THE LOGIC OF TIME AND THE CONTINUUM IN KANT'S CRITICAL PHILOSOPHY

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Abstract. We aim to show that Kant's theory of time is consistent by providing axioms whose models validate all *synthetic a priori* principles for time proposed in the *Critique of Pure Reason*. In this paper we focus on the distinction between time as form of intuition and time as formal intuition, for which Kant's own explanations are all too brief. We provide axioms that allow us to construct 'time as formal intuition' as a pair of continua, corresponding to time as 'inner sense' and the external representation of time as a line Both continua are replete with infinitesimals, which we use to elucidate an enigmatic discussion of 'rest' in the *Metaphysical foundations of natural science*. Our main formal tools are Alexandroff topologies, inverse systems and the ring of dual numbers.

§1. Introduction: Kant's theory of time. In the present work we provide a formalization of the theory of the temporal continuum developed by Immanuel Kant in the *Critique of Pure Reason*[1] and in other works of his critical period. This enterprise belongs to the more general project, which began in [Achourioti and van Lambalgen, 2011], of laying firm mathematical foundations for various aspects of Kant's Transcendental Philosophy.

Formalizing Kant's Transcendental Philosophy yields substantial gains. In particular, formalization helps to clarify the structure of the intricate web of concepts of Kant's philosophy, and is thus exegetically advantageous, since it makes the structure of Kant's argumentation explicit. The task of evaluating Kant's argumentation is thus simplified, and a more precise comparison of competing interpretations is made possible. The usefulness of mathematical formalization in this context is readily seen in the case of Kant's theory of time. While certainly crucial for the general argument of the first *Critique*, this theory is not obviously consistent, and its most difficult passages have given rise to multiple competing interpretations. Indeed, there have been fierce debates on how to interpret even some of the most fundamental concepts of kant's theory of time, such as the distinction between time as form and as formal intuition[2]. Our aim here is to use the resources of formal logic to show that Kant's theory of time is consistent and to shed some light on these controversial interpretative points.

The formalization that we are going to develop will be based for the most part on the properties which are ascribed to the intuition of time in the *Transcendental Aesthetic* and the *Transcendental Analytic*, in the *Critique of Pure Reason*. Here, as well as in other texts of his critical period, Kant provides an analysis of the synthetic *a priori* principles valid of our representation of time. These principles are *synthetic* because they are not conceptual truths. The term "*a priori*" has different senses, of which the

[1] Henceforth CPR; we make use of the Cambridge edition, see [**?**].

[2] Our take on this problem, from a formal perspective, is given in Section 10.

following is the most relevant for our present purpose: a principle is *a priori* if it is necessary and its necessity can be established without recourse to any particular sensory experience. "Necessary" is thus not to be interpreted as "true in all possible worlds". Logically speaking, necessary principles are preconditions for the possibility of making judgments that can be assinged a truth value. As such, they belong to what Kant calls "transcendental logic", and our aim is to show that these principles constitute a *formal* logic of time, as part of the more general project to present "transcendental logic" as a logic in the full modern sense.

Beyond strictly exegetical considerations, we also believe that Kant's theoretical philosophy is relevant to contemporary discussions in the philosophy of space and time, in the foundations of mathematics and physics, and in the cognitive sciences[3]. We thus hope that a formalization of Kant's theory of time could highlight its relevance to contemporary debates regarding the nature of time. We cannot of course treat exhaustively all of the aforementioned aspects in this paper. We will, however, draw such connections here and there in due course, sometimes more systematically, and give now some general remarks by way of illustration.

In the foundations of mathematics one can interpret Kant's theory of time as part of a tradition of thought about the concept of the continuum which, following [Feferman, 2009], we can term "phenomenological". Indeed, in the history of the concept of the continuum we can broadly ideantify two rival conceptions. On the one hand, one might take the continuum as built up from atomic, dimensionless entities, such as monads or points. The Cantorean continuum, a set of points on which various axioms of order or a distance function can be imposed, is an example of this conception of the continuum, which is the direct descendant of the medieval theory of the *compositio ex punctis* (see [Maier, 1966] and the discussion in [Grunbaum, 1977]). In opposition to this conceptualization of the continuum one might reject, on either philosophical or physical grounds, the "emergence" of macroscopically extended entities from actually infinite collections of dimensionless points, and pursue instead a "top-down" approach in which the continuum is an extended whole which "sticks together", and whose parts are themselves also continua. One might also object to the use, in the Cantorean approach, of actual infinities, and hold that the continuum must be given before its parts; these are then introduced by a process of division or approximation which is never exhausted, and is hence only "potential". This was Aristotle's view in the Physics, and it has been held in various forms by numerous philosophers and mathematicians in history[4]. The tension between these two different views of the continuum is already present in Euclid's *Elements*, where a point is defined as "that which has no parts" – an atomic entity – but at the same time it is specified that "the extremity of a line are points", which suggests that points "supervene" on the lines.

Kant's view of the time continuum belongs to the phenomenological tradition, and has indeed many notable similarities with the Aristotelian view of the continuum. In [Sachs, 1995], for instance, we find: "A point is like the now in time, i.e., not a part of it, but only the beginning, the end, or the division of time". In the sequel, we shall see that this characterization of the "point" and of the "now" is akin to Kant's, and we shall

[3] There is an increasing awareness in cognitive science that Kant's treatment of space, time and causality is still a rich source of ideas; see [Northoff, 2012] and [**?**] for recent illustrations.

[4] Kant, Hermann Weyl, C.S. Peirce, and Brouwer all share such concerns, albeit in different forms.

provide a formalization of it. From these considerations we can already conclude that it is *prima facie* inappropriate to model Kant's temporal continuum by means of the reals.

The synthetic *a priori* principles for time can often be read in two ways, as either cognitive or physical principles. This ambivalence can be explained by the fact that in Kant's philosophy physics and cognition are inexstricably intertwined, as is witnessed by the *Metaphysical Foundations of Natural Science* [**?**], where Kant attempted to derive as much as possible of physics *a priori* from the groundwork of the Critique. In this work we shall not delve deeply into the exact relation between physics and cognition within Kant's philosophy. We shall however make use of this ambivalence throughout, and intersperse remarks on the cognitive and physical interpretations of our formalism in the appropriate places. Indeed, we begin by listing Kant's *a priori* principles for time and we illustrate them using passages from the works of Isaac Barrow and Isaac Newton. Further remarks on the relevance of Kant's theory of the temporal continuum for the foundations of Physics can be found in Section 12.

1.1. Synthetic a priori principles for time.

1.1.1. Time is not an empirical concept that is somehow drawn from an experience. For simultaneity or succession would not themselves come into perception if the representation of time did not ground them a priori. Only under its presupposition can one represent that several things exist at one and the same time (simultaneously) or in different times (successively). (A30/B46)

Comment. One might think that the concept of time is acquired from natural timekeepers (in the 'starry heavens'). This leads to a relative concept of time, unless there is a way to compare the timekeepers; in other words, unless the temporal relations are *objective*. Even if time were a concept, that concept could not have empirical criteria for its applicability.

Newton: "It is possible that there is no uniform motion by which time may have an exact measure. All motions can be accelerated and retarded, but the flow of absolute time cannot be changed. The duration or perseverance of the existence of things is the same, whether their motions are rapid or slow or null; accordingly, duration is rightly distinguished from its sensible measures and is gathered from them by means of an astronomical equation"[5]

1.1.2. Time has only one dimension; different times are not simultaneous, but successive. (A31/B47)

Comment. To see that this is a synthetic judgment, one has to read 'times' as 'extended continua', not 'instants'. These 'times' can be compared to non-intersecting line segments, and simultaneity is a relation between such lines.

Isaac Barrow: in the first of his *Lectiones Geometricae* Barrow says that 'Time, abstractly speaking, is the continuance of each thing in its own Being'[6]. However [Arthur, 1995] observes that "since some things continue to exist longer than others, these times are durations with respect to the beings in question, and thus are relative measures"[7]; indeed, Barrow holds that 'Time, absolutely, on the other hand, is a quantum; admitting (in its own way) the principal affections of quantity: equality, inequality and proportiomn'[8]. Thus time is one-dimensional because this is the only way in which it can be absolute, or objective.

1.1.3. Time is no [...] general concept, but a pure form of sensible intuition. Different times are only parts of one and the same time. That representation, however, which can only be given through a single object, is an intuition. (A31-2/B47)

Comment. This is Kant's conclusion from 1.1.4. Concepts are by definition 'universal', meaning that they never pin down an object uniquely. This is because concepts (or at least basic concepts)

[5] See [Newton, 1999], Scholium to the definitions, p.410.

[6] See [Barrow, 1976], *Lectio I*. The same passage is quoted in [Arthur, 1995], p. 344.

[7] See [Arthur, 1995], p. 344.

[8] Ibid., p. 344.

arise by abstracting properties shared by different exemplars, which are compared in order to bring them under a concept. Since parts of time share all properties with time itself, and different times are parts of the one time, there is nothing to compare time with, hence time cannot be brought under a concept. The logical aspects of this situation are prominent in the following passage from the *Reflexionen*:

1.1.4. *Refl* **4425** *Spatium est quantum, sed non compositum*. For space does not arise through the positing of its parts, but the parts are only possible through space; likewise with time. The parts may well be considered *abstrahendo a caeteris*, but cannot be conceived *removendo caetera*; they can therefore be distinguished, but not separated, and the *divisio non est realis, sed logica*.
Comment. This rules out constructions like that of the Cantor set, obtained by iteratively removing the middle third of intervals. The Cantor set is totally disconnected; by contrast, our axioms will have as a consequence that the continuum is strongly connected, for various senses of 'strong'.

1.1.5. The infinitude of time signifies nothing more than that every determinate magnitude of time is only possible through limitations of a single time grounding it. The original representation time must therefore be given as unlimited. But where the parts themselves [...] can be determinately represented only through limitation, there the entire representation cannot be given through concepts [...] but immediate intuition must ground them. (A32/B47-8)
Comment. It is tempting to gloss this by saying time is potentially infinite in the same sense as the intuitionistic conception of $\mathbb{N}$; but this cannot be right, since the successor operation adds to the integers already constructed, whereas we cannot add to time. Constructing a new natural number would correspond to create a proper part of the one objective time. As Kant puts it in *Refl.* **4756** 'All given times are parts of a larger time. Infinity.' However, this notion of infinity is partly topological: the 'given time' (obtained by 'limiting' time) is like a closed subinterval of an open interval.

1.1.6. The property of magnitudes on account of which no part of them is the smallest (no part is simple) is called their continuity. Space and time are *quanta continua* because no part of them can be given except as enclosed between boundaries (points and instants), thus only in such a way that this part is again a space or a time. Space therefore consists only of spaces, time of times. Points and instants are only boundaries, i.e., mere places of their limitation; but places always presuppose those intuitions that limit or determine them, and from mere places, as components that could be given prior to space or time, neither space nor time can be composed. Magnitudes of this sort can also be called flowing, since the synthesis (of the productive imagination) in their generation is a progress in time, the continuity of which is customarily designated by the expression "flowing" ("elapsing"). All appearances whatsoever are accordingly continuous magnitudes [. . .] (A169-70/B211-2)
Comment. Continuity implies the possibility to construct arbitrarily fine subdivisions of time; but any concrete subdivision consists of a finite number of extended parts of time. To be able to iterate this construction, the parts of time must have the same continuity properties as time itself: 'Space therefore consists only of spaces, time of times.' Instants correspond to a kind of 'cut' in time, but time is not the set of cuts, unlike the unit (open) interval. Time exists as a whole prior to any subdivision – but how? Newton provides the ingredients for Kant's answer in *CPR*.
Newton: 'All things endure in so far as they remain the same at every time. The Duration of each thing flows, but its enduring substance does not flow, and is not changed with respect to before and after, but always remains the same. Its actions, however, are changed, but that these are changed, and are successively manifested according to the will of that which endures, argue perfection"[9]

1.1.7. The three modi of time are *persistence*, *succession* and *simultaneity* [...] Only through that which persists does existence in different parts of the temporal series acquire a magnitude,

[9] See [McGuire, 1978], p. 117.

which one calls duration. For in mere sequence alone existence is always disappearing and beginning, and never has the least magnitude. Without that which persists there is therefore no temporal relation. (A177/B219)

Comment. Kant will adopt Newton's linking of substance and persistence, but with a transcendental twist: without persistence no successiveness. Hence time cannot be modeled by a linear order only, one needs a timeless substrate as well. Given this, one can interpret successiveness as change of state, or *alteration*:

1.1.8. Here I add that the concept of alteration, and, with it, the concept of motion (as alteration of place), is only possible through and in the representation of time – that if this representation were not a priori (inner) intuition, then no concept, whatever it might be, could make comprehensible the possibility of an alteration, i.e., of a combination of contradictorily opposed predicates (e.g., a thing's being in a place and the not-being of the very same thing in the same place) in one and the same object. Only in time can both contradictorily opposed determinations in one thing be encountered, namely successively. Our concept of time therefore explains the possibility of as much synthetic a priori cognition as is presented by the general theory of motion, which is no less fruitful. (B48-9)

That is, now, the law of the continuity of all alteration, the ground of which is this: That neither time nor appearance in time consists of smallest parts, and that nevertheless in its alteration the state of thing passes through all these parts, as elements, to its second state. No difference of the real in appearance is the smallest, just as no difference in the magnitude of times is, and thus the new state of reality grows out of the first, in which it did not exist, through all the infinite degrees of reality, the differences between which are [arbitrarily small]. (A209/B254)

Comment. Here we have a necessary condition for the application of the category of causality; the latter says that the transition from state a to state b is given by a rule – here a function of time. The necessary condition expresses that the techniques of the differential and integral calculus can be applied – a small increment in the time variable leads to a small increment in the state variable.

1.1.9. *Refl.* **4756**[10] Is there an empty time before the world and in the world, i.e., are two different states separated by a time that is not filled through a continuous series of alterations[?] The instant in time can be filled, but in such a way that no time-series is indicated.
All parts of time are in turn times. The instant. Continuity.

Comment. The instant is not a point, but a collection of pairwise overlapping events – for example, episodes retrieved from memory ('synthesis of reproduction in imagination', A101). Furthermore binding features together to create an appearance ('synthesis of apprehension in an intuition', A99) is a physical process of non-zero duration. Of particular interest here are appearances that limit parts of time, such as reading the position of the hands of a clock – such limits are never dimensionless.

1.1.10. *Refl.* **4756** (continuation of 1.1.9) The infinite is greater than any number. Allness or totality (the all) is not to be understood in a series, nor to be comprehended in an aggregate. The infinite of continuation or juxtaposition. The infinitely small of composition or decomposition. Where the former is the condition, the latter does not occur. Infinite space and infinite past time are incomprehensible. In the world there is encompassment, process, and division into the infinite.

1.1.11. We cannot think of a line without drawing it in thought, we cannot think of a circle without de- scribing it, we cannot represent the three dimensions of space at all without placing three lines perpendicular to each other at the same point, and we cannot even represent time without, in drawing a straight line (which is to be the external figurative representation of time), attending merely to the action of the synthesis of the manifold through which we successively determine the inner sense, and thereby attending to the succession of this determination in inner sense. (B154).

[10] See [**?**] for an English translation of Kant's notes and fragments.

Comment. The idea that time needs an outer representation that supports all temporal concepts (successiveness, duration ...) is a striking anticipation of the recent finding that duration estimates are much more reliable when the relevant events are projected on a timeline; see Section 2.1 and Section 10. This need for an outer representation of time as "flowing" is already present in Barrow:

Barrow: "As a line, I say, is looked upon to be the Trace of a Point moving forward, being in some sort divisible by a Point, and may be divided by Motion one Way, viz. as to Length; so Time may be conceiv'd as the Trace of a Moment continually flowing, having some Kind of Divisibility from an Instant, and from a successive Flux ... And like as the Quantity of a Line consists of but one Length following the Motion; so the Quantity of Time pursues but one Succession streched out as it were in Length ... We therefore shall always express Time by a right Line"[11]

As we remarked above, these passages show that Kant's theory of time cannot be fully be captured by means of a set-theoretic construction of the continuum, by Dedekind cuts or equivalence classes of Cauchy sequences. The differences between the Kantian and the set-theoretic continuum become even more obvious when we consider how Kant glosses the properties of, e.g., infinity and continuity of time. Infinity (1.1.5) is explained in mereological terms. Continuity (1.1.6) is defined as the absence of simple parts, along with the primacy of extended parts over the points which bound them. In this respect, Michael Friedman has argued ([Friedman, 1992], p. 60ff) that Kant conflated the notion of density and continuity, noting that before Dedekind it was customary to explain both properties as "between any two points there is a third"; Onof and Schulting [Onof and Schulting, 2015] make the same point. We however remark that this definition of continuity would not be acceptable for Kant, since parts of time, and not points, are primitive entities in his ontology. Indeed, as we shall see starting from section 8, Kant's notion of continuity turns out to be much more refined.

Kant's rejection of boundaries as primitive entities appears in full at 1.1.9. An instant being a boundary separating two parts of time does not mean that the instant is empty or atomic, but that the series of alteration which the substance undergoes within that instant is not specified. This hints at the fact that the Kantian instant can itself be "put under a magnifying glass", and can thus be refined to reveal further parts of time. After all, since boundaries are constituted from parts of time, they must be extended (1.1.6); being extended, however, they must be infinitely divisible; thus any boundary can be split indefinitely. We shall see in the sequel that this aspect of Kant's notion of instant is captured in the formalization by using a family of temporal structures which are linked by mappings representing divisibility. Notice that these are not just sophistical subtleties, without consequences for the mathematical description of the temporal continuum. Indeed, we found that the Kantian continuum contains entities not present in the classical continuum, namely infinitesimals (see section [**?**]).

It is also important to highlight here the role that the categories, Kant's "pure concepts of the understanding", play in the constitution of time. Time, for example, is still seen as persisting through empty intervals, which is important as it allows Kant to talk of time elapsed, and hence of *duration* (1.1.7). Since the persistence of time and its measurability are seen as an effect of the category of substance, and its relation to the alteration of phaenomena is mediated by the category of causality (1.1.8), a full treatment of time would require formal analogues for the categories and for their role in the

[11] Cited in [Futch, 2008], p. 27.

constitution of the intuition of time. Here our aims are more modest, and thus we leave a formalization of the categories (in particular the categories of relation) to future work; see section 12.

We conclude this introduction by providing the reader with a brief outline of the structure of the paper. In Section 2 we introduce the basic language and the two first-order axiom systems we shall be using. These axiom systems purport to capture Kant's notion of "objective time", and Kant's description of the "modi" of time as "persistence, succession and simultaneity". In Section 3 we show that the axiom systems have the finite model property and we provide a notion of standard models for the axioms. In Section 4 we introduce the main topological tools we shall employ and which arise from Kant's notion of 'relations of succession' (b67N). In Section 5 we describe a construction of boundaries in terms of events that is faithful to Kant's passages presented above, and deal with the notion of the infinity of time. In Section 6 we show how to embed extended events in the space of boundaries which they generate. In Section 7 we explore the strong connectivity properties of the Kantian continuum and we provide a formal correlate for Kant's dictum that "parts of time can be distinguished, but not separated (*Refl. 4425*)". In Section 8 we formalize continuity and potential infinite divisibility in terms of inverse sequences of finite models of the axiom system and retraction maps, and hence provide a formal correlate to Kant's dictum that "the property of magnitudes on account of which no part of them is the smallest (no part is simple) is called their continuity" (A169/B211). Sections 9, 10 and 11.2 then combine the results from the preceding sections to provide a formal correlate of the Kantian notion of "time as an object" or "time as formal intuition", thus capturing Kant's dictum that "parts of time are times" and that "different times are all part of one and the same time". We conclude with Section 12, where we outline some directions of future work.

§2. Axiom systems for objective time.

2.1. Subjective and objective time. Since Kant does not tire of proclaiming that time is the form of sensibility and hence has no bearing upon things in themselves, one is easily led into thinking that time is in some sense subjective. There is an important sense in which this is not so, most easily explained using a spatial analogue: among spatial mental representations one may distinguish between those which are structured by a coordinate system centered upon the subject, and those where the coordinate system is centered upon an object in the visual field, which for this reason is represented as outside of the subject. One may call the first form of spatial representation 'subjective', and the second 'objective', even if one believes these are but different forms of mental representations. The same distinction applies to time; subjective time is centered upon the subject's *now*, with respect to which past, present and future are defined. As Kant puts it:

> Our apprehension of the manifold of appearance is always successive, and is therefore always changing. We can therefore never determine from this alone whether this manifold, as object of experience, is simultaneous or successive . . .

The phrase 'our apprehension of the manifold' refers to the *synthesis of apprehension in an intuition*: we cannot take in a sensory manifold at a glance, but must engage in a process of binding of features by 'running through [the sensory manifold] and holding together [the features found]' (A99). Linked to this first synthesis is the *synthesis of* *reproduction in imagination*. When admiring the beautiful façade of a large building our eyes scan the surface and the sensory impressions obtained until *now* are reproduced and combined to generate a coherent and stable building-object.[12] It is clear that the movement of the eyes determines the order in which features are perceived, which is to say that this notion of succession is subject-centered. The use of the terms "succession" may lead one to think that subjective time is linearly ordered, but that is not so: even transitivity fails. Take any relation of succession, for instance $B(a,b)$:= 'a begins after b begins', where a, b are two *events*. The subjective content of this relation goes beyond the purely temporal. The synthesis of reproduction says that a reproduction of b must be synthesized with a. If B is transitive, this implies that reproduction of impressions arbitrarily distant in the past is considered to be possible.

The examples of reproduction supplied by Kant suggest that reproduction operates locally, but not globally, whence the failure of transitivity. The picture is rather that of 'islands in time'; because of a developmental dissociation between the ability to remember past events and the ability to think of them as being arranged in a linear order, individual memories start off as unconnected:

> There is no evidence that events are automatically coded by the times of their occurrence or that memory is temporally organized . . . ; many older events are difficult to discriminate by their ages . . . but are still presumably episodic memories; and it seems likely that we are poor at remembering the internal order of some episodic memories. [. . .] What appear to be genuine episodic memories are more like "islands in time" than memories one reaches by mentally traveling through some temporally organized representation. [Friedman, 2007]

The quote strongly suggests that excluded middle in the form $B(a,b) \vee \neg B(a,b)$ will also fail, which removes yet another prerequisite for the linearity of time. It is possible to develop a formal theory of subjective time revolving around the idea of "islands in time" using maximal linearly ordered chains. We shall not do so here however, since Kant's main interest was the *internal* representation of time as centered upon *objects*. We shall call this 'objective time' for the sake of brevity, but it should be kept in mind that objective time is just as much a mental representation as subjective time is.

In the context of objective time events receive a slightly different interpretation. For Kant, the main question concerning time is one whose answer is announced in (A177/B219):

> As regards their existence, appearances stand a priori under rules of the determination of their relation to each other in *one* time.

Formally, this means that there is a function that maps appearances to their position in time. Let us call this mapping the *tenure* function. Temporal positions are determined by relations of simultaneity (or more broadly, overlap) and by causal relations. In both cases, the relevant relations can be expressed only if appearances are brought under concepts. If α is an appearance and C a concept, the *tenure* of (α, C) is a connected part of time during which C applies to α. The *tenure* of (α, C) will be called an *event*. Temporal relations such as $B(a,b)$ ("a begins after b begins") can now be expressed in terms of tenure events. Let α be an appearance (which is held fixed for the purposes of this argument), a = *tenure*$((\alpha, C_0))$, b = *tenure*$((\alpha, C_1))$, and suppose the state of α changes

[12] Kant remarks (A99) that the same binding processes take place when counting or constructing geometric figures and that "without it we could have a priori neither the representations of space nor of time".

from $\neg C_1$ to C_1 (Kant calls this an *alteration*, which is driven by the category of cause). Since $\neg C_1 \wedge C_0, C_1 \wedge C_0$ are incompatible, by B48-9 (cf. 1.1.8 above) we must have that $C_1 \wedge C_0$ holds after $\neg C_1 \wedge C_0$, whence we obtain $B(a, b)$ if the tenure of $\neg C_1 \wedge C_0$ is non-empty, while $\neg B(a, b)$ must hold if it is empty. Note that this form of excluded middle is dependend on the particular appearance α with respect to which tenure events are defined; it is not guaranteed that tenure events of different appearances are comparable with respect to B. In principle, we would have to index B with an appearance and the event structure associated to the appearance could be largely independent of other such event structures. Indeed, quote 1.1.4 by Isaac Barrow in Section 1 strongly suggests a situation of this kind. In CPR, a correspondence is established between these event structures – each centered upon a different object – via a notion of simultaneity (the "Category of Community"). Because the notion of simultaneity is symmetric, no object constitutes a privileged temporal perspective, which is the strongest sense in which time can be objective. Below we adopt unrestricted excluded middle for temporal relations to achieve the same effect; that is, the tenure events occurring in the relation need not be tied to the same appearance.

From a cognitive standpoint, the aforementioned role of the categories in the constitution of objective time is not innate. It is on the contrary developed in infancy by interacting with the environment, analogously to the way in which the transitivity and totality of the subjective temporal order arises, as we have seen above. This of course means that the representation of objective time itself must be acquired in development. Indeed, in children one observes the following:

(1) There is no reliable correlation between causal and temporal order (i.e. children do not object to backwards causation) and children are at chance at inferring temporal order of hidden events from causal premises. In Kant's terminology, one would say that these children still lack the *category of causality*;
(2) The order of events is sometimes encoded in the child's mind, but generally not accessible to reasoning (e.g. children find it difficult to recite an event sequence in reverse order). In Kant's terminology, one would say that children have the *subjective* time, but not yet the *objective* time. One can also say that, at least w.r.t. time, these children lack the *capacity to judge* [Longuenesse, 1998], which involves making temporal relationships explicit and reasoning about them.

These remarks are in agreement with Kant's notion that objective space and time are not innate (Kant would certainly have abhorred the idea) but are *originally acquired* by interacting with the environment[13].

Note finally that the capability of judging effectively about the order of events is just the simplest capacity which is required to have a representation of objective time. On top of it, it is also important to have the capacity of judging accurately the (comparative) duration of events. There is experimental evidence that this capacity is related to the possibility of representing such durations externally, ordering them on a line [Carelli and Forman, 2012]. This is in remarkable agreement with Kant's notion of the 'outer representation of time' mentioned in 1.1.11. In fact, one can think of diagrammatic realizations of the axioms in the Euclidean plane as external representations of time. Interestingly, one needs all of Euclid's postulates to obtain faithful realisations. We will return to this topic after the presentation of the axioms.

[13]See [Longuenesse, 1998], p. 222.

2.2. Relations between events. Having established that the primitive entities of Kant's theory of time must be tenure events of some appearance concepts, what temporal relations should be considered as holding between such events?

The second and third Analogies in the CPR seem to argue that succession and simultaneity are the relevant relations between events. Above we introduced the relation B for 'begins after', but there are other such relations of succession, for instance the relation P representing 'total precedence: $P(a, b)$ iff the "right boundary" of a is to the left of the "left boundary" of b. P would satisfy axioms such as irreflexivity and transitivity, as suggested by Russell [Russell, 1914], Walker [Walker, 1947], Kamp [Kamp, 1979] and Thomason [Thomason, 1989].[14] That P cannot be the right primitive for our purposes is apparent from the discussion in the previous section on the role of causality in alteration. A moment's reflection will show that this concept of precedence fits Hume, who conceived of causal chains as discrete, rather than Kant, who did not. Kant viewed effects in a causal chain as *alterations* – changes of state – and argued that such effects can in general be simultaneous with their causes, and moreover that *natura non facit saltus*: changes of state are continuous and themselves take time. This means that the left and right boundaries of an event (say the result of a state change), even though somewhat indefinite, will overlap with another event (the cause of that change).

The appropriate representation of precedence is therefore the pair of predicates $B(e, d)$ for 'e begins after d begins' and $E(e, d)$ for 'e ends before d ends'. We thus have a predicate B comparing left endpoints of events, and a predicate E comparing right endpoints. We have in addition a reflexive and symmetric predicate O for 'overlap', which in certain circumstances one can take to be transitive as well. In mereology, overlap would be interpreted roughly as 'having a common part'; in a temporal context 'overlap' can have a wider meaning, for instance $O(a, b)$ iff part of a is simultaneous with part of b. The difference is subtle but important. Events are really spatio-temporal, even though we study only the temporal component here. From a phenomenological perspecive, a definition of overlap in mereological terms suggests events occurring at the same place, whereas the second interpretation in terms of simultaneity relations is suggestive of distant events; since Kant's category of community in the CPR implies that simultaneity can be defined for events arbitrarily distant in space, the second interpretation seems to be the right one. We shall therefore adopt the relation O as a primitive.

We shall also make use, instead of the negations of (E, B, O), of their respective antonyms (positive predicates) $(\mathbb{E}, \mathbb{B}, \mathbb{O})$. For instance, the reflexive and transitive relation $\mathbb{B}(c, a)$ is read as 'c does not begin after a'. There are technical reasons, which we shall treat later, to use the antonyms of the basic predicates rather than their negations. There are also philosophical reasons, as the predicates $\mathbb{E}, \mathbb{B}, \mathbb{O}$ are instrumental in defining relations of simultaneity and of 'encompassment' between events, and thus have a philosophical standing of their own, which becomes even more evident if one considers the aforementioned possibility of relaxing the excluded middle axioms in the case of subjective time.

Finally, we also introduce two partial operations on events $\otimes_p, \otimes_f$. These operations have two distinct Kantian interpretations, one cognitive in terms of the synthesis of

[14] We are however indebted to Thomason [Thomason, 1989] for showing the power of (a modification of) Walker's axioms. One of the differences between Thomason's methods and ours is the use of topology, made possible by a different choice of primitive predicates.

apprehension in intuition, and one physical in terms of the category of community or interaction.

The cognitive interpretation conceives of events introduced by the operations as representing a possible 'potential' binding of other events: given a collection $A = \{a_0, ..., a_n\}$ of events and an event e in the range of the operations $\otimes_p, \otimes_f$, e binds A if $A = \{a \in W \mid a \preceq e\}$. Such events are therefore postulated *a priori* and do not depend upon sensibility, but they are instrumental in integrating the *a posteriori* events – what Kant called 'the comprehension of the manifold given in accordance with the form of sensibility in an intuitive representation.'[15]

The physical interpretation, on the other hand, understands events in the range of the operations as provided also *a priori* by the category of community or interaction, so that $a \otimes_f b$ is glossed as 'that part of b which can be causally influenced by a' and $a \otimes_p b$ is glossed as 'that part of a which can causally influence b'. The introduction of events of this type is then justified by appealing to the category of community, which states that all substances – and therefore all events – must be in thoroughgoing causal interaction[16].

The dual interpretation of the operations, in cognitive and physical terms, is an instance of the fundamental ambiguity in Kant's philosophy which we mentioned in Section 1, namely, that physics and cognition are in Kant fused together inextricably. It is not the purpose of the present paper to disentangle the connection between physics and cognition in Kant's works, thus we shall accept both interpretations of $\otimes_p, \otimes_f$ as valid.

DEFINITION 1. *An* event structure *is a tuple* $\mathcal{W} := (W; O, \mathbb{O}, E, \mathbb{E}, B, \mathbb{B}; \otimes_p, \otimes_f)$*, where* W *is a set of events,* $R \in \{O, \mathbb{O}, E, \mathbb{E}, B, \mathbb{B}\}$ *is a binary relation on* $W \times W$*, and* $\otimes_p, \otimes_f$ *are partial binary operations on* W*.*[17]

For ease of exposition we introduce the following notational conventions

DEFINITION 2. *Define the relation* $\preceq$ *by*

$$b \preceq a \Leftrightarrow \mathbb{B}(a, b) \wedge \mathbb{E}(a, b) \wedge O(a, b).$$

If $b \preceq a$ *holds, we say that* a covers b*.*

2.3. Two first order theories of objective time. For event structures $\mathcal{W} := (W; O, \mathbb{O}, E, \mathbb{E}, B, \mathbb{B}; \otimes_p, \otimes_f)$ we adopt the following axiom system GT, subdivided into groups of axioms having the same function, or at least contributing toward that function:

1. Negation represented positively
   (a) $E(a, b) \wedge \mathbb{E}(a, b) \rightarrow \perp$
   (b) $B(a, b) \wedge \mathbb{B}(a, b) \rightarrow \perp$

[15] Our axioms imply than any finite collection of events is bound by a transcendental event of this type, which is too strong if one reads this as implying that any two events are *actually* bound. If one reads the transcendental events as merely possible binding, however, this difficulty is avoided.

[16] It is most interesting that some region-based approaches to relativistic spacetimes and quantum gravity have considered similar operations on events, where the excluded middle axioms for $\mathbb{B}, \mathbb{E}$ – which we understood above as grounded on the category of community – would also be violated; see, for instance, [Christensen and Crane, 2005]. This draws an interesting connection between Kant's theory of time and the foundations of relativity, which will be a topic for futher investigations (see Section 12).

[17] An important difference with the literature referenced above is that the theory of these structures is developed without assuming from the outset finiteness of the domain; if finiteness is needed for a result, this will be indicated explicitly. Instead of assuming finiteness of the domain, we use the Alexandroff topology, which allows one to prove results for arbitrary domains that were previously obtained for finite domains only.

   (c) $O(a,b) \wedge \mathbb{O}(a,b) \to \bot$
2. Principles of excluded middle
   (a) $E(a,b) \vee \mathbb{E}(a,b)$
   (b) $B(a,b) \vee \mathbb{B}(a,b)$
   (c) $O(a,b) \vee \mathbb{O}(a,b)$
3. Reflexivity, symmetry
   (a) $O(a,a)$
   (b) $O(a,b) \to O(b,a)$
4. Linearity
   (a) $\mathbb{E}(a,b) \vee \mathbb{E}(b,a)$[18]
   (b) $\mathbb{B}(a,b) \vee \mathbb{B}(b,a)$
5. Conditions for overlap
   (a) $\mathbb{B}(a,b) \wedge \mathbb{E}(a,b) \to O(a,b)$
   (b) $\mathbb{B}(a,b) \wedge \mathbb{B}(b,a) \to O(a,b)$
   (c) $\mathbb{E}(a,b) \wedge \mathbb{E}(b,a) \to O(a,b)$
6. Transitivity
   (a) $\mathbb{E}(a,b) \wedge \mathbb{E}(b,c) \to \mathbb{E}(a,c)$
   (b) $\mathbb{B}(b,a) \wedge \mathbb{B}(c,b) \to \mathbb{B}(c,a)$
   (c) $O(c,a) \wedge O(c,b) \wedge E(a,b) \wedge \mathbb{E}(a,c) \to O(a,b)$
   (d) $O(c,a) \wedge O(c,b) \wedge B(a,b) \wedge \mathbb{B}(a,c) \to O(a,b)$
   (e) $O(c,a) \wedge O(c,b) \wedge \mathbb{E}(a,b) \wedge \mathbb{B}(a,c) \to O(a,b)$
   (f) $O(c,a) \wedge O(c,b) \wedge \mathbb{B}(a,b) \wedge \mathbb{E}(a,c) \to O(a,b)$
   (g) $\mathbb{O}(c,a) \wedge \mathbb{O}(a,b) \wedge E(a,b) \wedge B(a,c) \to \mathbb{O}(c,b)$
   (h) $\mathbb{O}(c,a) \wedge \mathbb{O}(a,b) \wedge B(a,b) \wedge E(a,c) \to \mathbb{O}(c,b)$
7. Covering axiom[19]
   (a) $\exists c(a \preceq c \wedge b \preceq c)$
8. Partial binary operations $\otimes_p, \otimes_f$ on events. We only list the axioms for $\otimes_f$ as the dual axioms for $\otimes_p$ can be obtained from these by replacing $\otimes_p$ for $\otimes_f$ and $\mathbb{B}$ for $\mathbb{E}$.
   (a) $\mathbb{B}(a,b) \vee O(a,b) \leftrightarrow \exists y(a \otimes_f b = y)$
   (b) $\mathbb{B}(a, a \otimes_f b)$
   (c) $a \otimes_f b \preceq b$
   (d) $\mathbb{B}(a,b) \to a \otimes_f b = b$
   (e) $O(a,b) \wedge \mathbb{B}(b,a) \to \mathbb{B}(a \otimes_f b, a) \wedge \mathbb{E}(a \otimes_f b, b)$
   (f) $(a \otimes_f b) \otimes_f c = a \otimes_f (b \otimes_f c)$
   (g) $(a \otimes_f b) \otimes_p a = b \otimes_p a$

The above axioms will be referred to as GT, where 'GT' stands for 'geometry of time', a reference to Kant's insistence on the necessity of an 'outer' (geometric) representation of time (B154). We shall also make use of the following consequence of GT, which we call "exact covering lemma":

LEMMA 1. *Let $\mathcal{W}$ be an event structure and let $A \subseteq W \times W$ be a finite multiset of events, with enumeration $a_1, \ldots, a_n$ (possibly with repetitions) such that for all k,*

[18]Follows from anti-symmetry implied by intended meaning E, plus axiom 2a; can also be used to derive linearity of (objective) time. Implies $\mathbb{E}$ is reflexive.

[19]It follows that $\preceq$ is directed; see main text for the significance of this property.

$\mathbb{B}(a_1, a_k)$ *and* $\mathbb{E}(a_n, a_k)$. *Then there exists* c *such that for all* k, $a_k \preceq c$, $\mathbb{B}(a_1, c)$ *and* $\mathbb{E}(a_n, c)$.

PROOF. Consider a multiset of events A with cardinality $|A| \leq n$ for $n \in \omega$. A number n of applications of Axiom 7 yields an event d such that $a_k \preceq d$ for all $k \leq n$. Set then $c = (a_0 \otimes_f d) \otimes_f (a_n \otimes_p d)$. It is straightforward to check that the axioms for $\otimes_f, \otimes_p$ ensure that c satisfies the properties in the statement of the Lemma. $\dashv$

In the sequel we shall also work in the subsystem GT_0, consisting of GT minus the axiom groups 8 and axiom 7, in which the exact covering lemma 1 does not hold. We also make use of an intermediate axiom system $(GT_0 + 7)$, to which we do not impose an explicit name since it will only be used in Section 5.

The philosophical meaning of these distinctions is that such a "stratification" of axiom systems provides a formal correlate to the role which the unity of apperception plays in Kant's system. In particular, we argue that the axiom system GT_0, where all axioms are universal, captures Kant's notion of *form of intuition*, while the axiom system GT is essential to capture Kant's notion of *formal intuition*. Indeed, it is only GT that forces time to be an object in the Kantian sense: time as an object is a standard model with respect to which GT is complete[20]. Axiom system $(GT_0 + 7)$ is a middle ground between form and formal intuition, since axiom 7 alone implies, in finite models, the existence of a universal cover:

DEFINITION 3. *Let* $\mathcal{W}$ *be an event structure and* $c \in W$. *Then* c *is said to be a universal cover of* $\mathcal{W}$ *if* $\forall a : a \preceq c$.

The universal cover shall be interpreted as a correlate of the Kantian "I think"; hence $(GT_0 + 7)$ involves the action of the unity of apperception.

Undoubtedly, the reader will have drawn diagrams to clarify the meaning of the axioms and of Lemma 1. If we think of the universal cover c as the empty form of time, then the geometric content of the axioms is to construct orthogonal projections from events to c. Euclid's proposition 12 (Bk. I) shows how to construct orthogonal projections, and the theory of parallels (Bk. I, props. 27 – 32) shows that projection preserves the primitive relations and operations, as well as quantitative relationships. Although these constructions and proofs are simple, they require Euclid's five postulates (as well as his principles for comparison of magnitudes). In other words, geometric principles are involved in the synthesis of time as an object, i.e. as formal intuition (B161n). Kant was of course well aware of the importance of this connection: the concluding paragraph of his letter to Rehberg[21] reads

> The necessity of the connection of both sensible forms, space and time, in the determination of the objects of our intuition – so that time, if the subject makes it itself an object of its representation, must be represented as a line in order to cognize it as quantum, just as, conversely, a line can only be thought as a quantum by being constructed in time – this insight into the necessary connection of inner sense with outer sense even in the time-determination of our existence appears to me to aid in the proof of the objective reality of the representations of outer sense (contrary to psychological idealism), which, however, I cannot pursue further here.

[20] More precisely: complete with respect to the class of geometric sentences defined below.
[21] Kant's response to Rehberg's letter is Ak. [447] in his *Gesammelte Schriften*; see [**?**].

2.4. Comments on the axioms. One justification for the axioms is that they are necessary and sufficient to represent a finite event structure geometrically as a collection of open intervals in a linear order, the theme of section 6; here we consider philosophical justifications.

2.4.1. *Time is a priori: Kant versus Leibniz.* One of the crucial properties that Kant ascribes to time is that of being an *a priori* representation (property 1.1.1 in Section 1), meaning that time is a representation which is not abstracted from experience (as might any empirical concept), but is a condition for experience to be possible in the first place. The central aim of this work is that of providing a formalization for Kant's temporal continuum on the basis of the axiom system above and of further, second order axioms which shall be introduced later. This raises the question, in what sense can an axiomatization of the temporal continuum be *a priori*, in agreement with Kant's *dictum*?

This question is particularly pressing since the very idea of setting up axioms on events so that a linear order of instants can be constructed which represents the flow of time goes back to Russell [Russell, 1914, Russell, 1936], and, one may argue, ultimately to Leibniz' relational view of time. In the Leibniz-Clarke correspondence, Leibniz maintains that time is the system of relations of precedence or succession which hold among empirical events, the "order of succession" of objects. This order is abstracted from the experience of empirical objects themselves, in a way which in a sense predates Frege's abstraction principles, and is thus *a posteriori*. Space is defined in an analogous way as the "order of coexistence" of empirical objects; roughly speaking, humans form their concept of space by abstracting from empirical objects the "same place" relation, thereby defining space to be the class of all places[22]. As we have seen in Section 1, however, Kant was radically opposed to the idea that time might be in any way abstracted from experience in this sense, as he claimed that time cannot be perceived, and that in any case it must already be there for temporal relations between events to be determined. Since our axiomatic approach seems *prima facie* similar to that developed by Russell and Thomason on ultimately Leibnizian grounds, one might wonder whether our approach is indeed in Kant's spirit, or even whether the distinction itself is amenable to formal treatment.

In this respect we note that there is indeed an essential difference between our approach and that of Russell-Leibniz, which lies in the stratification of the axiom systems which we mentioned above. The Russell-Leibniz approach is bottom-up: it starts from a set of given events and their relations, and requires purely universal axioms for temporal relations, such as GT_0; these temporal relations on empirical events are prior to time. The properties of the temporal relations allow one to define points as certain subsets of the set of events, and a linear order on these points, which represents time. On such a view, time is a higher-order concept, which is abstracted from first-order event structures; this goes against the Kantian doctrine that temporal relations and temporal parts already presuppose time itself, one of the arguments for both the fact that time is *a priori* and undefinable. That this was not Kant's view can be seen, for instance, at A30/B46, where Kant is adamant that temporal relations have their intended interpretation (as relations between parts of time) only if time is already given (see 1.1.4).

[22] The Leibnizian relational view of space has been subject of formal treatment, in particular in the philosophy of physics; see, for instance, [Manders, 1982, Forbes, 1987, Mundy, 1983].

By contrast, our axiomatization must be understood in the top-down direction: given (a representation of) the whole of time our axioms describe properties of its parts. As *Refl* **4425** in section 1 makes clear, these parts are only virtual or potential, and cannot really be detached from the whole of time. This top-down approach requires the existential axiom 7, which ensures the existence of a universal cover in finite event structures, and also, as we shall see, implies strong connectivity properties of the Kantian continuum. Indeed, whereas in a continuum like $[0, 1]$ each $x \in (0, 1)$ induces a decomposition of $[0, 1]$ into disconnected components, omitting an event from a Kantian continuum doesn't destroy connectedness. Axiom 7 must therefore be interpreted as representing 'time as a whole', or "unbounded time", which must depend on our cognition since sensibility affords us only with bounded events. Events introduced by the covering axiom are thus of very different character from empirical events in that they are provided by cognition itself and not by sensibility. Hence while GT_0 deals with formal properties of those events that are given *a posteriori* (e.g. the event initiated by turning on the heater, and terminated by switching it off), $(GT_0 + 7)$ embodies the *a priori* action of the 'transcendental unity of apperception', the principle that governs all our cognitive functions, on the form of intuition.

Events introduced by the two operations $\otimes_f, \otimes_p$ must also be understood as postulated *a priori* in a similar fashion, as they embody a stronger use of the unity of apperception than $(GT_0 + 7)$. We already outlined the two possible interpretations of the operations in the previous Section, hence it suffices here to remark that both interpretations imply that the events introduced by the two operations cannot be given by sensibility. In the cognitive interpretation such events embody a form of potential binding of empirical events grounded on the synthesis of apprehension in intuition, which itself is an instance of the synthesis of the unity of apperception. In the physical interpretation they reflect the role of the category of community or interaction, which acts *a priori* on the sensible manifold (the empirical events) and is also, from a Kantian standpoint, a synthesis which is a "specialization" of the synthesis of the unity of apperception. In both cases, the events postulated by $\otimes_f, \otimes_p$ are provided by our own cognition.

We are therefore working within a layered system which imposes constraints of increasing strength on the existence of these *a priori* events, which can most accurately be called *transcendental events*, and which are different from the merely *empirical events* provided to us by sensibility. GT_0 does not require any transcendental event to exist and thus it captures merely the *form of intuition*. $(GT_0 + 7)$ requires the existence of covering events and thus embodies the unity of apperception in the guise of the "whole encompassing time" provided by the "I think". Finally, GT represents the system in which the unity of apperception acts fully on the manifold of sensibility, in the guise of the categories or the three-fold synthesis. This layered structure marks an essential difference between our approach and respect to the Leibniz-Russell approach.

Note moreover that our formal treatment of the potential infinite divisibility of time in section 8 and following relies on the notion of an inverse system of models of the axioms; the directedness condition on the index set of the inverse system is interpreted as a higher order version of the synthesis of the unity of apperception, which follows from the first order version embodied by the axiom system GT (see Sections 8 and 10).

2.4.2. *Objective time and excluded middle in GT_0 and GT.* Axioms 2a, 2b and 2c express the law of excluded middle for the primitive predicates, in positive form. We argued above that these laws embody a form of simultaneity that makes truly objective time possible. If we want a clean separation between subjective and objective time, axioms 4a, 4b have to go as well because they rely on axioms 2a, 2b. These axioms are independent of $GT_0 - 2a, 2b, 2c, 4a, 4b$, as can be seen from the following structure

LEMMA 2. *Consider an event structure $\mathcal{C}$ with domain $\{a, b, c\}$, with $O, \mathbb{E}, \mathbb{B}$ interpreted as reflexive, O as symmetric, and satisfying in addition:*

$$O(c, a), O(c, b), \mathbb{E}(c, a), \mathbb{E}(c, b), \mathbb{B}(c, a), \mathbb{B}(c, b).$$

The pairs $(a, b), (b, a)$ are not contained in any of $O, \mathbb{O}, E, \mathbb{E}, B, \mathbb{B}$. Then

$$\mathcal{C} \models GT_0 - 2a, 2b, 2c, 4a, 4b.$$

We add two observations that show the power of the excluded middle principles in the context of GT_0.

LEMMA 3. *(i) $\mathbb{E}(a, b) \wedge \mathbb{B}(a, b) \to O(a, b)$*
(ii) $\mathbb{O}(a, b) \to B(a, b) \vee E(a, b))$

PROOF. (i) Contraposition applied to axiom 5, followed by an application of 2a, 2b.
(ii) Apply axioms 1c, 2a, 2b. ⊣

LEMMA 4. $E(a, b) \to \mathbb{E}(b, a)$.

PROOF. Assume $E(a, b)$, then by axiom 1a $\neg\mathbb{E}(a, b)$, whence by axiom 4a, $\mathbb{E}(b, a)$. ⊣

2.4.3. *Syntactic structure of the axioms of GT_0 and GT.* All axioms except the LEM principles are (or can be brought) in 'geometric' form:

DEFINITION 4. *A formula is* positive primitive *if it is constructed from predicates $O, \mathbb{B}, \mathbb{E}$ and terms $\otimes_p, \otimes_f$ using only $\vee, \bigvee, \wedge, \exists, \perp$.*

DEFINITION 5. *A formula is* geometric *or* geometric implication *if it is of the form*

$$(\dagger)\forall \bar{x}(\theta(\bar{x}) \to \psi(\bar{x}))$$

where θ and ψ are positive primitive.

The LEM axioms express in explicit form that the models for GT_0, GT (minus LEM) are classical. However, if φ is geometric, then if $GT_0, GT \models \varphi$ (classically), then GT_0, GT (minus LEM) proves φ in intuitionistic logic. Further advantages of being geometric will appear as we go.

2.4.4. *Axiom group 5.* These axioms state conditions for events a, b to have simultaneous parts. They presuppose axioms 2a and 2b, i.e. the linearity and connectedness of time. If one wants to model time as consisting of "islands in time" (cf. p. 8) these axioms should be reformulated along the following lines

$$(\mathbb{E}(a, b) \vee \mathbb{E}(b, a)) \wedge \mathbb{B}(a, b) \wedge \mathbb{B}(b, a) \to O(a, b).$$

2.4.5. *Axiom group 6.* Axioms 6a and 6b express transitivity for the distinguished predicates $\mathbb{E}, \mathbb{B}$, which we know to be reflexive by axioms 4a, 4a. Transitivity is a strong principle in our context. An argument along the lines of the one given on p. 9 shows that $\mathbb{B}, \mathbb{E}$ are transitive w.r.t. a given object, but simultaneity (i.e. the category of community) must be invoked to obtain universal transitivity. The various transitivity properties are essential in capturing a defining characteristic of the Kantian continuum: that 'instants in time can be filled' (property 1.1.9 in Section 1).

The remaining axioms of group 6 express conditional transitivity for O and $\mathbb{O}$. Given the close connection between overlap and simultaneity, special conditions under which O is transitive are to be expected. For $\mathbb{O}$ this is less clear. One way to understand, e.g. 6g is to introduce the abbreviation P for a transitive relation of 'total precedence' by $P(a,b) := \mathbb{O}(a,b) \wedge E(a,b)$; by applying lemma 3 to $\mathbb{O}(a,c) \wedge B(a,c)$ we obtain $\mathbb{O}(a,c) \wedge E(c,a)$ which is equivalent to $P(c,a)$, whence by the assumed transitivity of P, $P(c,b)$ and $\mathbb{O}(c,b)$. As explained above, there are Kantian reasons for not taking P to be primitive, hence we work with transitivity axiom 6g instead. We do have some properties of our chosen primitives that would follow trivially had we included P among the primitives. For instance the following property is a translation of $P(a,b) \wedge O(a,c) \rightarrow B(b,c)$ into our language:

LEMMA 5. $\mathbb{O}(a,b) \wedge E(a,b) \wedge \mathbb{B}(a,c) \rightarrow \mathbb{O}(a,c)$.

Together with axioms 1a, 1a and 1c, the principles of excluded middle also allow us to reformulate the order axioms in GT_0 in terms of the distinguished predicates $O, \mathbb{E}, \mathbb{B}$. For example

LEMMA 6. *Given axioms 2a, 2b, 2c, 1a, 1a and 1c:*
(i) axiom 6c is equivalent to

$$O(c,a) \wedge O(c,b) \wedge \mathbb{E}(a,c) \rightarrow \mathbb{E}(a,b) \vee O(a,b).$$

(ii) axiom 6g is equivalent to

$$O(c,b) \rightarrow O(c,a) \vee O(a,b) \vee \mathbb{E}(a,b) \vee \mathbb{B}(a,c)$$

Obviously the modified principles lack intuitive appeal, but they have the right preservation properties when we come to consider inverse systems of event structures and their limits.

2.4.6. *The covering axiom.* Enough has been said already about the meaning of axiom 7. In the context of the "islands in time", however, this axiom has to be replaced by the following:

$$(\mathbb{E}(a,b) \vee \mathbb{E}(b,a)) \wedge (\mathbb{B}(a,b) \vee \mathbb{B}(b,a)) \rightarrow \exists c(a, b \preceq c).$$

§3. Finite model property of GT_0 and GT.

The axioms collected in GT_0 are universal, as befits axioms concerned with the *form* of intuition. As a consequence, GT_0 is in a sense complete with respect to finite models. The same, however, holds for GT. To give a precise formulation of the result, we consider the class of *geometric formulas* defined in section 2.4.3, which were argued to be the formal analogue of Kant's judgments in [Achourioti and van Lambalgen, 2011].

THEOREM 1. *Let φ be a geometric implication in the signature of GT. Then $GT \models \varphi$ iff φ holds on all finite models of GT.*

PROOF. The direction from left to right is trivial. For the direction from right to left, we prove the contrapositive. Assume $GT \not\models \varphi(\bar{x}, \bar{y})$, where $\varphi(\bar{x}, \bar{y})$ is of the form

$\forall \bar{y}(\theta(\bar{x}, \bar{y}) \rightarrow \psi(\bar{x}, \bar{y}))$ for $\theta(\bar{x}, \bar{y}), \psi(\bar{x}, \bar{y})$ positive primitive formulas. Then for some countable structure $\mathcal{M}$,

$$\mathcal{M} \models GT + \exists \bar{y}(\theta(\bar{x}, \bar{y}) \wedge \neg \psi(\bar{x}, \bar{y})).$$

Thus there must be a tuple $\bar{a}$ of objects of $\mathcal{M}$ such that $\mathcal{M} \models \theta(\bar{a}), \neg\psi(\bar{a})$. Since the tuple $\bar{a}$ is finite, Lemma 1 provides us with an object c which covers every object of $\bar{a}$. We can now define a submodel $\mathcal{M}'$ of $\mathcal{M}$ having as objects of the domain the objects in $\bar{a}$, the covering event c, and all events which can be obtained from these by closing under the operations $\otimes_p, \otimes_f$. It is then straightforward to check that $\mathcal{M}'$ is a submodel of $\mathcal{M}$ and that the equational theory of $\otimes_p, \otimes_f$ forces $Dom(\mathcal{M}')$ to be finite. Hence $\mathcal{M}'$ is the desired finite model. $\dashv$

Note that the formula expressing the existence of a universal cover $\exists x \forall y(y \preceq x)$, is true on all finite models of GT_0 (and GT) but not on arbitrary models, showing that Theorem 1 above cannot be extended beyond geometric formulas.

3.1. Standard models. In GT, the relation O has a strong, constructive, interpretation as overlap (formalized using the axioms of group 8), while GT_0 also allows a weaker interpretation of O as a proximity, the relation of being 'infinitesimally close'. We need not give the axioms for proximity, because there is a theorem that completely characterizes proximity on $[0,1]$: $U, V \subseteq [0,1]$ are in proximity if the closure of U intersects the closure of V. To highlight the constructive interpretation of O in GT, let us denote with $I([0,1])$ the set of all open intervals of $[0,1]$, and with $\uparrow i, \downarrow i$ the upset and downset of any $i \in I([0,1])$ with respect to the natural order of $[0,1]$. We then have the following:

LEMMA 7. *Consider the event structure $\mathcal{W}$ with domain $I([0,1])$ defined by letting for any $i, j \in I([0,1])$:*

1. $\mathbb{E}(i,j)$ *if* $j \subseteq \downarrow i$
2. $\mathbb{B}(i,j)$ *if* $j \subseteq \uparrow i$
3. $O(i,j)$ *if* $cl(i) \cap cl(j) \neq \emptyset$
4. $i \otimes_f j = \uparrow i \cap j$ *if* $\uparrow i \cap j \neq \emptyset$*, undefined otherwise*
5. $i \otimes_p j = \downarrow i \cap j$ *if* $\downarrow i \cap j \neq \emptyset$*, undefined otherwise*

Where B, E are defined as negations of $\mathbb{B}, \mathbb{E}$. Then $\mathcal{W} \models GT_0$ but $\mathcal{W} \not\models GT$

PROOF. The first part is easily verified, the second part follows from the failure of the axioms for $\otimes_p, \otimes_f$. Indeed, we have $O([0,x),(x,1])$, but $(x,1] \otimes_f [0,x)$ is undefined. $\dashv$

Models in which O means overlap, and not just proximity, have a concrete interpretation in the structure of rational open intervals:

LEMMA 8. *Let Ω be the event structure having as domain $I([0,1]) \cap \mathbb{Q}$ and as definition of the predicates $\mathbb{E}, \mathbb{B}$ and operations $\otimes_p, \otimes_f$ the definitions of Lemma 7, and define $O(i,j)$ iff $i \cap j \neq \emptyset$. Then $\Omega \models GT$.*

On this basis one obtains the following representation result:

THEOREM 2. *Given a finite event structure $\mathcal{W} \models GT$, there is a finite substructure $\mathcal{V}$ of Ω which is a model of GT and is isomorphic to $\mathcal{W}$.*

§4. Time as a form of intuition. According to Kant, time as a "form of intuition" deals merely with 'relations of succession' (B67n). These relations of succession can be interpreted as the predicates $(E, B, O, \mathbb{E}, \mathbb{B}, \mathbb{O})$ of an event structure. These relations have a topological interpretation, which links them closely to those properties of time listed by Kant that concern its topology: time is infinitely divisible, time does not consist of instants, time is not a 'mere sequence' but is 'persistent', alterations are always gradual and continuous, and instants arise only as boundary points (which may themselves be extended). This list makes clear that Kantian time cannot be represented by the real number line, with the topology generated by the open intervals. In fact, no single topology appears to be sufficient for deriving all features of the Kantian continuum. In this section we shall introduce a variety of topologies and topological notions which we will use in the remainder of the paper. We start with some general remarks, useful to fix notation.

Given a topological space (X, τ), one defines the *specialization ordering* $\lhd \subseteq X \times X$ of (X, τ) by letting $x \lhd y$ if $U \in \tau, x \in U$ implies $y \in U$. The specialization ordering is always a preorder, it is a poset if the topological space is T_0, and it is the trivial ordering if the topological space is T_1 (all points of the space are closed).

Let now $(X, \leq)$ be a preordered set, i.e. $\leq \subseteq X \times X$ is a reflexive and transitive relation, and let $x \in X$. The *downset* of the point x in X is the set $\downarrow x = \{y \in X \mid y \leq x\}$; the *upset* of x is defined similarly. The *order open initial segment* generated by x is the set $(\leftarrow, x) = \{y \in X \mid y \leq x, y \neq x\}$, while the *order open final segment* generated by x is the set $(x, \rightarrow) = \{y \in X \mid y \geq x, y \neq x\}$. *Ordered closed* final and initial segments are defined similarly. Note that if $(X, \leq)$ is a linear order, the order topology on X is generated by the subbasis of all order open final and initial segments. Finally, we say that $(X, \leq)$ is a *complete linear order* if it antisymmetric, $x \leq y \vee y \leq x$ for any $x, y \in X$, and for any $Z \subseteq X$ and $U := \{b \in X \mid \forall x \in Z(x \leq b\}$ non-empty it holds that U has a $\leq$-minimal element.

We now turn to the topologies that can be defined from our primitive relations.

4.1. Topologies from $\mathbb{B}, \mathbb{E}$. The relations $\mathbb{B}(a, b), \mathbb{E}(a, b)$ are reflexive and transitive, and thus lend themselves to the following construction

DEFINITION 6. *Let R be a reflexive and transitive relation on a set X. $G \subseteq X$ is* R-upwards closed *if $a \in G, R(a, b) \Rightarrow b \in G$. We omit reference to R when it is clear from the context. Arbitrary unions and intersections of upwards closed sets are again upwards closed. The upwards closed sets will be called* open, *and their complements, the downwards closed sets, will be called* closed. *The collection of open sets is called the* Alexandroff topology.

DEFINITION 7. *An R-open set G is sub-basic if it is of the form* $\{e \mid R(a, e)\}$.

The relations $\mathbb{B}(a, b), \mathbb{E}(a, b)$ define Alexandroff topologies on an event structure. When considering the space of events equipped with two different topologies we sometimes use the term 'bitopological space'.

4.2. Proximity and overlap: relations of simultaneity. The relations $O, \mathbb{O}$ also have topological meaning. Let $\mathcal{W}$ be an event structure and let $A, B \subseteq W$ be sets of events. If we define AOB iff $\exists a \in A \exists b \in B\ O(a, b)$, then O is a kind of proximity relation on $\mathcal{PW}$. We shall also be interested in its dual $A\mathbb{O}B \Leftrightarrow \forall a \in A \forall b \in B\ \mathbb{O}(a, b)$, which one might call an apartness relation on $\mathcal{PW}$. Although the relations $O, \mathbb{O}$ are symmetric, it is helpful to consider them as asymmetric operations on open sets, where the asymmetry derives from the existence of the two topologies $\mathbb{B}, \mathbb{E}$.

DEFINITION 8. *Let U be $\mathbb{E}$ open, and V be $\mathbb{B}$ open, and define:*
(i) $U\mathbb{O} = \{b \mid \forall a \in U\ \mathbb{O}(a, b)\}$
(ii) $\mathbb{O}V = \{a \mid \forall b \in V\ \mathbb{O}(a, b)\}$

LEMMA 9. *(LEM) (i) $U\mathbb{O}$ is $\mathbb{B}$ open. (ii) $\mathbb{O}V$ is $\mathbb{E}$ open.*

PROOF. Let $b \in U\mathbb{O}$ and $\mathbb{B}(b, c)$. Choose $a \in U$; to show $\mathbb{O}(a, c)$. We have $\mathbb{O}(a, b)$, which implies $E(b, a)$ or $B(b, a)$. The first possibility implies $b \in U$, quod non. But $B(b, a)$ together with $\mathbb{O}(a, b)$ and $\mathbb{B}(b, c)$ implies $\mathbb{O}(a, c)$. (ii) is proved similarly. ⊣

Thus the relation $\mathbb{O}$ can be used to define two unary operations, one on the lattice of $\mathbb{E}$-open sets of an event structure $\mathcal{W}$, and the other on the lattice of $\mathbb{B}$-open sets. These operations will prove useful to provide a definition of Kantian boundary.

4.3. Topologies from the covering relation. For the defined relation $\preceq$ (definition 2) we have

LEMMA 10. *The relation $\preceq$ as defined in 2 is transitive and reflexive.*

PROOF. From axioms 5, 3a, 4b, 4a, 6b, 6a. ⊣ ⊣

In the sequel we shall often make use of the self-duality of the Alexandroff topologies generated by $\preceq$, and use both the topology whose closed sets are the upsets and the topology whose closed sets are the downsets. It will always be clear from the context which topology we are employing.

§5. Boundaries and limitations. The topologies defined in the previous section have a temporal meaning: the open sets of $\mathbb{E}$ represents the past, the up-sets of $\preceq$ the present and the open sets of $\mathbb{B}$ the future. Our ultimate aim is to show that the set of events can be given the structure of a one dimensional continuum, which may have some instants, all of which arise as boundaries. We argue as follows. Kant conceives of instants as limitations or boundaries. A boundary, however, is a topological concept, not an order-theoretic one. Informally, a temporal boundary in an event structure $\mathcal{W}$ determines a set of events $Past \subseteq W$ in the past of that boundary, and likewise a set of events $Fut \subseteq W$ which all lie in the future of the boundary. This does not mean that events in *Fut* are still to come; the division into *Past* and *Fut* is relative to the domain W of $\mathcal{W}$, and all $e \in W$ may be situated in the past from the standpoint of the *now*. Temporal progression in the sense of *coming to be* will be represented not by a single event structure, but by a system of event structures linked by continuous maps. We will remind the reader of this observation later, to explain some otherwise puzzling results.

According to the topologies defined in the previous section, the fact that the past and the future of a boundaries are respectively $\mathbb{E}$-open and $\mathbb{B}$-open means that $a \in$ *Past*$, \mathbb{E}(a, b) \Rightarrow b \in$ *Past* and $a \in$ *Fut*$, \mathbb{B}(a, b) \Rightarrow b \in$ *Fut*. Furthermore, under the given interpretation $a \in$ *Past* and $b \in$ *Fut* implies $\mathbb{O}(a, b)$, which in turn implies that *Past* and *Fut* are (set theoretically) disjoint. The complement of *Past* $\cup$ *Fut* can be viewed as a representation of the temporal boundary between *Past* and *Fut*, which we might as well call the present (*Pres*). These considerations are summarized in the following preliminary definition.

DEFINITION 9. *Given an event structure $\mathcal{W} := (W; O, \mathbb{O}, E, \mathbb{E}, B, \mathbb{B}; \otimes_f, \otimes_p)$, an* instant *or* boundary *in $\mathcal{W}$ is a triple $(Past, Pres, Fut)$ such that $Past \cup Pres \cup Fut = W$, Past is $\mathbb{E}$-open, and Fut is $\mathbb{B}$-open.*

LEMMA 11. *Pres, the complement of $Past \cup Fut$, is $\preceq$-closed, in the sense that $d \in Pres$ and $d \preceq c$ implies $c \in Pres$.*

5.1. One dimensionality of time. Kant states that 'Time has only one dimension' (A31/B47), however it is by no means obvious that time is one-dimensional, let alone that a linear order on instants can be constructed from events. For Kant, time is prior to its parts, which have the same structure as time itself – in particular there are no smallest parts allowing the construction of time as a set. As we have seen

> Space therefore consists only of spaces, time of times. Points and instants are only boundaries, i.e., mere places of their limitation; but places always presuppose those intuitions that limit or determine them, and from mere places, as components that could be given prior to space or time, neither space nor time can be composed. (CPR A170/B212)

Our aim can be stated as follows: define boundaries such that they define parts of time, are linearly ordered, but do not generate time. We now have to investigate whether the axioms GT_0 imply that the collection of instants $(Past, Pres, Fut)$ can be linearly ordered. The axioms in group 4 – the linearity axioms, which can be informally derived from the law of excluded middle for $\mathbb{B}$, $\mathbb{E}$ – imply

LEMMA 12. *For any two $\mathbb{E}$-open sets Past, Past': Past $\subseteq$ Past' or Past' $\subseteq$ Past and similarly for $\mathbb{B}$-open sets Fut, Fut'.*

Since pasts are linearly ordered, one may attempt to define a linear order $<$ on instants by letting

DEFINITION 10. $(Past, Pres, Fut) < (Past', Pres', Fut')$ *if* $Past \subseteq Past'$.

This suggestion does not work for all instants as defined in definition 9. In fact, given *Past* only, *Fut* can be chosen independently subject to the constraint that *Past* and *Fut* are $\mathbb{O}$-apart, and it follows that the event structure is in a loose sense two-dimensional. This problem can be avoided if *Fut* is somehow completely determined by *Past*. As we shall see, this issue is connected to the nature of the boundary between *Past* and *Fut*, i.e. *Pres*. For instance, the present should not contain any pair of events a, b such that $\mathbb{O}(a, b)$; if there were such a pair (a, b), the present could be split into two parts, one containing a but not b, the other containing b but not a, and hence it would not be really a boundary.

We introduce now three kinds of boundaries, depending on whether past and/or present and/or future are inhabited.

5.2. Two-sided boundaries.

DEFINITION 11. *A two-sided* boundary *is a triple $(Past, Pres, Fut)$ such that:*

1. *$Past \cup Pres \cup Fut = W$*
2. *Past is $\mathbb{E}$-open and nonempty, and Fut is $\mathbb{B}$-open and nonempty*
3. *Past and Fut are $\mathbb{O}$-apart*
4. *Pres is non-empty, and for all $a, b \in$ Pres: $O(a, b)$*
5. *for all $c \in$ Pres there is $a \in$ Past with $O(c, a)$, and likewise for Pres and Fut*

We obtain pairs (*Past*, *Fut*) satisfying the specifications of definition 11 by a fixpoint construction, starting from a pair (U, V), U $\mathbb{E}$-open, V $\mathbb{B}$-open. This fixpoint construction really amounts to the definition of two interrelated closure operators on two lattices. The first closure operator is defined on the lattice of sets open in the $\mathbb{E}$-topology, and maps $\mathbb{E}$-open sets to $\mathbb{E}$-open sets. The second is defined on the lattice of sets open in the $\mathbb{B}$ topology, in an analogous way. A two-sided boundary will then be represented as a pair $(Past, Fut)$ where $Past, Fut$ are closed w.r.t. the former and the latter closure operator, respectively. Since these closure operators are defined on the lattice of open sets, in order to avoid confusion we rather refer to sets which are closed with respect to these closure operators as fixpoints.

THEOREM 3. *Let U range over $\mathbb{E}$-opens, and V over $\mathbb{B}$-opens.*

1. *the mapping $L : U \mapsto \mathbb{O}(U\mathbb{O})$ is monotone on the complete lattice of $\mathbb{E}$ opens: $U \subseteq U'$ entails $\mathbb{O}(U\mathbb{O}) \subseteq \mathbb{O}(U'\mathbb{O})$*
2. *the mapping $L : U \mapsto \mathbb{O}(U\mathbb{O})$ is extensive: $U \subseteq \mathbb{O}(U\mathbb{O})$*
3. *the mapping $R : V \mapsto (\mathbb{O}V)\mathbb{O}$ is monotone and extensive*
4. *for any U we have that the least fixpoint above U is $L(U) = L(L(U))$, and similarly for V*
5. *if U is a L-fixpoint, then $V = U\mathbb{O}$ is an R-fixpoint and we have $\mathbb{O}V = U$. Analogously if V is a R-fixpoint, then $U = \mathbb{O}V$ is an L-fixpoint.*
6. *the set of L-fixpoints (resp. R fixpoints) is a complete linear order*
7. *$W, \emptyset$ are fixpoints of both operators.*

PROOF. For claim 1 one checks that $U \subseteq U'$ entails $U'\mathbb{O} \subseteq U\mathbb{O}$, which in turn entails $\mathbb{O}(U\mathbb{O}) \subseteq \mathbb{O}(U'\mathbb{O})$; these latter sets are $\mathbb{E}$ open by Lemma 9. For claim 2 we note that Lemma 9 implies $\mathbb{O}(U\mathbb{O})$ is $\mathbb{E}$ open. Choose $a \in U, b \in U\mathbb{O}$, then by definition $\mathbb{O}(a, b)$, hence $a \in \mathbb{O}(U\mathbb{O})$. Claim 3 is proven similarly. To prove claim 4, observe that by claim 1 we have

$$\mathbb{O}(U\mathbb{O}) \subseteq \mathbb{O}(\mathbb{O}(U\mathbb{O})\mathbb{O})$$

and by claim 3, setting $V = U\mathbb{O}$, we obtain the converse inclusion, whence (using LEM)

$$\mathbb{O}(U\mathbb{O}) = \mathbb{O}(\mathbb{O}(U\mathbb{O})\mathbb{O}).$$

To prove claim 5 we note that $U = \mathbb{O}(U\mathbb{O})$ implies $(\mathbb{O}V)\mathbb{O} = (\mathbb{O}(U\mathbb{O}))\mathbb{O} = U\mathbb{O} = V$. To prove claim 6, let $U = \bigcup_i U_i$ be a union of fixpoints of L. Then U need not be a fixpoint, but $\mathbb{O}(\bigcup_i U_i\mathbb{O})$ is the least fixpoint larger than the U_i. Furthermore, given a pair (U, V) we can construct two increasing sequences, where $\subseteq$ is interpreted coordinate-wise:

$$(U, V) \subseteq (U, U\mathbb{O}) \subseteq (\mathbb{O}(U\mathbb{O}), U\mathbb{O}),$$

and

$$(U, V) \subseteq (\mathbb{O}V, V) \subseteq (\mathbb{O}V, (\mathbb{O}V)\mathbb{O}).$$

Since the $\mathbb{E}$ opens are linearly ordered by $\subseteq$ (axioms 4), we may fuse the two sequences by ordering them linearly according to the first coordinate

$$(U, V) \leq (\mathbb{O}(U\mathbb{O}), V) \leq (\mathbb{O}V, V) \leq ((\mathbb{O}V), (\mathbb{O}V)\mathbb{O}),$$

which gives (by theorem 3, property 5), for any $\mathbb{E}$-open U, least and greatest extensions that qualify as *Past*s. The proof of claim 7 is straightforward. ⊣

We can now define boundaries of an event structure in terms of fixpoints by considering the operations R, L as a single operation (L, R) on the product lattice of the $\mathbb{E}$-open and $\mathbb{B}$-open sets:

DEFINITION 12. *Let $\mathcal{W}$ be an event structure. A pair (Past, Fut) is a fixpoint of the operation (L, R).*

LEMMA 13. *Let a, b with $\mathbb{O}(a, b), E(a, b)$ be given, then there exist a pair (Past, Fut) with $a \in$ Past, $b \in$ Fut.*

PROOF. We construct a least fixpoint separating a and b. Let $U_a = \{c \mid \mathbb{E}(a, c)\}$. Since $\mathbb{O}(a, b)$, $b \in U_a\mathbb{O}$, $b \notin \mathbb{O}(U_a\mathbb{O})$ = Past. Since *Fut* is obtained as the least fixpoint above $U_a\mathbb{O}$, $b \in$ *Fut*. ⊣

THEOREM 4. *Assume (Past, Fut) is as in definition 12. Define $F = ($Past $\cup$ Fut$)^c$, then:*

1. *F is non-empty*
2. *O is transitive on F*
3. *for all $c \in F$ there is $a \in$ Past with $O(c, a)$, and likewise for F and Fut*
4. *if we let Pres $= F$, (Past, Pres, Fut) is a boundary in the sense of definition 11.*

PROOF. For 1 choose $a \in$ *Past*, $b \in$ *Fut*, and pick a cover c satisfying $a, b \preceq c$; then $c \in F$. For 2, if for $c, d \in F$: $\mathbb{O}(c, d)$, the preceding lemma shows that the pair (*Past*, *Fut*) would not be a fixpoint. For 3 choose $c \in F$ with $c \in$ *Past*$\mathbb{O}$. By theorem 3 property 5, *Past*$\mathbb{O}$ is an R fixpoint, whence we must have $c \in$ *Fut* . ⊣

COROLLARY 1. *Every boundary (Past, Pres, Fut) in the sense of definition 11 is obtainable by the procedure given in the proof of theorem 3.*

PROOF. The condition implies that *Past*$\mathbb{O}$ $=$ *Fut* and *Past* $= \mathbb{O}$*Fut*, hence *Past* $= \mathbb{O}$(*Past*$\mathbb{O}$); and likewise for *Fut*. ⊣

5.3. One-sided boundaries. We next consider the case of 'one-sided' boundaries, which can occur in the presence of a universal cover:

DEFINITION 13. *Let w be a universal cover. If there exists c with $B(c, w) \wedge \mathbb{E}(c, w)$, then the 'right present' Pres$_r$ is the set $\{c \mid \mathbb{E}(c, w)\}$, which is $\mathbb{E}$ closed (as well as $\mathbb{B}$ closed). The complement of this set is the maximal bounded past, which is $\mathbb{E}$ open. Similarly one defines the 'left present' and the corresponding maximal bounded future.*

LEMMA 14. *$Pres_r$ = $\{c \mid \mathbb{E}(c, w)\}$ is $\mathbb{E}$ closed, hence its complement is $\mathbb{E}$ open. This gives a boundary of the form $(Past, Pres_r, \emptyset)$. The complementary boundary $Pres_l$ is $(\emptyset, Pres_r, Past)$.*

If the left and right presents exist, the inclusion ordering on the set of pasts represents the set of boundaries as a closed interval.

DEFINITION 14. *Let $\mathcal{W}$ be an event structure. The space of boundaries $\mathcal{B}(\mathcal{W})$ of $\mathcal{W}$ is the set of two-sided and one-sided boundaries, ordered under inclusion of pasts as in Definition 10.*

The left and right boundaries of an event structure $\mathcal{W}$ have somewhat a special status, since they do not conform to Definition 12 – they are not fixpoints – and they will be used to formalize Kant's notion of the infinity of time (see 5.6). We shall see in section 10 that for the purposes of this work we need only consider *complete* event structures, which are such that the presents of their one-sided boundaries are generated by what we call a *minimal covering event*, or an *infinitesimal event*. See Section 10, Theorem 9 and its associated Corollary.

THEOREM 5. *The set of boundaries of an event structure determines a complete linear order by means of the inclusion ordering on the Past's.*

PROOF. The inclusion order is linear by axiom 2a. The second half follows Theorem 3 (vi). $\dashv$

COROLLARY 2. *(ZF) The set of boundaries of an event structure is Hausdorff and compact.*

PROOF. The first observation is standard. Compactness follows from the fact that in a complete linear order, each closed interval is compact, combined with theorem 5. $\dashv$

We conclude this section with a lemma which shall prove useful later:

LEMMA 15. *Let $\mathcal{W}$ be a model of GT and let $x, y \in \mathcal{B}(\mathcal{W})$ be such that $x < y$. Then there exists an event $a \in \mathcal{W}$ such that $a \in Past(y) \cap Fut(x)$.*

PROOF. Since $x < y$ we must have that $Past(x) \subset Past(y)$ hence there exists $a \in Past(y), a \notin Past(x)$. If $a \in Fut(x)$ we are done. Otherwise $a \notin Fut(x), a \in Pres(x)$. Then $Fut(x)$ cannot be empty because $x < y$ and so $Fut(y) \subset Fut(x)$, and $O(a, c)$ for $c \in Fut(x)$. We then let $b = a \otimes_p c$ and we are done. $\dashv$

5.4. Formal boundaries. Purely formally, we can introduce triples (*Past*, *Pres*, *Fut*), where two entries are equal to $\emptyset$, hence the remaining entry must equal W.

DEFINITION 15. *A formal boundary is a triple of the form $(\emptyset, \emptyset, W)$ or of the form $(W, \emptyset, \emptyset)$.*[23]

Formal boundaries define formal intervals, which are not determined by their elements (they may not have any). They can be viewed as extensions of the linear order of boundaries, but they lie strictly beyond the left and right boundaries. Section 5.6 will clarify these remarks.

5.5. Fixpoints and geometric formulas. GT is a first-order theory of events, but we have been mostly concerned with higher order constructions: pasts are sets of events, boundaries are triples of such sets, and the linear order is a set of pairs of these triples. Since our focus is the linear continuum derived from event structures, the logical and set theoretic principles involved in the construction have to be scrutinized for Kantian content (or lack thereof). Here it is essential to recall that events are only virtual parts of time; they cannot be detached and collected into a set by means of a comprehension axiom. To have a set of events is to have a rule (or rules) for marking these events on the timeline. Since there is an intimate connection between constructive rules and geometric formulas, we have to investigate whether the sets of interest – e.g. the least

[23]These are called 'end-gaps' in topology.

fixpoint of the operation $U \mapsto \mathbb{O}(U\mathbb{O})$ – are somehow definable geometrically. Using the principles of excluded middle, we may write $c \in \mathbb{O}(U\mathbb{O})$ as

$$\forall b(O(b,c) \to \exists a \in U\ O(a,b)).$$

Suppose we start with a $\mathbb{E}$ open U_0 which is given by a geometric formula. Applying the operation, we obtain the $\mathbb{E}$ open U_1 defined by

$$U_1 = \{c \mid \forall b(O(b,c) \to \exists a \in U_0\ O(a,b))\},$$

from which it follows that U_1 is also given by a geometric formula.. Moreover U_1 is a fixpoint, hence the rule is determined in two steps. As we'll see later, geometric formulas have preservation properties that are important in dealing with boundaries on infinite event structures.

5.6. Infinity of time. If an event structure does not have one-sided boundaries, the linear order of boundaries is unlimited, and thus has some features of Kant's construal of the infinity of time. But let's look again at the relevant passage:

> The infinitude of time signifies nothing more than that every determinate magnitude of time is only possible through limitations of a single time grounding it. The original representation of time must there be given as unlimited. But where the parts themselves [...] can be determinately represented only through limitation, there the entire representation cannot be given through concepts [...] but immediate intuition must ground them. (A31-2/B47-8)

We interpret 'magnitude of time' as a function defined on intervals determined by boundaries. Given Kant's concept of number, a 'determinate magnitude' is a function that takes only rational values; since magnitudes must be continuous, and the values of the 'determinate magnitude' are closed points, the domain of the magnitude must consist of closed intervals. Let us follow Kant in deeming an infinite past to be incomprehensible (see 1.1.10); thus we assume a one-sided boundary on the left, which we may call 0. The time elapsed until *now* corresponds to a closed interval, with *now* represented as a one-sided right boundary. Obviously time does not stop *now*; on the other hand, all events are contained in the closed interval just defined, so that technically the future is empty. The solution is to consider *the same set of events* W as defining a formal open interval, as in definition 15.

The linear order of boundaries is extended by putting $(W, \emptyset, \emptyset)$ on the right. The interval $[0, (W, \emptyset, \emptyset))$ is the unlimited time, in which boundaries can be created that define temporal magnitudes. The new boundary expresses that all events, also those that have not yet occurred, are in its past. By construction, $(W, \emptyset, \emptyset)$ differs from the original right boundary rbd, even though the interval (rbd,$(W, \emptyset, \emptyset)$) thus defined is empty, hence is what is called a formally open interval. We change the topology by declaring $(W, \emptyset, \emptyset)$ a closed point. As a consequence the original set of boundaries is now open, not clopen. The new topological space is formal noncompact: a covering of the extended space cannot be reduced to a covering of the original space, because a formally open interval of the form $(x, (W, \emptyset, \emptyset)]$ (where x ranges over boundaries) is different from $(x, rbd]$ (where rbd is the original right boundary), even if they are extensionally equivalent. At the same time, this 'decompactification' of the closed linear order of boundaries allows us to continue to use compactness as a technical tool.

Our next task is to investigate if and when such linear orders can be considered to be continua. The reader may be forgiven for thinking that boundaries are just Dedekind cuts called by another name, and therefore Kant's continuum, formalised with our tools, cannot be too different from $\mathbb{R}$. But boundaries are not analogous to Dedekind cuts; we shall explain this in Section 10 and 11.2.

§6. Events as open intervals. Theorem 5 raises the following question: how is the original event structure related to the linear order obtained via boundaries? Ideally one should be able to map any event e to an open interval such that the topologically meaningful relations $O, \mathbb{B}, \mathbb{E}$ are preserved. The following definition owes its justification that two of our topologies are determined by the preorders $\mathbb{B}, \mathbb{E}$.

DEFINITION 16. *Let $\mathcal{W}_1, \mathcal{W}_0$ be event structures. A function $f : \mathcal{W}_1 \longrightarrow \mathcal{W}_0$ is an* event homomorphism*, or simply an* e-homomorphism*, if it preserves $O, \mathbb{B}, \mathbb{E}$, and, in GT, also the operations $\otimes_p, \otimes_f$.*

Note that an e-homomorphism must preserve the covering relation $\preceq$. We employ the term "e-homomorphism" rather than the customary term "homomorphism" to highlight the fact that our homomorphisms between event structures can at the same time be seen as maps which are continuous with respect to the $\mathbb{E}$ and $\mathbb{B}$ topologies.

The construction of a e-homomorphism f from an event structure $\mathcal{W}$ to a canonical event structure induced by its space of boundaries $\mathcal{B}(\mathcal{W})$ can be effected as follows:

DEFINITION 17. *Let $\mathcal{W} \models GT_0$ be an event structure and let $S \in W$ be a subset of $\mathcal{W}$. We define the left and right endpoints of S, denoted as $l(S), r(S)$ for $l(S), r(S) \in \mathcal{B}(\mathcal{W})$, letting:*

- $l(S) = \bigvee\{x \in \mathcal{B}(\mathcal{W}) \mid S \subseteq Fut(x)\}$
- $r(S) = \bigwedge\{x \in \mathcal{B}(\mathcal{W}) \mid S \subseteq Past(x)\}$

If $S = \{a\}$ for some $a \in \mathcal{W}$, then we denote $l(\{a\})$ simply as $l(a)$, and similarly for r.

The boundaries $l(S), r(S)$ of Definition 17 are well defined since by Theorem 5 the linear order of boundaries is a complete lattice. Moreover we have:

LEMMA 16. *Let $\mathcal{W} \models GT_0$, $S \subseteq W$. Then $S \subseteq Fut(l(S))$, $S \subseteq Past(r(S))$*

PROOF. Consider an event structure $\mathcal{W}$ and fix $S \subseteq W$; we show the claim for $l(S)$ since for $r(S)$ the reasoning is symmetric. If $\bigvee\{x \in \mathcal{B}(\mathcal{W}) \mid S \subseteq Fut(x)\}$ is either a formal boundary or a left boundary then the claim follows trivially. Otherwise let $L = \{Past(x) \mid x \in \mathcal{B}(\mathcal{W}), S \subseteq Fut(x)\}$. Then $\bigvee\{x \in \mathcal{B}(\mathcal{W}) \mid S \subseteq Fut(x)\} = \mathbb{O}(\bigcup L\mathbb{O})$ by Theorem 3. Since $S \subseteq Fut(x)$ for all x, we have $\mathbb{O}(a, b)$ for any $a \in S$, $b \in L$ and hence $S \subseteq L\mathbb{O}$, hence $\mathbb{O}(a, c)$ for any $a \in S$, $c \in \mathbb{O}(L\mathbb{O})$, thus no element a of S can be in the present of $\bigvee\{x \in \mathcal{B}(\mathcal{W}) \mid a \in Fut(x)\}$ because of Theorem 4, condition 3, which implies any $a \in S$ must be in the future of $\bigvee\{x \in \mathcal{B}(\mathcal{W}) \mid S \subseteq Fut(x)\} = l(S)$. The reasoning for $r(S)$ is similar. $\dashv$

The considerations above would allow us to see the events in an event structure $\mathcal{W}$ as intervals of the linear order of boundaries $\mathcal{B}(\mathcal{W})$ by considering the map $f : \mathcal{W} \to \{(x, y) \mid x, y \in \mathcal{B}(\mathcal{W}), x \leq y\}$ defined by letting $a \mapsto (l_a, r_a)$. One could also show that if the set $\{(x, y) \mid x, y \in \mathcal{B}(\mathcal{W}), x \leq y\}$ is endowed with relations so as to turn it into an event structure in the obvious way then the map f is a e-homomorphism. This suggests that one ought to be able to set up a pair of functors between the category of event structures and e-homomorphisms on one side, and a suitable category of linear orders, possibly equipped with additional structure, on the other side. An approach along these lines has been given in [Thomason, 1989], which provides functors between a category of event structures and homomorphisms, and a category of linear orders and monotone multi-valued maps. In light of the developments of Section 10, however, the most promising approach in our setting is to consider not merely a category of linear orders, but a suitable category of linearly ordered topological spaces, and to employ the theory of locales as developed in pointfree topology to obtain suitable morphisms. Indeed, our e-homomorphisms can be seen as an instance of the "approximate maps" which have been introduced in [Banaschewski and Pultr, 2010] as a representation of localic morphisms. We shall not pursue further this line of research here, since our main focus in this work is the study, in Sections 9 and following, of a concrete model of the axiom system GT which we take as a formal correlate of Kant's formal intuition of time. We hope however to address the problem of the most appropriate categorical setting for the present results in a future work.

§7. Time as a connected continuum. Continuity and connectedness occur in various guises in the list of synthetic a priori principles for time. In particular, we find the following characterizations of continuity and connectedness:

1. parts of times are themselves times
2. it is impossible to 'detach' a part of time from its encompassing whole[24]
3. time is divisible to infinity[25]
4. there are no jumps or leaps from one state of a substance to another, without intermediate transitions in between[26]
5. time is not a 'mere series' and duration is a continuous magnitude

These aspects of continuity and connectedness are all manifestations of Kant's notion of parthood; in this section we analyze how this notion bears on the connectedness properties of event structures, while in the upcoming sections we shall focus on the relation between parthood and the space of boundaries.

In the context of point-set topology, the definition of connectedness is a kind of indecomposability condition: a topological space X is connected there are no disjoint non-empty open sets U, V such that $X = U \cup V$. Kant's notion of indecomposability is much stronger than this, but first we must define what it means for a space to be connected in a bi-topological setting. Since the $\mathbb{E}$ open sets are linearly ordered by inclusion, any event structure is trivially connected in the $\mathbb{E}$ topology (and likewise for the $\mathbb{B}$ topology). To be able to say something more interesting we therefore need both topologies:

DEFINITION 18. *The event structure $\mathcal{W}$ is* connected *if there are no non-empty U, V such that U is $\mathbb{E}$-open, V is $\mathbb{B}$-open, $U \cap V = \emptyset$ and $U \cup V = W$.*

[24] "parts of time can be distinguished, but not separated" (*Refl.* 4425)

[25] In CPR A524/B552 Kant states that the division of something which is given *as a whole* in intuition must proceed to infinity, even though the division can never reduce the whole to simple parts. Since time, as stated in the Aesthetic, is given as a whole and is not composed from its parts, we can infer that time is divisible to infinity, i.e., ever smaller subdivision of times can be introduced.

[26] 'there is nothing simple in appearance, hence no immediate transition from one determinate state (not of its boundary) into another' (*Refl.* **4756**). In *Refl.* **4756** we find: 'a hiatus, a cleft, is a lack of interconnection among appearances, where their transition is missing'

In particular, if the event structure $\mathcal{W}$ is connected, it cannot be written as *Past* $\cup$ *Fut*. As a consequence of the covering axiom, we obtain:

LEMMA 17. *Let $\mathcal{W}$ be a model of GT. Then $\mathcal{W}$ is connected.*

The covering axiom implies still stronger forms of connectedness. We now focus attention on sets closed in both topologies. As we have seen, the *Pres* component of a boundary is such a set. It follows from Lemma 17 that such sets must be non-empty. But we also have

LEMMA 18. *The intersection of any two $Pres_1$, $Pres_2$ is non-empty.*

PROOF. Choose $a \in Pres_1, b \in Pres_2$. If $a = b$, we are done. Otherwise, choose c with $a, b \preceq c$. By lemma 11, $c \in Pres_1 \cap Pres_2$. ⊣

This result suggests that a stronger notion of connectedness is more appropriate for event structures:

DEFINITION 19. *A topological space is* ultraconnected *if any two non-empty closed sets have non-empty intersection.*

In our bi-topological setting, this concept is non-trivial only for sets closed in both topologies, hence in the preceding definition 'closed' will be taken in this sense; this is equivalent to stipulating that the downwards closed sets with respect to $\preceq$ are open; in other words, that the set of virtual parts generated by a given event is open. In these terms, an event structure $\mathcal{W}$ is ultraconnected if for sets of events U_1, U_2 downward closed under the relation 'is a virtual part of', $\mathcal{W} = U_1 \cup U_2$ implies that for some i, $\mathcal{W} = U_1$. If we consider an open set generated by an event as defining a submodel of $\mathcal{W}$, then ultraconnectedness expresses that parts of time cannot be represented merely as submodels of a given $\mathcal{W}$. We shall see in the sequel that precisely because parts are virtual, parts of time represented in $\mathcal{W}$ must correspond to an extension of $\mathcal{W}$.

LEMMA 19. *Event structures with the $\preceq$ topology are ultraconnected.*

In finite event structures, the above lemma reduces to the statement that there exists a universal cover, namely, an event $w \in W$ such that $w \in Pres(x)$ for any $x \in \mathcal{B}(\mathcal{W})$.

§8. Refinement and divisibility. The purpose of this section is to introduce a mathematical correlate to Kant's notion of potential infinite divisibility of time, with the aim of capturing Kant's notion of continuity:

> The property of magnitudes on account of which no part of them is the smallest (no part is simple) is called their continuity. (A169/B211)

From the passages presented in Section 1 it emerges that there are two main aspects to this notion of continuity 'in the small', which are both grounded on Kant's notion of parthood. The first aspect is indeed that "time does not consist of smallest parts" (property 1.1.6, Section 1), which means that any event in an event structure can be further divided into subevents. The second aspect is that "instants can be filled" (property 1.1.9, Section 1), which seems to imply that boundaries in time *can* be only "approximations" of points, and thus *can* be further refined. The use of modal expressions such as "can be divided" or "can be refined" is important here, as Kant's notion of continuity is tied to his notion of potential infinity: every event structure is finite, and determines a finite linear order of boundaries, but it can be further analyzed *ad infinitum*.

We shall formalize the notion of potential infinite divisibility by means of retraction maps. For this purpose we shall need the full set of axioms, GT. The fundamental idea is that given an event structure $\mathcal{W}_0 \models GT$, we can extend $\mathcal{W}_0$ to an event structure $\mathcal{W}_1 \models GT$ by adding "splittings" of events and of boundaries, in such a way that $\mathcal{W}_1$ "extends" $\mathcal{W}_0$ by the addition of more events and thus of more boundaries. Moreover, since parts of time are virtual only, it must be that the original undivided whole coexists with its parts after the division has occurred; no parts of time must be lost in the division procedure, which means that $\mathcal{W}_0$ must be a submodel of $\mathcal{W}_1$, and the map splitting events and boundaries must be surjective and equal to the identity on $\mathcal{W}_0$, i.e., a retraction map. We start with some preliminary definitions:

LEMMA 20. *Let $\mathcal{W}_1$ be a model of GT_0 and let W_0 be a substructure of $\mathcal{W}_1$. Then* $W_0 \models GT_0$

PROOF. Obvious, since GT_0 is a universal theory. $\dashv$

Next, we define the notion of retraction maps which we shall be using:

DEFINITION 20. *Let $\mathcal{W}_1$ be a model of GT_0, and let $\mathcal{W}_0$ be a substructure of $\mathcal{W}_1$. A map $f : \mathcal{W}_1 \to \mathcal{W}_0$ is a* $\mathbb{E}$-retraction *if it is surjective, equal to the identity on $\mathcal{W}_0$ and it preserves $\mathbb{E}$ and O. In this case $\mathcal{W}_0$ is called a* $\mathbb{E}$-retract *of $\mathcal{W}_1$. A* $\mathbb{B}$-retraction *is defined analogously. If f is both a $\mathbb{B}$ and a $\mathbb{E}$ retraction, then we say it is simply a* retraction*, and $\mathcal{W}_0$ is simply a* retract*.*

Consider now an event structure $\mathcal{W}_1 \models GT_0$ and a substructure $\mathcal{W}_0$ of $\mathcal{W}_1$ (hence $\mathcal{W}_0 \models GT_0$), where we assume that $\mathcal{W}_0$ is finite but impose no restrictions on $\mathcal{W}_1$. When is one guaranteed that $\mathcal{W}_0$ is a retract of $\mathcal{W}_1$? A first approximation to the answer is given by the following:

DEFINITION 21. *Let $\mathcal{W}_1, \mathcal{W}_0$ be models of GT_0 such that $\mathcal{W}_0$ is a submodel of $\mathcal{W}_1$, and let $a \in \mathcal{W}_1, a \notin \mathcal{W}_0$. A $\mathbb{E}$-limit of a in $\mathcal{W}_0$ is any event $b \in \mathcal{W}_0$ minimal with the property that $\mathcal{W}_1 \models \mathbb{E}(b, a)$. A $\mathbb{B}$-limit of a is defined analogously.*

LEMMA 21. *Let $\mathcal{W}_0, \mathcal{W}_1$ be two models of GT_0, with $\mathcal{W}_0$ a finite substructure of $\mathcal{W}_1$. Let $f_{\mathbb{E}} : \mathcal{W}_1 \to \mathcal{W}_0$ be defined as follows. If $a \in \mathcal{W}_0$, $f_{\mathbb{E}}$ is the identity. If $a \notin \mathcal{W}_0$, let $f_{\mathbb{E}}(a)$ be any $\mathbb{E}$-limit of a if one exists, and let $f_{\mathbb{E}}(a)$ be the universal cover of $\mathcal{W}_0$ otherwise. Then $f_{\mathbb{E}}$ is a $\mathbb{E}$ retraction of $\mathcal{W}_1$ onto $\mathcal{W}_0$. A $\mathbb{B}$ retraction of $\mathcal{W}_1$ onto $\mathcal{W}_0$ can be defined similarly.*

Thus, if $\mathcal{W}_1, \mathcal{W}_0$ are models of GT_0 and $\mathcal{W}_0$ is a finite substructure of $\mathcal{W}_1$ we have that $\mathcal{W}_0$ is both a $\mathbb{B}$ and a $\mathbb{E}$ retract of $\mathcal{W}_1$. The question is now whether the two $\mathbb{E}$ and $\mathbb{B}$ retractions defined in the previous lemma can be combined, to yield a full retraction of $\mathcal{W}_1$ to $\mathcal{W}_0$. Only if we can obtain a full retraction can we understand $\mathcal{W}_1$ as generated from $\mathcal{W}_0$ by means of a process of subdivision of events. A moment's thought reveals however that this is in general not possible unless we assume that $\mathcal{W}_1, \mathcal{W}_0$ are models of GT. In this case, we obtain:

LEMMA 22. *Let $\mathcal{W}_1$ be a model of GT and let $\mathcal{W}_0$ be a finite submodel of $\mathcal{W}_1$. Define a map $f : \mathcal{W}_1 \to \mathcal{W}_0$ as follows:*

$$\begin{cases} f(a) = a & \textit{if } a \in \mathcal{W}_0 \\ f(a) = (m_{\mathbb{B}} \otimes_f c(m_{\mathbb{B}}, m_{\mathbb{E}})) \otimes_f (m_{\mathbb{E}} \otimes_p c(m_{\mathbb{B}}, m_{\mathbb{E}})) & \textit{otherwise} \end{cases}$$

Where $m_{\mathbb{E}}$ (resp. $m_{\mathbb{B}}$) is any $\mathbb{E}$-limit (resp. $\mathbb{B}$-limit) of a if one such event exists, and it is the universal cover of $\mathcal{W}_0$ otherwise, and $c(m_{\mathbb{B}}, m_{\mathbb{E}})$ is any event which covers $m_{\mathbb{E}}, m_{\mathbb{B}}$ in $\mathcal{W}_0$ (Axiom 7). Then f is a retraction such that $a \preceq f(a)$ for any $a \in \mathcal{W}_1$ whose $\mathbb{E}, \mathbb{B}$ limits exist.

Note that if $B(m_{\mathbb{E}}, m_{\mathbb{B}})$, then $f(a)$ is just the exact cover of $m_{\mathbb{B}}, m_{\mathbb{E}}$. Note moreover that if $\mathcal{W}_1$ has a universal cover c and $c \in \mathcal{W}_0$, then $\mathbb{E}, \mathbb{B}$ limits will always exist for any $a \in \mathcal{W}_1$, and hence $a \preceq f(a)$ for any $a \in \mathcal{W}_1$.

The above result then tells us that we can regard a model $\mathcal{W}_1$ of GT as having been obtained from a submodel $\mathcal{W}_0$ by a process of adding new events, and hence, possibly, new boundaries. The philosophical upshot of this situation is the following: if we consider only time as a form of intuition, i.e., we only consider models of GT_0, then it is not possible in general to define retraction maps and model infinite potential divisibility. This becomes possible only if we have the additional power of the unity of apperception, in the form of the axioms for GT, which introduce transcendental events and thus allow any event in the bigger model to be mapped to an event which covers it in the submodel, preserving both $\mathbb{B}$ and $\mathbb{E}$. Infinite potential divisibility thus requires time as a formal intuition, and not merely the form of intuition.

The next step towards modelling potential infinite divisibility is to impose additional constraints on the retraction maps defined above, so that these maps can be really interpreted as "refining", or "splitting", events and boundaries of $\mathcal{W}_0$. We shall achieve this with the following definition:

DEFINITION 22. *Let $\mathcal{W}_1, \mathcal{W}_0$ be models of GT and let $f : \mathcal{W}_1 \to \mathcal{W}_0$ be a retraction map. For a given $a \in W_0$, we say that f splits a if there are $c, d \in W_1$ such that $\mathbb{O}(c,d)$ and $f(c) = f(d) = a$. For a given boundary $x \in \mathcal{B}(\mathcal{W}_0)$, we say that f splits x if there exists $c \in \mathcal{W}_1$ such that $f(c) \in Pres(x)$ and $\mathbb{O}(c,d)$ for any $d \in f^{-1}(Past(x) \cup Fut(x))$*

It is important to realize the philosophical difference between splitting an event and splitting a boundary. While adding splittings of events formalizes the potential refinement of parts of time and it ensures that new boundaries are introduced, it still does not ensure that the boundaries existing in $\mathcal{W}_0$ are “split’ or refined. Indeed, a boundary $x \in \mathcal{B}(\mathcal{W}_0)$ might be such that $\mathbb{O}(f^{-1}(Past(x))\mathbb{O}) = \mathbb{O}f^{-1}(Fut(x))$, in which case it corresponds to a unique boundary in $\mathcal{B}(\mathcal{W}_1)$. We have remarked above that Kant talks about divisions of parts of time and of boundaries in modal terms. This is to be interpreted as reflecting a notion of potentiality: parts of time *can* always be subdivided to infinity, and boundaries *can* always be split, since there are no simple parts in time. In light of these considerations we obtain the following picture of infinite divisibility:

DEFINITION 23. *Infinite divisibility is represented by a countable (inverse) sequence $\mathcal{S}$ of retracts related by retractions.*

$$\ldots\ldots \mathcal{W}_3 \longrightarrow_{f_{32}} \mathcal{W}_2 \longrightarrow_{f_{21}} \mathcal{W}_1 \longrightarrow_{f_{10}} \mathcal{W}_0$$

satisfying the following conditions:

1. *each $\mathcal{W}_n$ is finite*
2. *for any event $a \in W_n$ there exists $m \geq n$ such that f_{mn} splits a*
3. *for any boundary $x \in \mathcal{B}(\mathcal{W}_n)$ there exists $m \geq n$ such that f_{mn} splits x*

In section 10 we shall investigate the notion of limits of generalizations of such sequences, called inverse systems, and argue that the space of boundaries on the limits of such inverse systems provide an accurate formalization of the Kantian continuum.

§9. Unity and universality. The final aim of the present work, as we outlined in the introduction, is to provide a formal correlate for Kant's notion of time as formal intuition, i.e., time as a unique and fully determined object through which 'all concepts of time first become possible' (B161n). In the following three sections we shall pursue this aim, taking as our point of departure the collection of finite models of GT, which we conceive as a formal correlate for Kant's notion of "parts of time". While we do not have here the space to provide an extensive analysis of the concept of time as formal intuition, before we delve into the formal work it is expedient to provide the reader with a brief elucidation of this concept, so as to clarify the philosophical relevance of the mathematical developments in the remainder of the paper.

Time as formal intuition is, as we already remarked, time as a fully determined, or, in Kantian terms, "thoroughgoingly determined" object. This means that time as a formal intuition provides the ground that allows for the objective determination of any possible experience with respect to any temporal concept, i.e., it represents the condition for the possibility of objective temporal determination of any possible experience. We take a broad view of 'concepts of time' here; Kant probably means not just what he calls 'relations in time' (namely succession and simultaneity), but the whole array of concepts that are used to formulate the synthetic a priori principles for time, for example concepts of part and whole, temporal duration, and the idea that time can only be represented externally, as a line.

According to Kant, for example, if we consider any possible experience of succession of perceptions, we are immediately aware that the judgment regarding the temporal order of such perceptions is merely subjective, unless it is subsumed under a universal rule which makes this succession objective - and, ultimately, able to be communicated. An objective succession then requires subsumption of the perceptions under the category of causality. This subsumption, however, requires itself a manifold on which it can be applied, and in particular it requires a temporal intuition which can encompass any possible succession of perceptions - not just actual experiences - and which supports the formulation of judgments of objective temporal succession; the consequences of this objective temporal determination are then fully determined by the properties of this all-encompassing temporal intuition. We provide a formal correlate to the thoroughgoing determination of time as formal intuition with respect to judgments of temporal order by means of Theorem 9 in this Section. In section 11.2 it will be demonstrated how duration and the external representtion of time as a line require time as an object, synthesised by the unity of apperception.

It is important to remark here that time as formal intuition, satisfying all the *a priori* principles we listed in Section 1, is itself, according to what Kant says in the famous footnote at B161n of the second edition of the CPR, the product of a synthesis. This synthesis does not proceed according to empirical concepts, of course, but is *a priori* and it is such that it 'precedes all concepts" - also all concepts of the understanding - and generates a unified time; indeed, this synthesis is not only such that 'concepts of space and time first become possible" through it, but also such that 'through it (as the understanding determines the sensibility) space or time are first given as intuitions".

This synthesis thus generates space and time as *formal* intuitions, as opposed to *forms* of intuition. The interpretation of the nature of this synthesis has been hotly debated in the literature; we understand the mathematical results in this paper as broadly supporting Longuenesse's interpretation of this synthesis as the synthesis of the unity of apperception (see [Longuenesse, 1998]), which brings unity to the scattered manifolds of the form of intuition by relating all of them to the same subject, the 'I think" which, Kant says, must be able to accompany all my representations.

In the proposed formalism the synthesis of the unity of apperception has two formal correlates. At the first-order level we have axiom 7 and the axioms for $\otimes_p, \otimes_f$. These axioms together ensure that certain events necessarily exist which need not be given by empirical intuition. We termed these events "transcendental events"; they represent the first, most primitive "effect of the understanding on sensibility", in that the understanding posits these events in order to unify and connect the empirically given events. At the second-order level we have the directedness condition on inverse systems of finite models of GT, which we shall use in the following sections to formalize Kant's temporal continuum, and which was already argued to be a formal correlate of the unity of apperception in [Achourioti and van Lambalgen, 2011]. This condition ensures that any two "local times", i.e., any two possible temporal experiences, can be seen as parts of a temporal experience which is a whole encompassing both. This in turn implies that all temporal experiences must be able to be embedded into one global time, the inverse limit, which satisfies very specific properties, formalizing the idea that time as formal intuition is the ground allowing for objective determination of any possible experience with respect to the "concepts of time".

The reader will find it instructive to imagine how our experience would be like if these conditions were not in place. Renouncing the unity of apperception would leave one with "islands of time" (cf. 8), collections of event structures each accompanied by its own distinct consciousness; a being described by such a collection would, echoing Kant, be "multicoloured", the consciousness (the "I") accompanying this fleeting bunch of events having no relation whatsoever to events just past or about to come.

After these brief philosophical remarks, we are now ready to delve into the technical developments.

9.1. Inverse systems of parts of time and their limits. The guiding intuitions of the constructions that follow are that 'Parts of time are times' (A169/B211) and that 'Different times are only parts of one and the same time.' (A31-2/B47). We interpret the former as meaning that there exist finite families of 'parts of time', which obey the same axioms as time itself ('are times'). These parts of time contain both empirical and transcendental events, and hence are interpreted in our framework as models of GT. The second quote we understand to mean that there exists a unified time ('one and the same time') which is in some sense universal. In Section 8, on infinite divisibility, we introduced ways of creating parts of time, parts of parts of time, and so forth; we then argued that the resulting structures are related by retractions (see lemma 22). In light of these considerations we obtain the following:

DEFINITION 24. *Let T be a set. A family of parts of time indexed by T is a family $\{\mathcal{W}_t \mid t \in T\}$ of finite models of GT together with a set of retractions*

$$f_{vs} : \mathcal{W}_v \to \mathcal{W}_s$$

satisfying:

(i) f_{ss} *is the identity*
(ii) $f_{ts}(f_{vt}(a)) = f_{vs}(a)$ *('parts of parts are parts').*

The index set thus carries a partial order $\leq$, defined by letting $s \leq t$ for $s, t \in T$ iff there exists a retraction map $f_{ts} : \mathcal{W}_t \to \mathcal{W}_s$.

Our families of parts of time are instances of well known mathematical structures, known as 'inverse systems' or 'projective systems', or - in category theoretic terms - diagrams in the category of models of GT. We shall be interested in the inverse limit of such systems, and for this construction it is essential to be precise about what predicates are preserved by the maps. Since our maps are retractions, they are continuous in the sense of Section 6; hence, given the signature $\{E, B, O, \mathbb{B}, \mathbb{E}, \mathbb{O}; \otimes_p, \otimes_f\}$, a map f_{st} preserves $\mathbb{E}, \mathbb{B}, O, \otimes_p, \otimes_f$.

Consider now any two first order models M_1, M_0 and a map $f : M_1 \to M_0$. We say that a formula $\phi(\bar{x})$ is preserved by f if for any sequence of objects $\bar{a}$ from M_1, $M_1 \models \phi(\bar{a})$ implies $M_0 \models \phi(f\bar{a})$. We then have the following results:

LEMMA 23 (Lyndon). *Let $\phi(\bar{x})$ be a positive primitive formula and let $f : \mathcal{W}_1 \to \mathcal{W}_0$ be an e-homomorphism. Then f preserves $\phi(\bar{x})$. This holds in particular when f is a retraction.*

For geometric *sentences* in the distinguished signature we obtain a similar result, making essential use of the fact that the maps we consider are retractions:

LEMMA 24. *Let ϕ be the geometric sentence $\forall \bar{x}(\psi(\bar{x}, \bar{y}) \to \chi(\bar{x}, \bar{y}))$ and let $f : \mathcal{W}_1 \to \mathcal{W}_0$ be a retraction map. Then f preserves ϕ.*

PROOF. Assume that $\mathcal{W}_1 \models \phi$. We need to show that $\mathcal{W}_0 \models \phi$. Assume then that $\mathcal{W}_0 \models \psi(\bar{a})$ for some $\bar{a}$ in $\mathcal{W}_0$. Since the map f is a retraction this means that $\mathcal{W}_0$ is a submodel of $\mathcal{W}_1$, hence there is an embedding of $\mathcal{W}_0$ into $\mathcal{W}_1$. Thus $\mathcal{W}_1 \models \psi(\bar{a})$ and hence $\mathcal{W}_1 \models \chi(\bar{a})$ since $\mathcal{W}_1 \models \phi$. Since f is a e-homomorphism from Lemma 23 we have $\mathcal{W}_0 \models \chi(f\bar{a})$ and thus $\mathcal{W}_0 \models \chi(\bar{a})$ since $f\bar{a} = \bar{a}$. Hence $\mathcal{W}_0 \models \phi$ $\dashv$

Note that the above result cannot be extended to arbitrary geometric formulas, even in the presence of retraction maps.

These results have important consequences, because we now aim at imposing an additional condition on the index set T of a family of parts of time: this index set T should be a directed partial order, in the sense that for any $s, t \in T$ there must be a $u \geq s, t$. The requirement of directedness is crucial in order to obtain a well behaved notion of an inverse limit. Moreover, it also provides us with a formal correlate for the unity of apperception, since it implies that for any two parts of time $\mathcal{W}_s, \mathcal{W}_t$ there must be a part of time $\mathcal{W}_u$ which refines them both and which has both as submodels. From the results above, it follows that if the order on the index set is to be directed, all indexed models must satisfy the same geometric theory restricted to the distinguished vocabulary. Note that, e.g., the sentence

$$\exists x_1 ... \exists x_n (\mathbb{O}(x_1, x_2) \wedge E(x_1, x_2) \wedge \ldots \wedge \mathbb{O}(x_{n-1}, x_n) \wedge E(x_{n-1}, x_n)$$

, expressing the existence of an antichain of length n, is not in this theory.

LEMMA 25. *Let $(T, \leq, \{\mathcal{W}_s \mid s \in T\}, f_{ts})$ be a family of parts of time, each satisfying the same geometric extension of GT. Then $\leq$ is directed.*

PROOF. Let $\mathcal{W}_s, \mathcal{W}_t$ such that they satisfy the same geometric extension $\mathcal{G}$ of GT. By means of a 'dynamic proof' (see Coquand [**?**]), we can construct a finite model $\mathcal{W}_u$ of $\mathcal{G}$ such that $\mathcal{W}_s, \mathcal{W}_t$ are submodels of $\mathcal{W}_u$. By Lemma 22, there are maps $f_{us} : \mathcal{W}_u \to \mathcal{W}_s$ and $f_{u,t} : \mathcal{W}_u \to \mathcal{W}_t$ which are retractions. By Definition 24, T is directed. ⊣

Now that we have introduced inverse systems of finite models of GT as formal correlates of parts of time related via potential divisibility and the unity of apperception, we introduce the notion of a limit of such an inverse system. This notion will be used to provide a formal correlate to Kant's notion of time as a *formal intuition*, time as an object of which all times are but parts.

DEFINITION 25. *Let* $(T, \leq, \{\mathcal{W}_s \mid s \in T\}, f_{ts})$ *be an inverse system, and let* W_s *be the domain of* $\mathcal{W}_s$. *Let* $V \subseteq \Pi_{s \in T} W_s$ *the set of all* ξ *such that for* $s \geq t$, $h_{st}(\xi(s)) = \xi(t)$. *Define a model* $\mathcal{V}$ *with domain* V *and signature* σ *by putting for relation symbols* $R \in \sigma$: $\mathcal{V} \models R(\xi^1, \xi^2, \dots)$ *if for all* $s \in T$, $\mathcal{W}_s \models R(\xi^1_s, \xi^2_s, \dots)$. *An analogous definition applies to the function symbols.* $\mathcal{V}$ *is called the* inverse limit *of the given inverse system.*

THEOREM 6. *Let* $(T, \leq, \{\mathcal{W}_s \mid s \in T\}, f_{ts})$ *be an inverse system of finite event structures. Then the inverse limit* $\mathcal{V}$ *is non-empty.*

PROOF. The textbook proof of this results relies on equipping the finite event structures with the (auxiliary) discrete topology, and using the result that the limit of an inverse system of compact Hausdorff spaces is non-empty. We do not need to recur to this approach, however, since our maps are not mere e-homomorphisms but retractions, and hence they are surjective; thus an element of the inverse limit can be constructed by simple induction. ⊣

LEMMA 26. *Let* $(T, \leq, \{\mathcal{W}_s \mid s \in T\}, f_{ts}, \mathcal{V})$ *be an inverse system of finite models of* GT. *Then the projection* π_s *defined by* $\xi \in \mathcal{V} \mapsto \pi_s(\xi) := \xi(s)$ *is a e-homomorphism, satisfying* $s \geq t$: $h_{st}(\pi_s(\xi)) = \pi_t(\xi)$. *If the e-homomorphisms* h_{st} *are retractions, so are the projections.*

The following result states the standard categorical universal property of inverse limits:

THEOREM 7. *[Universality] Let* $(T, \leq, \{\mathcal{W}_s \mid s \in T\}, f_{ts}, \mathcal{V})$ *be an inverse system of finite models of* GT. *If* $(\mathcal{N}, \rho_s)$ *is an object satisfying the same diagrams as* $(\mathcal{V}, \pi_s)$, *then there exists a unique e-homomorphic map* $\iota : \mathcal{N} \longrightarrow \mathcal{V}$ *satisfying* $\rho_s = \iota \circ \pi_s$.

A natural question which arises at this point is whether the inverse limit of an inverse system of finite models of GT, i.e., of a family of parts of time, can itself be regarded as a model of GT.

LEMMA 27. *Let* $(T, \leq, \{\mathcal{W}_s \mid s \in T\}, f_{ts}, \mathcal{V})$ *be an inverse system of finite models of* GT, *and let* $\phi(\bar{x})$ *be a positive primitive formula in the distinguished vocabulary. Then for any tuple* $\bar{a}$ *of objects of* $\mathcal{V}$, $\mathcal{V} \models \phi(\bar{a})$ *if and only if* $\mathcal{W}_s \models \phi(\pi_s(\bar{a}))$ *for every* $s \in T$.

PROOF. The proof, by induction on the complexity of $\phi(\bar{x})$, can be found in [Achourioti and van Lambalgen, 2011] ⊣

Since Lemma 24 implies that only geometric *sentences* are preserved by retraction, we cannot expect to obtain a result as Lemma 27 for geometric formulas. We obtain, however, the following weaker result:

LEMMA 28. *Let $(T, \leq, \{\mathcal{W}_s \mid s \in T\}, f_{ts}, \mathcal{V})$ be an inverse system of finite models of GT, and let $\phi(\bar{x}, \bar{y})$ be a geometric formula in the distinguished vocabulary. Then for any tuple $\bar{a}$ of objects of $\mathcal{V}$, $\mathcal{V} \models \phi(\bar{a})$ if and only if $\mathcal{W}_s \models \phi(\pi_s(\bar{a}))$ for all $s \in S$, $S \subseteq T$ a cofinal subset of T.*

PROOF. Let $\forall \bar{x}(\psi(\bar{x}, \bar{y}) \rightarrow \chi(\bar{x}, \bar{y}))$ be a geometric formula in the distinguished vocabulary, and let $\bar{a}$ be a tuple of objects from $\mathcal{V}$ such that $\mathcal{V} \models \psi(\bar{a}) \rightarrow \chi(\bar{a})$. Then $\mathcal{V} \models \neg\psi(\bar{a}) \vee \chi(\bar{a})$, hence either $\mathcal{V} \models \neg\psi(\bar{a})$ or $\mathcal{V} \models \chi(\bar{a})$. In the former case because of Lemma 27 there must be an index $i \in T$ such that $\mathcal{W}_i \models \neg\psi(\pi_t\bar{a})$; since negations of positive primitive formulas in the distinguished vocabulary are preserved upwards, we then have $\mathcal{W}_j \models \neg\psi(\pi_s\bar{a})$, and hence $\mathcal{W}_j \models \psi(\pi_s\bar{a}) \rightarrow \chi(\pi_s\bar{a})$ for all $j \geq i$, which is a cofinal set of models. If $\mathcal{V} \models \chi(\bar{a})$ then by Lemma 27 we obtain $\mathcal{W}_t \models \chi(\pi_t\bar{a})$ for all $t \in T$, hence $\mathcal{W}_t \models \psi(\pi_t\bar{a}) \rightarrow \chi(\pi_t\bar{a})$ for all $t \in T$, which is obviously cofinal. For the direction from right to left, let $\bar{a}$ be a tuple of objects from $\mathcal{V}$ such that $\mathcal{W}_s \models \neg\psi(\pi_s\bar{a}) \vee \chi(\pi_s\bar{a})$ for all $s \in S$, $S \subseteq T$ a cofinal set of indices. Clearly, if there exists $s' \in S$ with $\mathcal{W}_{s'} \models \neg\psi(\pi'_s\bar{a})$ then $\mathcal{V} \models \neg\psi(\bar{a})$ since $\mathcal{V}$ retract to $\mathcal{W}_{s'}$, and we are done. Otherwise $\mathcal{W}_s \models \chi(\pi_s\bar{a})$ for all $s \in S$, hence $\mathcal{W}_t \models \chi(\pi_t\bar{a})$ for any $t \in T$ since S is cofinal in T, hence by Lemma 27 we obtain $\mathcal{V} \models \chi(\bar{a})$ and the result follows. $\dashv$

For geometric sentences, we obtain:

LEMMA 29. *Let $(T, \leq, \{\mathcal{W}_s \mid s \in T\}, f_{ts}, \mathcal{V})$ be an inverse system of finite models of GT, and let $\phi(\bar{x}, \bar{y})$ be a geometric sentence. Then $\mathcal{V} \models \phi(\bar{x}, \bar{y})$ iff $\mathcal{W}_s \models \phi(\bar{x}, \bar{y})$ for all $s \in T$*

PROOF. Since the projection maps π_s are retractions, the direction from left to right follows straightforwardly from Lemma 24. For the direction from right to left follows striaghtforwardly from Lemma 28. $\dashv$

Hence, we obtain that the inverse limit $\mathcal{V}$ of an inverse system of finite models of GT is itself a finite model of GT, since GT is a geometric theory. In category theoretic terms, this means that the category of models of GT and e-homomorphisms with respect to the distinguished signature has directed limits.

The next step is to study the nature of these limits in more detail. We start with some considerations on the relation between the expressive power of the language of GT and the topology induced on the inverse limit by the projection maps.

9.2. Expanding the language. The $O, \mathbb{E}, \mathbb{B}$ vocabulary is not sufficiently rich to serve as a language with which to describe the inverse limit. We add distinguished monadic predicates $U_0, U_1, U_2 \ldots ; V_0, V_1, V_2 \ldots ; C_0, C_1, C_2 \ldots$ to the signature of GT satisfying the following axiom schemes:

1. $\forall y \forall x(U_i(x) \wedge \mathbb{E}(x, y) \rightarrow U_i(y))$
2. $\forall y \forall x(V_i(x) \wedge \mathbb{B}(x, y) \rightarrow V_i(y))$
3. $\forall y \forall x(C_i(x) \wedge \mathbb{B}(y, x) \wedge \mathbb{E}(y, x) \rightarrow C_i(y))$

We assume that for each $\mathcal{W}_t$, all basic open sets of $\mathcal{W}_t$ of both the $\mathbb{E}, \mathbb{B}$ topologies are represented by a predicate. The interpretation of U_i on the inverse limit is thus an $\mathbb{E}$-open set, and likewise for the other predicates.[27] To allow GT to express facts about the topology on the limit generated by the basis defined by the U_i, V_j, we enlarge its logical vocabulary with the infinitary operations $\bigwedge, \bigvee$ and define infinitary geometric formulas by allowing $\bigwedge, \bigvee$ in positive primitive formulas. The expanded language allows us to express that for $I \subseteq \mathbb{N}$,

$$\bigwedge_{i \in I} U_i, \bigvee_{i \in I} U_i$$

are $\mathbb{E}$-open. If $\varphi(x)$ is a (possibly infinitary) formula defining an $\mathbb{E}$-open set, then

$$\forall u(\forall v(\varphi(v) \rightarrow \mathbb{O}(u, v)) \rightarrow \mathbb{O}(u, x))$$

defines the past generated by φ; as observed previously, if φ is positive primitive, the formula defining the past is geometric

$$\forall u(O(u, x) \rightarrow \exists v(\varphi(v) \wedge O(u, v))).$$

We will abbreviate this formula as

$$\mathbb{O}(\varphi\mathbb{O})(x).$$

The pasts determine a subtopology of the topology on the limit; the following lemma ensures that the subtopology of the pasts is still Alexandroff:

LEMMA 30. *Define $R(x, y)$ iff $x \in \mathbb{O}(\{y\}\mathbb{O})$. Then*
(i) R is transitive and reflexive
(ii) if U is a fixpoint, $y \in U$ and $R(x, y)$ then $x \in U$.

PROOF. Since $x \in \mathbb{O}(\{x\}\mathbb{O})$, we have $R(x, x)$. Assume $R(x, y) \wedge R(y, z)$, i.e.

$$x \in \mathbb{O}(\{y\}\mathbb{O}) \wedge y \in \mathbb{O}(\{z\}\mathbb{O}).$$

Since $\mathbb{O}(\{z\}\mathbb{O})$ is a fixpoint, we have

$$x \in \mathbb{O}(\{y\}\mathbb{O}) \subseteq \mathbb{O}(\mathbb{O}(\{z\}\mathbb{O})\mathbb{O}) = \mathbb{O}(\{z\}\mathbb{O}).$$

It follows that $R(x, y)$.
Assume $y \in U = \mathbb{O}(U\mathbb{O})$, then $x \in \mathbb{O}(\{y\}\mathbb{O}) \subseteq \mathbb{O}(U\mathbb{O}) = U$. ⊣

Hence the subtopology of the pasts is also Alexandroff, and not second-countable in the inverse limit considered.

We may now define a boundary as a formula $\beta(x) = \varphi(x) \vee \nu(x) \vee \psi(x)$ such that φ defines a past, ψ the corresponding future and $\nu(x) \leftrightarrow \neg(\varphi(x) \vee \psi(x))$ the present.

DEFINITION 26. *We define a linear order $<$ on β-formulas by putting $\beta < \beta'$ if $\forall x(\varphi(x) \rightarrow \varphi'(x))$.*

[27] The textbook way of inducing a topology on the inverse limit is to define an induced basis consisting of sets $\pi_t^{-1}(U)$ for open $U \subseteq \mathcal{W}_t$. This topology is second-countable. We employ a finer topology generated by the interpretations of the distinguished predicates on the limit. The new open sets are countable intersections of open sets from the induced basis. This topology, though first-countable, is no longer second-countable.

Since the inverse limit $\mathcal{V}$ of an inverse system of finite models of GT is itself a model of GT, one can construct the space of boundaries $\mathcal{B}(\mathcal{V})$ on $\mathcal{V}$. The construction of this space of boundaries can now be expressed wholly within the language using the above results; in particular, the linear order on the boundaries induced by the inclusion of pasts can now be expressed in terms of β formulas as above outlined.

9.3. Universality of GT. In the discussion above we have considered arbitrary inverse systems of finite models of GT, i.e., arbitrary families of parts of time. In order to achieve the required universality result for time as formal intuition, however, we need to consider the inverse system of *all* finite models of GT. The Kantian justification for considering this rather special inverse system lies in Kant's notion of a "possible experience". In formal terms, a possible (temporal) experience is nothing else that a finite model of GT. Since time as formal intuition is the all-encompassing time in which all possible experiences must be able to find their temporal determination, this means that the inverse limit of the inverse system of *all* finite models of GT is the best tool to understand the universality of time as formal intuition.

Let then $(\mathcal{W}_t, T, \leq, f_{vs})$ be a countable geometrically complete family of finite models[28] of GT, which is closed under divisibility, and let $\mathcal{V}$ be the inverse limit of such family. We then have

THEOREM 8. *The following hold:*

(i) $\mathcal{V} \models GT$

(ii) *for geometric sentences* φ *in the distinguished vocabulary* φ, $\mathcal{V} \models \varphi$ *iff* $GT \vdash \varphi$

Note moreover that the inverse limit is universal for $(\mathcal{W}_t, T, \leq, f_{vs})$ in the sense of Theorem 7 above.

§10. Time as formal intuition: the continuum. The purpose of this section is to investigate the structure of boundaries on inverse limits of parts of time more in detail; we shall see that the continuum that emerges in the limit has very special properties, which make the Kantian continuum quite different from the classical continuum of the reals.

10.1. The direct limit and the structure on the inverse limit. Consider an inverse system $(\mathcal{W}_t, T, \leq, f_{vs})$ of finite models of GT. We say that this inverse system satisfies *infinite divisibility* if it satisfies the conditions given in Definition 23 of Section 8. Given any inverse system of finite models of GT, not necessarily satisfying infinite divisibility, we immediately obtain that the space of boundaries $\mathcal{B}(\mathcal{V})$ on the limit $\mathcal{V}$ is compact Hausdorff because of Corollary 2 in Section 5. Moreover, since any map $f_{vs} : \mathcal{W}_v \to \mathcal{W}_s$ is a retraction map, this implies, as we have seen, that $\mathcal{W}_v$ embeds in $\mathcal{W}_s$, i.e., there exists a map $e_{sv} : \mathcal{W}_s \to \mathcal{W}_v$ which is an embedding. In our setting, since $\mathcal{W}_s$ is a submodel of $\mathcal{W}_v$, this embedding map is particularly simple as it is just the identity. We are therefore entitled to consider the *direct system* $(\mathcal{W}_t, T, \sqsubseteq, e_{sv})$ where $\sqsubseteq = OP(\leq)$ is the opposite of $\leq$, i.e., $t \sqsubseteq s$ iff $s \leq t$ for any $s, t \in T$, and e_{sv} is the identity embedding. For this direct system it is possible to define a direct limit $\mathcal{D}$ in the usual way (see [Hodges, 1997]). We then have:

[28]This is shorthand for: each geometric sentence not derivable from GT is represented by a countermodel in the family.

LEMMA 31. *Let $(\mathcal{W}_t, T, \leq, f_{vs})$ be an inverse system of finite models of GT and let $\mathcal{V}$ be its inverse limit. The direct limit $\mathcal{D}$ of the direct system $(\mathcal{W}_t, T, \sqsubseteq, e_{sv})$ is a countable model of GT, it is isomorphic to a submodel of $\mathcal{V}$, and hence it is a retract of $\mathcal{V}$.*

PROOF. The fact that $\mathcal{D}$ is a model of GT follows because $\mathcal{D}$ is the direct limit of a direct system of models of a geometric theory, and geometric formulas (actually, all Π_2 formulas) are preserved by direct systems ([Hodges, 1997], Theorem 2.4.6). Countability follows because all of the $\mathcal{W}_t$ are finite. To see that $\mathcal{D}$ is isomorphic to a substructure of $\mathcal{V}$ it suffices to consider the map $e : \mathcal{D} \to \mathcal{V}$ defined by mapping any element $a^\sim$ in the domain of $\mathcal{D}$ for $a \in \mathcal{W}_t$ to the thread $c \in \mathcal{V}$ defined by letting $c_s = f_{ts}(a)$ if $s \leq t$, and $c_s = a$ otherwise. This is easily checked to be an isomorphism with respect to the signature $(\mathbb{B}, \mathbb{E}, O, \otimes_p, \otimes_f)$. Since $\mathcal{D} \models GT$, moreover, this substructure is also a model of GT. Finally, because of Lemma 22, $\mathcal{D}$ is a retract of $\mathcal{V}$. ⊣

The direct limit $\mathcal{D}$ is homeomorphic to the submodel of $\mathcal{V}$ given by all the threads $\xi \in \mathcal{V}$ which become eventually constant, i.e., those threads such that there exists $t' \in T$ with $\xi(t) = a$ for all $t \geq t'$. We shall abuse our notation and call this submodel of $\mathcal{V}$ also $\mathcal{D}$.

We now start to direct our attention to the description of inverse limits obtained from inverse systems satisfying infinite divisibility, and to the elucidation of the properties which their spaces of boundaries satisfy. We start with a preservation Lemma.

LEMMA 32. *Let $\mathcal{W}_1, \mathcal{W}_0$ be models of GT and let $f : \mathcal{W}_1 \to \mathcal{W}_0$ be a retraction map. Then f preserves the following geometric formula:*

$$\forall x(x \preceq y \to \mathbb{E}(x, y) \vee \mathbb{B}(x, y))$$

PROOF. Assume that $\mathcal{W}_1 \models \forall x(x \preceq b \to \mathbb{E}(x, b) \vee \mathbb{B}(x, b))$ for some b in $\mathcal{W}_1$. We need to show that $\mathcal{W}_0 \models \forall x(x \preceq f(b) \to \mathbb{E}(x, f(b)) \vee \mathbb{B}(x, f(b)))$. Let then a be an object in $\mathcal{W}_0$ such that $\mathcal{W}_0 \models a \preceq f(b)$, and assume towards a contradiction that $\mathcal{W}_0 \models E(a, f(b)), B(a, f(b))$. Since the predicates E, B are reflected by e-homomorphisms, we must have that $\mathcal{W}_1 \models E(a, b), B(a, b)$, and thus $\mathcal{W}_1 \models a \preceq b$. Since we assumed that $\mathcal{W}_1 \models \forall x(x \preceq b \to \mathbb{E}(x, b) \vee \mathbb{B}(x, b))$, however, we obtain a contradiction. Thus either $\mathcal{W}_0 \models \mathbb{E}(a, f(b))$ or $\mathcal{W}_0 \models \mathbb{B}(a, f(b))$, which concludes the proof. ⊣

The lemma above can be understood as follows. Define an auxiliary relation $a \ll b$ on events of an event structure, where $a \ll b$ is defined as $B(a, b) \wedge E(a, b)$. The relation $\ll$ can be interpreted as the "well inside" relation from topology; we recall that in topology an open set A is well inside another open set B if the closure of A is contained in B. The formula of Lemma 32 then states the property of an event of being such that whenever another event is covered by it, this latter event is not well inside it. The lemma then shows that this property is preserved downwards by e-homomorphisms.

Downward preservation of the formula in Lemma 32 is of interest since as we have seen in general geometric *formulas* are not preserved downward by e-homomorphisms, or even by retractions. Indeed, the reader can check that the following stronger notion is not preserved downwards by e-homomorphisms:

DEFINITION 27. *Let $\mathcal{W}$ be a model of GT and let μ be an object of $\mathcal{W}$. We say that μ is $\preceq$-minimal if $\mathcal{W} \models \forall x(x \preceq \mu \to \mu \preceq x)$. Given an event structure $\mathcal{W}$ we denote its set of $\preceq$-minimal events with $\mathcal{I}(\mathcal{W})$*

If $\mathcal{W}$ is an event structure we can define an equivalence relation $e \subseteq \mathcal{I}(\mathcal{W}) \times \mathcal{I}(\mathcal{W})$ by letting $e(\mu, \nu)$ if $\mu \preceq \nu$. We denote with μ^e the equivalence class of μ. Then we can endow the set $\{\mu^e \mid \mu \in \mathcal{I}(\mathcal{W})\}$ with a linear order by letting $\mu^e \leq \nu^e$ if $\mathbb{E}(\nu^e, \mu^e)$. We shall abuse our notation and denote such linear order by $(\mathcal{I}(\mathcal{W}), \leq)$, even though, strictly speaking, the order is on the equivalence classes μ^e obtained as above.

The notion of a $\preceq$-minimal event allows us to introduce the notion of a complete event structure, that is, an event structure in which any event is the exact cover of two minimal $\preceq$-events.

DEFINITION 28. *Let $\mathcal{W}$ be a fnite model of GT. $\mathcal{W}$ is said to be complete if for any event $a \in \mathcal{W}$ we have $a \preceq e, e \preceq a$ for $e = \mu \vee \mu'$ the exact cover of two distinct $\preceq$-minimal events $\mu, \mu' \in \mathcal{I}(\mathcal{W})$*

THEOREM 9. *Let $\mathcal{W}$ be a finite model of GT. Then there exists a complete event structure $H(\mathcal{W})$ and a retraction map $h : H(\mathcal{W}) \to \mathcal{W}$*

PROOF. Let $\mathcal{W}$ be a model of GT; we construct $H(\mathcal{W})$ in two steps as follows. Define two equivalence relations $\sim_{\mathbb{B}}, \sim_{\mathbb{E}}$ on $\mathcal{W}$ by letting $a \sim_{\mathbb{B}} b$ if $\mathbb{B}(a,b), \mathbb{B}(b,a)$; then clearly $\sim_{\mathbb{B}}$ is an equivalence relation, and $\sim_{\mathbb{E}}$ is defined analogously. It is also clear that the relations $\mathbb{B}, \mathbb{E}$ linearly order the sets of equivalence classes under $\sim_{\mathbb{B}}, \sim_{\mathbb{E}}$ respectively. We first extend $\mathcal{W}$ to an event structure $\mathcal{W}'$ as follows. For any equivalence class under $\sim^{\mathbb{B}}$ such that there exists no $\preceq$-minimal event in it, let a be any event which is minimal with respect to $\preceq$ in the equivalence class; such event must exist because $\mathcal{W}$ is finite, and all events in the equivalence class are comparable with respect to $\preceq$. We then extend $\mathcal{W}$ by adding an event μ satisfying: $\mathbb{B}(\mu, a), \mathbb{B}(a, \mu)$ and $\mathbb{O}(\mu, c)$ for any c with $c^{\sim_{\mathbb{B}}} > a^{\sim_{\mathbb{B}}}$. It is easily checked that μ is a $\preceq$-minimal event covered by a (and hence by any event in $a^{\sim_{\mathbb{B}}}$). A similar procedure can be performed for $\sim_{\mathbb{E}}$ equivalence classes. Closing the resulting structure under the operations $\otimes_p, \otimes_f$ yields a finite model $\mathcal{W}'$ of GT, such that any event a in the model is covered and itself covers $\mu(a^{\sim_{\mathbb{B}}}) \vee \mu(a^{\sim_{\mathbb{E}}})$, where $\mu(a^{\sim_{\mathbb{B}}})$ denotes any $\preceq$-minimal event in $a^{\sim_{\mathbb{B}}}$ and similarly for $\mu(a^{\sim_{\mathbb{E}}})$. Clearly $\mathcal{W}$ is a submodel of $H(\mathcal{W})$, hence by Lemma 22 there must be a retraction map $h : H(\mathcal{W}) \to \mathcal{W}$. $\dashv$

The following corollary of Theorem 9 makes it clear that the inverse system of all finite models of GT can be replaced by the inverse system of all *complete* finite models of GT. This also implies that, for our purposes, it suffices to consider event structures whose one-sided boundaries (see Section 5) are defined by $\preceq$-minimal events.

COROLLARY 3. *The set of all complete finite models of GT forms an inverse system which is cofinal with the inverse system of all finite models of GT; the two inverse systems have then isomorphic limits.*

We now show how to characterize $\preceq$-minimal events as maximally overlapping sets of events. The following results will be of use presently:

LEMMA 33. *Let $\mathcal{W}$ be a model of GT and let $\mu \in \mathcal{W}$ be a $\preceq$-minimal event, $x \in \mathcal{B}(\mathcal{W})$. Then either $\mu \in Past(x)$ or $\mu \in Fut(x)$.*

PROOF. Let $\mathcal{W}, a, x$ be as in the statement of the lemma. We show that a cannot be in $Pres(x)$, which then means that it is either in $Past(x)$ or $Fut(x)$. If $a \in Pres(x)$ then there must be c, d such that $c \in Past(x), d \in Fut(x)$ with $O(a,c), O(a,d)$. But then $a \otimes_p c$, $a \otimes_f d$ are defined and we have $\mathbb{O}(a \otimes_p c, a \otimes_f d), a \otimes_p c \preceq a, a \otimes_f d \preceq a$, hence $a \otimes_p c, a \otimes_f d$ are strictly covered by a, which is a contradiction. ⊣

DEFINITION 29. *Let $\mathcal{W}$ be a model of GT_0 and let $\Sigma \subseteq \mathcal{W}$. We say that Σ is a maximal overlapping set of events if any two events in Σ overlap and Σ is maximal with this property, i.e., for any $a \notin \Sigma$ there exists $b \in \Sigma$ with $\mathbb{O}(a, b)$.*

Given an inverse system $(T, \leq, \mathcal{W}_s, f_{ts}, \mathcal{V})$ of finite models of GT, there exists a tight correspondence between the maximal overlapping sets of events of $\mathcal{V}$ and the set $\mathcal{I}(\mathcal{V})$ of $\preceq$-minimal events of $\mathcal{V}$, which is explicated as follows:

LEMMA 34. *Let $(T, \leq, \mathcal{W}_s, f_{ts}, \mathcal{V})$ be an inverse system of finite models of GT. Then the following hold:*

(i) *For any event θ of $\mathcal{V}$, the set $\{\mu \in \mathcal{I}(\mathcal{V}) \mid \mu \preceq \theta\}$ is in bijective correspondence with the set of all maximal overlapping sets of events in the submodel of GT generated by a*
(ii) *for any two boundaries $x, y \in \mathcal{B}(\mathcal{V})$ with $x < y$ there exists a $\preceq$-minimal event μ with $\mu \in Fut(x) \cap Past(y)$*

PROOF. Let $(T, \leq, \mathcal{W}_s, f_{ts})$ be an inverse system of finite models of GT, let $\mathcal{V}$ be its inverse limit, and let θ be an event of $\mathcal{V}$. To prove (i), equip the limit $\mathcal{V}$ with the Alexandroff topology in which the closed sets are the $\preceq$ downward closed sets; this topology has as a closed basis all sets of the form $\downarrow_{\preceq} \xi = \{\xi' \preceq \xi\}$ for all $\xi \in \mathcal{V}$. This topology on the limit is compact, because it is refined by the topology on the limit obtained by equipping the finite event structures with the discrete topology.

Consider now $\downarrow\theta = \{\xi \in \mathcal{V} \mid \xi \preceq \theta\}$, the submodel of GT generated by θ, equipped with the subspace topology. Since $\downarrow\theta$ is closed in $\mathcal{V}$, the subspace topology is also compact. Let now Σ be a maximal overlapping set of events in $\downarrow a$, and consider the family $\mathcal{F} = \{\downarrow\sigma \mid \sigma \in \Sigma\}$. The sets in this family are closed in the subspace topology; the fact that Σ is a pairwise overlapping set of events, along with the $\otimes_p, \otimes_f$ axioms of GT, ensure that $\mathcal{F}$ has the finite intersection property. By compactness then $\bigcap \mathcal{F} \neq \emptyset$, which means that there must be a $\mu \in \bigcap \mathcal{F}$. Clearly, μ is a $\preceq$-minimal covering event of $\downarrow a$ by maximality of Σ. It is straightforward to check that this construction yields, for two distinct $\Sigma, \Sigma' \subseteq \downarrow a$, two $\preceq$-minimal events μ, μ' with $\mathbb{O}(\mu, \mu')$; hence the correspondence is injective. Moreover, for any $\preceq$-minimal $\mu \in \downarrow a$, $\uparrow\mu$ is a maximal overlapping set of events of $\downarrow a$, hence the correspondence is surjective. We therefore obtain the required bijection. Note also that the $\preceq$-minimal events in $\downarrow a$ are still $\preceq$-minimal in $\mathcal{V}$.

For the second claim, let $x, y \in \mathcal{B}(\mathcal{V})$ with $x < y$. By Lemma 15, there must be an event $\xi \in Past(y) \cap Fut(x)$; the proof of (i) then yields the desired result. ⊣

The results above show that $\preceq$-minimal events correspond in our setting to the maximal overlapping classes of events which Russell employed in [Russell, 1936] to construct instants from events. We shall see in the sequel (Definition 30) that Walker instants and $\preceq$-minimal events can be combined to provide a formal correlate for the Kantian continuum. The results in this paper then suggest that the difference between the Walker and the Russell construction of time from events can be better understood in terms of ordered topological spaces; see the treatment in Section 10.2 and the remarks in Section 12.

Let now $(L, \leq)$ be a totally ordered set. Recall that a jump of L is a pair of elements $x, y \in L$ such that $x < y$ and there exists no $z \in L$ with $x < z < y$. The following result shows that if we construct the linear order of boundaries $\mathcal{B}(\mathcal{V})$ on the inverse limit of an inverse system of finite models of GT, the jumps of the linear order are in one to one correspondence with the $\preceq$-minimal events:

LEMMA 35. *Let $(\mathcal{W}_t, T, \leq, f_{vs})$ be an inverse system of parts of time and let $\mathcal{V}$ be its inverse limit. Then there exists a bijection j from the set of jumps of $\mathcal{B}(\mathcal{V})$ into $\mathcal{I}(\mathcal{V})$*

PROOF. Let $(T, \leq, \mathcal{W}_s, f_{ts}, \mathcal{V})$ be an inverse system of finite models of GT. We construct a bijection between the set of jumps of $\mathcal{B}(\mathcal{V})$ and $\mathcal{I}(\mathcal{V})$ as follows. Let $x, y \in \mathcal{B}(\mathcal{V})$ such that (x, y) is a jump in $\mathcal{B}(\mathcal{V})$. Then by (ii) of Lemma 34 there exists a $\preceq$-minimal event μ with $\mu \in Fut(x) \cap Past(y)$. It is then straightforward to check that the event μ is unique up to equivalence since $Past(y) \cap Fut(x)$ is a cluster under O.

We thus obtain a $\preceq$-minimal thread μ such that $\mu \in Past(y) \cap Fut(x)$. We can thus define a map j from the set of gaps of $\mathcal{B}(\mathcal{V})$ to $\mathcal{I}(\mathcal{V})$ by associating to any gap (x, y) the $\preceq$-minimal thread constructed as above. Clearly, if $(x, y), (z, w)$ are two distinct gaps, then it must hold that the minimal threads associated to $(x, y), (z, w)$ are distinct, hence the map is injective. To see that j is surjective consider a minimal thread $\xi \in \mathcal{B}(\mathcal{V})$, and consider the interval $(l(\xi), r(\xi))$. We showed in Lemma 16 that it must be the case that $\xi \in Fut(l(\xi))$ and $\xi \in Past(r(\xi))$. Moreover, $l(\xi), r(\xi)$ are maximal with the property of having ξ in the past and future, respectively. Hence any boundary y with $l(\xi) < y < r(\xi)$ must be such that $\xi \in Pres(y)$. However, by Lemma 33 this is impossible. Hence there cannot be any boundary y strictly between $l(\xi), r(\xi)$, which means that $(l(\xi), r(\xi))$ is a jump; moreover, it is clear that $j(l(\xi), r(\xi)) = \xi$ and we are done. $\dashv$

Note that the above proof can be extended to an order isomorphism between the set of jumps of $\mathcal{B}(\mathcal{V})$ and $(\mathcal{I}(\mathcal{V}), \leq)$. The following result shows that the linear order of boundaries on an inverse limit $\mathcal{V}$ is separable:

LEMMA 36. *Let $(T, \leq, \mathcal{W}_s, f_{ts}, \mathcal{V})$ be an inverse system of finite models of GT and let $\mathcal{V}$ be its inverse limit. Then $\mathcal{B}(\mathcal{V})$ is separable, and it is second countable only if there are countably many jumps.*

PROOF. Let $(T, \leq, \mathcal{W}_s, f_{ts})$ be an inverse system of finite models of GT and let $\mathcal{V}$ be its inverse limit. Since T is countable we again assume for simplicity that the inverse system is an inverse sequence.

In order to prove separability of $\mathcal{B}(\mathcal{W})$ consider the set $Q \subseteq \mathcal{B}(\mathcal{V})$ defined by letting Q be the set of all points $q \in \mathcal{B}(\mathcal{V})$ such that $q = r(\pi_n^{-1} Past(x))$ for some $n \in T, x \in \mathcal{B}(\mathcal{W}_n)$. The set Q is countable, since $\bigcup_{n \in T} \mathcal{B}(\mathcal{W}_t)$ is countable.

We first show the following claim: let $x, y \in \mathcal{B}(\mathcal{V})$ be such that (x, y) is not a jump, then there exists $q \in Q$ with $x < q < y$.

To prove this claim, assume (x, y) is not a jump. Then there must be $z \in \mathcal{B}(\mathcal{V})$ with $x < z < y$. We can then apply Lemma 15 and obtain threads ξ, ξ' such that $\xi \in Past(z), \xi \in Fut(x)$ and $\xi' \in Past(y), \xi' \in Fut(z)$. Since $\xi \in Past(z), \xi' \in Fut(z)$ we must have that $\mathbb{O}(\xi, \xi')$. Hence there must be a least $n \in T$ such that $\mathcal{W}_n \models \mathbb{O}(\xi_n, \xi'_n)$, and thus the set $\{x \in \mathcal{B}(\mathcal{W}_n) \mid \xi_n \in Past(x), \xi'_n \in Fut(x)\}$ is not empty and finite; let w be the minimal element of this set of boundaries. Clearly we have that $\xi \in \pi_n^{-1}(Past(w))$. Moreover we have that $\mathbb{O}(\zeta, \xi')$ for any $\zeta \in \pi_n^{-1} Past(w)$;

this follows because $\xi'_n \in Fut(w)$ and hence $\xi' \in \pi_n^{-1}(\xi'_n) \subseteq \pi_n^{-1}Fut(w)$, and $Past(w), Fut(w)$ are $\mathbb{O}$-remote.

Since $\xi \in \pi_n^{-1}(Past(w))$ and $\mathbb{O}(\zeta, \xi')$ for any $\zeta \in \pi_n^{-1}(Past(w))$, it must be the case that $\xi \in \mathbb{O}(\pi_n^{-1}(Past(w))\mathbb{O})$ and $\xi' \in \pi_n^{-1}(Past(w))\mathbb{O}$, that is, $r(\pi_n^{-1}(Past(w))$ must have ξ in the past and ξ' in the future. Thus if we let $q = r(\pi_n^{-1}(Past(w))$ we must have $x < q < y$ and we are done.

We can now show separability of the order topology on $\mathcal{B}(\mathcal{V})$ by showing that any nonempty basic open set of the order topology contains a boundary from Q. Consider a basic open set of the form $\{(x, y) \mid x, y \in \mathcal{B}(\mathcal{V})\}$. If the pair (x, y) defines a jump, then this basic open set is actually the emptyset, and we are done. Otherwise we just apply the result above to obtain a boundary $q \in Q$ which lies strictly between x and y, and we are done. Hence the space $\mathcal{B}(\mathcal{V})$ is separable.

Suppose now that there are countably many jumps. Then the families of open sets $\{(q, \leftarrow) \mid q \in Q\}$, $\{(q, \rightarrow) \mid q \in Q\}$, along with the sets $(x, \rightarrow), (y, \leftarrow)$ for (x, y) a jump provide us with a countable subbasis for the topology on $\mathcal{B}(\mathcal{V})$, and hence the space is second countable. $\dashv$

It is now time to focus our attention on inverse systems of finite models of GT which satisfy the requirement of infinite divisibility, in the sense of Definition 23. We begin with the following result:

LEMMA 37. *Let $(T, \leq, \mathcal{W}_s, f_{ts})$ be an inverse system satisfying infinite divisibility and let $\mathcal{V}$ be its inverse limit. Then for any $\xi, \xi' \in \mathcal{V}$, if $\mathbb{O}(\xi, \xi')$ then there exists ξ'' between ξ and ξ', i.e., $\mathbb{O}(\xi, \xi''), \mathbb{O}(\xi', \xi'')$ and $\mathbb{B}(\xi, \xi''), \mathbb{E}(\xi', \xi'')$. Moreover, ξ'' can be taken to be a thread $\xi'' \in \mathcal{D}$.*

PROOF. Let $\xi, \xi' \in \mathcal{V}$ be such that $\mathbb{O}(\xi, \xi')$, and let s be the least index such that $\mathcal{W}_s \models \mathbb{O}(\xi_s, \xi'_s)$. If there exists an event $a \in \mathcal{W}_s$ between ξ_s, ξ'_s we are done, since the eventually constant thread γ defined by a will be between ξ and ξ' in $\mathcal{V}$. Otherwise there is a boundary v with $\xi_s \in Past(v), \xi'_s \in Fut(v)$; by infinite divisibility and the definition of retraction it is straightforward to check that there must be $t \geq s$ and an event $a \in \mathcal{W}_t$ such that a is between ξ_t and ξ'_t in $\mathcal{W}_t$, and we can take the eventually constant thread defined by a as above. $\dashv$

The result above states a sort of density for event structures, which the limit $\mathcal{V}$ of an infinite divisibility inverse system satisfies: given any two events which do not overlap, a third event can be found in between which does not overlap with both.

We now focus our attention on the set $\mathcal{I}(\mathcal{V})$ of $\preceq$-minimal events on an infinite divisibility inverse limit. The following result will be needed in order to characterize the continuum arising from general infinite divisibility inverse systems:

LEMMA 38. *Let $(T, \leq, \mathcal{W}_s, f_{ts})$ be an inverse system satisfying infinite divisibility and let $\mathcal{V}$ be its inverse limit. Then $\mathcal{I}(\mathcal{V})$ is uncountable, and moreover for any $\xi \in \mathcal{D}$, the set $\{\mu \in \mathcal{I}(\mathcal{V}) \mid \mu \preceq \xi\}$ is uncountable.*

PROOF. Let $(T, \leq, \mathcal{W}_s, f_{ts})$ be an infinite divisibility inverse system with inverse limit $\mathcal{V}$, and choose for simplicity a cofinal sequence $\mathcal{S}$ of the form $\mathcal{W}_0 \leftarrow \mathcal{W}_1 \leftarrow \cdots$; the limit $\mathcal{V}'$ of $\mathcal{S}$ is then isomorphic to $\mathcal{V}$. Now choose any $\preceq$-minimal $a \in \mathcal{W}_0$; such an event must exist because $\mathcal{W}_0$ is finite. By infinite divisibility let j be the least index such that there are $c, d \in \mathcal{W}_j$ which split a. Then we have $f_{ji}(e) = a$ for any $e \in \mathcal{W}_j$, $e \preceq a$ and any $i < j$. Since $\mathbb{O}(c, d)$ there must be distinct $\preceq$-minimal events

$a_0 \preceq c, a_1 \preceq d$ such that $f_{ji}(a_0) = f_{ji}(a_1) = a$ for any $i < j$. Define then an initial segment $f_0 : [0, j] \to \bigcup_{i \leq j} \mathcal{W}_i$ by letting $f_0(i) = f_{ji}(a_0)$ for all $i \leq j$, and similarly for f_1. Recursive iteration of this construction yields an uncountable set of threads $\{f_\sigma \mid \sigma \in 2^\omega\}$ of $\mathcal{V}$ such that for any f_σ, we have $\pi_s(f_\sigma)$ is a $\preceq$-minimal event in $\mathcal{W}_s$ for all $s \in T$, i.e., $\mathcal{W}_s \models \forall x(x \preceq \pi_s(f_\sigma) \to \pi_s(f_\sigma) \preceq x)$ for all $s \in T$. We can now apply the right to left direction of Lemma 28 to conclude that f_σ is a $\preceq$-minimal event in $\mathcal{V}'$ for any $\sigma \in 2^\omega$, and we obtain the first part of the claim. The second part of the claim can be most easily seen as follows: consider any thread $\xi \in \mathcal{D}'$. Then there must be a least index i such that $\xi_j = c$ for all $i \geq j$. Consider then the cofinal sequence $\mathcal{S}'$ obtained from $\mathcal{S}$ by taking all $\{\mathcal{W}_j \mid j \geq i\}$, and choose any $\preceq$-minimal $a \preceq c$ in $\mathcal{W}_i$; again, such a must exist by finiteness. Applying the above construction starting from a then yields uncountably many $\preceq$-minimal events covered by ξ. $\dashv$

Since the space of boundaries on an event structure $\mathcal{W}$ is a compact and separable linear order, we can now make use of the characterization of this type of orders which has been given in [Ostaszewski, 1974]. More specifically, for any total order $(L, \leq)$, let us define two equivalence relations $\equiv, \sim \subseteq L \times L$ on L as follows. We let $x \equiv y$ if the cardinality of the set of points between x, y is countable, where a point z is between x, y if $x \leq z \leq y$ or $y \leq z \leq x$. We then let $x \sim y$ if $x = y$ or if $x^{\equiv} = \{x, y\}$. Note that for any two points $x, y \in L$, if $x \sim y$ then (x, y) defines a jump in L, since given any $z \in L$ with $x < z < y$, $x \equiv y$ implies $x \equiv z, z \equiv y$ and this would mean that $z \in x^{\equiv}$, which is impossible. These equivalence relations can be used to characterize the total order of boundaries $\mathcal{B}(\mathcal{V})$ on the inverse limit $\mathcal{V}$:

THEOREM 10. *Let $(T, \leq, \mathcal{W}_s, f_{ts})$ be an inverse system satisfying infinite divisibility and let $\mathcal{V}$ be its inverse limit. Then the following hold:*

(i) *For any $x \in \mathcal{B}(\mathcal{V})$, x has either an immediate predecessor x^-, or an immediate successor x^+, but not both.*
(ii) *For any $x, y \in \mathcal{B}(\mathcal{V})$, if y has no immediate predecessor and $x < y$, then there are uncountably many boundaries between x and y (similarly if y has no immediate successor and $y < x$)*

PROOF. Let $(T, \leq, \mathcal{W}_s, f_{ts})$ be an inverse system satisfying infinite divisibility, $\mathcal{V}$ be its inverse limit, and let $x \in \mathcal{B}(\mathcal{V})$. We first show that x cannot have both an immediate predecessor x^- and an immediate successor x^+. Suppose otherwise. Then $(x^-, x), (x, x^+)$ define jumps and by Lemma 35 there are $\preceq$-minimal events μ, μ' with $\mu \in Past(x) \cap Fut(x^-), \mu' \in Past(x^+) \cap Fut(x)$, hence $\mathbb{O}(\mu, \mu')$. Since $\mathcal{I}(\mathcal{V})$ is dense in $\mathcal{V}$ (Lemma 38), there must be a $\preceq$-minimal event μ'' between μ and μ'; since μ'' cannot be in $Pres(x)$ (Lemma 33), then it is either in $Past(x)$ or in $Fut(x)$. In the former case we obtain a contradiction since then there must be $y \in \mathcal{B}(\mathcal{V})$ with $\mu \in Past(y), \mu'' \in Fut(y)$ and hence x^- is not an immediate predecessor of x; in the latter case the argument is analogous. Hence x cannot have both an immediate predecessor and an immediate successor.

We now show that x must have either an immediate predecessor or an immediate successor. Consider then $Pres(x)$; since it is a pairwise overlapping set of events, it can be extended to a maximal overlapping set of events Σ. The proof of Lemma 34 (i) then implies that there exists a unique (up to equivalence) $\preceq$-minimal event μ which is covered by every event in Σ. Lemma 33 then implies that either $\mu \in Past(x)$ or $\mu \in Fut(x)$. Without loss of generality assume $\mu \in Past(x)$, and consider

$l(\mu) = \bigvee y \in \mathcal{B}(\mathcal{V}) \mid \mu \in Fut(y)$; recall from Proposition 16 that $\mu \in Fut(l(\mu))$. We claim that $l(\mu)$ is an immediate predecessor of x; in this case this means that $x = r(\mu)$. Suppose then that $l(\mu) < r(\mu) < x$. Then by Lemma 34 (ii), there exists $\mu' \in Past(x) \cap Fut(y)$, and hence $\mathbb{O}(\mu, \mu')$. It is now straightforward to check that, since $\mu' \in Past(x)$, then $\mu' \preceq \sigma$ for any $\sigma \in \Sigma$, which is impossble because μ is unique up to equivalence. Hence $x = r(\mu)$ and we are done.

To show (ii) let $x, y \in \mathcal{B}(\mathcal{V})$ be such that $x < y$ and y has no immediate predecessor. We first show that there must be an eventually constant thread $\gamma \in \mathcal{D}$ with $\gamma \in Fut(x) \cap Past(y)$. Indeed, since (x, y) is not a jump there is z with $x < z < y$ and, by the proof of Lemma 36, there are threads ξ, ξ' with $\xi \in Past(z) \cap Fut(x), \xi' \in Past(y) \cap Fut(z)$. Hence by Lemma 37 there must be an eventually constant thread $\gamma \in \mathcal{D}$ between ξ, ξ', and hence $\gamma \in Fut(x) \cap Past(y)$; by Lemma 38 there must then be uncountably many $\preceq$-minimal events covered by γ, which yield uncountably many boundaries between x and y by taking $\{l(\mu) \mid \mu \preceq \gamma\}$. $\dashv$

COROLLARY 4. *Let $(T, \leq, \mathcal{W}_s, f_{ts})$ be an inverse system satisfying infinite divisibility and let $\mathcal{V}$ be its inverse limit. Then $|x^{\equiv}| = 2$ for any $x \in \mathcal{B}(\mathcal{V})$.*

Given an event structure $\mathcal{W}$, we can endow the set of equivalence classes $\mathcal{B}(\mathcal{W})^{\sim} = \{x^{\sim} \mid x \in \mathcal{B}(\mathcal{W})\}$ with a linear order, by letting $x^{\sim} \leq y^{\sim}$ if $x \leq y$ or $x^{\sim} = y^{\sim}$. Note that this linear order is still a complete lattice; the join of a subset $S \subseteq \mathcal{B}(\mathcal{W})^{\sim}$ can be defined as $(\bigvee_{x^{\sim} \in S} x)^{\sim}$, and similarly for the meet. We then have:

THEOREM 11. *Let $(T, \leq, \mathcal{W}_s, f_{ts})$ be an inverse system satisfying infinite divisibility and let $\mathcal{V}$ be its inverse limit. Then $(\mathcal{B}(\mathcal{V})^{\sim}, \leq)$ is homeomorphic to $[0, 1]$.*

PROOF. The order topology of $(\mathcal{B}(\mathcal{V})^{\sim}, \leq)$ is compact because $\leq$ is a lattice. Moreover, we have that $(\mathcal{B}(\mathcal{V})^{\sim}, \leq)$ does not have jumps, i.e., it is dense. One can now check that the set $\{q^{\sim} \mid q \in Q\}$, where Q is the countable set defined in the proof of Lemma 36, is a countable dense set which is also dense in $\mathcal{B}(\mathcal{V})^{\sim}$. Since any countable dense linear order without endpoints is order isomorphic to $(0, 1) \cap \mathbb{Q}$, we have that there is an isomorphism between $Q^{\sim}$ and $(0, 1) \cap \mathbb{Q}$. Because of compactness of $(\mathcal{B}(\mathcal{V})^{\sim}, \leq)$ this isomorphism can be extended to an isomorphism between $(\mathcal{B}(\mathcal{V})^{\sim}, \leq)$ and $[0, 1]$. $\dashv$

The result above shows that we can recover the unit interval $[0, 1]$ by means of the set of equivalence classes $\mathcal{B}(\mathcal{V})^{\sim}$. This result, however, obliterates the additional structure contained in $\mathcal{B}(\mathcal{V})$. Indeed, $\mathcal{B}(\mathcal{V})$ can be seen as splitting any point x of $[0, 1]$ into two parts, which have a $\preceq$-minimal event in between. Indeed, again following [Ostaszewski, 1974], we obtain:

COROLLARY 5. *Let $(T, \leq, \mathcal{W}_s, f_{ts})$ be an infinite divisibility inverse system and let $\mathcal{V}$ be its inverse limit. Then $\mathcal{B}(\mathcal{V})$ is isomorphic to $[0, 1] \times \{0, 1\}$ with the lexicographic ordering.*

PROOF. The claim follows from Theorem 11, Lemma and the main Theorem of [Ostaszewski, 1974]. $\dashv$

10.2. Inexhaustibility and connectedness. It is to be noted at this point that the above results imply that the order topology on $\mathcal{B}(\mathcal{V})$ is disconnected: indeed, any jump (x, y) defines a decomposition $(y, \leftarrow), (x, \rightarrow)$ of the space. This result would rightly seem to the reader to be at odds with Kant's insistence on the high degree of connectivity of the temporal continuum.

One may observe, however, that the discrepancy arises only because the structure $\mathcal{B}(\mathcal{V})$ disregards completely the role of the $\preceq$-minimal events, which, as we shall see in Section 11.2, provide an accurate formalization of Kant's infinitesimals, and which act as a 'glue' keeping the Walker boundaries together. Indeed, any two boundaries of $\mathcal{B}(\mathcal{V})$ which correspond to the splitting of a real number in $[0,1]$ have such a $\preceq$-minimal event in between. Philosophically, the abundance of $\preceq$-minimal events in $\mathcal{V}$ points to the inexhaustibility of the continuum; indeed, the limit $\mathcal{V}$ is not an absolute endpoint of potential infinite divisibility, since each $\preceq$-minimal event represents further endless possibilities of division. This picture of the continuum is very much in keeping with Brouwer's view on the matter; in his dissertation, Brouwer describes the "ur-intuition", the basic intuition of all of mathematics, in the following terms:

> [The ur-intuition is] the substratum of all perception of change, which is divested of all quality, a unity of continuous and discrete, a possibility of the thinking together of several units, connected by a "between", which never exhausts itself by the interpolation of new units.[29]

The same principles are echoed in Hermann Weyl's writings on the continuum; in [Weyl, 1932], for instance, we find the following:

> 1. An individual point in [a continuum] is non-independent, i.e., is pure nothingness when taken by itself, and exists only as a 'point of transition' (which, of course, can in no way be understood mathematically);
> 2. it is due to the essence of time (and not to contingent imperfections in our medium) that a fixed temporal point cannot be exhibited in any way, that always only an approximate, never an exact determination is possible.

If the $\preceq$-minimal events of $\mathcal{V}$ are taken into account in the construction of the continuum, we also obtain quite a different picture with regards to connectedness:

DEFINITION 30. *Let $\mathcal{W}$ be a model of GT, and define $X = \mathcal{B}(\mathcal{W}) \cup \mathcal{I}(\mathcal{W})$. Then the total orders on $\mathcal{B}(\mathcal{W}), \mathcal{I}(\mathcal{W})$ can be extended to a total order on X, by letting, for any $x \in \mathcal{B}(\mathcal{W}), y \in \mathcal{I}(\mathcal{W})$, $x \leq y$ iff $y \in Fut(x)$, and $y \leq x$ iff $y \in Past(x)$. We define a topology τ on X by letting τ be the topology induced by the order topology of $(\mathcal{B}(\mathcal{W}), \leq)$. We denote the linearly ordered topological space $(X, \leq, \tau)$ so defined as $\mathcal{C}(\mathcal{W})$.*

LEMMA 39. *Let $\mathcal{W}$ be an event structure and let $\mathcal{C}(\mathcal{W})$ as in Definition 30. Then any point $x \in X$ such that $x \in \mathcal{I}(\mathcal{W})$ is open, and any point $x \in X$ such that $x \in \mathcal{B}(\mathcal{W})$ is closed.*

Lemma 39 then yields the following result[30]:

[29] For the reference of this quote and an interesting treatment of Brouwer's "flowing" continuum, see [van Dalen, 2009].

[30] The proof of Lemma 40 relies on concepts from a field of topology called digital topology [Kong and Rosenfeld, 1989]. Let us recall that given a topological space (X, τ) we denote with $\lhd$ its specialization ordering. A *selective space* is then a T_0 linearly ordered topological space $(X, \leq, \tau)$ such that τ has a subbasis of rays and such that $x \lhd y$ only if $x = y$ or x, y are adjacent in the ordering $\leq$. Selective spaces are studied in detail in [Kopperman et al., 1998], and originate from the study of the connected ordered topological spaces, also called COTS. A COTS is a topological space such that for any three points in the space, one is a cut point separating the other two; COTS were originally studied by Khalimsky, Kopperman

LEMMA 40. *Let $\mathcal{W}$ be an event structure. Then $\mathcal{C}(\mathcal{W})$ is a compact connected totally ordered topological space such that for any three points, one is a cut point separating the other two.*

The results above show that if we interpolate the linear order of $\preceq$-minimal events in the linear order of boundaries $\mathcal{B}(\mathcal{W})$ we obtain a compact connected totally ordered topological space, which is a close formal correlate of the Kantian continuum. We can however obtain an even closer formal rendition of Kant's time as formal intuition if we take the 'flowing magnitudes" in consideration.

§11. Duration, infinitesimals, magnitudes, external representation. In this section we formally introduce what we call the Kantian continuum, in which time is represented as an object both in inner sense and in outer sense, by structures which are very different yet intimately related. An important reason why time must be represented as an object, i.e. as formal intuition, is the need to possess a substrate on which continuous ('flowing') magnitudes can be defined, as well as metrics which uniformly assign duration to events. Michael Friedman [Friedman, 1992] has rightly pointed out that the expression 'flowing magnitude' ('*fliessende Grösse*') should be taken in the sense of Newton's *fluents*, independent variables which nonetheless vary continuously with time, and which therefore can be viewed as motions. Some of these are transcendental, e.g. drawing a line that is the external representation of time, and these motions are only required to be continuous. As a geometrical construction, drawing a line occurs in time as inner sense, as a function on the space of boundaries. These boundaries live on an inverse limit constructed under the guidance of the transcendental unity of apperception. This is time as inner sense, from which we somehow have to fashion the external representation of time in outer sense.

As we have seen, there exist very many jumps between boundaries, which seems at odds with the notion of a flowing magnitude, at least when the continuum is considered to be composed of boundaries; but this plethora of jumps turns out to be a blessing in disguise, because jumps are intrinsically bound up with infinitesimals, analogous to those used in synthetic differential geometry [**?**]. And, we will argue, a geometric line with infinitesimals is an excellent candidate for what Kant called the 'external representation of time':
(i) the continuity of drawing is represented by 'micro-continuity': an infinitesimal increment in time produces an infinitesimal displacement; this comes much closer to capturing the notion of flow than the static $\epsilon - \delta$ definition
(ii) combining theorem 12 below with the material on events as open intervals from section 6 leads one to view events as determined by diffuse boundaries on the geometric line; as Kant wrote: 'the instant in time can be filled' – this precisely what the construction given below does. Furthermore, the partial external representations of time (event

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and others (see [Khalimsky et al., 1990]). Any COTS which is locally connected and T_0 is a selective space such any point is either open or closed; conversely, any selective space such that any point is either closed or open is a COTS. Thus, the continuum we obtain in Definition 30 is a COTS. This fact highlights that there is a close relation between the theory of well formed ordered spaces (of which COTS are a special case), as studied by [Kopperman et al., 1998], and event structures as studied in this paper. Indeed, the axiom system GT can be understood as providing a first order theory for well formed spaces; the operations $\otimes_p, \otimes_f$, for instance, axiomatize the property of well formed spaces that upsets and downsets of open sets are open (see [Kopperman et al., 1998], Lemma 3.3.).

structures as introduced in section 2) can be unified in a single representation. We note here that time represented in outer sense – the drawing of a line – proceeds under the guidance of the transcendental synthesis of the imagination, or *synthesis speciosa*. The letter to Rehberg nicely captures the interplay between inner and outer sense:

> The necessity of the connection of both sensible forms, space and time, in the determination of the objects of our intuition – so that time, if the subject makes it itself an object of its representation, must be represented as a line in order to cognize it as quantum, just as, conversely, a line can only be thought as a quantum by being constructed in time . . .

Of especial importance is the remark that a quantum of time can be determined only via the external representation of time, which defines the unit which in turn determines the quantum (or 'magnitude') of time.

11.1. External representation of time. To construct the inner and outer components of the Kantian continuum, we will represent the inverse limit (time as inner sense) in a ring with infinitesimals extending the reals: the so-called *dual numbers* (external representation of time). We start with the latter.

Consider the polynomial ring $\mathbb{R}[X]$ and take the quotient $\mathbb{R}[X]/(X^2)$ of $\mathbb{R}[X]$ by the ideal (X^2). In this structure, $X^2 = 0$ and each element can be written as $q + Xr$, for real q, r, hence dual numbers are a 2-dimensional vector space over $\mathbb{R}$. Multiplication is given by

$$(q + Xr)(s + Xt) = qs + X(rs + qt).$$

Elements of the form Xr $(r > 0)$ satisfy $(Xr)^2 = 0$; such elements will be called *nilpotents* or *infinitesimals*. Multiplication, as well as vector addition and scalar multiplication, are continuous, hence $\mathbb{R}[X]/(X^2)$ is a topological vector space.[31] For our purposes we need a different topology on $\mathcal{R} := \mathbb{R}[X]/(X^2)$, generated by pre-orders $\leq, \geq$ which are compatible with the ring operations and leave the position of the nilpotents undecided. We only give the properties of $\leq$ (which is analogous to $\mathbb{E}$, while $\geq$ is analogous to $\mathbb{B}$)

1. $x \leq y$ implies $x + z \leq y + z$
2. $x \leq y$, $0 \leq r$ implies $xr \leq yr$
3. $0 \leq 1$
4. if e is nilpotent, $0 \leq d \wedge d \leq 0$

LEMMA 41. *(i) $\leq$ is reflexive, transitive, not anti-symmetric*
(ii) if d, e are nilpotent, then by transitivity $e \leq d \wedge d \leq e$,
(iii) more generally, if r is a real, d, e nilpotent, then $r + e \leq r + d \wedge r + d \leq r + e$,
(iv) $\leq$ restricted to the reals is a linear order

Each real thus has a cloud of infinitesimals surrounding it; these will be seen to correspond to boundaries on the inverse limit.

DEFINITION 31. *The Alexandroff topology τ on $\mathcal{R}$ is generated by the two classes of downsets $\{c \mid c \leq b\}, \{d \mid d \geq a\}$. The open intervals $\ll r, s \gg$ are defined as the*

[31] The multiplicative inverse $\frac{1}{X}$ of X is not defined, otherwise

$$\frac{1}{X}^{2} = \frac{1}{0} \in \mathbb{R}.$$

Hence $\mathbb{R}[X]/(X^2)$ is not a field.

intersection $\{c \mid c \leq s\} \cap \{d \mid d \geq r\}$. *It makes sense to call these intervals open, not closed, because the endpoints* r, s *are actually clouds of infinitesimally close points without a well-determined position.*

LEMMA 42. $(\mathcal{R}, \tau)$ *is connected, non-Hausdorff, not second countable, and no point is a cutpoint.*

11.2. Duration and persistence: from external representation to inner sense. Duration is first determined on the external representation, by a semi-norm on $(\mathcal{R}, \tau)$:

DEFINITION 32. *Let* $x = a + d, y = c + e$ *(*d, e *nilpotent), then the* semi-norm $\| x - y \|$ *is defined by*

$$\sqrt{(a-c)^2 + k^2d^2} = \sqrt{(a-c)^2};$$

here we use that in $\mathcal{R}$ *any nilpotent can be represented as* rX, r *a real.*

LEMMA 43. $\| x - y \|$ *satisfies*
(i) if r *is a real, then* $\| xr - yr \| = r \| x - y \|$ *(homogeneity)*
(ii) if $z = a + e$, e *nilpotent,* $\| (x - z) - (y - z) \| = \| x - y \|$ *(translation invariance)*
(iii) $x = 0$ *implies* $\| x \| = 0$, *but not conversely (as one would require for a norm).*

DEFINITION 33. *The semi-norm* $\| x - y \|$ *defines a homogeneous translation invariant pseudo-metric on* $\mathcal{R}$ *by putting* $\delta(x, y) = \| x - y \|$. *The topology generated by* δ *is coarser than the topology* τ, *which is not pseudo-metrisable.*

We use δ to generate a pseudo-metric on $\mathcal{B}(\mathcal{V})$. We have shown above that $\mathcal{B}(\mathcal{V})$, the space of boundaries, supports an equivalence relation $\sim$ each of whose equivalence classes is ordered and has two elements. Let q be a quotient map such that for each equivalence class $\{x, y\}$ with $x < y$ $q(x) = q(y) \in \mathcal{J}$, where $\mathcal{J}$ is a structure homeomorphic to the closed unit interval. In the following, we shall disregard the endpoints, since the *open* unit interval is homeomorphic to $\mathbb{R}$ via a map η.

DEFINITION 34. *Given the pseudo-metric* δ, *define the pseudo-metric* $\rho(x, z)$ *on* $\mathcal{B}(\mathcal{V})$ *by*

$$\rho(x, z) = \delta(\eta(q(x)), \eta(q(z))) = \| \eta(q(x)) - \eta(q(z)) \| .$$

ρ *can be extended to* $\mathcal{B}(\mathcal{V}) \cup \mathcal{I}(\mathcal{V})$ *by putting, for any jump* $x < y$ *and mutually covering* $\preceq$*-minimal events* z *berween* x *and* y: $\rho(x, z) = \rho(y, z) = 0$.

$\mathcal{B}(\mathcal{V})$ is not connected, which renders the expression $\rho(x, z)$ meaningless if x, z are in disconnected components, since disconnected components do not have a determinate temporal distance; therefore one needs $\mathcal{B}(\mathcal{V}) \cup \mathcal{I}(\mathcal{V})$. Kant expresses this by saying: ‘For in mere sequence alone existence is always disappearing and beginning, and never has the least magnitude. (A177/B219)’ In other words, duration presupposes persistence.

11.3. From time as inner sense to external representation. We have shown above that $\mathcal{B}(\mathcal{V})$, the space of boundaries, supports an equivalence relation $\sim$ each of whose equivalence classes $\{x, y\}$ is ordered (say as $x < y$) Let $q : \mathcal{B}(\mathcal{V}) \longrightarrow \mathcal{J}, \eta : \mathcal{J} \longrightarrow \mathcal{R}$ be as above. In between any $x < y$ there are mutually covering $\preceq$-minimal events. There exists an injective map ι that maps a $\preceq$-minimal event between x and y (with $\eta(q(x)) = \eta(q(y)) = r$) to some $r + e \in \mathcal{R}$, where e is nilpotent. This map is structure preserving in the sense that

$$\iota(\mu) \preceq \iota(\nu) \wedge \iota(\mu) \succeq \iota(\nu), \Rightarrow \iota(\mu) \geq \iota(\nu) \wedge \iota(\mu) \leq \iota(\nu).$$

11.4. Main theorem. The preceding considerations are summarised in the following

THEOREM 12. *((i)Consider the structure $\mathcal{B}(\mathcal{V}) \cup \mathcal{I}(\mathcal{V})$ obtained from the inverse limit $\mathcal{V}$, where $\mathcal{B}(\mathcal{V})$ is the set of boundaries (with relations $\leq, \geq$) and $\mathcal{I}(\mathcal{V})$ the set of $\preceq$-minimal events (with relations $\preceq, \succeq$). Then there exists an 'almost injective' homomorphism*

$$h : \mathcal{B}(\mathcal{V}) \cup \mathcal{I}(\mathcal{V}) \longrightarrow \mathcal{R},$$

i.e. h is injective except that it maps the extremities of a jump to a single real.
(ii) Conversely, duration as defined by the semi-norm on $\mathcal{R}$ can be lifted to $\mathcal{B}(\mathcal{V}) \cup \mathcal{I}(\mathcal{V})$.

$\mathcal{R}$ is the external representation of time viewed as completed. for this construction to work, we need time as an object, synthesised under the guidance of the unity of apperception, issuing in an inverse limit.

The actual construction of $\mathcal{R}$ can be represented by means of flowing magnitudes. Here, we remind the reader of the discussion of 'inexhaustibility' in section 10.2: $\preceq$-minimal events are not minimal in any absolute sense: they can be split in event structures projecting to the inverse limit. As a consequence, $\preceq$-minimal events that mutually cover each other (are 'simultaneous') may no longer be simultaneous in a refining retraction.

We are now in a position to formalise Kant's notions of 'flowing magnitude'. The first notion represents the drawing of a line in pure intuition.

DEFINITION 35. *A flowing magnitude in the sense of B155n is an 'almost injective' function*[32]

$$f : \mathcal{B}(\mathcal{V}) \cup \mathcal{I}(\mathcal{V}) \longrightarrow \mathcal{R}$$

which is continuous w.r.t. the Alexandroff topologies .

These flowing magnitudes map $\preceq$-minimal events to infinitesimals in $\mathcal{R}$ and are 'micro-continuous' in the sense that function values are infinitesimally close for infinitesimally close arguments.

Our next definition introduces flowing in the sense of actual physical movement:

DEFINITION 36. *A flowing magnitude in the sense of the Metaphysical foundations is an 'almost injective' function*

$$f : \mathcal{B}(\mathcal{V}) \cup \mathcal{I}(\mathcal{V}) \longrightarrow \mathcal{R}$$

which is continuous and is 'infinitesimally linear: if $x < y$ is a jump then $f(x) = f(y) \in \mathbb{R}$ and there exists a unique real b such that for all $\preceq$-minimal events ν between x and y,

$$f(\nu) = f(x) + b.\nu.$$

b of course represents the derivative f' of f at x.

These magnitudes provide an important bridge between the first *Critique* and the *Metaphysical foundations of natural science*. B155n announces

[32] See theorem 12 for the meaning of 'almost injective'.

> Motion of an object in space does not belong in a pure science, thus also not in geometry; for that something is movable cannot be cognized a priori but only through experience. But motion, as description of a space, is a pure act of the successive synthesis of the manifold in outer intuition in general through the productive imagination, and belongs not only to geometry but even to transcendental philosophy.

and this point is developed in the first chapter ('Phoronomy') of the *Metaphysical foundations*:

> In phoronomy, since I am acquainted with with matter through no other property but its movability, and may thus consider it only as a point, motion can only be considered as the describing of a space - in such a way, however, that I attend not solely, as in geometry, to the space described, but also to the time in which, and thus to the speed with which, a point describes the space.

Surprisingly, speed, hence differentiation, is treated in the *Metaphysical foundations* with a liberal sprinkling of infinitesimals. While discussing the change in velocity of an object projected upward from a point A, and reversing direction of motion at point B, Kant raises the question whether the object can be said to be at rest at point B, and answers affirmatively, with the following argument:

> The reason for this lies in the circumstance that the motion [of this object] is not thought of as uniform at a given speed but rather first as uniformly slowed down and thereafter as uniformly accelerated. Thus, the speed at point B is not completely diminished, but only to a degree that is smaller than any given speed. With this speed, therefore, the body would, if it were to be viewed always as still rising ... uniformly traverse with a mere moment of speed (the resistance of gravity here being set aside) a space smaller that any given space in any given time no matter how large. And hence it would absolutely not change its place (for any possible experience) in all eternity. It is therefore put into a state of enduring presence at the same place – i.e., of rest – even though this is immediately annulled because of the continual influence of gravity (i.e., the change of this state). (Ak 4:486)

Let q be a magnitude in the sense of definition 36, representing the height of the object, so that q' represents the speed. We have $q(a + b\epsilon) = q(a) + q'(a)b\epsilon$. Kant claims that at the turning point B the speed $q'(B)b\epsilon$ is a non-zero infinitesimal representing the 'mere moment of speed' and $q'(a)b\epsilon$ the distance traversed in real time b: 'a space smaller that any given space in any given time no matter how large.' If we equate'possible experience (of motion)' with the pseudo-metric, then since $\| q'(B)b\epsilon \| = 0$, $\delta(q(B + b\epsilon), q(B)) = 0$ for all b, and we have 'hence it would absolutely not change its place (for any possible experience) in all eternity. It is therefore put into a state of enduring presence at the same place – i.e., of rest.'

§12. Further work. We shall conclude this paper by mentioning briefly some salient directions for future work.

A first problem is that of providing a full-fledged formalization of Kant's "threefold" synthesis of appearances, and of the role of the categories in synthesizing such appearances. This requires taking into account both the objective temporal order of appearances and the role which concepts play in constituting and re-identifying such appearances. We have touched upon these issues in section 2.1, where it was seen that the relations and the axioms characterizing objective time can be justified from a Kantian standpoint if we conceive of events as tenures of concepts. In order to make the argument formally precise, thus providing an account of the role of the categories in the constitution of (temporal) experience, one needs to augment the present formal system with ways to deal with concepts and logical relations among these concepts, so that events might be distinguished intensionally.

A detailed formalization of the synthesis by the categories might also shed light on the relation between our formalization and works in the foundations of relativity theory along the lines of [Walker, 1948]. In particular, if events are conceived as tenure events of attributes of substances, and are thus indexed by these substances, then temporal comparability and linearity are unproblematically satisfied only by sets of events indexed by the same substance, as we pointed out in 2.1[33]. In order to extend the validity of these axioms to arbitrary events, the category of community must be invoked, ensuring the *instantaneous* interaction of substances and thus the temporal comparability of their attribute events. This general setup bears a strong resemblance with the formal theory in [Walker, 1948], and we believe that the study of the relation between these two frameworks might allow one to recast Kant's theory of space and time as a Newtonian, "time-slice" causal theory of spacetime[34].

A similar problem is that of relating our formalization with contemporary work in the foundations of relativistic physics within the causal set tradition[35]. In [Sorkin, 1991] Sorkin showed that any compact Hausdorff space can be approximated by finite T_0 spaces and inverse limits. Similar results were obtained independently by Kopperman and Wilson in [Kopperman and Wilson, 1997], working in the field of digital topology, and much earlier by Flachsmeyer [Flachsmeyer, 1961]. Examining the way in which these authors approximate compact Hausdorff spaces by means of finite T_0 spaces, and in particular how the unit interval is approximated in [Kopperman and Wilson, 1997], one notices a close relation with our infinite divisibility inverse systems. It is then natural to ask whether it is possible to generalize the construction presented in this paper to approximate by means of inverse systems more general classes of compact Hausdroff spaces. We believe that the most promising route to achieve such a result is to relate the present work to the theory of locales in point-free topology [**?**]. We plan to address this question, along with its application to the foundations of relativity, in a future work.

REFERENCES

[Achourioti and van Lambalgen, 2011]Achourioti, T. and van Lambalgen, M. (2011). A formalization of kant's transcendental logic. *The Review of Symbolic Logic*, 4(02):254–289.

[Arthur, 1995]Arthur, R. T. (1995). Newton's fluxions and equably flowing time. *Studies in History and Philosophy of Science Part A*, 26(2):323–351.

[Banaschewski and Pultr, 2010]Banaschewski, B. and Pultr, A. (2010). Approximate maps, filter monad, and a representation of localic maps. *Archivum Mathematicum*, 46(4):285–298.

[Barrow, 1976]Barrow, I. (1976). *Lectiones geometricae*. Georg Olms Verlag.

[Carelli and Forman, 2012]Carelli, M. G. and Forman, H. (2012). Representation of multiple durations in children and adults. *Child Development Research*, 2011.

[Christensen and Crane, 2005]Christensen, J. D. and Crane, L. (2005). Causal sites as quantum geometry. *Journal of mathematical physics*, 46(12):122502.

[33]The quote 1.1.4 by Isaac Barrow presented in section 1 captures the essential point regarding these axioms.

[34]For a treatment of causal theories of spacetime, see Winnie's [Winnie, 1977].

[35]See [Reid, 1999] for an introduction to causal sets.

[Feferman, 2009]Feferman, S. (2009). Conceptions of the continuum. *Intellectica*, 51:169–189.

[Flachsmeyer, 1961]Flachsmeyer, J. (1961). Zur spektralentwicklung topologischer räume. *Mathematische Annalen*, 144(3):253–274.

[Forbes, 1987]Forbes, G. (1987). Places as possibilities of location. *Nous*, pages 295–318.

[Friedman, 1992]Friedman, M. (1992). *Kant and the exact sciences*. Harvard University Press.

[Friedman, 2007]Friedman, W. J. (2007). The meaning of "time" in episodic memory and mental time travel. *Behavioral and Brain Sciences*, 30(03):323–323.

[Futch, 2008]Futch, M. (2008). *Leibniz's metaphysics of time and space*, volume 258 of *Boston Studies in the Philosophy of Science*. Springer Science & Business Media.

[Grunbaum, 1977]Grunbaum, A. (1977). Absolute and relational theories of space and space-time. *Minnesota Studies in the Philosophy of Science Minneapolis, Minn.*, 8:303–373.

[Hodges, 1997]Hodges, W. (1997). *A shorter model theory*. Cambridge university press.

[Kamp, 1979]Kamp, H. (1979). Events, instants and temporal reference. In *Semantics from different points of view*, pages 376–418. Springer.

[Khalimsky et al., 1990]Khalimsky, E., Kopperman, R., and Meyer, P. R. (1990). Computer graphics and connected topologies on finite ordered sets. *Topology and its Applications*, 36(1):1–17.

[Kong and Rosenfeld, 1989]Kong, T. Y. and Rosenfeld, A. (1989). Digital topology: Introduction and survey. *Computer Vision, Graphics, and Image Processing*, 48(3):357–393.

[Kopperman et al., 1998]Kopperman, R., Kronheimer, E., and Wilson, R. (1998). Topologies on totally ordered sets. *Topology and its Applications*, 90(1):165–185.

[Kopperman and Wilson, 1997]Kopperman, R. and Wilson, R. (1997). Finite approximation of compact hausdorff spaces. In *Topology Proceedings*, volume 22, page 175.

[Longuenesse, 1998]Longuenesse, B. (1998). *Kant and the Capacity to Judge*. Princeton University Press.

[Maier, 1966]Maier, A. (1966). *Die Vorläufer Galileis im 14. Jahrhundert*, volume 1. Edizioni di Storia e letteratura.

[Manders, 1982]Manders, K. L. (1982). On the space-time ontology of physical theories. *Philosophy of Science*, pages 575–590.

[McGuire, 1978]McGuire, J. E. (1978). Newton on place, time, and god: An unpublished source. *British Journal for the History of Science*, 11(2):114–129.

[Mundy, 1983]Mundy, B. (1983). Relational theories of euclidean space and minkowski spacetime. *Philosophy of Science*, pages 205–226.

[Newton, 1999]Newton, I. (1999). *The Principia: mathematical principles of natural philosophy*. Univ of California Press.

[Northoff, 2012]Northoff, G. (2012). Immanuel kant's mind and the brain's resting state. *Trends in cognitive sciences*, 16(7):356–359.

[Onof and Schulting, 2015]Onof, C. and Schulting, D. (2015). Space as form of intuition and as formal intuition: On the note to b160 in kant's critique of pure reason. *Philosophical Review*, 124(1):1–58.

[Ostaszewski, 1974]Ostaszewski, A. J. (1974). A characterization of compact, separable, ordered spaces. *Journal of the London Mathematical Society*, 2(4):758–760.

[Reid, 1999]Reid, D. D. (1999). Introduction to causal sets: an alternate view of spacetime structure. *arXiv preprint gr-qc/9909075*.

[Russell, 1914]Russell, B. (1914). *Our Knowledge of the External World as a Field for Scientific Method in Philosophy*. Routledge.

[Russell, 1936]Russell, B. (1936). On order in time. *Proc. Camb. Phil. Soc.*, 32:216–228.

[Sachs, 1995]Sachs, J. (1995). *Aristotle's Physics: a guided study*. Rutgers University Press.

[Sorkin, 1991]Sorkin, R. D. (1991). Finitary substitute for continuous topology. *International Journal of Theoretical Physics*, 30(7):923–947.

[Thomason, 1989]Thomason, S. K. (1989). Free construction of time from events. *Journal of Philosophical Logic*, 18(1):43–67.

[van Dalen, 2009]van Dalen, D. (2009). The return of the flowing continuum. *Intellectica*, 1:51.

[Walker, 1947]Walker, A. G. (1947). Durées et instants. *Revue Scientifique*, 85:131–134.

[Walker, 1948]Walker, A. G. (1948). Foundations of relativity, parts i and ii. *Proc. Royal Soc. Edinburgh (Section A)*, 62:319–335.

[Weyl, 1932]Weyl, H. (1932). *Das Kontinuum: Kritische Untersuchungen über die Grundlagen der Analysis*. W. de Gruyter.

[Winnie, 1977]Winnie, J. A. (1977). The causal theory of space-time. In Earman, J., Glymour, C., and Stachel, J., editors, *Foundations of Space-Time Theories*. University of Minnesota Press.

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