Then the axiom of choice will be partially valid to the extent exactly defined by that degree of entanglement. Consequently the quantum computer in any degree of entanglement cannot accomplish the reordering right, to the utterance and it will hang up in general. However in fact the quantum computer locked in the "Chinese room" will be replaced by another quantum computer including both the interviewers outside the "Chinese room" and the former inside it. The latter quantum computer will give the answer for the former always as it includes the "users" of it while the interviewers will think that the answer is given by the locked one, which will "hang up" and thus mute.

What turns out to be the case? As if a quantum computer cannot be locked in a "Chinese room" as a law of nature. However it will fail in the "Chinese room test" in a paradoxical way: It will give the right answer always even ostensibly "locked" while a human being can be really deprived of the context outside the "Chinese room" and thus to give a different answer in general.

The "Chinese room" allows of a curious visualization of the invariance to the axiom of choice as to a quantum computer: The invariance means that the "hung-up" quantum computer inside and the entire quantum computer including that inside together with the interviewers can be discussed as equivalent as quantum systems described in Hilbert space(s). The same can be formulated also so: The quantum computer is only a single one, and all including mankind is within it necessarily.

7. Conclusion

The two strategies to infinity originates from the definitive properties of the totality: "to be all". It is complete being all. However also it is in incomplete because it includes incompleteness within it being all. This seems to be contradictory, but it is very fruitful, in fact, for the definition of the totality rigorously as just that entity determinable by the property to be both complete and incomplete in a consistent way:

The totality contains its externality within it necessarily being all in definition. Thus, any element of the totality has to be doubled by a “twin” both identical and “complementary” (in the generalized sense of Niels Bohr) to the former twin. The “latter twin” as if represents a certain element of the externality of the totality after it has to be inside.

Therefore, the both “twin” dual spaces of the separable complex Hilbert space utilized by quantum mechanics as its basic mathematical formalism are fundamentally necessary as far as quantum mechanics is inherently holistic and thus referred to the whole and its totality.

The totality in mathematics is meant by the concept of infinity. One notices an isomorphism between the opposition “completeness – incompleteness” in both foundations of mathematics and quantum mechanics. The strategy of quantum mechanics to be complete can be interpreted directly onto the problem of how mathematics can be complete. The transition of the strategy is grounded by that isomorphism underlain in turn by the concept of the totality (though being a philosophical one properly).

Once the strategy of completeness is transferred to the foundations of mathematics, it can be visualized by Russell’s paradox of the “barber” as if doubled to himself by a “barber twin”: all elements of the totality are similar to Russell’s “barber”. However this is not contradictory, but productive to be defined any entity as an element of the totality. Quantum mechanics (furthermore be complete) is able to demonstrate how by means of both “twin” spaces of the separable complex Hilbert space.

Another pathway to the same vision to the completeness of mathematics can suggests Henkin’s proposition in virtue of Löb’s proof.

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