In Defence of Discrete Plural Logic (or How to Avoid Logical Overmedication When Dealing with Internally Singularized Pluralities)

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Abstract
In recent decades, plural logic has established itself as a well-respected member of the extensions of first-order classical logic. In the present paper, I draw attention to the fact that among the examples that are commonly given in order to motivate the need for this new logical system, there are some in which the elements of the plurality in question are internally singularized (e.g. 'Whitehead and Russell wrote *Principia Mathematica*'), while in others they are not (e.g. 'Some philosophers wrote *Principia Mathematica*'). Then, building on previous work, I point to a subsystem of plural logic in which inferences concerning examples of the first type can be adequately dealt with. I notice that such a subsystem (here called 'discrete plural logic') is in reality a mere variant of first-order logic as standardly formulated, and highlight the fact that it is axiomatizable while full plural logic is not. Finally, I urge that greater attention be paid to discrete plural logic and that discrete plurals are not used in order to motivate the introduction of full-fledged plural logic—or, at least, not without remarking that they can also be adequately dealt with in a considerably simpler system.

Keywords
Classical first-order logic, discrete plurality, singular logic, solid plural logic, solid plurality.

1 Introduction: a distinction that deserves more attention

In the recent (and rapidly growing) literature on plural logic, there is a distinction that deserves more attention than it has received: the distinction between plural terms whose referents are explicitly singularized (i.e. terms such as 'Whitehead and Russell' and 'two men'), and plural terms whose referents are not explicitly singularized (i.e. terms such as 'the authors of *Principia Mathematica*' and 'some men'). Let us call the former 'discrete plural terms' and the latter 'solid plural terms'.

Both discrete plural terms and solid plural terms pose a challenge to classical first-order logic, especially in connection with collective predication. Thus, statements such as 'twenty men surrounded the fort' and 'some men surrounded the fort' (featuring a discrete and a solid plural term, respectively) are difficult to formalize within the language of pure first-order logic, because the property 'surrounded the fort' is predicated collectively of the men in question—i.e. it is not reducible to a sum of individual predications, one for each man.

However, the fact remains that in terms such as 'Whitehead and Russell', 'two men' and 'twenty men' (i.e. in discrete plural terms) the reference to the plurality carries with it an explicit division of the plurality into individual components, whereas in terms such as 'some men' and 'the authors of *Principia Mathematica*' (i.e. in solid plural terms) this is not the case. Should this difference be taken into account at the time of prescribing a logical theory to deal with each kind of term and the inferences in which they occur? I believe it should and I will try to show why in the course of the present paper.

I shall begin, in Section 2, by pointing to three notable contributions to plural logic in which examples of discrete plural terms are given a significant role in the motivation of this logical theory, as well as three other notable contributions in which they are not. Then, in Section 3, I shall compare the notion of discrete plural term with other notions occurring in the literature (in particular, the notion of 'list of singular terms' and the notion of 'term formed by conjunction') and I shall point out the similarities and differences between them. In Section 4, I shall give an outline of a system of discrete plural logic, based on a paper published by me in 2010 under the name 'G.F. Díez', which has gone generally unnoticed. In sections 5 and 6, I shall highlight the fact that discrete plural logic is axiomatizable while full plural logic is not. Finally, in Section 7, I shall conclude that prescribing the use of full plural logic in order to deal with discrete collective predication is a case of 'logical overmedication', and I shall urge that discrete plural logic receives greater attention from theorists both of plural logic and classical first-order logic.

2 The use of examples of discrete and solid plural terms in the literature

Many plural logicians rely on examples of discrete plural terms, as well as solid plural terms, in order to motivate the introduction of this logical theory. Applying such a distinction, these authors can be described as maintaining that both kinds of plural terms (discrete and solid) justify the introduction of plural logic, in view of the inadequacy of standard first-order logic to deal with them. We shall mention three instances of this, among the most important contributions to the subject: Yi (2005), McKay (2006) and Oliver and Smiley (2016).

Yi (2005: 460) claims, indeed, that

> The Fregean systems, Frege's system and its descendants, cannot deal with the logic of the *plural constructions* (in short, *plurals*) of natural languages.

and illustrates this claim with the following examples:

> *Venus and Serena* are tennis players, and *they* won a U.S. Open doubles title.
> *Venus and Serena* are *the females who won a Wimbledon doubles title in 2000.*
> **There are some** tennis players who won a U.S. Open doubles title.
> *The females who won a Wimbledon doubles title in 2000* won a U.S. Open doubles title.
> (Italics, as well as boldface and underlining, are always as in the original.)

As we can see, Yi appeals to the discrete plural term 'Venus and Serena' as well as to the solid plural terms 'they', 'some tennis players' and 'the females who won a Wimbledon doubles title in 2000', and contends that 'the Fregean systems' are inadequate to deal with them, in the contexts given.

Our second case in point is McKay (2006: 5), who likewise blames standard first-order logic for its inability to give a satisfactory analysis of plurals, using as illustrations both solid and discrete plural terms:

> Standard first-order logic does not provide adequate resources for properly representing many ordinary things that we say.
> Arnie, Bob and Carlos are shipmates.
> …
> Many predicates can be true of some things without being true of any of them. For example:

> They are shipmates (classmates, fraternity brothers)
> They are meeting together
> They lifted a piano
> They are surrounding a building
> They come from many different countries
> They weight over 500 pounds
> Standard systems of logic provide no place for such predication.

According to McKay, then, the resources of standard first-order logic are inadequate for representing a plural term such as 'they' (solid), as much as a plural term such as 'Arnie, Bob and Carlos' (discrete), in the contexts given.

Our third example is Oliver and Smiley (2016: 1–2), where we read:

> The pluralist strategy, the one we favour, designs a new philosophical and formal logic to accommodate plural terms, plural predicates, and multivalued functions …
> Consider:
>
> (1) Whitehead and Russell were logicians
>
> (2) Whitehead and Russell wrote *Principia Mathematica*.
>
> … Our own view is, we think, simple and commonsensical: 'Whitehead and Russell' denotes two men, Whitehead and Russell, both in (1) and (2). The example witnesses the foundational thesis of this book, namely that there is such a thing as a *plural denotation*, a semantic relation holding between linguistic expressions—definite count-noun phrases or *terms*—and things, which is plural in the sense that a particular term may denote several things at once, not just one or perhaps none.

The use of the discrete plural 'Whitehead and Russell' as a paradigmatic example of a plural term is ubiquitous throughout Oliver and Smiley's book. Along with this example, examples of solid plural terms are also present, of course: 'the men who wrote *Principia Mathematica*', 'The Brontë Sisters', 'some soldiers', etc. (cf. Oliver and Smiley 2016: 2–4).

Another prominent example in this book is 'Tim and Alex met in the pub and had a pint', featuring the discrete plural term 'Tim and Alex'. This example appears on p. 36 as well as on the back cover of the book, as a caption of a picture of the authors, Alex Oliver and Timothy Smiley, in which they are indeed in a pub and having a pint. On that cover, we read:

> The authors argue that plural phenomena need to be taken seriously and that the only viable response is to adopt a plural logic, a logic based on plural denotation.

Hence, as we can see, Oliver and Smiley rely quite heavily on examples of discrete plural terms in promoting plural logic, putting them at the centre of their argument in favour of this new logical system.

Not all publications on plural logic, however, give discrete plural terms the same prominence as these three. There are other notable contributions, such as those from Burgess (2004), Rayo (2007) and Linnebo (2017), in which examples of discrete plural terms are, as a matter of fact, absent or virtually absent from the discussion.

3 Discrete plural terms versus plural terms formed by conjunction

Our category of discrete plural terms bears some similarity to Yi's 'complex terms' (2005: 477), to McKay's 'compound terms' (2006: 57), also called 'conjunctive terms' (93) and 'conjoined terms' (97), and to Oliver and Smiley's 'lists' (2016: 1) and 'lists of singular terms' (10). It is not identical, however, to any of them.

To start with, it is clear that a conjunction of terms will not count as discrete unless the terms put in conjunction are themselves discrete. Thus, for example, the plural term 'the British and the French' is a conjunction of two terms (and hence a 'list' and a 'complex term') without being discrete, because it carries with it no indication as to how many members the plurality has—i.e. how many British and French people there are.

On the other hand, our category of discrete plural terms does not coincide, either, with that of a 'conjunction of singular terms'. The term 'two men', for example, is a discrete plural (it points to a plurality that is explicitly atomized) without being a list, a complex or a conjunction of terms of any kind.

Besides, it should be noted that neither Yi nor McKay nor Oliver and Smiley introduce the categories just mentioned with a view to devising a system, simpler than full plural logic, in which to deal with those elementary plurals. In fact, they do not point to that possibility; and Oliver and Smiley, in particular, take a stance against it: "[a]ny adequate account of lists must include plural as well as singular terms" (2004: 609).

As far as I know, there is only one place in which the exact notion of discrete plurality is introduced, albeit under different terminology, and a logic laid out—a fragment of full plural logic—specifically to deal with it. This is my own paper, Díez (2010), of which I shall give a summary in the next two sections.

4 Basics of discrete plural logic

For the sake of simplicity, in Díez 2010 I restricted attention to languages without descriptions and without function terms. In standard first-order logic, the only terms of such languages are individual constants and individual variables. Discrete plural logic starts off, then, by adding the stipulation that if 't_1', ..., 't_n' are terms (for $n \geq 2$), then '$[t_1, \ldots, t_n]$' is also a term. Terms obtained by one or more applications of this rule will be called 'compound terms', while individual constants and variables will be called 'singular terms'. As is obvious, compound terms are plural terms intended to stand for discrete pluralities—i.e. pluralities of objects as referred to by natural language discrete plural terms.

According to the definition just given, compound terms may have a nested structure, in correspondence to nested discrete plurals in natural language. An example of such a nested structure is

> Desmond and Molly and Jasper and Claire competed in the dance contest

meaning that each couple competed against each other, rather than each of the four dancers competing with the other three. Discrete plural logic does not contain, however, the sort of plural constants and plural variables that allow direct representation of natural language solid plurals, such as 'the Brontë sisters' or 'some men'. Plural constants and plural variables belong specifically to solid plural logic (i.e. to full-fledged plural logic).

On the other hand, as discrete plural logic is conducted in a finitary language, it lacks the resources to directly represent pluralities such as 'infinitely many natural numbers' or 'denumerably many elements'—let alone bigger ones, such as 'uncountably many points', etc. This imposes a limitation of size that adds to the informal definitions of discrete plurality and natural language discrete plural term with which we started: according to this, indeed, all plural terms referring to infinite pluralities, and all infinite pluralities themselves, must be taken to be solid. Hence, a discrete plurality will be, in sum, *a plurality of a known and finite number of objects*. And a natural language discrete plural term will be *one that makes reference to a discrete plurality by explicitly mentioning its members one by one, or by some other means indicating how many there are*.

Then, in order to define atomic formulas, discrete plural logic requires an n-place relation symbol to be followed by n terms, with the usual commas and brackets, just as in standard first-order logic. The fact that terms in discrete plural logic can be singular or compound suffices to make it possible to represent discrete collective predication in a natural way. Thus, we can put

$$W\ (\ [h,\ r],\ p\ )$$

for 'Whitehead and Russell wrote *Principia Mathematica*' (where 'W' is a two-place relation symbol standing for 'x wrote y' , '$[h,\ r]$' stands for Whitehead and Russell, and 'p' stands for *Principia Mathematica*). On the other hand, we again emphasize that solid collective predication (such as 'The Brontë sisters supported one another', an example from Oliver and Smiley (2016: 3) cannot be adequately represented in this language.

In discrete plural logic, it is not possible to directly quantify over a plurality of objects, given that the only variables available are individual variables. This is in contrast, once more, with what happens in the case of solid plural logic. However, if the plurality carries with it an explicit singularization of its components (i.e. if it is a discrete plurality), then we can use individual variables to quantify over the components of the plurality in question. Thus, we can put

$$\exists x\ \exists y\ (M(x)\ \&\ M(y)\ \&\ x \neq y\ \&\ W([x,\ y], p))$$

for 'two men wrote *Principia Mathematica*' (where 'M' stands for 'x is a man' and 'W' and 'p' are as before). And we can put

$$\exists x\ (M(x)\ \&\ x \neq r\ \&\ W([x,\ r], p))$$

for 'Russell wrote *Principia Mathematica* together with some other man'.

Given these elements, discrete plural logic needs only one additional item to be put to work, apart from the usual first-order deductive apparatus. This is the indiscernibility rule of inference (or axiom) for compound terms, which ensures that two discrete pluralities can be equated if and only if they have the same elements. According to this rule, two compound terms '$[t_1, \ldots, t_n]$' and '$[r_1, \ldots, r_m]$' can be equated if and only if each of 't_1', …, 't_n' can be equated with some of 'r_1', …, 'r_m' and each of 'r_1', …, 'r_m' can be equated with some of 't_1', …, 't_n'.

It is easy to see, then, that most first-order inferences concerning discrete pluralities will be accounted for in this system. Thus, this system allows us to account for the validity of inferences such as:

Whitehead and Russell wrote *PM*.
Hence, Russell and Whitehead wrote *PM*.

Whitehead, Russell and Whitehead wrote *PM*.
Hence, Whitehead and Russell wrote *PM*.

Whitehead and Russell were different men and wrote *PM*.
Hence, two men wrote *PM*.

Whitehead and Russell were different men and wrote *PM*.
Hence, Russell together with some other man wrote *PM*.

Etc.

(For more examples and details of actual deductions, see Díez 2010: 150–8).

5 The axiomatizability of discrete plural logic

The formal language I introduced in Díez 2010 lacked function symbols as well as the description operator and the 'among' symbol. The among symbol is a binary relation symbol common to plural logic languages (usually introduced as a logical symbol and often represented by '$\prec$'), which is intended to stand for the relation that holds between an object and a plurality to which that object belongs.

These three elements (function symbols, the description operator and the among symbol) can easily be added to the system described in the previous section, in order to account for further inferences pertaining to discrete plurals. The result will be a slightly more sophisticated version of discrete plural logic than the one I offered in Díez 2010.

The details of this extension are straightforward, except perhaps for the addition of 'among' as a logical symbol, which requires a subsidiary rule of inference (or axiom) governing its use. This would be a rule to the effect that '$[t_1, \ldots, t_n] \prec [r_1, \ldots, r_m]$' can be used in a deduction if and only if each of 't_1', …, 't_n' can be equated with some of 'r_1', …, 'r_m'. Both this rule (the 'among rule', as we could call it) and the indiscernibility rule are arguably logical rules of inference (or axioms). Then, armed with these additional elements, we will be able to account for the validity of some discrete plural inferences that lie outside the scope of the simplified system given above. An example of such an inference (featuring 'among' and a discrete plural description) is:

Whitehead and Russell were the two men who wrote *PM*.
Hence, Russell was among the two men who wrote *PM*.

In Díez (2010: 158–61), I gave a set-theoretic semantics for discrete plural logic in the simple version as well as an outline of the proofs of soundness and completeness for that system. These proofs, in turn, can be straightforwardly expanded to the sophisticated version, thus showing discrete plural logic to be axiomatizable in both versions (with and without function symbols, descriptions, the among symbol and its governing rule). In fact, the modifications needed in order to transform standard first-order logic into discrete plural logic (in either of the two versions) are so straightforward, that the latter can be regarded as a mere 'variant', so to speak, of the former.

Furthermore, it is arguable that in a discrete plural term, the basic vehicle of reference is the individual object (i.e. each of the components of the atomized plurality), rather than the plurality itself. According to this, a term such as 'Whitehead and Russell' would not be as genuinely plural as would 'the authors of *Principia Mathematica*', given that in the former but not the latter the plurality is referred to through an explicit division of the plurality into individual components. Seen in this light, discrete plural logic is also closer in spirit to classical first-order logic than to full plural logic.

Indeed, a plausible divide between the concept of a 'singular logic' and the concept of a 'plural logic' would be the principle that *plural logic, by contrast with singular logic, allows direct reference, predication and quantification over pluralities of objects* (cf. again Díez 2010: 153, paraphrased). According to such a principle, what we have been calling 'discrete plural logic' should be classified as a singular logic rather than as a plural logic. It was with this idea in mind that I first called this system 'first-order logic for itemized collections' (Díez 2010: 156), and if I am now suggesting to call it otherwise, it is only with a view to increasing its visibility.

Besides, there is a major feature that distinguishes discrete plural logic from solid plural logic at the metatheoretical level: the former is axiomatizable while the latter is not. Surprisingly enough, however, nearly all expositions in the field fail to take notice of this fact.

6 The unaxiomatizability of full plural logic

All discrete plural terms can be represented in full plural logic languages such as the ones set forth in Yi (2005, 2006), McKay (2006) and Oliver and Smiley (2016), although it must be noted that some of these languages allow a more natural representation than others. Thus, a term like 'Whitehead and Russell' can be represented in McKay's language in a most natural way by means of a compound term '$[h, r]$', just like ours (McKay 2006: 57–8). The same can be said of Yi's 'plenary plural language', which contains a term connective '@' (standing for 'and') with which we can render 'Whitehead and Russell' as '$[h@r]$' (Yi 2005: 477).

In Yi's 'meager plural language', however, the term connective '@' is eliminated in favour of the logical predicate '**H**' standing for 'is-one-of' (Yi's version of 'among'), together with the rest of his logical vocabulary (Yi 2006: 240–1). In such a language, a discrete plural term like 'Whitehead and Russell' will have to be represented by a sort of circumlocution such as 'a plurality to which Whitehead belongs, and Russell, and no-one else'.

As to the plural languages presented in Oliver and Smiley (2016), although they make room for the introduction of a term-forming function symbol 'and', they do not classify it as a logical symbol (164ff., 177ff., 221, 244). Hence, if we want to represent a term like 'Whitehead and Russell' in such languages, out of a pair of individual constants ('h' and 'r') and without resorting to any other extra-logical symbol, we will have to use their among symbol ('$\preceq$') and a circumlocution akin to the one we have just given in relation to Yi's meagre plural language (cf. Oliver and Smiley 2016: 108ff., 221ff., 244ff.).

Given the fact that discrete plural terms (and inferences concerning them) can be represented and accounted for in full plural logic, discrete plural logic cannot aspire to be more than a 'poor relative' of full plural logic. However—and here is where logical overmedication begins—there is no reason to justify the lack of attention that has been afforded to this distinctly relevant fragment of the bigger system. In particular, while we are routinely informed that full plural logic is not axiomatizable (e.g. Burgess 2004: 220, Yi 2006: 257, McKay 2006: 140–1, Oliver and Smiley 2016: 217), almost nobody cares to report the axiomatizability of the fragment corresponding to discrete plural logic (an exception is, again, Díez 2010: 153). Authors who rely on examples of discrete plurals in their defence of the need for plural logic, as those seen in Section 2, are especially liable to this criticism.

In the case of Oliver and Smiley (2016), in particular, the meticulousness with which they discuss many other issues, examining alternative proposals and sub-proposals to the smallest detail (cf. 9–12, 44–9, 65–72, 97–102, 120–8, etc.), makes it remarkably striking that they ignore such a simple strategy to deal with these basic plurals. Given that two examples of discrete plurals ('Whitehead and Russell' and 'Tim and Alex') have a most salient role in their book, one would expect them to do as much as mention this possibility of accounting for them.

7 Conclusion: avoiding logical overmedication

Propositional logic is a relatively uninteresting fragment of first-order logic (in the sense that it is much less useful), and yet it is customary for logic handbooks to discuss it and to highlight that it possesses a major metatheoretical property—decidability—that the whole system does not have. Something similar can be said of monadic first-order logic. This sort of information is helpful, because when we face the task of analysing the validity of an inference, we look, in general, for the simplest logical theory at our disposal that is suitable to do the job.[1]

The main aim of the present paper, then, is to urge that expositions of plural logic pay a similar tribute to the fragment that we have been calling 'discrete plural logic'. Thus, I want to urge that expositions of plural logic mention the existence of such a fragment, that

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[1] In fact, I am convinced that, sooner or later, it will become standard practice in expositions of plural logic to point to this axiomatizable fragment of the system (whether it is called 'discrete plural logic', 'first-order logic for itemized collections', 'first-order logic with compound terms' or something else). After all, the basic idea behind it is quite simple: *for dealing with itemized lists, we only need to introduce compound terms and an indiscernibility rule for them, and the resulting system is axiomatizable*. It will only take a few expositions of plural logic to include a remark to that effect, for the idea to seep through—it is only a matter of time. On the other hand, as I have just remarked, pointing to discrete plural logic as a relevant fragment of full plural logic does not undermine the importance of the latter, any more than using propositional logic undermines the importance of first-order logic. Thus, we can account for the validity of a basic inference such as

John is ill.

If John is ill, he stays at home.

Hence, John stays at home.

by an expedient as simple as two propositional letters (p for 'John is ill' and q for 'John stays at home') and a truth table, dispensing with the use of predicate letters and a predicate logic proof procedure. But by doing that we are not neglecting the importance of predicate logic, we are simply pointing to a simple solution for a simple problem.

they point to the class of plural inferences that can be adequately accounted for within it and that they highlight the fact that this subsystem of plural logic bears the property of axiomatizability, a property that full plural logic does not have. A complementary goal would be that some expositions of singular first-order logic were also enriched with the discussion of this minor extension, which enables us to deal with inferences concerning discrete collective predication at minimum cost. Neither of these two goals undermines, in my view, the importance of full plural logic and its essential role in dealing with solid plurals.

When we go to the doctor about a health problem, we expect to be given the simplest, least invasive treatment that can cure us. When we ask an engineer to build a bridge, we expect them to apply the simplest physical theory in order to design it (e.g. Newtonian mechanics rather than relativistic mechanics). Likewise, when we go to the logician to help us analyse the validity of certain inferences, we expect to be given the simplest logical theory that can adequately account for them. *Look for the simplest treatment that suits your needs*, would be the underlying principle. On these grounds, authors who appeal to examples of discrete pluralities in order to motivate the introduction of full plural logic should think twice about doing so, or should at least include a comment to the effect that those examples, in particular, can be dealt with in a much simpler system—one that does not require the introduction of plural constants, plural variables and plural quantifiers.

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