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Brouwer's intuition of twoity and constructions in separable mathematics

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Abstract My first aim in this paper is to use time diagrams in the style of Brentano to analyze constructions in Brouwer's separable mathematics more precisely. I argue that constructions must involve not only pairing and projecting as basic operations guaranteed by the intuition of twoity, as sometimes assumed in the literature, but also a recalling operation. My second aim is to argue that Brouwer's views on the intuition of twoity and arithmetic lead to an ontological explosion. Redeveloping the constructions of natural numbers and systems sketched in an appendix to Brouwer's Cambridge lectures, I observe that the only plausible way he can make some elementary arithmetic in his separable mathematics is by allowing for the same canonical number to be determined by multiple separable entities, resulting in an overabundant mathematical ontology.

1 Introduction

The only legitimate way of ensuring existence in the intuitionistic mathematical universe envisioned by Brouwer is through constructions in intuition (Brouwer, 1954, p.2). Intuition is given an exclusive ontological role in his intuitionistic program. It acts as the dividing line that marks what belongs to mathematics and it automatically grants all its existents the ontological status of mind-dependent entities constructed in intuition.

According to Brouwer, intuition is rooted in the perception of the movement of time in which a subject experiences a sensation, and, at a later stage in time another sensation, while at the same moment keeping the first one in the memory. This so-called "intuition of twoity" states moreover that the subject is capable of an abstract perception of the very structure of one thing that gives way to another, which, as Brouwer calls it, is the "empty twoity":

> [...] intuitionistic mathematics is an essentially languageless activity of the mind having its origin in the perception of a move of time. This perception of a move of time may be described as the falling apart of a life moment into two distinct things, one of which gives way to the other, but is retained by memory. If the twoity thus born is divested of all quality, it passes into the empty form of the common substratum of all twoities. And it is this common substratum, this empty form, which is the basic intuition of mathematics. (Brouwer, 1981, pp.4–5)

One important but still relatively understudied part of intuitionistic mathematics is the discrete fragment that arises out of this twoity phenomenon alone and serves as the foundation for the intuitionistic theory of the continuum. Brouwer calls this fragment "separable mathematics" because it is concerned exclusively with mental entities which are separable in the sense that their construction can be carried out by pairing two previously constructed separable entities into a new one from the intuition of twoity. The construction of all separable entities starts with the empty twoity and the two units which are elements of it because these are the basic systems of separable mathematics:[1]

> This empty two-ity and the two unities of which it is composed, constitute the basic mathematical systems. And the basic operation of mathematical construction is the mental creation of the two-ity of two mathematical systems previously acquired, and the consideration of this two-ity as a new mathematical system. (Brouwer, 1954, p.2)

Pairing is given the title of the "basic operation of mathematical construction". The quotation above continues with Brouwer remarking that this basic operation of pairing two separable entities and obtaining the pair as a new separable entity suffices to reconstruct systems corresponding to those of classical discrete mathematics. I will examine his thoughts on this correspondence with classical mathematics in Section 3.2. For Brouwer, the subject performs constructions of separable entities introspectively through pairing, relying on the continuous retention of the separable entities previously constructed in the memory:

> It is introspectively realized how this basic operation, continually displaying unaltered retention by memory, successively generates each natural number, the infinitely proceeding sequence of the natural numbers, arbitrary finite sequences and infinitely proceeding sequences of mathematical systems previously acquired, finally a continually extending stock of mathematical systems corresponding to "separable" systems of classical mathematics. (Brouwer, 1954, p.2)

Brouwer tells us here that separable entities come in different forms. The primary focus of this paper is on those identified as the natural numbers and their systems. One major concern, already expressed by van Stigt (1990, p.300), is that Brouwer never bothered to give a detailed account of how the construction of these separable entities is carried out. To the best of my knowledge, the most explicit but still rather vague mention of the idea is found in a sketch appended to his Cambridge lectures (Brouwer, 1981, p.90), to be discussed soon. Apparently, Brouwer sees these constructions as straightforward reflective exercises that need not be spelled out in detail using words since they must be "introspectively realized". However, to confirm the existence of separable entities in the intuitionistic mathematical universe we cannot simply appeal to Brouwer's authority. These existence claims must be supported by demonstrating in as much detail as possible how separable entities can be experienced assuming only the basic premises of construction that underlie the intuition of twoity.

[1]Systems in separable mathematics are roughly non-empty discrete sets.

My first aim in this paper is to identify these premises and make them explicit by putting forward a diagrammatic interpretation of Brouwer's construction of separable entities.[2] The pivotal idea of my interpretation is the adoption of a retentional model of time consciousness and the use of time diagrams in the style of Brentano to give a detailed description of the continuous retentions of the subject in a succession of stages of time. The retentional analysis of Brouwer's views goes back to van Atten (2006) and Tieszen (2008). My contribution is in the use of time diagrams to analyze the construction of a separable entity more precisely as a process based on pairing, projecting, and recalling of present sensations and retentions experienced in the inner time of the subject. With this I maintain that pairing, projecting, and recalling must all be "basic operations" of construction. Recently, van Atten (2015) has already suggested that projecting must be admitted as basic operation along with pairing. I extend his observation with the claim that a recalling operation is just as important to make Brouwer's constructions of his separable entities possible. This necessity is best seen through the precision of the diagrammatic interpretation. It provides the tools needed to represent constructions of separable entities and allow for their rigorous examination.

My second aim is to argue that Brouwer's views on the intuition of twoity and arithmetic result in a problem of ontological explosion, namely, the problem that arises when one is forced to admit an excessive amount of dubious entities into one's ontology. Using the diagrammatic interpretation to redevelop the constructions of natural numbers and systems originally sketched in an appendix to Brouwer's Cambridge lectures (Brouwer, 1981, p.90), I observe that, as a consequence of his treatment of addition as pairing, to make basic arithmetic workable in his separable mathematics, his ontology must admit different separable entities that all take up the role of the same "canonical number", namely, one, two, three, etc. The intuition of twoity distinguishes all these separable entities and therefore results in an ontological explosion manifested by a plurality of duplicate entities for each number. Borrowing a token-type distinction used by van Atten (2018), I contrast three notions of sameness underlying the types of separable entities, numbers, and canonical numbers. I conclude that identity of separable entities is finer than identity of canonical number.

The remainder of this paper is structured as follows. I launch the diagrammatic interpretation in Section 2 beginning with an overview of Brouwer's views on the intuition of twoity. I introduce Brentano's time diagrams to represent the retentional model assumed by Brouwer and I treat pairing, projecting, and recalling as basic operations diagrammatically. In Section 3, I utilize my interpretation to give diagrammatic constructions of the natural numbers building mainly on Brouwer's rough sketch from the Cambridge lectures. After a discussion of the problem of ontological explosion, I examine Brouwer's view of addition as pairing, explore the token-type distinction, and address the uniqueness of the canonical numbers. Finally, I observe that this ontological explosion is problematic for Brouwer because it challenges his claim that the intuition of twoity can produce a form of separable mathematics that corresponds to its classical counterpart. I draw some conclusions in Section 4.

[2]Due to space limitations, I must set aside the question of whether my diagrams represent constructions in a solipsistic mind or the idealized mind of a transcendental subject which allows for the possibility of communication to other minds equally equipped with all the mental inventory assumed by Brouwer. I refer the reader to (Placek, 1999, pp.5–9 and pp.22–27, ch.2, §4) for an excellent discussion of the problem of other minds in Brouwer and a defense of a non-solipsistic reading of his philosophy.

2 The diagrammatic interpretation

To the best of my knowledge, Brouwer's most explicit articulation of the construction of separable entities is given in a sketch appended to his Cambridge lectures (Brouwer, 1981). In this short appendix, dated in the early fifties, Brouwer says how the construction of the natural numbers and their systems is experienced out of the intuition of the empty twoity:

> The inner experience (roughly sketched):
> twoity;
> twoity stored and preserved aseptically by memory;
> twoity giving rise to the conception of invariable unity;
> twoity and unity giving rise to the conception of unity plus unity;
> threeity as twoity plus unity, and the sequence of natural numbers;
> mathematical systems conceived in such a way that a unity is a mathematical system and that two mathematical systems,
> stored and aseptically preserved by memory, apart from each other,
> can be added; etc. (Brouwer, 1981, p.90)

Drawing inspiration from Brentano's time diagrams, my goal in this section is to develop a diagrammatic interpretation aimed at providing a more rigorous explanation of the process of construction of these separable entities from Brouwer's intuition of twoity. The main thesis I wish to vindicate here is that the construction of separable entities requires not only pairing and projecting, but also recalling as the three basic operations. The subject starts a construction at the initial stage at which the empty twoity is created and transits to new stages through repeated application of the basic operations. To properly reach this conclusion, it will be necessary to motivate the retentional model of time consciousness that I claim to operate in the background of all these constructions. This will be my first task.

Before we proceed, two words of caution are necessary. First, my portrayal of constructions as processes may spark confusion, since, as observed by Sundholm (1983, pp.164–167), the term 'construction' is overloaded with different meanings in intuitionism. Because I am interested in separable mathematics alone, I reserve the term to the process of construction and call the things obtained as the result of a process of construction simply separable entities. It is in this sense that I regard constructions as processes carried out in the mind. My aim is to clarify what features they must have to support the sketch offered by Brouwer above.

Second, the intuition of twoity that can be witnessed in its mature form in Brouwer's writings in the early fifties can be traced back to his dissertation (Brouwer, 1907, p.9). One major difference is that in this initial formulation emphasis was given to a "between" that connects the two elements of our perception of two things in time. It was insisted that the discrete and continuum are inseparable primitive complements. After the introduction of choice sequences, Brouwer rarely speaks of continuity as an irreducible intuition, but still stresses that together with the empty twoity "between" is created (Brouwer, 1981, p.40). However, since our focus is on the discrete elements of intuitionistic mathematics, this connection by a between can simply be ignored in what follows.

2.1 Retentions in Brentano's time diagrams

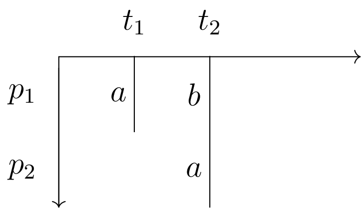
To begin with, in order to understand the intuition of twoity, we first need to look at its origin in the ordinary phenomenon of the movement of time. To give an example, suppose that you have experienced two distinct sensations, one immediately followed by another, like the sounds of two successive ticks of a clock. Brouwer maintains that after the first sensation is succeeded by the second one in the present moment, it is retained in memory:

> By a move of time a present sensation gives way to another present sensation in such a way that consciousness retains the former one as a past sensation and moreover, through this distinction between present and past, recedes from both and from stillness, and becomes *mind*. (Brouwer, 1948, p.1235)

Note that what is involved here is not scientific but internal time, the temporal framework within which our experiences take place in consciousness. Brouwer is careful to point out this distinction in his dissertation (Brouwer, 1907, p.99, fn.2). Thus, the quoted passage is suggesting that at each moment a subject is not only aware of the sensation that is now at the present stage of time consciousness. Because they are capable of retention, they are also aware of the sensation that previously affected them in the former present stage, before it was taken over by the new sensation, thus becoming the immediate past stage.

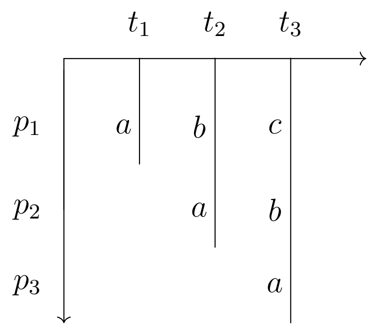
Brouwer has more to say and we will see what comes after this passage soon. Now I want to emphasize that, up to this point, Brouwer simply takes for granted a retentional model of time consciousness, as van Atten (2006, p.125) and Tieszen (2008, pp.81–82) note. More precisely, I believe the features of time consciousness needed in the background to support Brouwer's claim above are those of a retentional model in the style of Brentano.[3] According to the retentional model of time consciousness, a subject's experience of succession is understood in terms of experiences retained in a sequence of temporal stages.

It is reported that Brentano originally argued for the retentional model of time consciousness when explaining how the perception of a melody requires the awareness of a series of tonal retentions in his 1873 Würzburg lectures (Stumpf, 1976, p.38). Brentano insisted that in the absence of the awareness of past tones as retentions, subjects could only perceive each tone in isolation and never the melody in its unity. Brentano then described the succession of sensations in inner time consciousness in terms of a diagram like the following:



[3] Before Brentano, we find a short outline of a retentional model in Kant (1998, A102). It is hard to tell to what extent Brentano's description of time consciousness was influenced by Kant's.

The horizontal line depicts the flow of sensations experienced at present for the subject. It represents the passage of a former now to a new now. Here a stands for a sensation affecting the subject at an initial present stage of time t_1 and b indicates another sensation that is then experienced at the subsequent present stage t_2. At this new present stage t_2, the subject becomes aware of the former stage t_1 as the immediate past and the sensation a that was then experienced is kept as a retention. In the diagram, p_1 describes the sensations that are experienced in the present moment at stages t_1 and t_2, while p_2 expresses what has been experienced in the past at t_1 and kept as a retention at t_2. Retentions are illustrated going downward to stress that they keep sinking into the past as the subject has new experiences. For example, here is a diagram of the experience of three sensations a, b, and c:



To be exact, Brentano did not include labels for the stages of time t_i nor retentions p_j. But it will be helpful to include this information in the diagrams for the sake of clarity because from this point on I will be using such time diagrams to illustrate Brouwer's views. It will be important to keep in mind at what stage something has been constructed and what is available to the subject in the present p_1 and the immediate past p_2 as the first retention.

Finally, I should note that Husserl (1964, §36) has further developed Brentano's ideas with his retentional-protentional model of time consciousness. Unlike retentions that involve past experiences retained in the consciousness, protentions are understood as anticipations of a future moment that is yet to be experienced. It can be argued, however, that protentions play a smaller role in Brouwer's constructions and only seem to enter into play to allow for the indefinite iteration of the twoity (van Atten, 2006, p.125). The only kind of protentions that are invoked in this paper are found in our implicit postulate that the subject can always continue a construction finished at the present by going to a next stage. The so-called horizon of possible experiences then consist of all the "protained" separable entities that can be constructed in all further stages from a given present stage. [4] I shall only illustrate retentions in my diagrams, for I am only interested in constructions that have been carried out.

2.2 The basic operation of pairing

Now that my intention to understand retentions diagrammatically is clear, let me return to the examination of Brouwer's views on the intuition of twoity, more specifically at the point where the quotation given earlier stopped. I have claimed that so far there is nothing

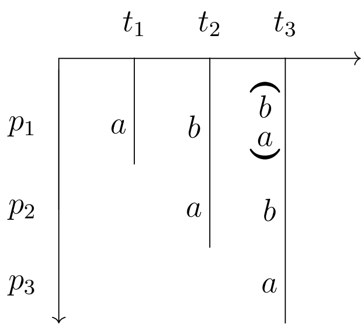
[4]See also van Atten (2018, p.1594). For an in-depth study of protentions in a Husserlian account of the constructions of natural numbers and sets, see Tieszen (1989, pp.107–108, p.137, pp.147–148).

new to the assertion made therein, for they simply assume a retentional model. Indeed, the innovative component of Brouwer's thought starts with his next assertion that the subject also has the ability to experience the past and present sensations together as new object. The quote examined earlier continues immediately with this observation:

> As mind it takes the function of a subject experiencing the present as well as the past sensation as object. And by reiteration of this two-ity phenomenon, the object can extend to a world of sensations of motley plurality. (Brouwer, 1948, p.1235)

Brouwer states that this synthesis of two distinct things into a new unified object, a twoity, is founded on the possibility of thinking them together (Brouwer, 1907, p.8, p.119). This thinking together of two distinct things, which I refer to as pairing, is the basic operation of mathematical construction alluded by Brouwer (1954, p.2) in the introduction. This is how a twoity is "born" in the consciousness of the subject (Brouwer, 1981, p.4). This means that the pair of a first and second sensations, the twoity, affects the subject as a third experience that is distinct from either one of the two sensations that gave rise to it.

How should we understand this pairing operation diagrammatically? Firstly, to introduce some useful notation, if a and b represent the past and present experiences that affected a subject at t_1 and t_2 respectively, I shall say that $(a\ b)$ is the twoity, or the paired experiences, obtained by the thinking-together of a and b in this respective order. Secondly, as $(a\ b)$ itself is a new experience, it must take place in the present moment at new stage t_3 that follows immediately after t_2, like in the previous diagram. A new experience also means a new retention. At the new stage t_3 the pair $(a\ b)$ is now experienced at present p_1, so the sensation b formerly experienced in the present p_1 at t_2 is now retained in the immediate past p_2, and a dives even further into the past as p_3. This is shown in the following diagram:



Note, however, that this is not the basic intuition of mathematics yet. The pair $(a\ b)$ just formed still possesses all the material content it acquired from the past and present sensations a and b and needs to be divested of all quality first (Brouwer, 1981, p.5). It only passes into the empty form that enjoys the highest status of "empty twoity" when all the content of the paired sensations is abstracted away. This "one thing which gives way to another" is the abstract structure that is common to all pairs of sensory experiences. This is what the empty twoity, the first separable entity constructed by the subject in their inner time, is.

Now, some terminology and an important observation are in order. I shall illustrate the empty twoity as a pair of faded gray and black dots (• •) to remind the reader that the first and second units, the elements of the empty twoity, must be distinct from each other. If both units turned out to be identical there would be, for obvious reasons, no perception of "one thing which gives way to another" to begin with and thus no empty twoity. The twoity phenomenon only allows for distinct things to be paired, for otherwise there would be no twoity but unity. This observation will be crucial to our discussion of the uniqueness of numbers in Section 3. For the lack of better term, • may be called the "then" and • the "now" unit.

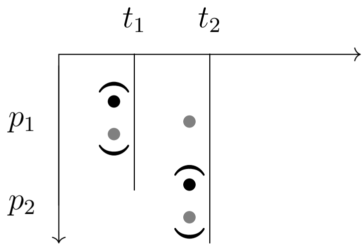
2.3 Projecting and recalling as additional basic operations

Pairing is needed as a basic operation of construction to provide the empty twoity. Now let me motivate the need of additional projecting and recalling operations, starting with the former. Brouwer sees the empty twoity as the starting point of the construction of all mathematics. As pointed out by van Atten (2004, p.5), Brouwer can be found already in his dissertation arguing that the intuition of twoity even precedes that of unity itself, adding that unity arises only at a later stage by projection on the first element of a twoity:

> F. Meyer [...] says that one thing is sufficient, because the circumstance that I think of it can be added as a second thing; this is false, for exactly this adding (i.e. setting it while the former is retained) presupposes the intuition of two-ity; only afterwards this simplest mathematical system is projected on the first thing and the ego which thinks the thing. (Brouwer, 1907, p.179, fn.1)

This is a point of considerable importance. Recall that we saw in the introduction that pairing is a basic operation of mathematical construction (Brouwer, 1954, p.2). It allows us to construct separable entities from the empty twoity and its then and now units by retention. But how are these two units constructed since the empty twoity is originally the first separable entity? There has to be another basic operation, projecting, which allow us to experience either the first or second element of the empty twoity, and, more generally, of any other pair. The need of projection to separate one element out of a twoity in intuition is first hinted at in van Atten (2015, p.19). It needs to be regarded as a basic operation on a par with pairing to get the construction of separable entities off the ground with the formation of units.

How is projecting generally used to construct separable entities? Returning to our diagrams, let me discuss what should be the proper interpretation of Brouwer's claim of the precedence of twoity over unity in the quoted passage from his dissertation. Suppose that, by abstraction on a pair of sensations $(a\ b)$, a subject experienced the empty twoity (• •) at an initial stage, as already explained in the previous subsection. Clearly, this prior pair of sensations must be kept as a retention, but, since it is irrelevant for construction purposes, I shall leave it and its two elements out of the diagram. From this point on our diagrams will always start with the empty twoity at the initial stage t_1. Then, at the next stage t_2, the subject may project the first element of the empty twoity, as Brouwer wanted. In this case it becomes what is experienced in the present and the empty twoity is retained in the past:



A more detailed description of pairing and projecting will be given in Section 2.4. For now, this suffices to show that pairing and projecting are both needed as basic operations of construction that allow a subject to pass from a certain stage to a following one. At this subsequent stage a possibly new separable entity is constructed while keeping what was constructed at the immediately preceding stage as a retention. Yet, closer examination reveals that to interpret the constructions of numbers and systems sketched in the appendix to the Cambridge lectures (Brouwer, 1981, p.90) a third basic operation of recalling is required.

The need for a third basic operation has not been recognized in the literature, as far as I know. This is not surprising, I would say, because some of the nuances needed to make sense of Brouwer's rather sloppy views on construction can only be seen through the lens of a rigorous framework such as the one provided by the diagrammatic interpretation.[5] I will motivate the necessity to go beyond pairing and projecting with a simple example regarding systems and then propose what I think is the best basic operation to fill this gap. The following is a paraphrase of how Brouwer (1981, p.90) states systems are constructed in his sketch:

- a unit is a system;
- two systems, retained apart from each other, can be "added" to form a new one.

The use of scare quotes here is important. The reader would do well to keep in mind that adding in the separable context means pairing. This is evident from how Brouwer describes adding as "setting it while the former is retained" while he is emphasizing the importance of pairing in his dissertation (Brouwer, 1907, p.179, fn.1). But this is not the point I want to stress at the moment with my discussion of systems. I shall go back to it later in Section 3.2.

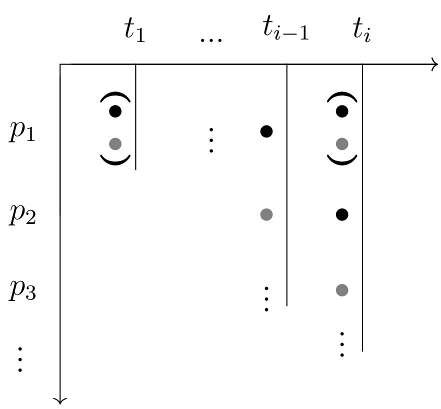
Rather, I wish to draw attention to the fact that Brouwer is presuming that the empty twoity (● ●) can be shown to be a system from this characterization of system constructions. Indeed, Brouwer contended in the second quoted passage from the introduction that the basic systems of mathematics are the empty twoity and its then and now units (Brouwer, 1954, p.2). But, if

[5]Another natural approach to the study of constructions is through Kripke schema (see van Atten (2018)). Due to its formulation in the intuitionistic theory of sequences of natural numbers, it has the advantage that it can go beyond separable mathematics. But, to my mind, this alternative is more directly amenable to the analysis of the construction of true propositions (e.g. the study of weak counterexamples in van Atten (2018, §4)) rather than that of objects like separable entities, our focus in this paper. This distinction corresponds to that between the intuition *that* a propositions is true and intuition *of* an object (see Tieszen (1989, ch.1,§3)).

we are to take Brouwer at his word, then the subject has to be able to construct the empty twoity by actually pairing the then and now units to show it is a system. I claim that to meet this condition the subject must carry out a new construction of the empty twoity that differs from its original construction in which, as explained in Section 2.2, the empty twoity is created directly by abstraction on pairs of sensible experiences. Some remarks on what makes a construction different from another construction are needed to justify my claim.

Since we regard constructions as processes consisting of certain operations (see Section 2.4), two constructions may result in the same separable entity but still be distinct from each other if the operations involved in them are different. This intensional view of construction is not clearly stated in Brouwer's writings, but is shared by van Atten (2018, p.1597). The original construction of the empty twoity by abstraction from a pair of sensations is then different from a construction of the empty twoity (• •) by pairing • and •. They must be distinguished because no pairing of units is present in the original construction. In this direct creation of the empty twoity by abstraction, what is paired are sensations still invested with material content and not units. If Brouwer admitted that units be paired already at this stage he would contradict his premise that the empty twoity is genetically the first separable entity. Thus, the original construction does not suffice to show that the empty twoity is a system, at least not if Brouwer's proposed system constructions should be taken seriously.

So the only way to show that the empty twoity is a system is to "reconstruct" it as a pair of its two units after it has been originally "born" at the initial stage t_1. To be more precise, the subject must carry out a construction where, from t_1 with (• •) in p_1, as usual, they arrive again at (• •) in p_1 but at a later stage t_i, for some $i > 1$, after a pairing operation. Given that it must be obtained as the result of the pairing of the two units • and • in this order, at t_{i-1} the subject must experience • in the present p_1 and • in the immediate past p_2. This is roughly how this reconstruction should look like diagrammatically:

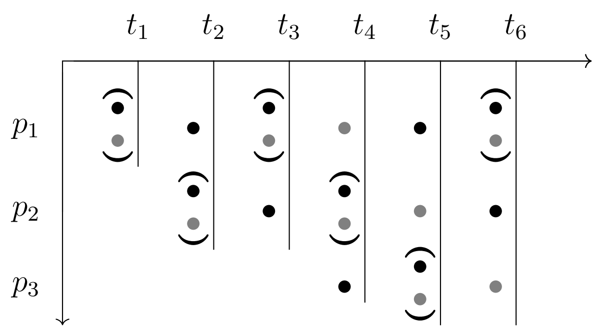


To my mind, such a reconstruction of the twoity is impossible if the subject only has access to pairing and projecting as basic operations. Units can only be obtained by projection on pairs with units as their first or second elements. But the pair a unit is projected out from would have to occur immediately below the unit as a retention (see the penultimate diagram), making it impossible to pair a unit with another unit. Obviously we are missing a basic operation of construction if something as simple as pairing two units is an illegal move.

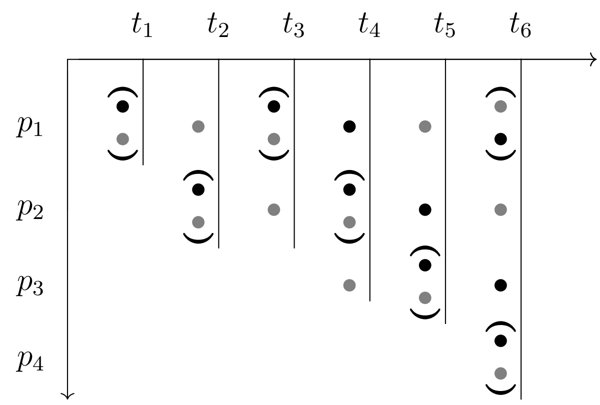
As far as I am aware, Brouwer has never addressed the issue in print. Yet the solution that fits his ideas most naturally seems to be to introduce a recalling operation. The idea of recalling is that a subject should be able to bring back to the present retentions deep down in the past—after all, what would be the purpose of storing something in the memory if it cannot be brought back to mind? If subjects were not capable of recalling as part of the intuition of twoity, then any separable entity that was not constructed in the present or immediate past, that is, in the first two rows of a diagram, would no longer be accessible to the subject, unless they construct them once again. Let me give an example. As it will be seen in Section 3, Brouwer constructs successors of numbers by pairing them with a unit. Suppose after a subject constructed, say, a one-hundred-ity by incessantly pairing the empty twoity with units ninety eight times, they decide to take a break. During their break they happen to be affected by two other experiences, perhaps the perception of a cup of coffee on the table and then a bird on the window. After this disastrous interruption, they would be unable to resume the construction from the point where they stopped. They must begin the construction of the one-hundred-one-ity from scratch starting with the empty twoity.

It could be argued in Brouwer's defense that he does not have to worry about such distractions in his constructions because the example I gave above happens to be an empirical one. But this would be to miss the general point I am making. At any stage t_i, the subject is only able to pair the separable entities in the immediate past p_2 and present p_1. After successive pairing and projecting from the empty twoity, the subject will always eventually lose access to older constructions retained in the further past p_j, for $j > 2$. The constructions of threeity and fourity I propose in the next section, for example, show how quickly previously constructed numbers fall out of the scope of pairing if a recalling operation is not used.

I will therefore admit recalling as a basic operation of construction. With recalling, it is easy for the subject to reconstruct the empty twoity by first projecting its second element, bringing the pair back to mind, projecting its second element, recalling the retained second element, and then pairing them both once a past and present unit are obtained in succession:



I do not want to give the false impression that recalling is only useful for reconstructions. Some separable entities only become constructible after the introduction of recalling. The best example is the construction of the "reverse" empty twoity (● ●). It is very similar to the reconstruction of the empty twoity described above, except for the order of the projections and the fact that the empty twoity is retained in a distant past p_4 at the furthest stage t_6:



With recalling, a subject may pass to a new stage without having a new experience. In other words, at the end of a construction the number stages t_i might be greater than the number of present and retained experiences p_js. Therefore, in general we have $i \geq j$.

2.4 General outline of the interpretation

I have argued that the initial stage from which all separable mathematics can be developed is the stage of time t_1 at which the empty twoity is first created by abstraction directly from a pair of sensations invested with material content. Constructions are then processes in which from this initial stage the subject arrives at a final intended stage t_n. At each stage stage t_i, the subject can transit to a next stage t_{i+1}, where $i < n$, constructing one separable entity as the result of an application of the basic operations of pairing, projecting, and recalling. It is finally time to give a more general description of all the elements of my interpretation:

Start. *The subject has constructed* $(\bullet\ \bullet)$ *in the present* p_1 *at stage* t_1*, with no retentions. This is the starting point of the construction of all other separable entities.*

Pairing. *Given two separable entities* α *and* β *constructed at stage* t_i*, with* β *in the present* p_1 *and* α *as a retention in the immediate past* p_2*, the subject may form a new pair* $(\alpha\ \beta)$*, in this respective order, that is experienced in the present* p_1 *at the next stage* t_{i+1}*, retaining* β *in the immediate past* p_2 *and* α *in the further past* p_3*. Other retentions* p_j *at* t_i*, for* $j > 2$*, are also retained at the new stage* t_{i+1} *but one level further in the past* p_{j+1}*.*

Projecting. *Given a separable entity* $(\alpha\ \beta)$ *at stage* t_i *in the present* p_1*, the subject may project its first or second element, therefore having either* α *or* β *in the present* p_1 *at the next stage* t_{i+1}*, while* $(\alpha\ \beta)$ *is retained in the immediate past* p_2*. Any retentions* p_j *at* t_i*, for* $j > 0$*, are also retained at the new stage* t_{i+1} *but one level further in the past* p_{j+1}*.*

Recalling. *Given one separable entity* α *at stage* t_i *in the present* p_1*, and any sequence of retentions in which* β *occurs in* p_j *some point down the line, the subject may transit to a next stage* t_{i+1} *with* β *in the present* p_1*, followed immediately by* α *in the immediate past* p_2*. All other former retentions are preserved in the same order. That is, each retention in the past level* p_k *at* t_i *also occurs at* t_{i+1} *one level further* p_{k+1} *if* $k < j$ *or in the same level* p_k *if* $k > j$*.*

If we wanted to be absolutely rigorous in our diagrams we could have specific labels at every non-initial stage t_i, for $i > 1$, to mark the operation used to arrive at it. I shall omit them for simplicity but still treat constructions that only differ by their use of operations as different constructions (see the fourity construction in Section 3.1 for a good example).

I regard this diagrammatic interpretation of construction as nothing but a more careful expression of Brouwer's (1954) claim that the empty twoity and its two units generate all separable mathematics through pairing and continuous retention. There Brouwer wrongly declared that pairing was the only operation of construction. But the two units can only be extracted from the empty twoity if projecting is admitted as a basic operation. So, I believe van Atten (2015, p.19) is right in his recognition of the importance of projecting. But one crucial observation that has been neglected until now is that a third basic operation, recalling, is necessary to reproduce Brouwer's constructions. What is most interesting about recalling is not that it serves to resume older constructions, as expected of an operation of this sort. It is that its presence is required to reconstruct the empty twoity as a pair of two units (and thus to show that it is a system) and that it results in constructions of new separable entities that would not be constructible otherwise like the reverse empty twoity.

3 The problem of ontological explosion

Ontological explosion refers to the admission of an overabundant ontology. To begin with, commitment to a Meinongnian view of nonexistent objects leads to an explosion of entities, as any imaginable object becomes a constituent part of one's ontology. This is perhaps the most extreme example of what an ontological explosion looks like. Serious worries about the phenomenon of explosion of entities are expressed by Sosa (1999), though in the context of a general background that is not limited to mathematical objects. In this paper I wish to concentrate on cases of explosion concerning the admission of duplicate entities, more specifically within a mathematical ontology. Ontological explosion in this sense refers to the existence of multiple entities that are all identified with a same mathematical object. The admission of two different objects that serve as the number three, for instance. The existence of duplicates is clearly problematic if we expect mathematical objects to be unique.

Questions about different entities playing the role of a single mathematical object may be reminiscent of the problem of identification famously raised by Benacerraf (1965) against the reduction of natural numbers to some particular sets. However, there is one crucial difference that sets the problem of ontological explosion apart. The problem of identification is about which entity is the one that should be identified with a certain mathematical object. One asks for example which set, $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ or $\{\{\{\emptyset\}\}\}$, actually identifies the number 3. Indeed, the possibility of having both identifications is even explicitly rejected as absurd.[6] The problem of ontological explosion is about the existence of multiple entities that are all equally identified with a same mathematical object. There is no question about which one identifies the mathematical object given that all of them do. The problem is simply a matter of ensuring the uniqueness of mathematical objects in the presence of duplicates.

[6]See alternative (A) in (Benacerraf, 1965, p.56).

Brouwer's ontology is affected by this phenomenon of ontological explosion. More precisely, the totality of all separable entities that can be constructed assuming the intuition of twoity must contain duplicates of the canonical natural numbers: one, two, three, etc. I will expose this explosion of entities after a diagrammatic analysis of the sketch of the construction of the canonical natural numbers given in the Cambridge lectures (Brouwer, 1981, p.90) and the study of a very elementary fragment of arithmetic that can be done with these constructed numbers using addition drawing from Brouwer's (1907) dissertation. The problem of ontological explosion here is roughly that the canonical natural numbers cannot be uniquely determined according to Brouwer's views if addition is to be well-defined for every natural number in his separable mathematics. To be exact, since, as it will be seen, addition amounts to pairing, and pairing is only allowed for two different separable entities, to represent the addition of a number with itself the universe of Brouwer's separable mathematics must necessarily "explode" and contain several duplicates that serve as a single number.

3.1 The canonical natural numbers

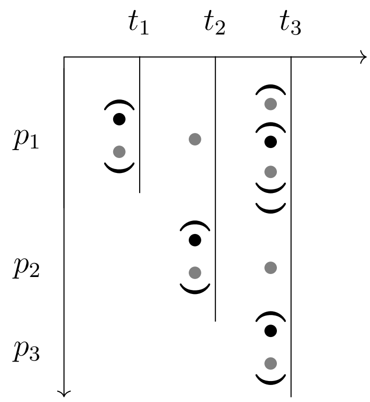
Instead of one, two, three, and so on, I will follow Brouwer's jargon and refer to the canonical natural numbers constructed from his views on intuition as unity, twoity, threeity, and so on. This terminological distinction will be proven useful after we demonstrate that his canonical natural numbers are not uniquely determined, for we can more naturally use the indefinite article and speak of a unity, a twoity, a threeity, and so on. I will still occasionally speak of the numbers one, two, three, and so on when I want to emphasize uniqueness.

The scheme that Brouwer proposes to describe the construction of the canonical natural numbers in the appended sketch from the Cambridge lectures (Brouwer, 1981, p.90) can be explained by means of the diagrammatic interpretation and its three basic operations. This scheme of construction can be more briefly rephrased in the following way:

- twoity;
- twoity retained;
- twoity giving rise to unity;
- twoity and unity giving rise to unity plus unity;
- threeity as twoity plus unity;
- the remaining canonical natural numbers.

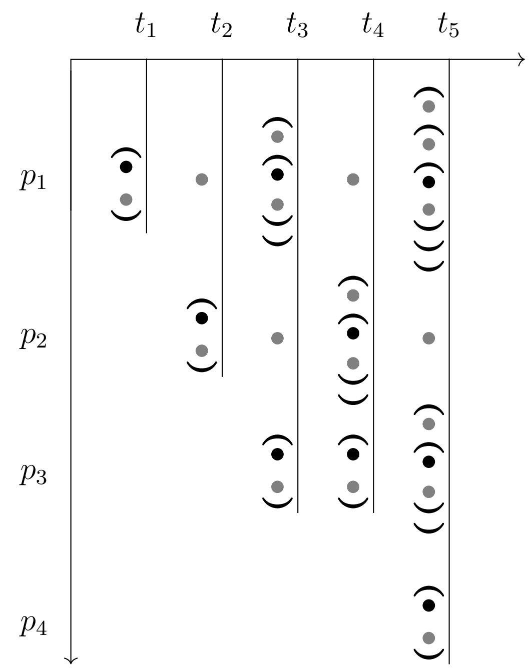
Twoity in this scheme clearly refers to the empty twoity (• •) and not an arbitrary twoity of sensations still invested with material content and in need of abstraction. I take that what Brouwer then means by "twoity giving rise to unity" should be evident from the dissertation passage we examined earlier on the precedence of pairs (Brouwer, 1907, p.179, fn.1). Twoity gives rise to unity by means of first projection to construct the unity •. Therefore the process "twoity, twoity retained, and twoity giving rise to unity" when explained in detail is exactly the construction of the unity • illustrated in our fourth diagram. The only difference is that here Brouwer is vague about which unit should be projected out.

Brouwer's proposed construction of threeity shows that pairing underlies the process of "twoity and unity giving rise to unity plus unity". After the first element • is projected out of the empty twoity (• •) at t_2 and the subject becomes aware of it in the present p_1, they can also perceive the empty twoity as a retention in the immediate past p_2. Therefore, they may be affected by another experience of two distinct things in time, past and present, "unity plus unity", just by thinking of them together. This forms ((• •) •) as "twoity plus unity", which, in other words, is a threeity. It is experienced at the next stage t_3 in the present p_1, where • dives in the past p_2 and (• •) in the even further past p_3. Notice how Brouwer carefully writes "twoity" and "unity" in this respective order to emphasize that the twoity is experienced in a more distant past than unity. In diagrammatic terms we have:



Surprisingly, the construction of the remaining canonical numbers is a bit tricky. First of all, according to this scheme, Brouwer seems to want to define fourity as threeity as unity, that is, as the separable entity (((• •) •) •). How should the subject proceed to construct it after finishing the construction of threeity at t_3? If we look at the diagram above we will see that the subject cannot simply pair the two separable entities in the present p_1 and immediate past p_2. That would result in another separable entity, (• ((• •) •)). It does correspond one-to-one with (((• •) •) •), but it is not in agreement with Brouwer's proposal.

Instead, Brouwer would presumably say that we need to construct a "threeity plus unity". Given that the subject has already constructed ((• •) •) in the present p_1 at stage t_3, they just need to become aware of the unity • in the present p_1 at the next stage t_4. The construction is completed by pairing the threeity retained in p_2 and unity in p_1, resulting in the experience of "threeity plus unity" in the present moment p_1 of the newest stage t_5:



Notice that two different operations allow the subject to become aware of the unity • after constructing the threeity and move from t_3 to t_4: one is to project the second element of $((\bullet\ \bullet)\ \bullet)$ to obtain • in the present p_1; the other is to recall • to bring it back to the present p_1. Either way the result is the same. Because we suppress operation labels in our illustrations for simplicity, both constructions give us diagrams that visually look the same, despite the distinct operations used to arrive at the unity in each case.

What about the construction of all other canonical numbers? It is not hard to see that, like the fourity, any other "$(n + 1)$-ity", for $n \geq 3$, can be constructed analogously either by second projection on the n-ity or by recalling the unity and then pairing with the n-ity. What is the earliest stage in time the subject can construct $(n+1)$-ity? Well, it always takes two additional stages to project/recall the unity and then pair it with n-ity. So, at the beginning, the empty twoity is constructed at t_1 by stipulation, and the construction of unity and threeity can be finished as early as at stages t_2 and t_3, respectively. Then, as a rule, for any other canonical numbers we have that, assuming an arbitrary n-ity can be constructed as early as at stage t_k, for any $n \geq 3$, then $(n+1)$-ity can be constructed at t_{k+2}. To give an example, the construction of fourty-two-ity along these lines requires at least $3 + 39 \times 2 = 81$ stages! It should be noted that the threeity is an exception because the subject can obtain a unit by the first projection and construct threeity directly by pairing the empty twoity and the unit without having to appeal to neither projecting nor recalling. Despite its exceptional status, threeity does fit a more general pattern of having the pair $(n\ \bullet)$ as the successor of n-ity.

I trust that with this we have explained how "the sequence of natural numbers" can be obtained by a subject under our diagrammatic interpretation. But let me finish this subsection with an additional remark about system constructions: we can tell that not only the empty twoity and its two units, but all the remaining canonical numbers, from their construction, are systems as well, for so is the pair $(\alpha\ \beta)$ given any two systems α and β.

3.2 Why addition leads to ontological explosion

Brouwer's sketch addresses the construction of his natural numbers, but leaves out the explanation of their arithmetical operations. How should addition, multiplication, and exponentiation of his natural numbers, constructed as we just shown, be understood? In the course of this investigation we will quickly come across an explosion of entities. The phenomenon also arises from multiplication and exponentiation, but since they are defined via addition we can pinpoint Brouwer's account of addition as the root of the problem.

I mentioned in Section 2.3 that Brouwer treats adding as pairing. This suggests, for instance, not only that fourity $(((\textcolor{gray}{\bullet}\ \bullet)\ \textcolor{gray}{\bullet})\ \textcolor{gray}{\bullet})$ is defined as threeity plus unity, but also that $(\textcolor{gray}{\bullet}\ ((\textcolor{gray}{\bullet}\ \bullet)\ \textcolor{gray}{\bullet}))$, a separable entity we met before, is actually unity plus threeity. From this pairing view of addition we can infer that Brouwer understood multiplications and exponentiation of numbers in terms of repeated pairing of separable entities. A brief look at Brouwer's dissertation will be particularly illuminating to substantiate this claim. In fact, he opens the dissertation with a three-page long explanation of how numeric equality amounts to one-to-one correspondence and how addition, multiplication, and exponentiation can be defined by means of repeated counting. Brouwer tells us, in particular, that:

> It follows that any fixed set of signs, once counted, will produce the same 'natural number' if it is counted in a different order, that is to say, the sequence of ordinal numbers to which it is brought into a one-to-one correspondence, will be interrupted at the same number (Fundamental Theorem of Arithmetic).
>
> By $3 + 4$ I mean the following: First count up to 3, then count on, but let the elements after 3 correspond one-to-one with the sequence of ordinals $1 \ldots 4$. It follows from the fundamental theorem of arithmetic: $3 + 4 = 4 + 3$. Likewise $(3 + 4) + 5 = 3 + (4 + 5)$. (Brouwer, 1907, pp.4–5)

Brouwer's explanation alludes to a linguistic account of counting in terms of written signs, but there is little doubt that the constructions here are those of pairing of separable entities. This is later suggested by Brouwer (1907, p.77) in his dissertation, when he claims that for some fundamental parts of mathematics it has been shown how they can be "built up from units of perception, by simple juxtaposition [...] while at every stage in the process complete systems which have been constructed before can be taken as new units."

In the passage above, Brouwer appeals to one-to-one correspondence to conclude that threeity plus fourity equals fourity plus threeity and derive similar equalities. This may seem to go against our underlying assumption, made here and before in Section 3.1, that, to return to a familiar example, fourity $(((\textcolor{gray}{\bullet}\ \bullet)\ \textcolor{gray}{\bullet})\ \textcolor{gray}{\bullet})$ and unity plus threeity $(\textcolor{gray}{\bullet}\ ((\textcolor{gray}{\bullet}\ \bullet)\ \textcolor{gray}{\bullet}))$ are distinct separable entities. It is crucial to distinguish at this point between numeric equality and identity of separable entities. The empty twoity $(\textcolor{gray}{\bullet}\ \bullet)$ is not identical to the "reverse" empty twoity $(\bullet\ \textcolor{gray}{\bullet})$, though definitely numerically equal to it given the obvious one-to-one correspondence between units. If these were identical separable entities then the identity of $\textcolor{gray}{\bullet}$ and $\bullet$ would follow. But the very phenomenon of twoity is founded on the realization of both things being different. Thus, numeric equality is a coarser relation that forgets the distinction between the units $\bullet$ and $\bullet$ and only sees how many units a separable entity consists of. Van Atten (2018, p.1597) has recently introduced a distinction between tokens and types to deal with general cases of similarity in Brouwer's constructions. He writes:

> At the most concrete level, construction processes occurring at different times are for that reason different processes. But we may come to see that processes that are different in this sense have various things in common, and we may therefore see them as instantiations or tokens of the same type of construction process. The same can be done for constructions in the other two senses, constructed objects and the objectified processes. For example, this allows us to observe that an act in which we construct the number 2 and an act in which we construct the number 3 have in common that the objects constructed in them are of the same type, that of natural number. In the extreme case, we may even come to identify processes with one another, and identify the objects constructed in them. This is the sense in which we can say, for example, that when constructing the number 2 time and again, each time we carry out the same construction process in which we construct the same object. (van Atten, 2018, p.1597)

Using this token-type distinction, we may then say that $\bullet$ and $\bullet$ are different as tokens of the type of separable entities but identical as tokens of the type of natural numbers. So, one possibility to articulate the distinction more fully is to define the types of separable entities and natural numbers with their respective criteria of token identity as follows:

Definition 3.1. *The type of separable entities has as tokens all separable entities. Identity between separable entities is the smallest reflexive relation.*

Definition 3.2. *The type of natural numbers has as tokens all separable entities identified by one-to-one correspondence, that is, the smallest equivalence relation* $1\text{-}1$ *such that* $\bullet$ $1\text{-}1$ $\bullet$ *and* $(\alpha\ \beta)$ $1\text{-}1$ $(\beta\ \alpha)$ *and* $(\alpha\ \beta)$ $1\text{-}1$ $(\alpha'\ \beta')$*, for* α $1\text{-}1$ α' *and* β $1\text{-}1$ β'.

Identity of separable entities is simply identity of tokens in the type of separable entities, while, numeric equality, which is just one-to-one correspondence, amounts to identity of tokens in the type of natural numbers. Are these the only notions of sameness that Brouwer needs to be careful about when doing separable mathematics? I think not. To show how an ontological explosion occurs in Brouwer, let me call attention to a third notion of identity that lies in the middle ground between identity of separable entities and numeric equality.

I call this third notion "identity of canonical numbers". By a canonical number I mean the numbers unity, twoity, threeity and so on that, as seen in Section 3.1, Brouwer defined in the sketch appended to the Cambridge lectures (Brouwer, 1981, p.80). Judging from how his proposed constructions go, we may alternatively say that a number is canonical just in case it is unity or a pair consisting of a canonical number plus unity.

Identity of canonical numbers is finer than numeric equality. To give a simple example, threeity, which is in fact twoity plus unity, is canonical but unity plus twoity is not. The one-to-one correspondence relation simply collapses the canonical and non-canonical distinction by only looking at the number of units in a separable entity.

Yet, identity of separable entities must be even finer than identity of canonical numbers. This is why an explosion occurs in the ontology of Brouwer's separable mathematics. Brouwer seems to be driven into a corner whenever we ask what it means for a number to be added by itself due to his view of addition as pairing. Even a trivial theorem such as "fourity is numerically equal to twoity plus twoity" needs some considerable degree of interpretation because it is unclear what separable entity "twoity plus twoity" is. Of course, it cannot be $(({\color{gray}\bullet}\ \bullet)\ ({\color{gray}\bullet}\ \bullet))$, as pairing two identical things is not allowed. For a legitimate separable entity, Brouwer's only way out is to pair two distinct pairs of distinct units. So "twoity plus twoity" can be either $(({\color{gray}\bullet}\ \bullet)\ (\bullet\ {\color{gray}\bullet}))$ or $((\bullet\ {\color{gray}\bullet})\ ({\color{gray}\bullet}\ \bullet))$ but nothing else. This shows that Brouwer must admit both the empty twoity $({\color{gray}\bullet}\ \bullet)$ and its reverse $(\bullet\ {\color{gray}\bullet})$ as different separable entities that play the same role of his canonical number two. If only one of them were recognized as twoity, there would be no way to express "twoity plus twoity" in his separable mathematics.

So both $({\color{gray}\bullet}\ \bullet)$ and $(\bullet\ {\color{gray}\bullet})$ need to be identified as Brouwer's canonical number two. Because of how the other canonical numbers are defined, this overabundance of identifications is immediately carried over to them too. Since twoity is unity plus unity, both units ${\color{gray}\bullet}$ and $\bullet$ need to serve as unity. Once we observe that threeity is defined as twoity plus unity, we see that $(({\color{gray}\bullet}\ \bullet)\ {\color{gray}\bullet})$, $((\bullet\ {\color{gray}\bullet})\ {\color{gray}\bullet})$, $(({\color{gray}\bullet}\ \bullet)\ \bullet)$, and $((\bullet\ {\color{gray}\bullet})\ \bullet)$ all must be simultaneously admitted as the canonical number three, threeity, in separable mathematics. This duplication of canonical numbers goes on indefinitely and grows exponentially, since every other $(n+1)$-ity is defined as n-ity plus unity and thus there are twice as many $(n+1)$-ities as n-ities. I propose as a definition of the type of canonical numbers and its corresponding identity relation:

Definition 3.3. *The type of canonical natural numbers has as tokens ${\color{gray}\bullet}$ and $\bullet$, and, if α is a token, so are $(\alpha\ {\color{gray}\bullet})$ and $(\alpha\ \bullet)$, but nothing else. Tokens are identified by the smallest equivalence relation $\equiv$ such that ${\color{gray}\bullet} \equiv \bullet$ and $(\alpha\ \beta) \equiv (\alpha'\ \beta')$ provided that $\alpha \equiv \alpha'$ and $\beta \equiv \beta'$.*

Thus, to make addition possible as pairing, the universe of Brouwer's separable mathematics has to contain exactly two different separable entities that function as the numbers one and two and, moreover, for $n \geq 2$, 2^n different separable entities for each number $n+1$. Separable mathematics is thus based on an overabundant ontology that admits $2,199,023,255,552$ different separable entities that are all just as well identified with the number 42!

The ontological explosion that Brouwer falls victim of therefore has to do with the discrepancy between the fine-grained identity of separable entities and the slightly coarser identity of canonical numbers (which is still finer than numeric equality). It says that the canonical numbers cannot be unique in his separable mathematics as long as elementary arithmetic operations like addition are well-defined for all natural numbers.

I think this ontological explosion reveals a deep fact about where Brouwer's intuitionistic mathematics starts deviating from classical mathematics: we do not even have to look as far as its unusual treatment of the continuum. The deviation already occurs at the level of the discrete where the uniqueness of Brouwer's canonical numbers cannot be guaranteed.

Why is this ontological explosion something Brouwer would want to avoid? In Section 1 we mentioned Brouwer's claim that the intuition of twoity can reconstruct systems that correspond to those of classical discrete mathematics (Brouwer, 1954, p.2). The problem of ontological explosion indicates that the correspondence envisaged by Brouwer cannot be an exact one for systems of natural numbers. In Brouwer's defense, he does actually state elsewhere that classical discrete mathematics can be rebuilt in a "suitably modified form", indicating that in his eyes the correspondence does not have to be exact:

> Inner experience reveals how, by unlimited unfolding of the basic intuition, much of [classical] 'separable' mathematics can be rebuilt in a suitably modified form. (Brouwer, 1981, p.5)

It is hard to tell what are the suitable modifications Brouwer has in mind. But certainly a system where there are 2 number ones, 2 number twos, and 2^n number $n+1$s, for $n \geq 2$, does not seem to be a suitable modification of the classical system of natural numbers, which ensures the uniqueness of each natural number. In other writings, Brouwer replaces the expression "suitably modified form" with "slightly modified form" (Brouwer, 1952, p.141). But, again, a modification with consequences of this magnitude cannot be a slight one. Brouwer's views on intuition result in an overabundant ontology of separable mathematical objects and a structure that is hardly recognizable as that of the natural numbers. Brouwer thought that the intuition of twoity could reproduce all classical discrete mathematical systems. The problem of ontological explosion we exposed cast doubts on this general claim.

Is there a way out of the ontological explosion for Brouwer? To circumvent the problem, he could try to ground his separable mathematics in a type of canonical natural numbers, such as the type from Definition 3.3. It would then be stipulated that to each canonical number there corresponds exactly one token, so separable mathematics could be done by operating with tokens of this type instead of separable entities directly. Unfortunately, at the fundamental level the ontological explosion would remain problematic for Brouwer because the subject can never completely forget about the distinctions that two identical canonical numbers have as separable entities. If the subject could disregard their difference, then (• •) and (• •), for example, should be indistinguishable in all contexts and no extra care would be necessary when substituting one for another in a separable entity. Yet, the acknowledgment that ((• •) (• •)) is not a separable object but ((• •) (• •)) is reveals that this is mistaken.

4 Concluding remarks

I proposed a diagrammatic interpretation of Brouwer's constructions of separable entities based on time diagrams in the style of Brentano. One essential element of my interpretation is the observation that constructions need not only pairing and projecting as basic operations, as is sometimes assumed, but also a recalling operation. I also provided a diagrammatic reconstruction of the constructions out of the intuition of the empty twoity sketched by Brouwer in an appendix to his Cambridge lectures (Brouwer, 1981, p.90), concentrating on the canonical natural numbers. An explosion of canonical numbers threatens Brouwer's separable mathematics because of his treatment of addition as pairing. Brouwer can only make certain elementary arithmetic operations, like adding a number to itself, if he allows for the same canonical number to be determined by more than one separable entity. The token-type distinction can be helpful to state the problem more accurately with the type of canonical numbers. But the fact remains that Brouwer must admit that his canonical numbers are not unique as separable entities. This appears to be a problem for Brouwer, who thought his separable mathematics corresponds in some sense to its classical counterpart.

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